Chapter 1

The geometry of Euclidean Space

We consider the basic operations on vectors in 3 and 3 dim. space: vector addition, scalar multiplication, dot product and cross product. In section **??** we generalize these notions to n dim'l space.

1.1 Vectors in 2, 3 dim space

Definition 1.1.1. A vector in $\mathbb{R}^n, n = 2, 3$ is an ordered pair(triple) of real numbers, such as

$$(a_1, a_2), \text{ or } (a_1, a_2, a_3).$$

Here the numbers a_1, a_2 are called x **coordinate**, y **coordinate** or x **component**, y **component** of (a_1, a_2). The point $(0, 0)$ is called the **origin** and denoted by O.

$\mathbf{a} = (a_1, a_2)$ or $\mathbf{a} = (a_1, a_2, a_3)$ will be standard notation for vectors. A point P is represented by ordered pairs of real numbers (a_1, a_3) called **Cartesian coordinate)** of P.

Vectors are identified with points in the plane or space.

$$\mathbb{R}^2 = \{(a_1, a_2) \mid a_1 \in \mathbb{R}, a_2 \in \mathbb{R}\}$$

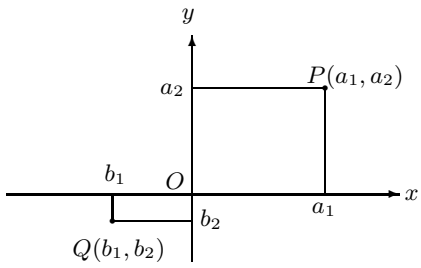


Figure 1.1: Coordinate plane

Vector addition and scalar multiplication-algebraic view

The operation of addition can be extended to $\mathbb{R}^3$. Given two triples, $\mathbf{a} = (a_1, a_2, a_3), \mathbf{b} = (b_1, b_2, b_3)$, we define

$$\mathbf{a} + \mathbf{b} = (a_1, a_2, a_3) + (b_1, b_2, b_3) = (a_1 + b_1, a_2 + b_2, a_3 + b_3) = \mathbf{b} + \mathbf{a}$$

to be the **sum** of (a_1, a_2, a_3) and (b_1, b_2, b_3). The vector $\mathbf{0} = (0, 0, 0)$ is the **zero element**.

Scalar multiple of a vector. For real s and vector $\mathbf{v}$, $s\mathbf{v}$ is the vector having magnitude $|s|$, having same direction as $\mathbf{v}$ when $s > 0$, opposite direction when $s < 0$. Here s is called **scalar** $s\mathbf{v}$ is the **scalar multiple** of $\mathbf{v}$. Componentwise, it is written as

$$s(v_1, v_2, v_3) = (sv_1, sv_2, sv_3).$$

Example 1.1.2. Show that $(-s)\mathbf{v} = -(s\mathbf{v})$ for any scalar s and vector $\mathbf{v}$.

Commutative law and associate law for additions:

(i) $\mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u}$ (commutative law)

(ii) $(\mathbf{u} + \mathbf{v}) + \mathbf{w} = \mathbf{u} + (\mathbf{v} + \mathbf{w})$ (associate law)

(iii) $-(u_1, u_2, u_3) = (-u_1, -u_2, -u_3)$ (additive inverse)

The **difference** is defined as

$$(a_1, a_2, a_3) - (b_1, b_2, b_3) = (a_1 - b_1, a_2 - b_2, a_3 - b_3).$$

For any real α, and $\mathbf{u} = (u_1, u_2, u_3)$ in $\mathbb{R}^3$, **scalar multiple** $\alpha\mathbf{u}$ is defined as

$$\alpha\mathbf{u} = (\alpha u_1, \alpha u_2, \alpha u_3).$$

Additions and scalar multiplication has the following properties:

(i) $(\alpha\beta)\mathbf{u} = \alpha(\beta\mathbf{u})$ (associate law)

(ii) $(\alpha + \beta)\mathbf{u} = \alpha\mathbf{u} + \beta\mathbf{u}$ (distributive law)

(iii) $\alpha(\mathbf{u} + \mathbf{v}) = \alpha\mathbf{u} + \alpha\mathbf{v}$ (distributive law)

(iv) $\alpha(0, 0, 0) = (0, 0, 0)$ (property of 0)

(v) $0\mathbf{u} = \mathbf{0}$ (property of 0)

(vi) $1\mathbf{u} = \mathbf{u}$ (property of 1)

Vectors-Geometric view

We associate a vector $\mathbf{a}$ with a point (a_1, a_2, a_3) in the space. Thus, we can visualize it with a position vector $\mathbf{a} = (a_1, a_2, a_3)$. One can also interpret a **vector** as **directed line segment having an initial point** (Usually the initial point is at the origin), i.e, a line segment with specified *magnitude* and *direction*, and initial point at the origin. Vectors are usually denoted by boldface such as $\mathbf{a}$ or $\vec{a}$.

