Chapter 1

The geometry of Euclidean Space

We consider the basic operations on vectors in 3 and 3 dim. space: vector addition, scalar multiplication, dot product and cross product. In section ?? we generalize these notions to n dim'l space.

1.1 Vectors in 2, 3 dim space

Definition 1.1.1. A vector in $\mathbb{R}^n, n = 2, 3$ is an ordered pair(triple) of real numbers, such as

$$(a_1, a_2), \text{ or } (a_1, a_2, a_3).$$

Here the numbers $a_1, a_2$ are called $x$ coordinate, $y$ coordinate or $x$ component, $y$ component of $(a_1, a_2)$. The point $(0, 0)$ is called the origin and denoted by $O$.

$a = (a_1, a_2)$ or $a = (a_1, a_2, a_3)$ will be standard notation for vectors. A point $P$ is represented by ordered pairs of real numbers $(a_1, a_3)$ called Cartesian coordinate) of $P$.

Vectors are identified with points in the plane or space.

$$\mathbb{R}^2 = \{(a_1, a_2) \mid a_1 \in \mathbb{R}, a_2 \in \mathbb{R}\}$$
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Vector addition and scalar multiplication—algebraic view

The operation of addition can be extended to \( \mathbb{R}^3 \). Given two triples, \( \mathbf{a} = (a_1, a_2, a_3), \mathbf{b} = (b_1, b_2, b_3) \), we define

\[
\mathbf{a} + \mathbf{b} = (a_1, a_2, a_3) + (b_1, b_2, b_3) = (a_1 + b_1, a_2 + b_2, a_3 + b_3) = \mathbf{b} + \mathbf{a}
\]

to be the sum of \((a_1, a_2, a_3)\) and \((b_1, b_2, b_3)\). The vector \( \mathbf{0} = (0, 0, 0) \) is the zero element.

Scalar multiple of a vector. For real \( s \) and vector \( \mathbf{v} \), \( s\mathbf{v} \) is the vector having magnitude \(|s|\), having same direction as \( \mathbf{v} \) when \( s > 0 \), opposite direction when \( s < 0 \). Here \( s \) is called scalar \( s\mathbf{v} \) is the scalar multiple of \( \mathbf{v} \). Componentwise, it is written as

\[
s(v_1, v_2, v_3) = (sv_1, sv_2, sv_3).
\]

Example 1.1.2. Show that \((-s)\mathbf{v} = -(sv)\) for any scalar \( s \) and vector \( \mathbf{v} \).

Commutative law and associate law for additions:

(i) \( \mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u} \) \hspace{1cm} (commutative law)

(ii) \( (\mathbf{u} + \mathbf{v}) + \mathbf{w} = \mathbf{u} + (\mathbf{v} + \mathbf{w}) \) \hspace{1cm} (associate law)

(iii) \(-\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3) = (-\mathbf{u}_1, -\mathbf{u}_2, -\mathbf{u}_3) \) \hspace{1cm} (additive inverse)

The difference is defined as

\[
(a_1, a_2, a_3) - (b_1, b_2, b_3) = (a_1 - b_1, a_2 - b_2, a_3 - b_3).
\]
1.1. VECTORS IN 2, 3 DIM SPACE

For any real \( \alpha \), and \( \mathbf{u} = (u_1, u_2, u_3) \) in \( \mathbb{R}^3 \), scalar multiple \( \alpha \mathbf{u} \) is defined as

\[
\alpha \mathbf{u} = (\alpha u_1, \alpha u_2, \alpha u_3).
\]

Additions and scalar multiplication has the following properties:

(i) \( (\alpha \beta)\mathbf{u} = \alpha (\beta \mathbf{u}) \) (associate law)
(ii) \( (\alpha + \beta)\mathbf{u} = \alpha \mathbf{u} + \beta \mathbf{u} \) (distributive law)
(iii) \( \alpha (\mathbf{u} + \mathbf{v}) = \alpha \mathbf{u} + \alpha \mathbf{v} \) (distributive law)
(iv) \( \alpha (0, 0, 0) = (0, 0, 0) \) (property of 0 )
(v) \( 0 \mathbf{u} = \mathbf{0} \) (property of 0 )
(vi) \( 1 \mathbf{u} = \mathbf{u} \) (property of 1 )

Vectors-Geometric view

We associate a vector \( \mathbf{a} \) with a point \( (a_1, a_2, a_3) \) in the space. Thus, we can visualize it with a position vector \( \mathbf{a} = (a_1, a_2, a_3) \). One can also interpret a vector as directed line segment having an initial point (Usually the initial point is at the origin), i.e., a line segment with specified magnitude and direction, and initial point at the origin. Vectors are usually denoted by boldface such as \( \mathbf{a} \) or \( \mathbf{\tilde{a}} \).

