Chapter 1

Preliminary

1.1 2nd order linear p.d.e. in two variables

General 2nd order linear p.d.e. in two variables is given in the following form:

$$L[u] = Au_{xx} + 2Bu_{xy} + Cu_{yy} + Du_x + Eu_y + Fu = G, \text{ in } \Omega$$

where Ω is an open set in $\mathbb{R}^2$. According to the relations between coefficients, the p.d.es are classified into 3 categories, namely,

elliptic	if $AC - B^2$	$> 0, A$, C has the same sign and B is small
hyperbolic	if $AC - B^2$	< 0
parabolic	if $AC - B$	$= 0$

Furthermore, if the coefficients A, B and C are constant, it can be written as

$$[\frac{\partial}{\partial x}, \frac{\partial}{\partial y}] \begin{bmatrix} A & B \\ B & C \end{bmatrix} \begin{bmatrix} \frac{\partial u}{\partial x} \\ \frac{\partial u}{\partial y} \end{bmatrix} + Du_x + Eu_y + Fu = G.$$

Auxiliary condition

$$\begin{cases} \text{B.C. - Dirichlet, Neumann, Robin} \\ \text{I.C.} \\ \text{Interface Cond} \end{cases}$$

The condition $u = g_0$ on $\Gamma_0 \subset \partial\Omega$ is called the *Dirichlet B.C.*, the condition $\frac{\partial u}{\partial n} = g_1$ on $\Gamma_1 \subset \partial\Omega$ is called the *Neumann B.C.*, the condition $\alpha\frac{\partial u}{\partial n} + u = g_2$ on $\Gamma_2 \subset \partial\Omega$ is called the *Robin B.C.* If some of these conditions are mixed, we say it is a mixed B.C.

Dirichlet Problem

In general, 2nd order linear p.d.e. in $\mathbb{R}^d$ can be given in the following convenient form:

$$\begin{aligned} L[u] &= -\sum_{i,j=1}^d \frac{\partial}{\partial x_i}\left(a_{ij}\frac{\partial u}{\partial x_j}\right) + cu = -\nabla \cdot \mathcal{A}\nabla u + cu = f \text{ in } \Omega \\ &\text{BC's} \end{aligned} \tag{1.1}$$

$\mathcal{A} = (a_{ij})_{i,j=1}^d$ is the coefficient matrix. The equation will be elliptic if $\mathcal{A}$ is positive definite. Here u maybe electromagnetic potential, displacement of elastic membrane, temperature, concentration of chemical component, or pressure of a fluid(in porous media), etc.

Notations

$$\partial_i u = \frac{\partial u}{\partial x_i},\ \partial_{ij} u = \frac{\partial^2 u}{\partial x_j \partial x_i}, \Delta = (\partial_{11} + \cdots \partial_{dd})$$

so that

$$\nabla u = (\partial_1 u, \cdots, \partial_d u)^T, \quad \nabla \cdot \mathbf{v} = (\partial_1 v_1 + \cdots + \partial_d v_d)$$

represent, and a new vector field.

$$\Delta: \text{ Laplace operator } = \nabla \cdot \nabla = \nabla^2$$

$$C(\Omega), C^1(\Omega), C(\bar{\Omega}), C^k(\bar{\Omega}), C(\partial\Omega)$$

- behavior near boundary
- Equation (1.1) holds in an open set Ω.

Definition 1.1.1 (Classical solution)**.** *Assume $f \in C(\Omega)$, $g \in C(\partial\Omega)$. A function u is called a classical solution if $\in C^2(\Omega) \cap C(\bar{\Omega})$.*

We say a pde is "well posed" if a solution exists and the solution depends continuously on the data. There are basically two class of method to discretize it,

(1) Finite Difference method

(2) Finite Element method

1.2 The Maximum Principle

In this section, we assume L is symmetric positive definite, i.e, the matrix A is symmetric positive definite.