The elements in $\mathbb{R}^3$ are not only ordered triple of numbers, but are also regarded as vectors. We call a_1, a_2 and a_3 the **components** of $\mathbf{a}$. The triple $(0, 0, 0)$ is called **(zero vector)** denoted by $\mathbf{0}$ or $\vec{0}$.

Two vectors $\mathbf{a} = (a_1, a_2, a_3)$ and $\mathbf{b} = (b_1, b_2, b_3)$ are equal if $a_1 = b_1, a_2 = b_2$ and $a_3 = b_3$. Geometrically, this means they have the same direction and magnitude.

Geomteric representation of vectors

See Figure 1.3. The directed line segment PQ from P to Q is denoted by $\overrightarrow{PQ}$. P and Q are called **initial position** and **terminal position** respectively. The vector with tail at origin is called **position vector**. If we move the vector in parallel, we regard it as the same vector. In other words, a vector is determined by direction and magnitude. Referring the parallelogram $ABDC$ in Figure 1.3, we see $\overrightarrow{AB} = \overrightarrow{CD}$ and $\overrightarrow{AC} = \overrightarrow{BD}$.

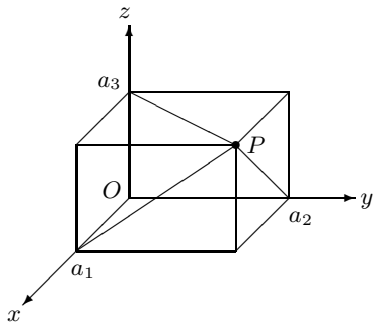


Figure 1.2: point $P(a_1, a_2, a_3)$

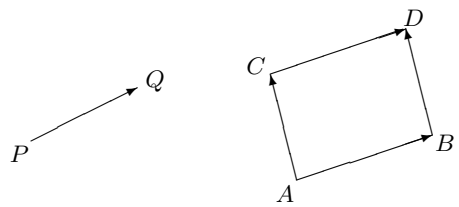


Figure 1.3: vector

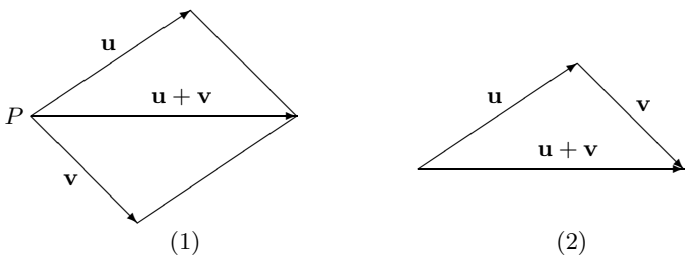


Figure 1.4: sum of two vectors

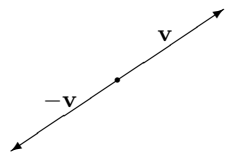


Figure 1.5: $\mathbf{v}$ and $-\mathbf{v}$

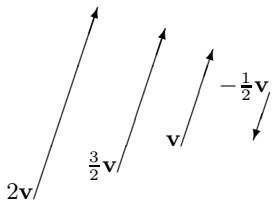


Figure 1.6: scalar multiple of $\mathbf{v}$

See figure 1.4 (1). If two vectors $\mathbf{u}$, $\mathbf{v}$ have same tail P, we define the sum of $\mathbf{u}$ and $\mathbf{v}$ as the vector $\mathbf{u}+\mathbf{v}$ having position at the opposite vertex of the parallelogram.