The elements in \( \mathbb{R}^3 \) are not only ordered triple of numbers, but are also regarded as vectors. We call \( a_1, a_2 \) and \( a_3 \) the components of \( \mathbf{a} \). The triple \( (0, 0, 0) \) is called (zero vector) denoted by \( \mathbf{0} \) or \( \mathbf{\tilde{0}} \).

Two vectors \( \mathbf{a} = (a_1, a_2, a_3) \) and \( \mathbf{b} = (b_1, b_2, b_3) \) are equal if \( a_1 = b_1, a_2 = b_2 \) and \( a_3 = b_3 \). Geometrically, this means they have the same direction and magnitude.

Geometric representation of vectors

See Figure 1.3. The directed line segment \( \overrightarrow{PQ} \) from \( P \) to \( Q \) is denoted by \( \overrightarrow{PQ} \). \( P \) and \( Q \) are called initial position and terminal position respectively. The vector with tail at origin is called position vector. If we move the vector in parallel, we regard it as the same vector. In other words, a vector is determined by direction and magnitude. Referring the parallelogram \( ABDC \) in Figure 1.3, we see \( \overrightarrow{AB} = \overrightarrow{CD} \) and \( \overrightarrow{AC} = \overrightarrow{BD} \).
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Figure 1.2: point \( P(a_1, a_2, a_3) \)

Figure 1.3: vector

Figure 1.4: sum of two vectors
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Figure 1.5: \( \mathbf{v} \) and \(-\mathbf{v}\)

Figure 1.6: scalar multiple of \( \mathbf{v} \)

See figure 1.4 (1). If two vectors \( \mathbf{u}, \mathbf{v} \) have same tail \( P \), we define the sum of \( \mathbf{u} \) and \( \mathbf{v} \) as the vector \( \mathbf{u} + \mathbf{v} \) having position at the opposite vertex of the parallelogram.

Alternative notation

\[
\mathbf{v} = (v_1, v_2, \cdots, v_n) = [v_1, v_2, \cdots, v_n]
\]

Figure 1.7: Addition
is called a row vector, while it can also be written as a column vector:

\[
\begin{bmatrix}
  v_1 \\
  v_2 \\
  \vdots \\
  v_n
\end{bmatrix}
\]  

(1.1)

Standard basis vectors

**Definition 1.1.3.** The following vectors \( \mathbf{i}, \mathbf{j}, \mathbf{k} \) are called (standard basis vector) of \( \mathbb{R}^3 \) (Figure 1.13).

\[
\begin{align*}
\mathbf{i} &= (1, 0, 0), \\
\mathbf{j} &= (0, 1, 0), \\
\mathbf{k} &= (0, 0, 1)
\end{align*}
\]

**Remark 1.1.4.** For a given \( \mathbf{v} = (a_1, a_2, a_3) \),

\[
(a_1, a_2, a_3) = a_1(1, 0, 0) + a_2(0, 1, 0) + a_3(0, 0, 1)
\]

we write \( \mathbf{v} = a_1 \mathbf{i} + a_2 \mathbf{j} + a_3 \mathbf{k} \).

### 1.2 Inner product, length, distance

**Dot product-Inner product**

**Definition 1.2.1.** Given two vectors \( \mathbf{a} = a_1 \mathbf{i} + a_2 \mathbf{j} + a_3 \mathbf{k} \) and \( \mathbf{b} = b_1 \mathbf{i} + b_2 \mathbf{j} + b_3 \mathbf{k} \) we define

\[
a_1 b_1 + a_2 b_2 + a_3 b_3
\]
to be the dot product or (inner product) of $a$ and $b$ and write $a \cdot b$.

**Definition 1.2.2.** If $u = (u_1, u_2, \cdots, u_n)$ is a vector in $\mathbb{R}^n$, then the length or the norm of $u$ is given by

$$||u|| = \sqrt{u_1^2 + \cdots + u_n^2} = \sqrt{u \cdot u}.$$ 

to be the dot product or (inner product) of $a$ and $b$ and write $a \cdot b$.