Theorem 1.2.1 (Maximum Principle)**.** *Assume A is positive definite symmetric, $c \geq 0$. Let u be the solution of elliptic p.d.e. given by*

$$\begin{aligned} L[u] &= -\sum_{ij} \frac{\partial}{\partial x_i}\left[a_{ij}\frac{\partial u}{\partial x_j}\right] + cu = -\nabla \mathcal{A} \nabla u + cu = f \leq 0 \ in\ \Omega \\ u &= g\ on\ \partial\Omega \end{aligned}$$

Then we have

$$|u(x,y)| \leq \max_{(x,y)\in\partial\Omega} |u(x,y)|,\ (x,y) \in \Omega$$

Proof. Assume $c > 0$. By positive definite, there exists orthogonal matrix P depending on (x, y) such that $P^T AP = \text{diag}\{d_1, d_2\}$ where $d_1, d_2 > 0$. If u has a positive maximum at some interior point $Q = (x^*, y^*)$ of Ω, then define

$$\begin{pmatrix} s \\ t \end{pmatrix} = P^T(x^*, y^*) \begin{pmatrix} x \\ y \end{pmatrix}$$

so that $L[u] = -\nabla_{(s,t)} P^T AP \nabla_{(s,t)} u + cu = 0$. At Q, $u_s(Q) = u_t(Q) = 0$, $u_{ss}(Q) \le 0$ and $u_{tt}(Q) \le 0$. Hence

$$L[u] = -d_1 u_{ss}(Q) - d_2 u_{tt}(Q) + c(Q)u(Q) = f \le 0$$

Remembering, $d_1 > 0$, $d_2 > 0$, $cu > 0$ this is a contradiction. Similarly, u cannot have negative minimum.

Now if $c \ge 0$ not $c > 0$ we consider a perturbation. Choose α so large that $L[e^{\alpha x}] = -(d_1\alpha^2 + d_2\alpha^2 - c)e^{\alpha x} < 0$ and let $v = u + Ee^{\alpha x}$.

$$L[v] = L[u] + EL[e^{\alpha x}] < 0 \quad \text{for all} \quad E > 0.$$

Suppose v has a pos. max. at Q, an interior point of Ω. Then $L[v] = -d_1 v_{ss}(Q) - d_2 v_{tt}(Q) + c(Q)v(Q) \ge 0$, a contradiction. Hence $u(x, y) \le v(x, y) < \max_{\partial\Omega}\{u + Ee^{\alpha x}\}$. Let $E \to 0$. Then

$$u(x, y) \le \max_{\partial\Omega} u.$$

□

Applying maximum principle to u and $-u$, we obtain the following result.

Corollary 1.2.1. *If*

$$\begin{aligned} L[u] &= 0 \quad && \text{in } \Omega \\ u &= 0 \quad && \text{on } \partial\Omega, \end{aligned}$$

then $u \equiv 0$.

As a consequence we have uniqueness of solution.

Corollary 1.2.2. *If* u_1, u_2 *satisfy*

$$\begin{aligned} L[u_i] &= f \quad && \text{in } \Omega \\ u_i &= g \quad && \text{on } \partial\Omega, \end{aligned}$$

then $u_1 = u_2$.

Corollary 1.2.3 (Continuous dependence of boundary data)**.** *If* u_1, u_2 *are the solutions of pde with two different boundary values. Then*

$$\sup_{\Omega} |u_1(x) - u_2(x)| \le \sup_{\partial\Omega} |u_1(x) - u_2(x)|.$$

1.3 Finite Difference Method

Let $u(x)$ be a function defined on $\Omega \subset \mathbb{R}^n$. Let $U_{i,j}$ be the function defined over discrete domain $\{(x_i, y_j)\}$ (such points are grid points) that may approximate $u_{i,j} = u(x_i, y_j)$. Such functions are called grid functions.

