Numerical PDE

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References

(1) Finite element method, by Axelsson and Barker

(2) Clae Johnson

(3) Iterative methods for solving linear systems by Anne GreenBaum

(4) Bramble, et al.

(5) Widlund et al.

Chapter 1

Finite Difference Method

1.1 2nd order linear p.d.e. in two variables

General 2nd order linear p.d.e. in two variables is given in the following form:

$$L[u] = Au_{xx} + 2Bu_{xy} + Cu_{yy} + Du_x + Eu_y + Fu = G$$

According to the relations between coefficients, the p.d.es are classified into 3 categories, namely,

elliptic	if $AC - B^2$	> 0	i.e., A, C has the same sign
hyperbolic	if $AC - B^2$	< 0	
parabolic	if $AC - B$	$= 0$	

Furthermore, if the coefficients A, B and C are constant, it can be written as

$$[\frac{\partial}{\partial x}, \frac{\partial}{\partial y}] \begin{bmatrix} A & B \\ B & C \end{bmatrix} \begin{bmatrix} \frac{\partial u}{\partial x} \\ \frac{\partial u}{\partial y} \end{bmatrix} + Du_x + Eu_y + Fu = G.$$

Auxiliary condition

$$\begin{cases} \text{B.C.} \\ \text{Interface Cond} \\ \text{I.C.} \end{cases}$$

we say "well posed" if a solution exists. There are basically two class of method to discretize it,

(1) Finite Difference method

(2) Finite Element method

1.2 Finite difference operator

Let $u(x)$ be a function defined on $\Omega \subset \mathbb{R}^n$. Let $U_{i,j}$ be the function defined over discrete domain $\{(x_i, y_j)\}$ such that $U_{i,j} = u(x_i, y_j)$. Such functions are called grid functions.
We introduce some operators on the grid functions.
Shift operator.

$$S^+U_i = U_{i+1}, \quad S^-U_i = U_{i-1}, \quad S^+S^+U_i = (S^+)^2U_i = U_{i+2}$$

In case of two variables,

$$S_x^+U_{i,j} = U_{i+1,j}, \quad S_x^-U_{i,j} = U_{i-1,j}$$

$$S_x^+S_y^-U_{i,j} = U_{i+1,j-1}.$$

Averaging operator

$$\begin{aligned}
\mu^+U_i &= \frac{U_{i+1}+U_i}{2}, \quad \text{right average} \\
\mu^-U_i &= \frac{U_i+U_{i-1}}{2}, \quad \text{left average} \\
\mu^0U_i &= \frac{U_{i+1}+U_{i-1}}{2}, \quad \text{central average}
\end{aligned}$$

Difference operator

$$\begin{aligned}
\delta^+U_i &= \frac{U_{i+1}-U_i}{h_{i+1}}, \quad \text{forward difference} \\
\delta^-U_i &= \frac{U_i-U_{i-1}}{h_i}, \quad \text{backward difference} \\
\delta^0U_i &= \frac{U_{i+1}-U_{i-1}}{h_i+h_{i+1}}, \quad \text{central difference} \\
\delta^2U_i &= \frac{2(\delta^+-\delta^-)}{h_i+h_{i+1}}, \quad \text{central 2nd difference}
\end{aligned}$$

Difference equation is an equation of type $F(\delta U_i, U_i) = 0$.
Difference scheme is a set of difference equations from which one can determine the unknown grid functions.

Example 1.2.1. Consider the following second order differential equation :

$$-u''(x) = f(x), u(a) = c, u(b) = d.$$

Given a mesh $a = x_0 < x_1 < \cdots < x_N = b, \Delta x_i = x_{i+1} - x_i = h$, we have

$$-\frac{U_{i-1} - 2U_i + U_{i+1}}{h^2} = f_i = f(x_i), \quad i = 1, \cdots N-1, U_0 = c, U_N = d$$

which determines U_i uniquely. We obtain an $(N-1) \times (N-1)$ matrix equations.

$$\begin{pmatrix} 2 & -1 & & & & \\ -1 & 2 & -1 & & & \\ & & \cdot & \cdot & \cdot & \\ & & & -1 & 2 & -1 \\ & & & & -1 & 2 \end{pmatrix} \begin{pmatrix} U_1 \\ \cdot \\ \cdot \\ \cdot \\ U_{N-1} \end{pmatrix} = h^2 \begin{pmatrix} f_1 \\ \cdot \\ \cdot \\ \cdot \\ f_{N-1} \end{pmatrix} + \begin{pmatrix} c \\ 0 \\ 0 \\ 0 \\ d \end{pmatrix}$$

Above equation can be written as $L_h U^h = F^h$, called a difference equation for a given differential equation.

Exercise 1.2.2. Write down a matrix equation for the same problem with second boundary condition changed to the normal derivative condition at b, i.e, $u'(b) = d$.

Example 1.2.3 (Heat equation). We consider

$$\begin{aligned} u_t &= \sigma u_{xx}, \quad \text{for } 0 < x < 1, \quad 0 < t < T \\ u(t,0) &= u(t,1) = 0 \\ u(0,x) &= g(x), \quad g(0) = g(1) = 0 \end{aligned}$$

Let $x_i = ih, i = 0, \cdots, N, \Delta x = 1/N$ and $t_n = n\Delta t, \Delta t = \frac{T}{J}$. Then we have the following difference scheme

$$\frac{U_i^{n+1} - U_i^n}{\Delta t} = \sigma \left[\frac{U_{i-1}^n - 2U_i^n + U_{i+1}^n}{\Delta x^2} \right],$$

for $i = 1, 2, \cdots, N-1$ and $n = 1, 2, \cdots, M-1$ where $U_i^n \approx u(t_i, x_n)$. From the boundary condition and initial condition we have

$$U_i^0 = g(x_i), U_0^n = 0, U_N^n = 0.$$

$$U_i^{n+1} = U_i^n + \frac{\sigma \Delta t}{\Delta x^2} \left[U_{i-1}^n - 2U_i^n + U_{i+1}^n \right].$$

In vector notation

$$U_h^{n+1} = U_h^n - \frac{\sigma \Delta t}{\Delta x^2} A U_h^n$$

where A is the same matrix as in example 1. If $n = 0$, right hand side is known. Thus

$$U_h^n = (I - \sigma \frac{\Delta t}{\Delta x^2} A)^n G, \quad G = (g(x_1), \cdots, g(x_{N-1}))^T.$$

This is called **forward Euler** or **explicit scheme**. If we change the right hand side to

$$\begin{aligned}\frac{U_i^{n+1}-U_i^n}{\Delta t} &= \sigma\left[\frac{U_{i-1}^{n+1}-2U_i^{n+1}+U_{i+1}^{n+1}}{\Delta x^2}\right]\\ U_i^{n+1} &= U_i^n+\frac{\sigma\Delta t}{\Delta x^2}\left[U_{i-1}^{n+1}-2U_i^{n+1}+U_{i+1}^{n+1}\right].\end{aligned}$$

$$(I+\sigma\frac{\Delta t}{\Delta x^2}A)^n U_h^n = G, \quad G=(g(x_1),\cdots,g(x_{N-1}))^T.$$

This is called **backward Euler** or **implicit scheme**.

1.2.1 Error of difference operator

For $u \in C^2$, use the Taylor expansion about x_i

$$\begin{aligned}u_{i+1} &= u(x_i+h_i)=u(x_i)+h_i u'(x_i)+\frac{h_i^2}{2}u''(\xi), \quad \xi\in(x_i,x_{i+1})\\ \therefore \frac{u_{i+1}-u_i}{h_i}-u'(x_i) &= \frac{h_i}{2}u''(\xi).\end{aligned}$$

Expand about x_{i+1},

$$u_i = u_{i+1}-h_i u'(x_i)+\frac{h_i}{2}u''(\theta).$$

These are first order accurate. To derive a second order scheme, expand about $x_{i+1/2}$,

$$\begin{aligned}u_{i+1} &= u_{i+1/2}+\frac{h_i}{2}u'(x_{i+1/2})+\frac{1}{2}(\frac{h_i}{2})^2u''(x_{i+1/2})+\frac{1}{6}(\frac{h_i}{2})^3u^{(3)}(\xi)\\ u_i &= u_{i+1/2}-\frac{h_i}{2}u'(x_{i+1/2})+\frac{1}{2}(\frac{h_i}{2})^2u''(x_{i+1/2})-\frac{1}{6}(\frac{h_i}{2})^3u^{(3)}(\xi).\end{aligned}$$

Subtracting,

$$\frac{u_{i+1}-u_i}{h_i}=u'(x_{i+1/2})+O(h_i^2).$$

Thus we obtain a second order approximation to $u'(x_{i+1/2})$. By translation, we have

$$\frac{u_{i+1}-u_{i-1}}{2h_i}-u'(x_i)=O(h_i^2/6) \quad \text{if } h_i=h_{i+1}.$$

H.W. Do the same for irregular mesh.(use weighted difference)

Assume $h_i = h_{i+1}$ and we substitute the solution of differential equation into the difference equation. Using $-u''=f$ we obtain

$$\frac{(-u_{i-1}+2u_i-u_{i+1})}{h^2}-f(x_i)$$

$$\begin{aligned}
&= \frac{1}{h^2}(-u_i + hu_i' - \frac{h^2}{2}u_i'' + \frac{h^3}{6}u^{(3)} - \frac{h^4}{24}u^{(4)}(\theta_1) + 2u_i) \\
&\quad + \frac{1}{h^2}(-u_i - hu_i' - \frac{h^2}{2}u_i'' - \frac{h^3}{6}u^{(3)} - \frac{h^4}{24}u^{(4)}(\theta_2)) - f(x_i) \\
&= -u_i'' - f(x_i) - \frac{h^2}{24}(u^{(4)}(\theta_1) + u^{(4)}(\theta_2)) \text{ truncation error} \\
&= \frac{h^2}{24}\max|u^{(4)}|.
\end{aligned}$$

We let $\tau_h = L_h u - F^h$ and call it the **truncation error(discretization error)**.