Alternative notation

$$\mathbf{v} = (v_1, v_2, \cdots, v_n) = [v_1, v_2, \cdots, v_n]$$

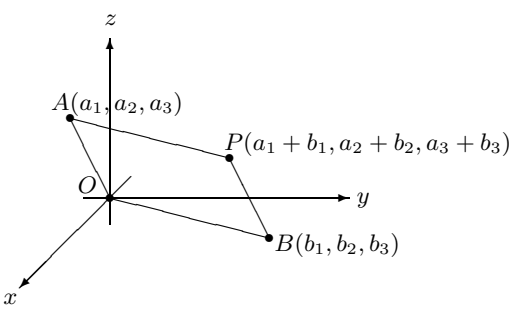


Figure 1.7: Addition

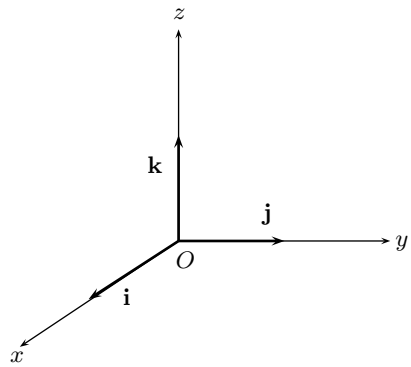


Figure 1.8: standard basis vector

is called a row vector, while it can also be written as a column vector:

$$\begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix} \tag{1.1}$$

Standard basis vectors

Definition 1.1.3. The following vectors $\mathbf{i}$, $\mathbf{j}$, $\mathbf{k}$ are called **(standard basis vector)** of $\mathbb{R}^3$ (Figure 1.13).

$$\mathbf{i} = (1, 0, 0),\ \mathbf{j} = (0, 1, 0),\ \mathbf{k} = (0, 0, 1)$$

Remark 1.1.4. For a given $\mathbf{v} = (a_1, a_2, a_3)$,

$$(a_1, a_2, a_3) = a_1(1, 0, 0) + a_2(0, 1, 0) + a_3(0, 0, 1)$$

we write $\mathbf{v} = a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k}$.

1.2 Inner product, length, distance

Dot product-Inner product

Definition 1.2.1. Given two vectors $\mathbf{a} = a_1\mathbf{i}+a_2\mathbf{j}+a_3\mathbf{k}$ and $\mathbf{b} = b_1\mathbf{i}+b_2\mathbf{j}+b_3\mathbf{k}$ we define

$$a_1b_1 + a_2b_2 + a_3b_3$$

to be the **dot product** or **(inner product)** of $\mathbf{a}$ and $\mathbf{b}$ and write $\mathbf{a} \cdot \mathbf{b}$.

Definition 1.2.2. If $\mathbf{u} = (u_1, u_2, \cdots, u_n)$ is a vector in $\mathbb{R}^n$, then the **length** or the **norm** of $\mathbf{u}$ is given by

$$\|\mathbf{u}\| = \sqrt{u_1^2 + \cdots u_n^2} = \sqrt{\mathbf{u} \cdot \mathbf{u}}.$$

to be the **dot product** or **(inner product)** of $\mathbf{a}$ and $\mathbf{b}$ and write $\mathbf{a} \cdot \mathbf{b}$.

Example 1.2.3. Let $\mathbf{a} = 2\mathbf{i} - 3\mathbf{j} + \mathbf{k}$, $\mathbf{b} = \mathbf{i} + 2\mathbf{j} - \mathbf{k}$. Find

(1) $\mathbf{a} \cdot \mathbf{a}$

(2) $\mathbf{a} \cdot \mathbf{b}$

(3) $\mathbf{a} \cdot (\mathbf{a} - 3\mathbf{b})$

(4) $(3\mathbf{a} + 2\mathbf{b}) \cdot (\mathbf{a} - \mathbf{b})$

Proposition 1.2.4 (Properties of Inner Product)**.** *For vectors* $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$ *and scalar* α*, the following hold:*

(1) $\mathbf{a} \cdot \mathbf{a} \geq 0$ *(equality holds only when* $\mathbf{a} = \mathbf{0}$*)*

(2) $\mathbf{a} \cdot \mathbf{b} = \mathbf{b} \cdot \mathbf{a}$

(3) $(\mathbf{a} + \mathbf{b}) \cdot \mathbf{c} = \mathbf{a} \cdot \mathbf{c} + \mathbf{b} \cdot \mathbf{c}$

(4) $(\alpha\mathbf{a}) \cdot \mathbf{b} = \alpha(\mathbf{a} \cdot \mathbf{b})$

(5) $\|\mathbf{a}\| = \sqrt{\mathbf{a} \cdot \mathbf{a}}$

Example 1.2.5. For $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$ we have the following.