Difference operator

$$\begin{aligned}
\partial^+ U_i &= \frac{U_{i+1} - U_i}{h_{i+1}}, \quad \text{forward difference} \\
\partial^- U_i &= \frac{U_i - U_{i-1}}{h_i}, \quad \text{backward difference} \\
\partial^0 U_i &= \frac{U_{i+1} - U_{i-1}}{h_i + h_{i+1}}, \quad \text{central difference} \\
\partial^2 U_i &= \frac{2(\partial^+ - \partial^-)}{h_i + h_{i+1}}, \quad \text{central 2nd difference}
\end{aligned}$$

Example 1.3.1. *Note that*

$$\begin{aligned}
\partial^+ U_i &= \frac{U_{i+1} - U_i}{h_{i+1}} = \partial^0 U_{i+1/2}, \quad \textit{central difference at } x_{i+1/2} \\
\partial^- U_i &= \frac{U_i - U_{i-1}}{h_i} = \partial^0 U_{i-1/2}, \quad \textit{central difference at } x_{i-1/2}
\end{aligned}$$

H.W 1. We can interpret $\partial^2 U_i$ as a central difference $2\frac{\partial^0 U_{i+1/2} - \partial^0 U_{i-1/2}}{h_i + h_{i+1}}$. Derive the truncation error.

Example 1.3.2. *Consider the following second order two point boundary value problem :*

$$-u''(x) = f(x), \; BC.\, u(a) = c, u(b) = d.$$

Assume a mesh $a = x_0 < x_1 < \cdots < x_N = b, \Delta x_i = x_{i+1} - x_i = h$. *Replacing the derivative by a difference quotient, we obtain*

$$-\frac{u_{i-1} - 2u_i + u_{i+1}}{h^2} + O(h^2) = f(x_i), \quad i = 1, \cdots N-1, u_0 = c, u_N = d$$

Dropping the error term, we obtain a system of linear equations in the approximate values U_i*:*

$$-\frac{U_{i-1} - 2U_i + U_{i+1}}{h^2} = f_i = f(x_i), \quad i = 1, \cdots N-1, U_0 = c, U_N = d.$$

This is an $(N-1) \times (N-1)$ *matrix equations.*

$$h^{-2} \begin{pmatrix} 2 & -1 & & & & \\ -1 & 2 & -1 & & & \\ & & \cdot & \cdot & \cdot & \\ & & & -1 & 2 & -1 \\ & & & & -1 & 2 \end{pmatrix} \begin{pmatrix} U_1 \\ \cdot \\ \cdot \\ \cdot \\ U_{N-1} \end{pmatrix} = \begin{pmatrix} f_1 \\ \cdot \\ \cdot \\ \cdot \\ f_{N-1} \end{pmatrix} + h^{-2} \begin{pmatrix} c \\ 0 \\ 0 \\ 0 \\ d \end{pmatrix}$$

Above equation can be written as $L_h U^h = F^h$, *where* $U^h = (U_1, \cdots, U_{N-1})$ *and* $F^h = (f_i) +$ *boundary terms. It is called a difference equation for a given differential equation.*

Exercise 1.3.1. *Write down a matrix equation for the same problem with second boundary condition changed to the normal derivative condition at* b, *i.e,* $u'(b) = d$. *If one uses first order difference for derivative, we lose accuracy.*

We need an extra equation in this case. There are several choices:

(1) Use first order backward difference scheme

$$\frac{U_N - U_{N-1}}{h} = d$$

and append this to the last eq.(first order)