Definition 1.2.4. We say a difference scheme is **consistent** if the truncation error approaches zero as h approaches zero, in other words, if $L_h u - f \to 0$ in some norm.

Truncation error measures how well the difference equation approximates the differential equation. But it does not measure the actual error in the solution.

Use of different quadrature for f. Instead of $f(x_i)$ we can use

$$\frac{1}{12}[f(x_{i-1}) + 10f(x_i) + f(x_{i+1})] = \frac{5}{6}f(x_i) + \frac{\mu_0}{6}f(x_i)$$

where $\mu_0 f(x_i)$ is the average of f which is $f(x_i) + O(h^2)$.
H.W Show for uniform grid(use $-u'' = f$)

$$\frac{-u_{i-1} + 2u_i - u_{i-1}}{h^2} = \frac{1}{12}[f(x_{i-1}) + 10f(x_i) + f(x_{i+1})] + Ch^4 \max|u^{(6)}(x)|.$$

Nonuniform grid(irregular mesh)

We use central difference scheme to get

$$\begin{aligned}
u'(x_{i+1/2}) &\approx \tfrac{u_{i+1}-u_i}{h_{i+1}} \text{ and } u'(x_{i-1/2}) \approx \tfrac{u_i - u_{i-1}}{h_i}. \quad \text{Thus,} \\
u''(x_i) &\approx (\tfrac{u_{i+1}-u_i}{h_{i+1}} - \tfrac{u_i-u_{i-1}}{h_i})/(\tfrac{h_i+h_{i+1}}{2}).
\end{aligned}$$

H.W. Find truncation error for $u''(x_i)$ in case of nonuniform grid.

$$\begin{aligned}
L_h u - f &= 2[-h_i u_{i+1} + (h_i + h_{i+1})u_i - h_{i+1}u_{i-1}]/h_i h_{i+1}(h_i + h_{i+1}) - f(x_i) \\
&= 2[-h_i(u_i + h_{i+1}u_i' + \frac{h_{i+1}^2}{2}u_i'' + \frac{h_{i+1}^3}{6}u_i^{(3)} + O(h^4)) + (h_i + h_{i+1})u_i \\
&\quad -h_{i+1}(u_i - h_i u_i' + \frac{h_i^2}{2}u_i'' - \frac{h_i^3}{6}u_i^{(3)} + O(h^4))]/h_i h_{i+1}(h_i + h_{i+1}) - f(x_i) \\
&= -u_i^{(2)} - f_i + \frac{1}{3}(h_{i+1} - h_i) + O(h_i^2 + h_{i+1}^2).
\end{aligned}$$

H.W Use $\frac{1}{3}[f(x_i) + f(x_{i+1}) + f(x_{i-1})]$ for the right hand side. What is the truncation error?

Definition 1.2.5. L_h is said to be **stable** if

$$\|U_h\| \leq C\|L_h U^h\| \leq C\|F^h\| \quad \text{for all } h > 0$$

where U^h is the solution of the difference equation, $L_h U^h = F^h$. Also note that L_h is stable if and only if L_h^{-1} is bounded.

Definition 1.2.6. A finite difference scheme is said to **converge** if

$$\|U^h - u\| \to 0 \quad \text{as } h \to 0.$$

$\|U^h - u\|$ is called a **discretization error.**

Theorem 1.2.7 (P. Lax). *Given a consistent scheme, stability is equivalent to convergence.*

Proof. Assume stability. From $L_h u - f = \tau^h, L_h U^h - F^h = 0$, we have $L_h(u - U^h) = \tau^h$. Thus,

$$\|u - U^h\| \leq C\|L_h(u - U^h)\| = C\|\tau^h\| \to 0.$$

Hence the scheme converges. Obviously a convergent scheme must be stable. From the theory of p.d.e, we know $\|u\| \leq C\|f\|$. Hence

$$\|U^h\| \leq \|U^h - u\| + \|u\| \leq O(\tau^h) + C\|f\| \leq C\|f\| \leq C\|F^h\|.$$

□

1.3 Elliptic equation

1.3.1 Basic finite difference method for elliptic equation

In this chapter, we only consider finite difference method. First consider the following elliptic problem:(Dirichlet problem by Finite Difference Method)

$$\begin{aligned} -\Delta u &= f \text{ in } \Omega \\ u &= g \text{ on } \partial\Omega \end{aligned}$$

(1) Approx. D.E. $-(u_{xx} + u_{yy}) = f$ at each interior mesh pt.

(2) The unknown function is to be approximated by a grid function u

(3) Replace the derivative by difference quotient.

$$\begin{aligned} u(x+h) &= u(x) + hu_x(x) + \tfrac{h^2}{2}u_{xx}(x) + \tfrac{h^3}{6}u_{xxx}(x) + O(h^4) \\ u(x-h) &= \ldots \end{aligned}$$

$$\frac{u(x+h) - 2u(x) + u(x-h)}{h^2} = u_{xx}(x) + O(h^2)$$

$$\begin{aligned} u_{xx}(x,y) &\doteq [u(x+h,y) - 2u(x,y) + u(x-h,y)]/h^2 \\ u_{yy}(x,y) &\doteq [u(x,y+h) - 2u(x,y) + u(x,y-h)]/h^2 \end{aligned}$$

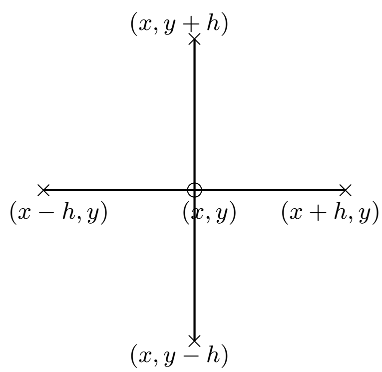

Figure 1.1: 5-point Stencil

This picture is called, Molecule, Stencil, Star, etc. For each point (interior mesh pt), approx $\nabla^2 u = \Delta u$ by 5-point stencil. By Girshgorin disc theorem, the matrix is nonsingular. $L[u]$ is called **differential operator** while $L_h[u]$ is called **finite difference operator**, e.g.,

$$L_h[u](x,y) = [-4u(x,y) + u(x+h,y) + u(x-h,y) + u(x,y+h) + u(x,y-h)]/h^2$$

or more generally,

$$L[u] = -[\frac{\partial}{\partial x}, \frac{\partial}{\partial y}]\,\mathrm{Diag}\{a_{11}, a_{22}\}\begin{bmatrix} \frac{\partial u}{\partial x} \\ \frac{\partial u}{\partial y} \end{bmatrix} + cu = -(a_{11}u_x)_x - (a_{22}u_y)_y + cu$$

Uniform meshes

$$\begin{aligned} u_x(x) &\doteq \tfrac{u(x+h) - u(x-h)}{2h} \\ (u_x)_x(x) &\doteq \tfrac{u_x(x+\frac{h}{2}) - u_x(x-\frac{h}{2})}{h} \qquad \text{Central difference} \end{aligned}$$

$$\begin{aligned} u_x(x+\tfrac{h}{2}) &= \tfrac{u(x+h)-u(x)}{h} \\ u_x(x-\tfrac{h}{2}) &= \tfrac{u(x)-u(x-h)}{h} \end{aligned}$$

$$(a_{11}u_x)_x = [(a_{11}u_x)(x+\frac{h}{2}) - (a_{11}u_x)(x-\frac{h}{2})]/h$$

Nonuniform meshes

$$\begin{aligned} u(x+h_2) &= u(x) + h_2 u_x(x) + \tfrac{h_2^2}{2}u_{xx} + \tfrac{h_2^3}{3!}u^{(3)} + \cdots \quad \times h_1 \\ u(x-h_1) &= u(x) - h_1 u_x(x) + \tfrac{h_1^2}{2}u_{xx} - \tfrac{h_2^3}{3!}u^{(3)} \cdots \quad \times h_2 \end{aligned}$$

$$\begin{aligned} & h_1 u(x+h_2) - h_2 u(x-h_1) \\ = \; & (h_1-h_2)u(x) + 2h_1h_2 u_x(x) + \frac{h_1h_2}{2}(h_2-h_1)u_{xx} + \cdots \end{aligned}$$

$$\therefore \qquad u_x(x) = \frac{h_1u(x+h_2) - h_2u(x-h_1) - (h_1-h_2)u(x)}{2h_1h_2} + O(h)$$

This is only first order accurate. To consider the second derivatives, we shall lose $O(h)$ accuracy. To get a second order method multiply two equations respectively by h_1^2, h_2^2 and subtract to get

$$\begin{aligned} & h_1^2u(x+h_2) - (h_1^2+h_2^2)u(x) - h_2^2u(x-h_1) \\ = \; & (h_2h_1^2 + h_1h_2^2)u_{xx}(x) + \left(\frac{h_2^3h_1^2}{6} + \frac{h_1^3h_2^2}{6}\right)\max|u'''|. \end{aligned}$$

Hence u_{xx} is seconder order accurate.

Assume the differential operator is of the form(with $\gamma > 0$)

$$L[u] \equiv -[u_{xx} + u_{yy}] + \gamma u = f$$

whose discretized form
$L_h[U] = a_0U(x,y) - a_1U(x+h,y) + \cdots = F(x,y)$

$$\frac{1}{h^2}\begin{pmatrix} 4+\gamma h^2 & -1 & -1 & 0 \\ -1 & 4+\gamma h & 0 & -1 \\ -1 & 0 & 4+\gamma h^2 & -1 \\ 0 & -1 & -1 & 4+r\gamma h^2 \end{pmatrix}\begin{pmatrix} U_1 \\ U_2 \\ U_3 \\ U_4 \end{pmatrix} = F$$

satisfies

(1) $L_h[u] = L[u] + O(h^2)$ as $h \to 0$. u is true solution.