(1) $(\mathbf{a} - \mathbf{b}) \cdot \mathbf{c} = \mathbf{a} \cdot \mathbf{c} - \mathbf{b} \cdot \mathbf{c}$

(2) $\mathbf{a} \cdot (\mathbf{b} + \mathbf{c}) = \mathbf{a} \cdot \mathbf{b} + \mathbf{a} \cdot \mathbf{c}$

(3) $\mathbf{a} \cdot (\mathbf{b} - \mathbf{c}) = \mathbf{a} \cdot \mathbf{b} - \mathbf{a} \cdot \mathbf{c}$

(4) $\mathbf{a} \cdot \mathbf{b} = \frac{1}{2}(\|\mathbf{a}\|^2 + \|\mathbf{b}\|^2 - \|\mathbf{a} - \mathbf{b}\|^2)$

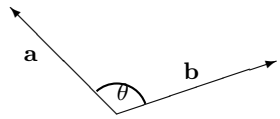


Figure 1.9: Angle between two vectors

Length of vectors

The **length, norm** of a vector $\mathbf{a} = (a_1, a_2, a_3)$ is

$$\sqrt{(a_1 - 0)^2 + (a_2 - 0)^2 + (a_3 - 0)^2} = \sqrt{a_1^2 + a_2^2 + a_3^2}$$

denoted by $\|\mathbf{a}\|$. Also we note that

$$\|\mathbf{a}\| = (\mathbf{a} \cdot \mathbf{a})^{1/2}.$$

Example 1.2.6. Find the lengths of the following vectors.

(1) $\mathbf{a} = (3, 2, 1)$

(2) $3\mathbf{i} - 4\mathbf{j} + \mathbf{k}$

Unit vectors

$$\mathbf{u} = \frac{\mathbf{u}}{\|\mathbf{u}\|}$$

Definition 1.2.7. A vector with norm 1 is called a **unit vector**. Any nonzero vector $\mathbf{u}$ can be made into a unit vector by setting $\mathbf{u}/\|\mathbf{u}\|$. This process is called a **normalization**.

Example 1.2.8. Normalize the followings.

(1) $\mathbf{i} + \mathbf{j} + \mathbf{k}$

(2) $3\mathbf{i} + 4\mathbf{k}$

(3) $a\mathbf{i} - \mathbf{j} + c\mathbf{k}$

Distance between two points

Definition 1.2.9. If $P = (u_1, u_2, u_3)$ and $Q = (v_1, v_2, v_3)$ then

$$d(\mathbf{u}, \mathbf{v}) := \|\mathbf{u} - \mathbf{v}\| = \sqrt{(u_1 - v_1)^2 + (u_2 - v_2)^2 + (u_3 - v_3)^2}$$

is called the **distance between** P and Q.

Angle between two vectors

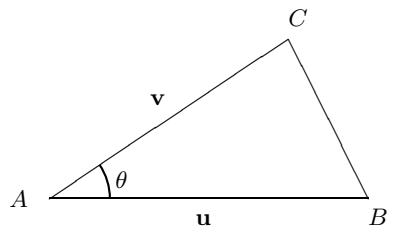
Proposition 1.2.10. *Let* $\mathbf{u}, \mathbf{v}$ *be two nonzero vectors in* $\mathbb{R}^2$ *or* $\mathbb{R}^3$ *and let* θ *be the angle between them. Then*

$$\mathbf{u} \cdot \mathbf{v} = \|\mathbf{u}\| \, \|\mathbf{v}\| \, \cos\theta.$$

Hence

$$\theta = \cos^{-1} \frac{\mathbf{u} \cdot \mathbf{v}}{\|\mathbf{u}\| \, \|\mathbf{v}\|}$$

Proof. Let $\mathbf{u} = \overrightarrow{AB}$, $\mathbf{v} = \overrightarrow{AC}$. Then $\mathbf{u} - \mathbf{v} = \overrightarrow{CB}$. Let$\angle CAB = \theta$. Then by



$$|BC|^2 = |AB|^2 + |AC|^2 - 2|AB| \, |AC| \, \cos\theta$$

Figure 1.10: law of cosine

the law of cosine (figure 1.10) we have

$$\|\mathbf{v} - \mathbf{u}\|^2 = \|\mathbf{u}\|^2 + \|\mathbf{v}\|^2 - 2\|\mathbf{u}\| \, \|\mathbf{v}\| \, \cos\theta.$$