(2) Assume the D.E. holds at the end point and use central difference equation by using a fictitious point U_{N+1} :

$$-\frac{1}{h^2}(U_{N-1} - 2U_N + U_{N+1}) = f(1) \tag{1.2}$$

$$\frac{1}{2h}(U_{N+1} - U_{N-1}) = d \tag{1.3}$$

Substitute the last eq. into first eq., we have

$$\frac{U_N - U_{N-1}}{h^2} = \frac{d}{h} + \frac{f(1)}{2}. \tag{1.4}$$

The matrix is still symmetric; Eq. (1.4) can be viewed as centered difference approximation to $u'(x_n - \frac{h}{2})$ and rhs as the first two terms of Taylor expansion

$$u'(x_n - \frac{h}{2}) = u'(x_n) - \frac{h}{2}u''(x_n) + \cdots$$

(3) Approximate $u'(1)$ by higher order scheme such as

$$\frac{u_{N-2} - 4u_{N-1} + 3u_N}{2h} = d.$$

In this case one has second order truncation error (Show it) but the matrix loses symmetry.

Exercise 1.3.2. *(1) Solve above D.E. (Dirichlet and Neumann) with* $f = 2 - 6x$ *so that* $u = x - x^2 + x^3$ *and the following BCs (with* $h = 1/n$, $n = 5, 10, 20, 40$*). Report the error* $\|u - u_h\|_\infty = \max_i |(u - u_h)(x_i)|$.

(a) $u(0) = 0, u(1) = 1$

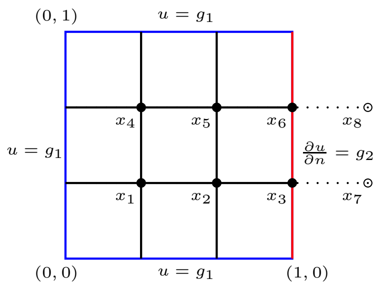

Figure 1.1: Grid for the Neumann problem

(b) $u(0) = 0, u'(1) = 2$

(2) Write down the stiffness matrix of 2D problem, $\Omega = [0,1] \times [0,1]$ *with Neumann condition at* $x = 1$ *on the unit square with* 3×3 *grid. i.e.,* $\frac{\partial u}{\partial n} = g_2$ *along* $x = 1$. *Label the node* x_1, x_2, x_3 *lexicographically from the bottom row.(excluding the boundary) There are two possibilities to treat the Neumann condition: One is to use backward difference. Another is to assume fictitious values and use central difference, then incorporate them into the five point stencil. In other words, use* $u_x \doteq \frac{u_7 - u_2}{2h} = g_2(1, \frac{1}{3})$ *and substitute into the stencil, the third equation becomes*

$$\frac{1}{h^2}(-2u_2 + 4u_3 - u_6) = (f + \frac{2}{h}g_2)(1, \frac{1}{3}).$$

1.3.1 Convergence of Finite Difference Method

For $u \in C^4$, use the Taylor expansion about x_i

$$\begin{aligned}
u(x_{i+1}) &= u(x_i) + h_i u'(x_i) + \frac{h_i^2}{2}u''(x_i) + \frac{h^3}{6}u^{(3)}(x_i) + \frac{h^4}{24}u^{(4)}(\xi_1), \quad \xi_1 \in (x_i, x_{i+1}) \\
u(x_{i-1}) &= u(x_i) - h_i u'(x_i) + \frac{h_i^2}{2}u''(x_i) - \frac{h^3}{6}u^{(3)}(x_i) + \frac{h^4}{24}u^{(4)}(\xi_2), \quad \xi_2 \in (x_i, x_{i+1}).
\end{aligned}$$

Assume $h_i = h_{i+1}$ and we substitute the solution of differential equation into the difference equation. Using $-u'' = f$ we obtain

$$
\begin{aligned}
& \frac{(-u_{i-1} + 2u_i - u_{i+1})}{h^2} - f(x_i) \\
= & \frac{1}{h^2}(-u_i + hu_i' - \frac{h^2}{2}u_i'' + \frac{h^3}{6}u^{(3)} - \frac{h^4}{24}u^{(4)}(\xi_1) + 2u_i) \\
& + \frac{1}{h^2}(-u_i - hu_i' - \frac{h^2}{2}u_i'' - \frac{h^3}{6}u^{(3)} - \frac{h^4}{24}u^{(4)}(\xi_2)) - f(x_i) \\
= & -u_i'' - f(x_i) - \frac{h^2}{24}(u^{(4)}(\xi_1) + u^{(4)}(\xi_2)) \text{ truncation error} \\
= & \frac{h^2}{12}\max|u^{(4)}|.
\end{aligned}
$$

Given a pde $Lu = f$ with B.C, we associate a finite difference scheme

$$L_h U^h = F^h.$$

We let $\tau_h = L_h u - F^h$ and call it the **truncation error**.