(2) $AU = F + Bdy,\ Au = [\Delta u - \gamma u + O(h^2)] + Brdy$

$$A(U-u) = O(h^2) = \varepsilon$$

Then the discretization error $U - u = A$ has the form $^{-1}\varepsilon$(depends on h) and satisfies

$$\|U-u\| \le \|A^{-1}\| \cdot \|\varepsilon\| \le \|A^{-1}\| O(h^2)$$

If we put $D = diagA = \{a_{11}, \ldots, a_{nn}\}$, then $D^{-1}A(U-u) = D^{-1}\varepsilon$. Write $D^{-1}A = I + B$, where B is off diagonal. Then we know $\|B\|_\infty = \frac{4}{4+\gamma h^2} < 1$ if $\gamma > 0$. Thus $(D^{-1}A)^{-1} = (I+B)^{-1}$ exists and

$$\|(D^{-1}A)^{-1}\|_\infty = \|(I+B)^{-1}\|_\infty \le \frac{1}{1-\|B\|_\infty} \le \frac{4+\gamma h^2}{\gamma h^2}.$$

Hence

$$\|U-u\|_\infty \le \|(D^{-1}A)^{-1}\|_\infty \cdot \|D^{-1}\varepsilon\|_\infty \le \frac{4+\gamma h^2}{\gamma h^2} \cdot \frac{h^2}{4+\gamma h^2} O(h^2) = O(h^2) \to 0$$

Thus, we have proved the following result.

Theorem 1.3.1. *Let*

(1) $u \in C^4(\Omega)$

(2) $r > 0$

(3) uniform mesh

Then $\|U-u\|_\infty = O(h^2)$ *as* $h \to 0$.

Generally, when A, B, C are not constant, we can still put the problem into a conservative form as follows:

$$\begin{aligned} L[u] &= Au_{xx} + 2Bu_{xy} + Cu_{yy} + Du_x + Eu_y + Fu + G = 0 \\ &= \nabla^T \cdot \begin{pmatrix} A & B \\ B & C \end{pmatrix} \nabla u - (A_x + B_y - D)u_x - (B_x + C_y - E)u_y + Fu + G, \end{aligned}$$

where $\nabla = \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}\right)$, so that $\nabla u = \begin{pmatrix} u_x \\ u_y \end{pmatrix}$. If $A_x + B_y - D = 0$ and $B_x + C_y - E$, it is self-adjoint.

Treating the cross term like u_{xy}

Assume $u_{xy} = u_{yx}$, we approximate $\frac{\partial}{\partial y}\frac{\partial u}{\partial x}$ by $\delta_y^0 \delta_x^0 U^h$ where $\delta_x^0 U^h(P) = \frac{U^h(E)-U^h(W)}{2\Delta x}$ is the central difference. Then from

$$\delta_y^0 U^h(P) = \frac{U^h(N) - U^h(S)}{2\Delta y}$$

and forward -backward difference formula we get

$$\delta_y^0 \delta_x^0 U^h = \frac{1}{2\Delta_y}\left[\frac{U^h(NE) - U^h(NW)}{2\Delta x} - \frac{U^h(SE) - U^h(SW)}{2\Delta x}\right]$$

Change of variable method to eliminate the cross term

One can transform the variable so that the resulting equation in new variable does not have cross term.

Lemma 1.3.2. *Let $s = s(x, y)$, $t = t(x, y)$ be a coordinate transform which is locally one-to-one onto. Denote its derivative by $\frac{\partial(s,t)}{\partial(x,y)} = P^T$, Jacobian matrix. Then we have*

$$\nabla_{(x,y)} u \quad = \begin{bmatrix} u_x \\ u_y \end{bmatrix} = \begin{bmatrix} u_s s_x + u_t t_x \\ u_s s_y + u_t t_y \end{bmatrix} = P \cdot \begin{bmatrix} u_s \\ u_t \end{bmatrix} = P \cdot \nabla_{(s,t)} \cdot u$$

In other words,

$$\nabla_{(x,y)} \quad = \begin{pmatrix} \partial/\partial x \\ \partial/\partial y \end{pmatrix} = \begin{pmatrix} \frac{\partial s}{\partial x} \cdot \frac{\partial}{\partial s} + \frac{\partial t}{\partial x} \cdot \frac{\partial}{\partial t} \\ \frac{\partial s}{\partial y} \cdot \frac{\partial}{\partial s} + \frac{\partial t}{\partial y} \cdot \frac{\partial}{\partial t} \end{pmatrix} = \begin{pmatrix} \frac{\partial s}{\partial x} & \frac{\partial t}{\partial x} \\ \frac{\partial s}{\partial y} & \frac{\partial t}{\partial y} \end{pmatrix} \begin{pmatrix} \partial/\partial s \\ \partial/\partial t \end{pmatrix} = P \cdot \nabla_{(s,t)}$$

Hence $\nabla_{(x,y)}^T = \nabla_{(s,t)}^T \cdot P^T$ and we see that

$$\nabla_{(x,y)}^T A \nabla_{(x,y)} u = \nabla_{(s,t)}^T P^T A P \nabla_{(s,t)} u.$$

If A is symmetric, there exists a P such that $P^T A P = \text{diagonal} = \{d_1, d_2\}$. If we choose $s(x, y)$, $t(x, y)$ so that $\frac{\partial(s,t)}{\partial(x,y)} = P^T$, then

$$\nabla_{(x,y)}^T A \nabla_{(x,y)} u = \frac{\partial}{\partial s}\left(d_1 \frac{\partial u}{\partial s}\right) + \frac{\partial}{\partial t}\left(d_2 \frac{\partial u}{\partial t}\right).$$

Example 1.3.3. Transform the problem $u_{xx} + 4u_{xy} + u_{yy} = 0$ so that it does not have cross term.

Since $A = \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}$, its eigenvalues are $3, -1$ with corresponding eigenvectors $(1,1)$ and $(1,-1)$, we see that with $P = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$, we have

$$P^T A P = \begin{pmatrix} 3 & 0 \\ 0 & -1 \end{pmatrix}.$$

Hence the transformed equation is

$$\frac{\partial}{\partial s}(3\frac{\partial u}{\partial s}) - \frac{\partial}{\partial t}(\frac{\partial u}{\partial t}) = 0.$$

$P^T = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} = \begin{pmatrix} \partial s/\partial x & \partial s/\partial y \\ \partial t/\partial x & \partial t/\partial y \end{pmatrix}$. $\therefore \quad s = x + y$, $t = x - y$. $\binom{s}{t} = P^T \binom{x}{y}$.

If $s =$ constant, $ds = s_x\, dx + s_y\, dy = 0$, so the $s =$ constant line is described as $\frac{dy}{dx} = -\frac{s_x}{s_y} = -\frac{P_{11}}{P_{12}}$. Likewise, if $t =$ constant, $dt = t_x dx + t_y dy = 0 \therefore \frac{dy}{dx} = -\frac{t_x}{t_y} = -\frac{P_{21}}{P_{22}}$. If $P = \begin{bmatrix} \cos\lambda & -\sin\lambda \\ \sin\lambda & \cos\lambda \end{bmatrix}$, $s = x\cos\lambda - y\sin\lambda$, $t = x\sin\lambda + y\cos\lambda$, $b\cot 2\lambda = \frac{c-a}{2}$ when A is $\begin{bmatrix} a & b \\ b & c \end{bmatrix}$.

1.3.2 Treatment of irregular boundaries(Dirichlet boundary conditions

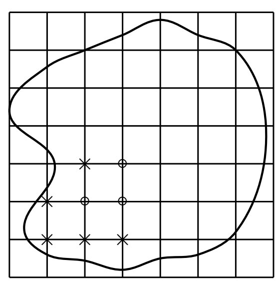

Figure 1.2: Ω_h, $\circ$ regular, $\times$ irregular

Let Ω be a domain with grid. Let Ω_h be the set of all grid points in Ω.

Definition 1.3.4. Two points P, Q on the grid are said to be **properly adjacent** if they are adjacent and the line segment connecting P, Q belongs to Ω.

Let Ω_h^* be the set of all **regular** points, i.e. the set of all points $P \in \Omega_h$ such that its four adjacent points belong to Ω_h and they are properly adjacent to P. Let $\Omega_h' = \Omega_h - \Omega_h^*$ be the set of all **irregular** points.

In the following, we let E be the east neighbor point of P in Ω_h and let W be the west neighborhood point of P in Ω_h, etc.

Method 1

We form the difference equation $L_h U^h = f^h$ at all regular points only while we let $U^h(P) = g(Q)$ if P is an irregular point. Here $Q = P$ if $P \in \partial\Omega$. Otherwise, Q is a point of $\partial\Omega$ closest to P. Here $U^h(P)$ is not an unknown.

Method 2(Collatz-linear interpolation)

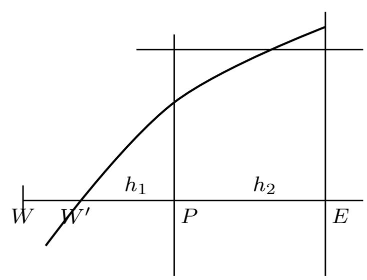

Figure 1.3: Near irregular boundary

We form $L_h U^h = f^h$ at all points of Ω_h as follows: First we form $L_h U^h = f^h$ at all regular points of Ω_h. If $P \in \partial\Omega_h$ is an irregular point lying near west part of $\partial\Omega$, take the point of intersection W' of the line segment EP with $\partial\Omega$. Then we let

$$U^h(P) = \frac{h_1}{h_1 + h_2} U^h(E) + \frac{h_2}{h_1 + h_2} u(W').$$

Now append this equation to the difference equation. If E happens to belong to $\partial\Omega$ also, then $U^h(P)$ is completely determined, hence we do not need to append it to the difference equation.

Remark 1.3.5. The last equation has nothing to do with the differential equation itself, thus it may break certain properties of matrix.