The left hand side is

$$\begin{aligned} \|\mathbf{u} - \mathbf{v}\|^2 &= (\mathbf{u} - \mathbf{v}) \cdot (\mathbf{u} - \mathbf{v}) \\ &= \mathbf{u} \cdot \mathbf{u} - \mathbf{u} \cdot \mathbf{v} - \mathbf{v} \cdot \mathbf{u} + \mathbf{v} \cdot \mathbf{v} \\ &= \|\mathbf{u}\|^2 - 2\,\mathbf{u} \cdot \mathbf{v} + \|\mathbf{v}\|^2. \end{aligned}$$

Hence we obtain

$$\|\mathbf{u}\|\,\|\mathbf{v}\|\,\cos\theta = \mathbf{u}\cdot\mathbf{v}.$$

□

Corollary 1.2.11. *Two nonzero vector* $\mathbf{u}$ *and* $\mathbf{v}$ *are* ***perpendicular, orthogonal*** *if and only if* $\mathbf{u}\cdot\mathbf{v} = 0$.

Example 1.2.12. Find the angle between $\mathbf{i}+\mathbf{j}+2\mathbf{k}$ and $-\mathbf{i}+2\mathbf{j}+\mathbf{k}$.

sol. By proposition 1.2.10,

$$\frac{(\mathbf{i}+\mathbf{j}+2\mathbf{k})\cdot(-\mathbf{i}+2\mathbf{j}+\mathbf{k})}{\|\mathbf{i}+\mathbf{j}+2\mathbf{k}\|\,\|-\mathbf{i}+2\mathbf{j}+\mathbf{k}\|} = \frac{-1+2+2}{\sqrt{1+1+4}\sqrt{1+4+1}} = \frac{3}{6} = \frac{1}{2}$$

Hence the angle is $\cos^{-1}(1/2) = \pi/3$.

■

Corollary 1.2.13. *Given two points* $A(u_1,u_2,u_3)$, $B(v_1,v_2,v_3)$, *the area of the triangle* OAB *is*

$$\frac{1}{2}\sqrt{(u_2v_3-u_3v_2)^2+(u_3v_1-u_1v_3)^2+(u_1v_2-u_2v_1)^2}$$

Proof. Let $\overrightarrow{OA} = \mathbf{u}$, $\overrightarrow{OB} = \mathbf{v}$, $\angle BOA = \theta$. Then the area of $\triangle OAB$ is

$$\begin{aligned}
&\frac{1}{2}|OA|\,|OB|\,\sin\theta\\
=&\frac{1}{2}\|\mathbf{u}\|\,\|\mathbf{v}\|\sqrt{1-\cos^2\theta}\\
=&\frac{1}{2}\sqrt{\|\mathbf{u}\|^2\,\|\mathbf{v}\|^2-(\mathbf{u}\cdot\mathbf{v})^2}\\
=&\frac{1}{2}\sqrt{(u_1^2+u_2^2+u_3^2)(v_1^2+v_2^2+v_3^2)-(u_1v_1+u_2v_2+u_3v_3)^2}\\
=&\frac{1}{2}\sqrt{(u_2v_3-u_3v_2)^2+(u_3v_1-u_1v_3)^2+(u_1v_2-u_2v_1)^2}
\end{aligned}$$

□

Theorem 1.2.14 (Cauchy-Schwarz inequality)**.** *For any two vectors* $\mathbf{u}$, $\mathbf{v}$

$$|\mathbf{u}\cdot\mathbf{v}| \le \|\mathbf{u}\|\,\|\mathbf{v}\|$$

holds, and the equality holds iff $\mathbf{u}$ *and* $\mathbf{v}$ *are parallel.*

Proof. We may assume $\mathbf{u}$, $\mathbf{v}$ are nonzero. Let θ be the angle between $\mathbf{u}$ and $\mathbf{v}$. Then by prop 1.2.10

$$|\mathbf{u}\cdot\mathbf{v}| = \|\mathbf{u}\|\,\|\mathbf{v}\|\,|\cos\theta| \le \|\mathbf{u}\|\,\|\mathbf{v}\|$$

holds. Since $\|\mathbf{u}\|\,\|\mathbf{v}\| \neq 0$, if equality holds $|\cos\theta| = 1$ i.e, $\theta = 0$ or π. Hence $\mathbf{a}$ and $\mathbf{b}$ are parallel. □