Definition 1.3.1. *We say a difference scheme is* ***consistent*** *if the truncation error approaches zero as h approaches zero, in other words, if*

$$\tau_h = L_h u - F^h \to 0$$

in some norm.

Truncation error measures how well the difference equation approximates the differential equation. But it does not measure the actual error in the solution. Let $\eta = u - U^h$ be the actual **discretization error.** Then we have

$$
\begin{aligned}
L_h \eta &= L_h(u - U^h) \qquad (1.5) \\
&= L_h u - F^h = \tau_h. \qquad (1.6)
\end{aligned}
$$

Definition 1.3.2. *L_h is said to be* ***stable*** *if L_h^{-1} is bounded, i.e, L_h is stable if there is a constant $C > 0$ independent of h such that*

$$\|U^h\| \le C\|F^h\| \quad \text{for all } h > 0.$$

Definition 1.3.3. *A finite difference scheme is said to* ***converge*** *if*

$$\|U^h - u\| \to 0 \qquad \text{as } h \to 0.$$

If $\|u - U^h\| = O(h^p)$ then we say the order of convergence is p.

Theorem 1.3.3 (P. Lax)**.** *Given a consistent scheme, stability is equivalent to convergence.*

Proof. Assume the stability. From $L_h u - f = \tau^h,\ L_h U^h - F^h = 0$, we have $L_h(u - U^h) = \tau^h$. Then,

$$\|u - U^h\| = \|L_h^{-1} \circ L_h(u - U^h)\| \le C\|L_h(u - U^h)\| = C\|\tau^h\| \to 0.$$

Hence the scheme converges. Now we show that a convergent scheme is stable. From the theory of p.d.e, we know $\|u\| \le C\|f\|$. Hence

$$\|U^h\| \le \|U^h - u\| + \|u\| \le C\|f\| + O(\tau^h) = C\|F^h\| + O(\tau^h).$$

□

1.4 FDM for Elliptic equation in 2D

Consider the following elliptic problem:

$$\begin{aligned} -\Delta u &= f \text{ in } \Omega \\ u &= g_1 \text{ on } \Gamma_1 \\ \frac{\partial u}{\partial n} &= g_2 \text{ on } \partial\Omega\backslash\Gamma_1. \end{aligned} \tag{1.7}$$

More generally, we may consider

$$L[u] = -(a_{11}u_x)_x - (a_{22}u_y)_y + cu = f, (with\ B.C.).$$

We solve it by the Finite Difference Method. We assume $\Omega = (0, a) \times (0, b)$ is a rectangular domain. Divide it by horizontal and vertical grid lines $x = ih_1(i = 1, \cdots, \ell - 1)$ and $y = jh_2(j = 1, \cdots, m - 1)$, where $h_1 = a/\ell$ and $h_2 = b/m$ for some integers ℓ, m. For simplicity we assume a, b are given so that $h := h_1 = h_2$. Let the discrete domain be defined as

$$\Omega_h = \{(ih, jh) | i = 1, \cdots, \ell - 1,\ j = 1, \cdots, m - 1\},$$

$\partial\Omega_h$ is obviously defined and we let $\bar{\Omega}_h = \Omega_h \cup \partial\Omega_h$. The F.D. discretization consists of the following steps:

(1) Approximate the D.E. $-(u_{xx} + u_{yy}) = f$ by a finite difference at each interior mesh pt.