Method 3(Shortley-Weller)

For an irregular point P, we set

$$\frac{\partial^2 u}{\partial x^2}(P) \doteq 2\left(\frac{U^h(E)-U^h(P)}{h_2}-\frac{U^h(P)-u(W')}{h_1}\right)/(h_1+h_2).$$

Advantage: This equation comes from the differential equation, thus preserving(hopefully) certain properties of the matrix(like positive definiteness, banded structure, diagonal dominance). But usually symmetry breaks down.

Method 4

Let P be an irregular point whose west neighbor point W lies outside of Ω_h. Assuming u is defined at W, we use extrapolation to get to get $U^h(W) = \alpha U^h(W') + \beta U^h(P)$, where W' is the point of intersection of the line segment from P with $\partial\Omega$. ($\alpha = \frac{h_2}{h_1}, \beta = -\frac{h_2-h_1}{h_1}$. $h_2 = h$ and h_1 is the distance from P to the boundary.) We substitute $U^h(W)$ in the difference equation. (It is called fictitious point method)

Neumann or Robin boundary condition(regular point)

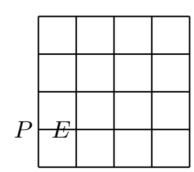

Figure 1.4: regular boundary point

Consider the boundary condition of type $\frac{\partial u}{\partial n} + \gamma u = g$ on $\partial\Omega$. Let P be a regular boundary point(boundary point lying on the grids). If the boundary is vertical line, then use one sided difference to get

$$\frac{U^h(P)-U^h(E)}{h} + \gamma(P)U^h(P) = g(P)$$

and append it to the difference equation.

If P is an irregular boundary point(figure 1.3), use $\frac{U^h(E)-U^h(W)}{2h}+\gamma(P)U^h(P) = g(P)$. Now solve it for $U^h(W)$ and substitute it into the difference equation at P to get a new equation.

$$\frac{1}{h^2}(-U^h(S) - U^h(W) + 4U^h(P) - U^h(N) - U^h(E)) = f(P)$$

If P is near corner do the same for north and south derivative.

Neumann or Robin boundary condition(irregular point)

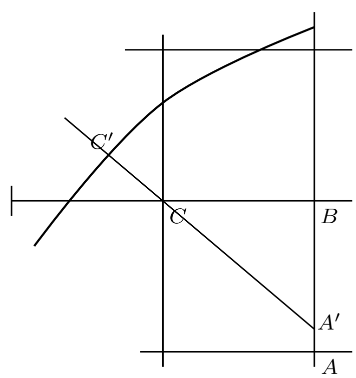

Figure 1.5: irregular boundary point

We let C be a grid point not in $\partial\Omega_h$. Draw a normal line to $\partial\Omega$ and let C' be the point of intersection with $\partial\Omega$. Now treat C as a grid point. Extend CC' to the closest grid line consisting of AB, letting A' denote the intersection of extension and the segment AB. Use

$$\begin{aligned}
\frac{\partial u}{\partial n}(C') &\doteq \frac{u(C)-U^h(A')}{\overline{CA'}} \\
U^h(C') &\doteq (1-\sigma)U^h(A') + \sigma U^h(C) \\
\Rightarrow \frac{U^h(C)-U^h(A')}{\overline{CA'}} + \sigma(C')U^h(C') &= g(C')
\end{aligned}$$

where $U^h(A')$ is obtained by interpolation.

$$U^h(A') = (1-\alpha)U^h(A) + \alpha U^h(B)$$

This is an equation involving unknowns $U^h(A), U^h(B)$ and $U^h(C)$.

Example 1.3.6.

$$-\nabla^2 u + (x^2+y^2)u = 40(2-x-y+4xy)e^{xy}, \quad 0 \le x, y \le 1$$

where $u(x,y) = [10-20\{(x-\frac{1}{2})^2+(y-\frac{1}{2})^2\}]e^{xy}$ on the boundary. Compare with

h	$= 0.2$	5×5 mesh	$\|U-u\|_\infty$	$= 0.0506$
	$= 0.1$	10×10		$= 0.0140$
	$= 0.05$	20×20		$= 0.0035$

error is $O(h^2)$.

Assuming the error is of the form $\|U-u\|_\infty = Mh^\alpha$, we see

$$\frac{\|U_h-u\|}{\|U_{\frac{h}{2}}-u\|} = \frac{Mh^\alpha}{M(\frac{h}{2})^\alpha} = 2^\alpha.$$

These are computable with u replaced by $U_{h_{min}}$ and

$$\alpha = \log\left[\frac{\|U_h-u\|}{\|U_{\frac{h}{2}}-u\|}\right] \Big/ \log 2$$

Theorem 1.3.7 (Maximum Principle). *Assume A is positive definite symmetric, $c \ge 0$. Let u be the solution of elliptic p.d.e. given by*

$$L[u] = -\sum_i \frac{\partial}{\partial x_i}\left[\sum_j a_{ij}\frac{\partial u}{\partial x_j}\right] + cu = 0 = -\nabla^T A \nabla u + cu.$$

Then for $(x,y) \in \operatorname{int}\Omega$

$$|u(x,y)| \le \max_{(x,y)\in\partial\Omega} |u(x,y)|$$

Proof. Assume $c > 0$. There exists orthogonal matrix P such that $P^TAP = \operatorname{diag}\{d_1, d_2\}$. If u has a positive maximum at some interior point $Q = (x^*, y^*)$ of Ω, then define

$$\begin{pmatrix} s \\ t \end{pmatrix} = P^T(x^*, y^*)\begin{pmatrix} x \\ y \end{pmatrix}$$

so that $L[u] = -\nabla_{(s,t)}P^TAP\nabla_{(s,t)}u + cu = 0$. At Q, $u_s(Q) = u_t(Q) = 0$, $u_{ss}(Q) \le 0$ and $u_{tt}(Q) \le 0$. Hence

$$L[u] = -(d_1u_s)_s(Q) - (d_2u_t)_t(Q) + c(Q)u(Q) = 0$$

Remembering, $d_1 > 0$, $d_2 > 0$, $cu > 0$ this is a contradiction. Similarly, u cannot have negative minimum.

Now if $c \geq 0$ not $c > 0$ we consider a perturbation. Choose α so large that $L[e^{\alpha x}] = -(d_1\alpha^2 + d_2\alpha^2 - c)e^{\alpha x} < 0$ and let $v = u + Ee^{\alpha x}$.

$$L[v] = L[u] + EL[e^{\alpha x}] < 0 \quad \text{for all} \quad E > 0.$$

Suppose v has a pos. max. at Q, an interior point of Ω. Then $L[v] = -d_1 v_{ss}(Q) - d_2 v_{tt}(Q) + c(Q)v(Q) \geq 0$, a contradiction. Hence $u(x,y) \leq v(x,y) < \max_{\partial\Omega}\{u + Ee^{\alpha x}\}$. Let $E \to 0$. Then

$$u(x,y) \leq \max_{\partial\Omega} u.$$

□

Applying maximum principle to u and $-u$, we obtain the following result.

Corollary 1.3.8. *If*

$$\begin{aligned} L[u] &= 0 \qquad \text{in } \Omega \\ u &= 0 \qquad \text{on } \partial\Omega, \end{aligned}$$

then $u \equiv 0$.

As a consequence we have uniqueness of solution.

Corollary 1.3.9. *If* u_1, u_2 *satisfy*

$$\begin{aligned} L[u_i] &= 0 \qquad \text{in } \Omega \\ u_i &= g \qquad \text{on } \partial\Omega, \end{aligned}$$

then $u_1 = u_2$.

A symmetric, positive definite system satisfying $L[u] \leq 0$ has a unique solution.

Theorem 1.3.10 (Discrete max. principle). *Let the grid function U satisfy the finite difference equation $L_h[U] \equiv \sum_Q A(P,Q)U(Q) = 0$, i.e,*

$$A(P,P)U(P) + A(P,E)U(E) + A(P,S)U(S) + A(P,N)U(N) + A(P,W)U(W) = 0$$

for each mesh point P, where coefficient $A(P,Q)$ are generated by finite difference method(e.g, p.955) and A is pos. def. weakly diagonally dominant.

Then

$$\min_{P\in\partial\Omega_h} U(P) \leq U(P^*) \leq \max_{P\in\partial\Omega_h} U(P), \quad \textit{for all } P* \in \operatorname{int}\Omega_h.$$

Proof. Solving for $U(P)$, we have

$$U(P) \quad = \tfrac{1}{A(P,P)} \sum_{Q \neq P} -A(P,Q)U(Q)$$

Thus

$$|U(P)| \leq \sum_{Q \neq P} \left| \frac{A(P,Q)}{A(P,P)} \right| \max_{Q \neq P} |U(Q)| \leq \max_{Q \neq P} |U(Q)|,$$

since $|A(P,P)| \geq \sum_{Q \neq P} |A(P,X)|$. □

Corollary 1.3.11.

$$\begin{aligned} L_h(U) &= 0 \qquad in\ \Omega \\ U &= 0 \qquad on\ \partial\Omega \end{aligned}$$

implies $U = 0$.

Theorem 1.3.12. *Discrete max. principle also hold if* $L_h[U] \leq 0$.

Proof. Assume $U(p_0) > U(p)$ for all $p \in \partial\Omega_h$, $p_0 \in \Omega_h$. Then

$$\Delta_h U(p_0) = \frac{1}{h_x^2}[U(p_1) + U(p_2)] + \frac{1}{h_y^2}[U(p_3) + U(p_4)] - 2(\frac{1}{h_x^2} + \frac{1}{h_y^2})U(p_0).$$

But $\Delta_h U = -L_h[U] \geq 0$. Hence

$$U(p_0) \leq \frac{1}{2(\frac{1}{h_x^2} + \frac{1}{h_y^2})} \left(\frac{1}{h_x^2}[U(p_1) + U(p_2)] + \frac{1}{h_y^2}[U(p_3) + U(p_4)] \right) \leq U(p_0)$$

which means $U(p_0) = U(p_\nu)$, $\nu = 1, 2, 3, 4$. Repeat the argument for each p_ν instead of p_0 until we arrive at the boundary point of Ω. Then we get

$$U(p) \equiv U(p_0), \quad \forall p \in \Omega_h \cup \partial\Omega_h\,,$$

which is a contradiction to the assumption. □

Note. Minimum principle is obtained when $L_h[U] \geq 0$.