**Example 1.2.3.** Let $a = 2i - 3j + k$, $b = i + 2j - k$. Find

1. $a \cdot a$
2. $a \cdot b$
3. $a \cdot (a - 3b)$
4. $(3a + 2b) \cdot (a - b)$

**Proposition 1.2.4** (Properties of Inner Product). For vectors $a$, $b$, $c$ and scalar $\alpha$, the following hold:

1. $a \cdot a \geq 0$ (equality holds only when $a = 0$)
2. $a \cdot b = b \cdot a$
3. $(a + b) \cdot c = a \cdot c + b \cdot c$
4. $(\alpha a) \cdot b = \alpha(a \cdot b)$
5. $||a|| = \sqrt{a \cdot a}$

**Example 1.2.5.** For $a$, $b$, $c$ we have the following.

1. $(a - b) \cdot c = a \cdot c - b \cdot c$
2. $a \cdot (b + c) = a \cdot b + a \cdot c$
3. $a \cdot (b - c) = a \cdot b - a \cdot c$
4. $a \cdot b = \frac{1}{2}(||a||^2 + ||b||^2 - ||a - b||^2)$
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![Figure 1.9: Angle between two vectors](image)

**Length of vectors**

The length, norm of a vector \( \mathbf{a} = (a_1, a_2, a_3) \) is

\[
\sqrt{(a_1 - 0)^2 + (a_2 - 0)^2 + (a_3 - 0)^2} = \sqrt{a_1^2 + a_2^2 + a_3^2}
\]

denoted by \( \|\mathbf{a}\| \). Also we note that

\[
\|\mathbf{a}\| = (\mathbf{a} \cdot \mathbf{a})^{1/2}.
\]

**Example 1.2.6.** Find the lengths of the following vectors.

1. \( \mathbf{a} = (3, 2, 1) \)
2. \( 3\mathbf{i} - 4\mathbf{j} + \mathbf{k} \)

**Unit vectors**

\[
\mathbf{u} = \frac{\mathbf{u}}{\|\mathbf{u}\|}
\]

**Definition 1.2.7.** A vector with norm 1 is called a unit vector. Any nonzero vector \( \mathbf{u} \) can be made into a unit vector by setting \( \mathbf{u}/\|\mathbf{u}\| \). This process is called a normalization.

**Example 1.2.8.** Normalize the followings.

1. \( \mathbf{i} + \mathbf{j} + \mathbf{k} \)
2. \( 3\mathbf{i} + 4\mathbf{k} \)
3. \( a\mathbf{i} - \mathbf{j} + c\mathbf{k} \)
Distance between two points

**Definition 1.2.9.** If \( P = (u_1, u_2, u_3) \) and \( Q = (v_1, v_2, v_3) \) then

\[
d(u, v) := \|u - v\| = \sqrt{(u_1 - v_1)^2 + (u_2 - v_2)^2 + (u_3 - v_3)^2}
\]

is called the distance between \( P \) and \( Q \).

Angle between two vectors

**Proposition 1.2.10.** Let \( u, v \) be two nonzero vectors in \( \mathbb{R}^2 \) or \( \mathbb{R}^3 \) and let \( \theta \) be the angle between them. Then

\[
u \cdot v = \|u\| \|v\| \cos \theta.
\]

Hence

\[
\theta = \cos^{-1} \frac{u \cdot v}{\|u\| \|v\|}
\]

**Proof.** Let \( u = \overrightarrow{AB}, v = \overrightarrow{AC} \). Then \( u - v = \overrightarrow{CB} \). Let \( \angle CAB = \theta \). Then by

![Figure 1.10: law of cosine](image)

the law of cosine (figure 1.10) we have

\[
\|v - u\|^2 = \|u\|^2 + \|v\|^2 - 2\|u\| \|v\| \cos \theta.
\]

The left hand side is

\[
\|u - v\|^2 = (u - v) \cdot (u - v)
\]

\[
= u \cdot u - u \cdot v - v \cdot u + v \cdot v
\]

\[
= \|u\|^2 - 2 \|u\| \cdot v + \|v\|^2.
\]
Hence we obtain
\[ \|u\| \|v\| \cos \theta = u \cdot v. \]

**Corollary 1.2.11.** Two nonzero vector \( u \) and \( v \) are **perpendicular**, orthogonal if and only if \( u \cdot v = 0 \).

**Example 1.2.12.** Find the angle between \( i + j + 2k \) and \(-i + 2j + k\).

**sol.** By proposition 1.2.10,
\[
\frac{(i + j + 2k) \cdot (-i + 2j + k)}{\|i + j + 2k\| \cdot |i + 2j + k|} = \frac{-1 + 2 + 2}{\sqrt{1 + 1 + 4} \sqrt{1 + 1 + 1}} = \frac{3}{6} = \frac{1}{2}
\]
Hence the angle is \( \cos^{-1}(1/2) = \pi/3 \).