(2) The unknown function u is replaced by the grid function U^h.

$$\begin{aligned} u(x+h) &= u(x) + hu_x(x) + \tfrac{h^2}{2}u_{xx}(x) + \tfrac{h^3}{6}u_{xxx}(x) + O(h^4) \\ u(x-h) &= \ldots \end{aligned}$$

$$\frac{u(x+h) - 2u(x) + u(x-h)}{h^2} = u_{xx}(x) + O(h^2)$$

$$\begin{aligned} u_{xx}(x,y) &\doteq [u(x+h,y) - 2u(x,y) + u(x-h,y)]/h^2 \\ u_{yy}(x,y) &\doteq [u(x,y+h) - 2u(x,y) + u(x,y-h)]/h^2 \end{aligned}$$

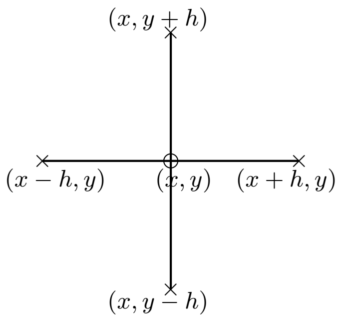

Figure 1.2: 5-point Stencil

For each point (interior mesh pt), approximate $\nabla^2 u = \Delta u$ by 5-point stencil. It is called a Molecule, Stencil, Star, etc. By Girshgorin disc theorem, the matrix is nonsingular. $L[u]$ is called **differential operator** while $L_h[u]$ is called **finite difference operator**, e.g.,

$$L_h[u](x,y) = [-4u(x,y)+u(x+h,y)+u(x-h,y)+u(x,y+h)+u(x,y-h)]/h^2$$

With uniform meshes

$$\begin{aligned} u_x(x) &\doteq \tfrac{u(x+h)-u(x-h)}{2h} \\ (u_x)_x(x) &\doteq \tfrac{u_x(x+\frac{h}{2})-u_x(x-\frac{h}{2})}{h} \qquad \text{Central difference} \end{aligned}$$

Note that

$$\begin{aligned} u_x(x+\tfrac{h}{2}) &\doteq \tfrac{u(x+h)-u(x)}{h} \\ u_x(x-\tfrac{h}{2}) &\doteq \tfrac{u(x)-u(x-h)}{h} \end{aligned}$$

For a problem with variable coefficients $a(x,y)$, we use central difference

$$(a_{11}u_x)_x \doteq [(a_{11}u_x)(x+\frac{h}{2}) - (a_{11}u_x)(x-\frac{h}{2})]/h$$

Assume the differential operator is of the form(with $c > 0$):

$$L[u] \equiv -[u_{xx} + u_{yy}] + cu = f.$$

The discretized equation is
$L_h[U^h] = F(x,y)$ where

$$L_h[U^h] = \frac{1}{h^2}\begin{pmatrix} 4+ch^2 & -1 & -1 & 0 \\ -1 & 4+ch & 0 & -1 \\ -1 & 0 & 4+ch^2 & -1 \\ 0 & -1 & -1 & 4+ch^2 \end{pmatrix}\begin{pmatrix} U_1 \\ U_2 \\ U_3 \\ U_4 \end{pmatrix}.$$

(1) For the true solution u, we have $L_h[u] = L[u] + O(h^2)$(truncation error) as $h \to 0$.

(2) $L_h U^h = F^h + Bdy$, $L_h u = [-\Delta u + cu + O(h^2)] + Bdy$.