Theorem 1.3.13. *Let u be the solution of $L[u] = -\nabla^T A \nabla u + cu = 0$ in Ω and $u = g$ on $\partial\Omega$ and let U be the finite sequence of grid functions satisfying $L_h(U) = 0$, where $L_h(u) = \mathcal{O}(h^\alpha)$, $\alpha > 0$(truncation error). If A is constant, diagonally dominant, positive definite, then $\|U - u\|_\infty = \mathcal{O}(h^\alpha)$ as $h \to 0$.*

Proof. Let $w = U - u$, then $L_h[w] = L_h[U] - L_h[u] = -L_h[u]$ and $w = 0$ on $\partial\Omega$. Let $s(x,y) \equiv r^2 - (x-x_0)^2 - (y-y_0)^2$ with $(x_0, y_0) \in \operatorname{int}\Omega$, r chosen so large that the circle $s = 0$ contains Ω. Then $L_h[s] = L[s]$ because s is quadratic. (compute it)

$$\begin{aligned} L[s] &= -\nabla^T A \nabla s + cs = -a_{11}s_{xx} - (a_{12}+a_{21})s_{xy} - a_{22}s_{yy} + cs \\ &= 2(a_{11}+a_{22}) + cs \geq 2(a_{11}+a_{22}). \end{aligned}$$

There exist $M > 0$ such that $|L_h[u]| \leq Mh^\alpha$ by the hypothesis $L_h[u] = \mathcal{O}(h^\alpha)$. We see that

$$L_h\left[\frac{Mh^\alpha s(x,y)}{2(a_{11}+a_{22})}\right] \geq Mh^\alpha \geq |L_h[u]|.$$

Also

$$L_h\left[\pm w - \frac{Mh^\alpha s(x,y)}{2(a_{11}+a_{22})}\right] = \pm L_h[w] - L_h[\quad] \leq \mp L_h[u] - Mh^\alpha \leq 0$$

(Recall $w = U - u$ and $L_h[w] = -L_h[u]$ and $w = 0$ on $\partial\Omega$)

Since the discrete maximum principle also holds if $L_h(U) = 0$ is replaced by $L_h[U] \leq 0$,

$$\max_{P\in\Omega}\left[\pm w - \frac{Mh^\alpha s}{2(a_{11}+a_{22})}\right] \leq \max_{P\in\partial\Omega}\left[\pm w - \frac{Mh^\alpha s}{2(a_{11}+a_{22})}\right] = -\frac{Mh^\alpha}{2(a_{11}+a_{22})}\min_{\partial\Omega} s \leq 0.$$

Thus $|w| \leq \frac{Mh^\alpha s}{2(a_{11}+a_{22})}$ and

$$\|u - U\|_\infty = \|w\|_\infty \leq \frac{Mh^\alpha}{2(a_{11}+a_{22})}\max_{\Omega} s \leq \frac{Mh^\alpha r^2}{2(a_{11}+a_{22})}.$$

□

1.3.3 Convection -diffusion equation

Consider another type of differential equation, namely a special case of convection diffusion equation.

$$-\epsilon u_{xx} + a u_x = f$$

$$u(0) = u_0,\ \ u(1) = u_1$$

or with Neumann condition $u'(1) = 0$ if $a(1) > 0$. If we use the central difference scheme for the first order derivative, we get

$$\epsilon\left(\frac{-U_{i-1}^h + 2U_i^h - U_{i+1}^h}{h^2}\right) + a\frac{U_{i+1}^h - U_{i-1}^h}{2h} = f_i^h$$

$$-\left(\frac{\epsilon}{h^2}+\frac{a}{2h}\right)U_{i-1}^h+\frac{2\epsilon}{h^2}U_i^h-\left(\frac{\epsilon}{h^2}-\frac{a}{2h}\right)U_{i+1}^h=f_i^h$$

Thus the sum of off diagonal elements is

$$\sum_{j\neq i}|a_{ij}|=\left|\frac{\epsilon}{h^2}+\frac{a}{2h}\right|+\left|\frac{\epsilon}{h^2}-\frac{a}{2h}\right|.$$

If a, h is fixed and $\epsilon \to 0$, it becomes a/h, while $a_{ii} = 2\epsilon/h^2 \to 0$. Thus the resulting matrix is not diagonally dominant and it causes a lot of problems. For example, the resulting linear system is not positive definite and hence it may be more difficult to solve. But, most importantly, the resulting discretization does not yield an accurate approximation to the problem. One way to fix this situation is to keep the Peclet number : $\frac{ah}{\epsilon} < 2$ so that the sum of off diagonal elements is less than $2\epsilon/h^2 = a_{ii}$. The disadvantage of this scheme is that small h enlarges the size of discrete equation.

1.3.4 Numerical Difficulties

(1) The iterative method may fail to converge

(2) The solution may exhibit oscillation which are physically unrealistic

(3) Taking small mesh size means large problem size which take more time to solve.

1.3.5 Upwind difference scheme

An alternative way to avoid this difficulty is to use backward difference for u_x for $a > 0$ and forward difference for u_x for $a < 0$. This method of choosing difference scheme is called **upwind difference scheme**. For $a > 0$

$$\epsilon\left(\frac{-U_{i-1}^h+2U_i^h-U_{i+1}^h}{h^2}\right)+a\frac{U_i^h-U_{i-1}^h}{h}=f_i^h$$

$$-\left(\frac{\epsilon}{h^2}+\frac{a}{h}\right)U_{i-1}^h+\left(\frac{2\epsilon}{h^2}+\frac{a}{h}\right)U_i^h-\left(\frac{\epsilon}{h^2}\right)U_{i+1}^h=f_i^h$$

The resulting system is irreducibly diagonally dominant, thus it is an M-matrix and hence Jacobi method works.

The following examples are taken from (K.W. Morton-Numerical Solution of convection-Diffusion problems, Chapman Hall, 1996)

Example 1.3.14.

$$-\epsilon\Delta u + \mathbf{b}\cdot u = 0 \text{ on } (0,1)\times(0,1)$$

with

$$\mathbf{b} = b(\cos\theta, \sin\theta)$$

for $0 \le \theta < \frac{\pi}{2}$ and discontinuous inflow boundary condition

$$u(0,y) = \begin{cases} 0, & y \in [0, \frac{1}{2}) \\ 1, & y \in (\frac{1}{2}] \end{cases}$$

and $\frac{\partial u}{\partial y} = 0$ on the rest of the boundary. This leads to an internal layer along $y = \frac{1}{2} + x\tan\theta$ and a boundary layer at $x = 1$ for $y > \frac{1}{2} + \tan\theta$ when $\tan\theta < \frac{1}{2}$.

Example 1.3.15 (Heat equation).

$$\begin{aligned} \frac{\partial u}{\partial t} + \mathbf{b}\cdot u &= \epsilon\Delta u \text{ on } \Omega\times(0,T) \\ u(x,y,0) &= u_0(x,y) \end{aligned}$$

where $u_0(x,y)$ is a circular cone type centered at $(1,0)$ with

$$\mathbf{b} = b(wy, -wx)$$

with exact inflow boundary are needed.

1.4 Parabolic p.d.e's

Consider a heat equation on a bar.

$$u_t = u_{xx}, \quad 0 \le x \le 1, \quad 0 < t \le T.$$

Theorem 1.4.1 (Maximum principle). *If u satisfies the above condition for $t \le T$, then*

$$\min\{f,g,h\} = m \le \min_{0\le x\le 1, 0\le t\le T} u \le \max_{0\le x\le 1, 0\le t\le T} u \le M = \max\{f,g,h\}$$

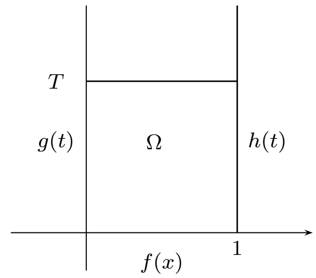

Figure 1.6: Domain

Proof. Put $v = u + Ex^2$, $E > 0$

$$\frac{\partial v}{\partial t} - \frac{\partial^2 v}{\partial x^2} = -2E < 0$$

If v attains a maximum at $Q \in \operatorname{int}\Omega$, then

$$\begin{aligned} v_t(Q) &= 0, \\ v_{xx}(Q) &\leq 0. \end{aligned}$$

Thus $(v_t - v_{xx})(Q) \geq 0$, a contradiction. Hence v has maximum at a boundary point of Ω

$$u(x,t) \leq v(x,t) \leq \max v(x,t) \leq M + E.$$

Since E was arbitrary, the proof is complete. For minimum, use $-E$ instead of E. □

More general parabolic p.d.e.

$$u_t = Au_{xx} + Du_x + Fu + G$$

$$\text{F.D.M}\begin{cases}\text{Explicit}\cdots\text{write down the values of grid function}\\ \text{Implicit}\cdots\ \text{variables implicitly representing the value}\end{cases}$$

Let the grid be given by

$$\begin{aligned} 0 &= x_0 < x_1 < x_2 < \cdots < x_{N+1} = 1, \quad x_i = ih, \quad \text{uniform grid} \\ 0 &= t_0 < t_1 < \cdots, \quad t_j = jk \end{aligned}$$

Explicit method

$$\begin{cases} \frac{U_{i,j+1}-U_{i,j}}{k} \doteq u_t \\ \frac{U_{i+1,j}-2U_{i,j}+U_{i-1,j}}{h^2} \doteq u_{xx} \end{cases}$$

$$\therefore \qquad U_{i,j+1} = \lambda U_{i-1,j} + (1-2\lambda)U_{i,j} + \lambda U_{i+1,j}$$

where $\lambda = k/h^2$.