Remark 1.2.15. The Cauchy-Schwarz inequality reads, componentwise, as

$$(ax+by+cz)^2 \le (a^2+b^2+c^2)(x^2+y^2+z^2)$$

Theorem 1.2.16 (Triangle inequality)**.** *For any two vector* $\mathbf{u}$*,* $\mathbf{v}$ *it holds that*

$$\|\mathbf{u}+\mathbf{v}\| \le \|\mathbf{u}\| + \|\mathbf{v}\|$$

and equality holds when $\mathbf{u}$*,* $\mathbf{v}$ *are parallel and having same direction.*

Proof.

$$\|\mathbf{u}+\mathbf{v}\|^2 = (\mathbf{u}+\mathbf{v})\cdot(\mathbf{u}+\mathbf{v}) = \|\mathbf{u}\|^2 + 2\mathbf{u}\cdot\mathbf{v} + \|\mathbf{v}\|^2$$

By C-S

$$\|\mathbf{u}+\mathbf{v}\|^2 \le \|\mathbf{u}\|^2 + 2\|\mathbf{u}\|\,\|\mathbf{v}\| + \|\mathbf{v}\|^2 = (\|\mathbf{u}\| + \|\mathbf{v}\|)^2$$

Equality holds iff

$$\mathbf{u}\cdot\mathbf{v} = \|\mathbf{u}\|\,\|\mathbf{v}\|$$

i.e, the angle is 0. □

Definition 1.2.17. If two vectors $\mathbf{u}$, $\mathbf{v}$ satisfy $\mathbf{u}\cdot\mathbf{v}b = 0$ then we say they are **orthogonal**(perpendicular).

Example 1.2.18. for any real θ two vectors are $\mathbf{i}_\theta = (\cos\theta)\mathbf{i} + (\sin\theta)\mathbf{j}$, $\mathbf{j}_\theta = -(\sin\theta)\mathbf{i} + (\cos\theta)\mathbf{j}$ orthogonal.

Example 1.2.19. Find a unit vector orthogonal to $2\mathbf{i}-\mathbf{j}+3\mathbf{k}$ and $\mathbf{i}+2\mathbf{j}+9\mathbf{k}$.

sol. Let $a\mathbf{i}+b\mathbf{j}+c\mathbf{k}$ be the desired vector. Then a, b, c are determined by

$$\begin{aligned} 2a - b + 3c &= 0 \text{ orthogonality} \\ a + 2b + 9c &= 0 \text{ orthogonality} \\ a^2 + b^2 + c^2 &= 1 \text{ unicity} \end{aligned}$$

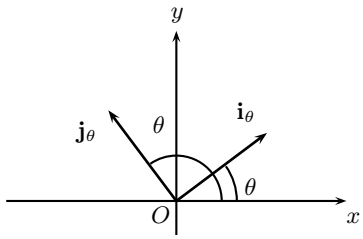


Figure 1.11: $\mathbf{i}_\theta$ and $\mathbf{j}_\theta$

Hence the desired vector is

$$\pm\frac{1}{\sqrt{19}}\left(3\mathbf{i}+3\mathbf{j}-\mathbf{k}\right).$$

$\square$

Triangle inequality

Theorem 1.2.20. *For any vectors* $\mathbf{a}, \mathbf{b}$*, we have*

$$\|\mathbf{a}+\mathbf{b}\| \leq \|\mathbf{a}\| + \|\mathbf{b}\|.$$

Use C-S.

1.3 Vector equations of lines and planes

Parametric equation of lines(Point-Direction form)

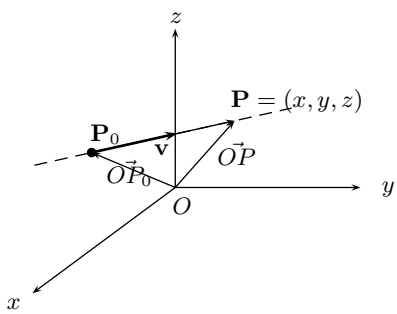


Figure 1.12: A line is determined by a point and a vector

The equation of the line ℓ through the tip of $\vec{OP_0}$ and pointing in the direction of $\vec{P_0P}$ is

$$\ell(t) = \vec{OP_0} + t\vec{P_0P} = \mathbf{x}_0 + t\mathbf{v}$$

where t takes all real values. In coordinate form, we have

$$\begin{aligned} x &= x_0 + at, \\ y &= y_0 + bt, \\ z &= z_0 + ct, \end{aligned}$$

where $\mathbf{x}_0 = (x_0, y_0, z_0)$ and $\mathbf{v} = (a, b, c)$.