**Corollary 1.2.13.** Given two points \( A(u_1, u_2, u_3), B(v_1, v_2, v_3) \), the area of the triangle \( OAB \) is
\[
\frac{1}{2} \sqrt{(u_2v_3 - u_3v_2)^2 + (u_3v_1 - u_1v_3)^2 + (u_1v_2 - u_2v_1)^2}
\]

**Proof.** Let \( \overrightarrow{OA} = u, \overrightarrow{OB} = v, \angle BOA = \theta \). Then the area of \( \triangle OAB \) is
\[
\frac{1}{2} |OA| |OB| \sin \theta
= \frac{1}{2} \|u\| \|v\| \sqrt{1 - \cos^2 \theta}
= \frac{1}{2} \sqrt{\|u\|^2 \|v\|^2 - (u \cdot v)^2}
= \frac{1}{2} \sqrt{(u_1^2 + u_2^2 + u_3^2)(v_1^2 + v_2^2 + v_3^2) - (u_1v_1 + u_2v_2 + u_3v_3)^2}
= \frac{1}{2} \sqrt{(u_2v_3 - u_3v_2)^2 + (u_3v_1 - u_1v_3)^2 + (u_1v_2 - u_2v_1)^2}
\]

**Theorem 1.2.14** (Cauchy-Schwarz inequality). For any two vectors \( u, v \)
\[
|u \cdot v| \leq \|u\| \|v\|
\]
holds, and the equality holds iff \( u \) and \( v \) are parallel.
Proof. We may assume \( u, v \) are nonzero. Let \( \theta \) be the angle between \( u \) and \( v \). Then by prop 1.2.10

\[
|u \cdot v| = \|u\| \|v\| |\cos \theta| \leq \|u\| \|v\|
\]

holds. Since \( \|u\| \|v\| \neq 0 \), if equality holds \( |\cos \theta| = 1 \) i.e, \( \theta = 0 \) or \( \pi \). Hence \( a \) and \( b \) are parallel.

**Remark 1.2.15.** The Cauchy-Schwarz inequality reads, componentwise, as

\[
(ax + by + cz)^2 \leq (a^2 + b^2 + c^2)(x^2 + y^2 + z^2)
\]

**Theorem 1.2.16** (Triangle inequality). For any two vector \( u, v \) it holds that

\[
\|u + v\| \leq \|u\| + \|v\|
\]

and equality holds when \( u, v \) are parallel and having same direction.

Proof.

\[
\|u + v\|^2 = (u + v) \cdot (u + v) = \|u\|^2 + 2u \cdot v + \|v\|^2
\]

By C-S

\[
\|u + v\|^2 \leq \|u\|^2 + 2\|u\| \|v\| + \|v\|^2 = (\|u\| + \|v\|)^2
\]

Equality holds iff

\[
u \cdot v = \|u\| \|v\|
\]

i.e, the angle is 0.

**Definition 1.2.17.** If two vectors \( u, v \) satisfy \( u \cdot vb = 0 \) then we say they are orthogonal (perpendicular).

**Example 1.2.18.** For any real \( \theta \) two vectors are \( i_\theta = (\cos \theta)i + (\sin \theta)j \), \( j_\theta = -(\sin \theta)i + (\cos \theta)j \) orthogonal.

**Example 1.2.19.** Find a unit vector orthogonal to \( 2i - j + 3k \) and \( i + 2j + 9k \).

**sol.** Let \( ai + bj + ck \) be the desired vector. Then \( a, b, c \) are determined by

\[
2a - b + 3c = 0 \text{ orthogonality}
\]

\[
a + 2b + 9c = 0 \text{ orthogonality}
\]

\[
a^2 + b^2 + c^2 = 1 \text{ unicity}
\]
Hence the desired vector is
\[ \pm \frac{1}{\sqrt{19}} (3i + 3j - k). \]

Triangle inequality

**Theorem 1.2.20.** For any vectors \( \mathbf{a}, \mathbf{b}, \) we have
\[
\| \mathbf{a} + \mathbf{b} \| \leq \| \mathbf{a} \| + \| \mathbf{b} \|.
\]

Use C-S.

### 1.3 Vector equations of lines and planes

**Parametric equation of lines (Point-Direction form)**

Figure 1.12: A line is determined by a point and a vector
1.3. VECTOR EQUATIONS OF LINES AND PLANES

The equation of the line $\ell$ through the tip of $\overrightarrow{O}P_0$ and pointing in the direction of $P_0\overrightarrow{P}$ is

$$\ell(t) = \overrightarrow{O}P_0 + t\overrightarrow{P_0}P = x_0 + tv$$

where $t$ takes all real values. In coordinate form, we have

$$x = x_0 + at,$$
$$y = y_0 + bt,$$
$$z = z_0 + ct,$$

where $x_0 = (x_0, y_0, z_0)$ and $v = (a, b, c)$.