Stability: Error doesn't accumulate. In this case **solution remains bounded** as time goes on.

Theorem 1.4.2. *If u is sufficiently smooth, then*

$$\left| u_{xx} - \frac{u(x+h,t) - 2u(x,t) + u(x-h,t)}{h^2} \right| = O(h^2) \quad as \quad h \to 0$$

and

$$\left| u_t - \frac{u(x,t+k) - u(x,t)}{k} \right| = O(k) \quad as \quad k \to 0$$

Theorem 1.4.3. *Suppose u is sufficiently smooth, and satisfies*

$$\begin{aligned} u_t &= u_{xx} \qquad 0 < x < 1, \quad t > 0 \\ u(x,0) &= f(x) \\ u(0,t) &= g(t) \\ u(1,t) &= h(t). \end{aligned}$$

If $U_{i,j}$ is the solution of the explicit finite difference scheme, then for $0 < \lambda \le \frac{1}{2}$,

$$\max_{i,j} |u_{i,j} - U_{i,j}| \doteq O(h^2 + k) \quad as \quad h, k \to 0,$$

i.e, finite difference solution converges to the true solution.

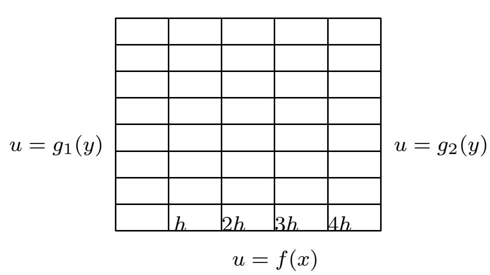

Figure 1.7:

Proof. Put $u_{ij} \equiv u(x_i, t_j)$. Then from

(1) $\dfrac{u_{i,j+1} - u_{i,j}}{k} = u_t + O(k)$

(2) $\dfrac{u_{i+1,j} - 2u_{i,j} + u_{i,j}}{h^2} = u_{xx} + O(h^2)$

we get

$$u_{i,j+1} = u_{i,j} + \frac{k}{h^2}(u_{i+1,j} - 2u_{i,j} + u_{i-1,j}) + k(O(k) + O(h^2)).$$

Hence

$$u_{i,j+1} = \lambda u_{i-1,j} + (1-2\lambda)u_{i,j} + \lambda u_{i+1,j} + Ck(k+h^2).$$

Let the discretization error be $w_{i,j} = u_{ij} - U_{ij}$ so that

$$w_{i,j+1} = \lambda w_{i-1,j} + (1-2\lambda)w_{i,j} + \lambda w_{i+1,j} + O(k^2 + kh^2).$$

Since $0 < \lambda \le \frac{1}{2}$, $0 \le 1 - 2\lambda < 1$, three coefficient are positive and their sum is 1. (convex combination) We see

$$|w_{i,j+1}| \le \lambda |w_{i-1,j}| + (1-2\lambda)|w_{i,j}| + \lambda |w_{i+1,j}| + M(k^2 + kh^2) \quad \text{for some} \quad M > 0.$$

If we define $\|w_j\| = \max_{1 \le i \le N} |w_{i,j}|$, then

$$\begin{aligned} \|w_{j+1}\| &\le \|w_j\| + M(k^2 + kh^2) \\ &\le \|w_{j-1}\| + 2M(k^2 + kh^2) \le \cdots \le \|w_0\| + (j+1)M(k^2 + kh^2). \end{aligned}$$

Since $\|w_0\| = 0$,

$$\|w_{j+1}\| \le (j+1)kM(k+h^2) \le TM(k+h^2), \quad (j+1)k \le T.$$

In fact,

$$M = \max_{0 \le x \le 1,\, 0 \le t \le T} \left(\frac{1}{2}|u_{tt}| + \frac{k^2}{12}|u_{xxxx}|\right).$$

□

Remark 1.4.4. If $\lambda > \frac{1}{2}$, the solution may not converge.

Exercise 1.4.5. Prove the formula is unstable for $\lambda > \frac{1}{2}$. Let

$$u(x,0) = \begin{cases} \varepsilon, & x = \frac{1}{2} \\ 0, & x \ne \frac{1}{2} \end{cases} \quad \text{with} \quad g = h = 0$$

$$\begin{aligned} U_{i,j+1} &= \lambda U_{i+1,j} + (1-2\lambda)U_{i,j} + \lambda U_{i-1,j}, \qquad \lambda = k/h^2 \\ |U_{i,j+1}| &= \lambda|U_{i+1,j}| + (2\lambda-1)|U_{i,j}| + \lambda|U_{i-1},j|, \quad 1 \le i \le N-1. \end{aligned}$$

Hence

$$\sum_{i=1}^{N-1} |U_{i,j+1}| = \lambda \sum_{i=1}^{N-2} |U_{i+1,j}| + (2\lambda-1) \sum_{i=1}^{N-1} |U_{i,j}| + \lambda \sum_{i=2}^{N} |U_{i-1,j}|,$$

since $U(x_i,t) = 0$, $i = 1, N$. Let $S(t_j) = \sum_{i=1}^{N} |U(i,j)|$. Then

$$S(t_{j+1}) = (4\lambda-1)S(t_j) = (4\lambda-1)^2 S(t_{j-1}) = \cdots = (4\lambda-1)^{j+1}S(0) = (4\lambda-1)^{j+1}\varepsilon.$$

Since the number of nonzero $U_{i,j}$ for each j is $2j+1$, there is a point (x_p, t_j) such that

$$|U(x_p,t_j)| \ge \frac{1}{2j+1} S(t_j) = \frac{1}{2j+1}(4\lambda-1)^j \cdot \varepsilon$$

which diverges as $j \to \infty$ since $4\lambda - 1 > 1$.

Considering the alternating sign, one can see the solution alternates: $j = 0$

$$U_{i,1} = (1-2\lambda)\epsilon, U_{i-1,1} = \lambda\epsilon, U_{i+1,1} = \lambda\epsilon$$

$$U_{i,2} = 2\lambda^2\epsilon + (1-2\lambda)^2\epsilon, U_{i-1,2} = (1-2\lambda)\epsilon + (1-2\lambda)\epsilon = 3\lambda\epsilon(1-2\lambda) < 0.$$

Stability of linear system

$$\begin{pmatrix} U_{1,j+1} \\ \vdots \\ U_{N-1,j+1} \end{pmatrix} = \begin{pmatrix} 1-2\lambda & \lambda & \dots & 0 \\ \lambda & 1-2\lambda & \ddots & \\ 0 & & \ddots & \lambda \\ & & \lambda & 1-2\lambda \end{pmatrix} \begin{pmatrix} U_{1,j} \\ \vdots \\ U_{N-1,j} \end{pmatrix} + \begin{pmatrix} g(t_j) \\ 0 \\ \vdots \\ 0 \\ h(t_j) \end{pmatrix}$$

In vector form, $\mathbf{U_{j+1}} = \mathbf{AU_j} + \mathbf{G_j}$. Assume $\mathbf{G_j} = \mathbf{0}$, $j = 1, 2, \dots$. Let μ be an eigenvalue of A. Then by G-disk theorem,

$$\begin{aligned} |1-2\lambda-\mu| &\le 2\lambda \\ -2\lambda &\le 1-2\lambda-\mu \le 2\lambda \\ -2\lambda &\le -1+2\lambda+\mu \le 2\lambda \\ 1-4\lambda &\le \mu \le 1 \end{aligned}$$

If $0 < \lambda \le \frac{1}{2}$, then $-1 \le \mu \le 1$, hence stable. If $\lambda > \frac{1}{2}$, then $|\mu| > 1$ is possible. So the scheme may be unstable. The following example show it is actually unstable.

Example 1.4.6 (Issacson, Keller). Try $v(x,t) = \mathrm{Re}(e^{i\alpha x - wt}) = \cos\alpha x \cdot e^{-wt}$.

$$\begin{aligned}
v_t - v_{xx} &\doteq \frac{v(x,t+\Delta t) - v(x,t)}{\Delta t} - \frac{v(x+\Delta x,t) - 2v(x,t) + v(x-\Delta x,t)}{\Delta x^2} \\
&= v(x,t)\left(\frac{e^{-w\Delta t} - 1}{\Delta t}\right) - \frac{\cos(\alpha x + \alpha\Delta x) - 2\cos\alpha x + \cos(\alpha x - \alpha\Delta x)}{\Delta x^2} e^{-wt} \\
&= v(x,t)\left(\frac{e^{-w\Delta t} - 1}{\Delta t} - \frac{2\cos\alpha\Delta x - 2}{\Delta x^2}\right) \\
&= v(x,t)\frac{1}{\Delta t}\{e^{-w\Delta t} - [(1-2\lambda) + 2\lambda\cos\alpha\Delta x]\} \\
&= v(x,t)\frac{1}{\Delta t}\left[e^{-w\Delta t} - \left(1 - 4\lambda\sin^2\frac{\alpha\Delta x}{2}\right)\right]
\end{aligned}$$

Thus v is a solution of the difference equation provided w and α satify $e^{-w\Delta t} = 1 - 4\lambda\sin^2\frac{\alpha\Delta x}{2}$.

With I.C. $v(x,0) = \cos\alpha x$, solution becomes

$$v(x,t) = \cos\alpha x e^{-wt} = \cos\alpha x\left(1 - 4\lambda\sin^2\frac{\alpha\Delta x}{2}\right)^{\frac{t}{\Delta t}}$$

Clearly, for all $\lambda \le \frac{1}{2}$, $|v(x,t)| \le 1$. However, if $\lambda > \frac{1}{2}$, then for some Δx, we have $\left|1 - 4\lambda\sin^2\frac{\alpha\Delta x}{2}\right| > 1$. So $v(x,t)$ becomes arbitrarily large for sufficiently large $t/\Delta t$. Since every even function has a cosine series, we may give any even function $f(x)$ of the form $f(x) = \sum_\nu \beta_\nu \cos(2^\nu \pi x)$ to get an unstable problem.