Example 1.3.1. (1) Find equation of line through $(2, 1, 5)$ in the direction of $4\mathbf{i} - 2\mathbf{j} + 5\mathbf{k}$.

(2) In what direction, the the line $x = 3t - 2, y = t - 1, z = 7t + 4$ points ?

sol. (1) $\mathbf{v} = (2, 1, 5) + t(4, -2, 5)$
(2) $(3, 1, 7) = 3\mathbf{i} + \mathbf{j} + 7\mathbf{k}$.

■

Example 1.3.2. Does the two lines $(x, y, z) = (t, -6t + 1, 2t - 8)$ and $(3t + 1, 2t, 0)$ intersect ?

sol. If two line intersect, we must have

$$(t_1, -6t_1, 2t_1 - 8) = (3t_2 + 1, 2t_2, 0)$$

for some numbers t_1, t_2.(Note: we have used two different parameters t_1 and t_2). But since the system of equation

$$\begin{aligned} t_1 &= 3t_2 + 1 \\ -6t_1 &= 2t_2 \\ 2t_1 - 8 &= 0 \end{aligned}$$

has no solution, the lines do not meet.

■

Two point form

We describe the equation of line through two points $\mathbf{x}_0, \mathbf{x}_1$.

The direction is given by $\mathbf{v} = \mathbf{x}_1 - \mathbf{x}_0$. So by point -direction form

$$\ell(t) = \mathbf{x}_0 + t(\mathbf{x}_1 - \mathbf{x}_0).$$

If $P = (x_1, y_1, z_1)$ is the tip of $\mathbf{x}_0$ and $Q = (x_2, y_2, z_2)$ is the tip of $\mathbf{x}_1$, then $\mathbf{v} = (x_2 - x_1, y_2 - y_1, z_2 - z_1)$. Writing it componentwise, we see

$$\begin{aligned} x &= x_1 + (x_2 - x_1)t \\ y &= y_1 + (y_2 - y_1)t \\ z &= z_1 + (z_2 - z_1)t \end{aligned}$$

Solving these for t, we see

$$t = \frac{x - x_1}{x_2 - x_1} = \frac{y - y_1}{y_2 - y_1} = \frac{z - z_1}{z_2 - z_1}$$

This is another equation of line.

Example 1.3.3. Find eq. of a line through $(2, 1, -3)$ and $(6, -1, -5)$.

Example 1.3.4. Find eq. of line segment between $(1, 1, -3)$ and $(2, -1, 0)$

Equation of a plane, Point-Normal form

Let $\mathbf{x}_0$ be a point in the plane and $\mathbf{n} = (A, B, C)$ be a nonzero normal vector to the plane. If $\mathbf{x}$ is any point in the plane, then

$$\mathbf{n} \cdot (\mathbf{x} - \mathbf{x}_0) = 0. \tag{1.2}$$

Or equivalently,

$$A(x - x_0) + B(y - y_0) + C(z - z_0) = 0. \tag{1.3}$$

Equation of a plane, vector and parametric form

Let $\mathbf{x}_0$ be a point in the plane W. If $\mathbf{v}_1, \mathbf{v}_2$ are two vectors positioned with their initial point at $\mathbf{x}_0$ in the plane then for any vector $\mathbf{x}$ in the plane the

vector $\mathbf{x}-\mathbf{x}_0$ lies in the plane W so that there exist two scalars t_1, t_2 such that

$$\mathbf{x} - \mathbf{x}_0 = t_1\mathbf{v}_1 + t_2\mathbf{v}_2. \tag{1.4}$$

This is a parametric equation of the plane.

If $\mathbf{v}_1 = (a_1, b_1, c_1), \mathbf{v}_2 = (a_2, b_2, c_2)$ the equation becomes

$$\begin{aligned} x &= x_0 + a_1t_1 + a_2t_2 \\ y &= y_0 + b_1t_1 + b_2t_2, \ (-\infty < t_1, t_2 < \infty) \ . \\ z &= z_0 + c_1t_1 + c_2t_2 \end{aligned} \tag{1.5}$$

Example 1.3.5. Find eq. of plane through the origin and is parallel to the vectors $\mathbf{v}_1 = (1, 1, -3)$ and $\mathbf{v}_2 = (2, -1, 0)$