**Example 1.3.1.**  
(1) Find equation of line through $(2,1,5)$ in the direction of $4\overrightarrow{i} - 2\overrightarrow{j} + 5\overrightarrow{k}$.

(2) In what direction, the the line $x = 3t - 2, y = t - 1, z = 7t + 4$ points ?

**sol.**  
(1) $v = (2,1,5) + t(4,-2,5)$

(2) $(3,1,7) = 3\overrightarrow{i} + \overrightarrow{j} + 7\overrightarrow{k}$.

**Example 1.3.2.** Does the two lines $(x,y,z) = (t,-6t+1,2t-8)$ and $(3t+1,2t,0)$ intersect ?

**sol.**  
If two line intersect, we must have

$$(t_1,-6t_1,2t_1-8) = (3t_2+1,2t_2,0)$$

for some numbers $t_1, t_2$. (Note: we have used two different parameters $t_1$ and $t_2$). But since the system of equation

$$t_1 = 3t_2 + 1$$
$$-6t_1 = 2t_2$$
$$2t_1 - 8 = 0$$

has no solution, the lines do not meet.
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Two point form

We describe the equation of line through two points \( \mathbf{x}_0, \mathbf{x}_1 \).

The direction is given by \( \mathbf{v} = \mathbf{x}_1 - \mathbf{x}_0 \). So by point-direction form

\[ \ell(t) = \mathbf{x}_0 + t(\mathbf{x}_1 - \mathbf{x}_0). \]

If \( P = (x_1, y_1, z_1) \) is the tip of \( \mathbf{x}_0 \) and \( Q = (x_2, y_2, z_2) \) is the tip of \( \mathbf{x}_1 \), then \( \mathbf{v} = (x_2 - x_1, y_2 - y_1, z_2 - z_1) \). Writing it componentwise, we see

\[ x = x_1 + (x_2 - x_1)t \]
\[ y = y_1 + (y_2 - y_1)t \]
\[ z = z_1 + (z_2 - z_1)t \]

Solving these for \( t \), we see

\[ t = \frac{x - x_1}{x_2 - x_1} = \frac{y - y_1}{y_2 - y_1} = \frac{z - z_1}{z_2 - z_1} \]

This is another equation of line.

Example 1.3.3. Find eq. of a line through \((2, 1, -3)\) and \((6, -1, -5)\).

Example 1.3.4. Find eq. of line segment between \((1, 1, -3)\) and \((2, -1, 0)\)

Equation of a plane, Point-Normal form

Let \( \mathbf{x}_0 \) be a point in the plane and \( \mathbf{n} = (A, B, C) \) be a nonzero normal vector to the plane. If \( \mathbf{x} \) is any point in the plane, then

\[ \mathbf{n} \cdot (\mathbf{x} - \mathbf{x}_0) = 0. \]

(1.2)

Or equivalently,

\[ A(x - x_0) + B(y - y_0) + C(z - z_0) = 0. \]

(1.3)

Equation of a plane, vector and parametric form

Let \( \mathbf{x}_0 \) be a point in the plane \( W \). If \( \mathbf{v}_1, \mathbf{v}_2 \) are two vectors positioned with their initial point at \( \mathbf{x}_0 \) in the plane then for any vector \( \mathbf{x} \) in the plane the
vector $\mathbf{x} - \mathbf{x}_0$ lies in the plane $W$ so that there exist two scalars $t_1, t_2$ such that

$$\mathbf{x} - \mathbf{x}_0 = t_1 \mathbf{v}_1 + t_2 \mathbf{v}_2. \quad (1.4)$$

This is a parametric equation of the plane.

If $\mathbf{v}_1 = (a_1, b_1, c_1), \mathbf{v}_2 = (a_2, b_2, c_2)$ the equation becomes

$$x = x_0 + a_1 t_1 + a_2 t_2$$
$$y = y_0 + b_1 t_1 + b_2 t_2, \quad (-\infty < t_1, t_2 < \infty). \quad (1.5)$$
$$z = z_0 + c_1 t_1 + c_2 t_2$$

**Example 1.3.5.** Find eq. of plane through the origin and is parallel to the vectors $\mathbf{v}_1 = (1, 1, -3)$ and $\mathbf{v}_2 = (2, -1, 0)$