Implicit Finite Difference Method.

Given a heat equation

$$\begin{aligned}
u_t &= u_{xx} \\
u(0,t) &= g(t), \quad t > 0 \\
u(1,t) &= h(t) \\
u(x,0) &= f(x), \quad 0 \le x \le 1.
\end{aligned}$$

We discretize it by implicit difference method.

$$\frac{U_{i,j+1} - U_{i,j}}{\Delta t} = \frac{U_{i+1,j+1} - 2U_{i,j+1} + U_{i-1,j+1}}{\Delta x^2} \qquad i = 1,\ldots,N-1.$$

Multiply by Δt, then with $\lambda = \Delta t/\Delta x^2$, we have

$$\begin{aligned}
U_{i,j+1} - U_{i,j} &= \lambda U_{i+1,j+1} - 2\lambda U_{i,j+1} + \lambda U_{i-1,j+1} \\
-U_{i,j} &= \lambda U_{i+1,j+1} - (1+2\lambda)U_{i,j+1} + \lambda U_{i-1,j+1} \qquad j = 1,\ldots,N-1.
\end{aligned}$$

This yields a system of $N-1$ unknowns in $\{U_{i,j+1}\}_{i=1}^{N-1}$.

$$\begin{bmatrix} -\lambda U_{0,j+1} \\ 0 \\ \vdots \\ 0 \\ -\lambda U_{N,j+1} \end{bmatrix} + \begin{bmatrix} -U_{1,j} \\ \vdots \\ -U_{N-1,j} \end{bmatrix} = - \begin{bmatrix} (1+2\lambda) & -\lambda & 0 & & \\ -\lambda & (1+2\lambda) & -\lambda & & \\ 0 & \ddots & \ddots & & \\ & \ddots & \ddots & \ddots & -\lambda \\ & 0 & & -\lambda & (1+2\lambda) \end{bmatrix}$$

$$\times \begin{bmatrix} U_{1,j+1} \\ \vdots \\ U_{N-1,j+1} \end{bmatrix}$$

Theorem 1.4.7. *The implicit finite difference scheme is stable for all $\lambda = \Delta t/\Delta x^2$. (solution remains bounded).*

Proof. For each j, let $U_{k(j),j}$ be chosen so that $|U_{k(j),j}| \geq |U_{i,j}|$, $i = 1, \ldots, N-1$. We choose $i_0 = k(j+1)$ in the following relation.

$$U_{i,j+1} = U_{i,j} + \lambda\{U_{i+1,j+1} - 2U_{i,j+1} + U_{i-1,j+1}\}.$$

Then

$$(1+2\lambda)U_{i_0,j+1} = U_{i_0,j} + \lambda\{U_{i_0+1,j+1} + U_{i_0-1,j+1}\}$$

for $1 \leq i \leq N-1$. Taking absolute values,

$$(1+2\lambda)|U_{i_0,j+1}| \leq |U_{i_0,j}| + \lambda(|U_{i_0+1,j+1}| + |U_{i_0-1,j+1}|) \leq |U_{i_0,j}| + 2\lambda|U_{i_0,j+1}|.$$

Thus $|U_{i_0,j+1}| \leq |U_{i_0,j}| \leq |U_{k(j),j}|$ and hence $|U_{i,j+1}| \leq |U_{i_0,j+1}| \leq |U_{k(j),j}|$ for $1 \leq i \leq N-1$, and $|U_{i,j+1}| \leq M = \max\{f, g, h\}$, for $i = 0$ or N, by boundary condition. Repeat the same procedure until we hit the boundary.

$$|U_{i,j+1}| \leq |U_{k(j),j}| \leq \cdots \leq |U_{k(0),0}| \leq M = \max(f, g, h)$$

□

Using the matrix formulation: We check the eigenvalues of the system

$$A\mathbf{U_{j+1}} = \mathbf{U_j} + \mathbf{G_j}$$

Eigenvalue of A satisfies $|\mu + (1+2\lambda)| \leq 2\lambda$ by G-disk theorem. From this, we see $|\mu| \geq 1$ and hence the eigenvalues of A^{-1} is less than one in absolute value. Thus

$$U_{j+1} \leq A^{-1}(U_j + G_j) = \cdots = A^{-j-1}U_0 + A^{-j-1}G_0 + A^{-j-2}G_1 + \cdots + A^{-1}G_j.$$

$$\|U_{j+1}\| \leq \|A^{-j-1}\| \, \|U_0\| + \|A^{-1}\| \cdot \frac{1}{1 - \|A^{-1}\|} \max \|G_j\|$$

remain bounded. **Note.** A does not have -1 as eigenvalues and all the eigenvalues are positive real.

Theorem 1.4.8. *For sufficiently smooth u, we have*

$$|u_{ij} - U_{ij}| = \mathcal{O}(h^2 + k) \quad \text{as} \quad h \quad \text{and} \quad k \to 0 \qquad \text{(for all } \lambda\text{)}$$

Proof. Let $u_{ij} = u(x_i, t_j)$ be the true solution. Then we have

$$\frac{u_{i,j+1} - u_{i,j}}{k} = \frac{1}{h^2}\{u_{i+1,j+1} - 2u_{i,j+1} + u_{i-1,j+1}\} + \mathcal{O}(h^2 + k)$$

Let $w_{i,j} = u_{i,j} - U_{i,j}$ be the discretization error. Then

$$\begin{aligned} w_{i,j+1} &= w_{i,j} + \lambda\{w_{i+1,j+1} - 2w_{i,j+1} + w_{i-1,j+1}\} + \mathcal{O}(kh^2 + k^2) \\ (1 + 2\lambda)w_{i,j+1} &= w_{i,j} + \lambda w_{i+1,j+1} + \lambda w_{i-1,j+1} + \mathcal{O}(kh^2 + k^2) \end{aligned}$$

Let $\|w_j\| = \max_i |w_{i,j}|$. Then

$$(1 + 2\lambda)|w_{i,j+1}| \leq \|w_j\| + 2\lambda\|w_{j+1}\| + \mathcal{O}(kh^2 + k^2)$$

and so

$$(1 + 2\lambda)\|w_{j+1}\| \leq \|w_j\| + 2\lambda\|w_{j+1}\| + \mathcal{O}(kh^2 + k^2).$$

Thus

$$\begin{aligned} \|w_{j+1}\| &\leq \|w_j\| + C(kh^2 + k^2) \\ &\leq \cdots \leq \|w_0\| + C(j + 1)k(k + h^2) \\ &\leq \|w_0\| + CT(k + h^2) = CT(k + h^2) \end{aligned}$$

for $t = (j + 1)k \leq T$. □

1.4.1 Discretization of parabolic p.d.e, General Case

Consider

$$\begin{aligned} c\frac{\partial u}{\partial t} &= \frac{\partial}{\partial x}(p\frac{\partial u}{\partial x}) + \frac{\partial}{\partial y}(p\frac{\partial u}{\partial y}) - \gamma u + f \quad \text{in } \Omega \times [0, T] \\ \text{I.C.} \quad u(x, y, 0) &= h(x, y) \text{ in } \Omega \\ \text{B.C.} \quad u(x, y, t) &= g(x, y, t) \text{ for } (x, y) \in \partial\Omega. \end{aligned}$$

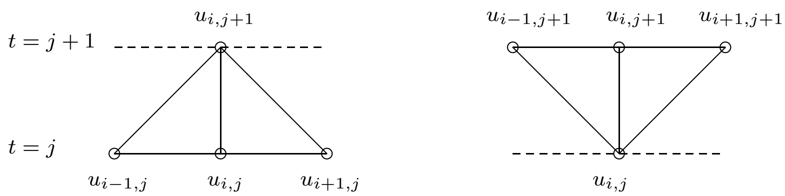

Figure 1.8: forward backward Euler method Stencil

where c, p, γ, f are functions of x, y and t. Assume

$$\begin{array}{ll} 0 < p_0 & \leq p(x,y,t) \leq p_1 \\ 0 & \leq \gamma(x,y,t) \leq \gamma \\ 0 < c_0 & \leq c(x,y,t) \leq c_1. \end{array}$$

With $U_{i,j}^n = U(x_i, y_j, t_n)$ we have

$$\begin{aligned} M_h U_{i,j}^n &= \frac{1}{\Delta x^2}\left(p_{i+1/2,j}^n U_{i+1,j}^n + p_{i-1/2,j}^n U_{i-1,j}^n - (p_{i+1/2,j}^n + p_{i-1/2,j}^n)U_{i,j}^n\right) \\ &+ \frac{1}{\Delta y^2}\left(p_{i,j+1/2}^n U_{i,j+1}^n + p_{i,j-1/2}^n U_{i,j-1}^n - (p_{i,j+1/2}^n + p_{i,j-1/2}^n)U_{i,j}^n\right) - \gamma_{i,j}^n U_{i,j}^n \end{aligned}$$

For $0 \leq \theta \leq 1$, we let

$$[\theta c_{ij}^{n+1} + (1-\theta)c_{ij}^n]\frac{U_{i,j}^{n+1} - U_{i,j}^n}{\Delta t} = \theta M_h U_{i,j}^{n+1} + (1-\theta)M_h U_{i,j}^n + \theta f_{ij}^{n+1} + (1-\theta) f_{ij}^n$$

For $\theta = 0$, we have

$$c_{ij}^n \frac{U_{i,j}^{n+1} - U_{i,j}^n}{\Delta t} = M_h U_{i,j}^n + f_{ij}^n \quad \text{forward Euler}$$

For $\theta = 1$, we have

$$c_{ij}^{n+1} \frac{U_{i,j}^{n+1} - U_{i,j}^n}{\Delta t} = M_h U_{i,j}^{n+1} + f_{ij}^{n+1} \quad \text{backward Euler}$$

For $\theta = \frac{1}{2}$, we have Crank-Nicolson.

Matrix formulations

For $\theta = 1$,

$$\left(\left(\frac{c_{ij}^{n+1}}{\Delta t}\right) I - M_h\right)\vec{U}^{n+1} = \left(\frac{c_{ij}^{n+1}}{\Delta t}\right)\vec{U}^n + \vec{F}^n$$

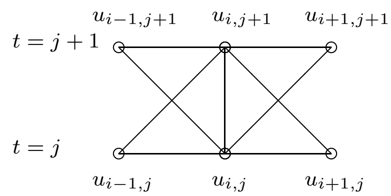

Figure 1.9: Crank-Nicolson method Stencil

For $\theta = \frac{1}{2}$,

$$\left(\left(\frac{c_{ij}^{n+1}}{\Delta t}\right) I - \frac{1}{2} M_h\right) \vec{U}^{n+1} = \left(\frac{\bar{c}_{ij}}{\Delta t} + \frac{1}{2} M_h\right) \vec{U}^n + \frac{1}{2}(\vec{F}^n + \vec{F}^{n+1})$$

where $\bar{c}_{ij} = \frac{1}{2}(c_{ij}^{n+1} + c_{ij}^n)$.

Exercise 1.4.9. (1) Consider a heat equation with $k = 1$

$$\begin{aligned} u_t &= k u_{xx} \qquad 0 < x < 1, \quad t > 0 \\ u(x,0) &= f(x) \\ u(0,t) &= g(t) \\ u(1,t) &= h(t). \end{aligned}$$

where $k > 0$ is constant. If $f(x) = \cos \pi x$, $g(t) = e^{-\frac{kt}{\pi^2}}$, $h(t) = -e^{-\frac{kt}{\pi^2}}$ it has solution $u = e^{-\frac{kt}{\pi^2}} \cos \pi x$. Use the following method to compute numerical solution up to $T = 1.0$. Check $\|u - U\|_2$ or $\|u - U\|_\infty$ for each time step $j\Delta t$.

(a) Explicit FDM with $h = \frac{1}{10}, \frac{1}{20}$, for $k = \lambda h^2$ where $\lambda = 0.2, 0.4$ and 0.6.

(b) Implicit FDM with $h = \frac{1}{10}, \frac{1}{20}$, for $k = \lambda h^2$ where $\lambda = 0.2, 0.4$, 0.6, 0.8 and 1.6 .

You can use either Gauss-Seidel type of iteration method or LU-decomposition to solve the system of equations arising in the implicit method.

1.5 Hyperbolic Equation

$$\begin{cases} u_t + a_{11} u_x + a_{12} v_x = b_1 \\ v_t + a_{21} u_x + a_{22} v_x = b_2 \end{cases} \tag{1.1}$$

where $a_{i,j}$ and u, v and function of (x,t).

$$\begin{aligned} u(x,0) &= f(x) \qquad t \geq 0 \\ v(x,0) &= g(x) \qquad -\infty < x < \infty \end{aligned}$$

Let $w = [u,v]^T$, $b = [b_1, b_2]^T$, $A = \{a_{ij}\}$. Then the D.E. is of the form

$$W_t + AW_x = b.$$

Equation (2.94) is called hyperbolic, if there exist a P such that $P^{-1}AP = diag\{\lambda_1, \lambda_2\}$ where $\lambda_i(x,t)$ are real and distinct. Let $Z = [z_1, z_2]^T$ be related to W by $W = PZ$, then

$$\begin{aligned} P_tZ + PZ_t + A(P_xZ + PZ_x) &= b \\ PZ_t + APZ_x &= b - (P_t + AP_x)Z \\ \therefore \quad Z_t + P^{-1}APZ_x &= P^{-1}\{b - (P_t + AP_x)Z\} = \beta(x,t,Z). \end{aligned}$$

Componentwise,

$$(Z_i)_t + \lambda_i(Z_i)_x = \beta_i, \qquad i = 1,2.$$

Let $x_i(t)$ be the solution of the o.d.e.

$$\frac{dx_i}{dt} = \lambda_i(x_i,t) \quad \text{such that} \quad x_i(t^*) = x^*$$

Let $Z_i(t) \equiv Z_i(x_i(t),t)$ be defined along the curve $\frac{dx_i}{dt} = \lambda_i(x_i,t)$ (called characteristics). Then

$$\frac{dZ_i}{dt} = \frac{\partial Z_i}{\partial x}\cdot\frac{dx_i}{dt} + \frac{\partial Z_i}{\partial t} = \lambda_i(x_i(t),t)\frac{\partial Z_i}{\partial x} + \frac{\partial Z_i}{\partial t} = \beta_i(x_i,t,Z)$$

Thus $Z_i(x_i(t),t)$ solve the o.d.e (p.d.e on the characterstics) with

$$Z_i(0) = Z_i(x_i(0),0) = (P^{-1}W)_i(x_i(0),0).$$

The shaded part is called the "Domain of dependence" of (x^*, t^*) and its base is called the "interval of dependence".

A necessary condition for convergence: The numerical domain of dependence must contain the analytic domain of dependence.

If the grid point is only in the inner region of the domain of dependence, then changing f by $f + \delta$, g by $d + \delta$ near the boundary yields the same solution (numerical).

Example 1.5.1. Plucked string of wave

$$\begin{cases} u_t = cv_x \\ v_t = cu_x \end{cases}$$

$$\begin{array}{ll} u(x,0) & = f(x) \\ v(x,0) & = \frac{1}{c}\int_0^x G(\sigma)\,d\sigma = g(x) \end{array}$$

$$\begin{cases} u_{tt} - c^2 u_{xx} = 0 \\ u(x,0) = f(x) \\ u_t(x,0) = cv_x(x,0) = G(x) \end{cases}$$

$$\begin{bmatrix} u \\ v \end{bmatrix}_t + \begin{bmatrix} 0 & -c \\ -c & 0 \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix}_x = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

Eigenvalues of A are $\pm c$. Corresponding to the eigenvector $\binom{1}{-1},\binom{1}{1}$ Therefore, $P = \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}$.

$$P^{-1}AP = \begin{pmatrix} c & 0 \\ 0 & -c \end{pmatrix} \quad \text{and} \quad P^{-1} = \frac{1}{2}\begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}$$

If $W = PZ$, then

$$Z_t + \begin{pmatrix} c & 0 \\ 0 & -c \end{pmatrix} Z_x = 0 \Rightarrow \begin{array}{l} \dfrac{\partial z_1}{\partial t} + c\dfrac{\partial z_1}{\partial x} = 0 \\ \dfrac{\partial z_2}{\partial t} - c\dfrac{\partial z_2}{\partial x} = 0 \end{array} \quad x(t^*) = x^*$$

Along $\frac{dx_1}{dt} = c$, $\frac{dx_2}{dt} = -c$, $\frac{dz_i}{dt} = 0$. Thus

$$\begin{array}{llll} x_1(t) & = ct + x^* - ct^* & \Rightarrow\ Z_1(ct + x^* - ct^*, t) & = Z_1(x^* - ct^*, 0) \\ x_2(t) & = -ct + x^* + ct^* & \Rightarrow\ Z_2(-ct + x^* + ct^*, t) & = Z_2(x^* + ct^*, 0) \end{array}$$

$$Z = P^{-1}W = \frac{1}{2}\begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} u \\ v \end{pmatrix}$$

$$\begin{array}{lll} Z_1(x,t) & = & \frac{1}{2}[u(x,t) - v(x,t)] = \frac{1}{2}[f(x^* - ct^*) - g(x^* - ct^*)] \\ Z_2(x,t) & = & \frac{1}{2}[u(x,t) + v(x,t)] = \frac{1}{2}[(f(x^* + ct^*) + g(x^* + ct^*)] \end{array}$$

Hence

$$\begin{bmatrix} u \\ v \end{bmatrix}_{(x^*,t^*)} = P \cdot Z = \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} \frac{1}{2} \begin{bmatrix} f(x^* - ct^*) - g(x^* - ct^*) \\ f(x^* + ct^*) + g(x^* + ct^*) \end{bmatrix}$$
$$= \frac{1}{2} \begin{bmatrix} f(x^* - ct^*) - g(x^* - ct^*) + f(x^* + ct^*) + g(x^* + ct^*) \\ -f(x^* - ct^*) + g(x^* - ct^*) + f(x^* + ct^*) + g(x^* + ct^*) \end{bmatrix}$$

These are called D'Alembert solutions.

1.5.1 Method of Characteristics

Numerical procedure "See R.S. Varga" or "Y. Gregory" Ch16

$$\begin{aligned} \tfrac{dx_i}{dt} &= \lambda_i(x_i, t), \qquad i = 1, 2, \\ \tfrac{dZ_i}{dt} &= \beta_i(x_i, t, Z_1, Z_2) \end{aligned}$$

Assume $Z_i(t, x)$ is known at t-th level (say by interpolation).
1st step: Find $P_1(\tilde{x}_1, t)$, $P_2(\tilde{x}_2, t)$ by

$$\frac{x^* - \tilde{x}_i}{\Delta t} = \lambda_i(x^*, t^*), \qquad i = 1, 2 \qquad \text{(Backward)}$$

2nd step:

$$\frac{Z_i(P^*) - Z_i(P_i)}{\Delta t} = \beta_i(P_i, Z_1(P_i), Z_2(P_i)) \qquad \text{(Forward)}$$

solve for $Z_i(P^*)$, $i = 1, 2$.

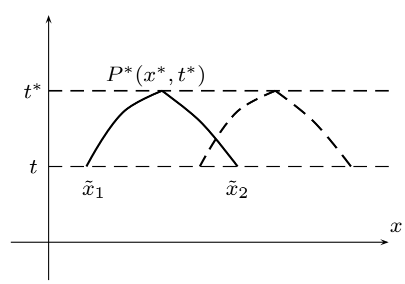

Figure 1.10: Find $\tilde{x}_1, \tilde{x}_2$