We have
$$L_h(U^h - u) = \tau_h = O(h^2).$$
Let A be the matrix representation of L_h then with abuse of notations(e.g, u is also treated as a vector of finite entries), the discretization error $U^h - u = A^{-1}\tau_h$ satisfies
$$\|U^h - u\| \le \|A^{-1}\| \cdot \|\tau_h\| \le \|A^{-1}\| O(h^2).$$
If we can show $\|A^{-1}\|$ is bounded independent of h, we can show the convergence of the scheme. We proceed as follows: If we put $D = diagA = \{a_{11}, \dots, a_{nn}\}$, then $D^{-1}A(U^h - u) = D^{-1}\tau_h$. Write $D^{-1}A = I + B$, where B is off diagonal. Then we know $\|B\|_\infty = \frac{4}{4+ch^2} < 1$ since $c > 0$. Thus $(D^{-1}A)^{-1} = (I + B)^{-1}$ exists and
$$\|(D^{-1}A)^{-1}\|_\infty = \|(I + B)^{-1}\|_\infty \le \frac{1}{1 - \|B\|_\infty} \le \frac{4 + ch^2}{ch^2}.$$
Hence
$$\|U^h - u\|_\infty \le \|(D^{-1}A)^{-1}\|_\infty \cdot \|D^{-1}\tau_h\|_\infty \le \frac{4 + ch^2}{ch^2} \cdot \frac{h^2}{4 + ch^2} O(h^2) = O(h^2) \to 0.$$
Thus, we have proved the following result.

Theorem 1.4.1 (Convergence of FDM -special case)**.** *Assume*

(1) $u \in C^4(\Omega)$

(2) $c > 0$

(3) uniform mesh

Then $\|U^h - u\|_\infty = O(h^2)$ *as* $h \to 0$.

Note that FDM requires high regularity.

Checking the order of convergence

Assuming the error is of the form $\|U^h - u\| = Mh^\alpha$ for some norm $\|\cdot\|$, we see
$$\frac{\|U^h - u\|}{\|U^{h_0} - u\|} = \frac{Mh^\alpha}{Mh_0^\alpha} = (\frac{h}{h_0})^\alpha.$$
Hence
$$\alpha = \log\left[\frac{\|U^h - u\|}{\|U^{h_0} - u\|}\right] \Big/ \log(\frac{h}{h_0}).$$
If the exact solution u is not known, we replace u by $U_{h_{min}}$ for sufficiently small h_{min}. Typically, we take $h_0 = h/2$.

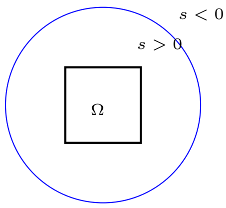

Figure 1.3: Region for proving Error estimate

1.5 Sobolev Spaces

Multi-index

For $\alpha = (\alpha_1, \cdots, \alpha_d), (\alpha_i \in \mathbb{Z}^+)$, let $|\alpha| = \sum_i \alpha_i$ and

$$\mathbf{x}^\alpha = x_1^{\alpha_1} \cdots x_d^{\alpha_d}, \ \partial^\alpha u = \partial_1^{\alpha_1} \cdots \partial_d^{\alpha_d} u = \frac{\partial^{|\alpha|} u}{\partial x_1^{\alpha_1} \cdots \partial x_d^{\alpha_d}}.$$

Weak Derivative

We denote the inner product $\int_\Omega u(x)v(x)dx$ on $L^2(\Omega)$ by (u, v).

Definition 1.5.1. *(1) Let $u \in L^2(\Omega)$. Given a multi-index α, we say u has a weak derivative (of order α) if there exists a function $v \in L^2(\Omega)$ such that*

$$\int_\Omega v\phi \, dx = (-1)^{|\alpha|} \int_\Omega u \partial^\alpha \phi \, dx \tag{1.8}$$

for all $\phi \in C_0^\infty(\Omega)$. In this case we write $\partial^\alpha u = v$. Such a derivative is unique in $L^2(\Omega)$.

(2) For a vector field $\mathbf{q} = (L^2(\Omega))^d$, we say the weak divergence of $\mathbf{q}$ is well defined if there exist a function $v \in L^2(\Omega)$ satisfying

$$\int_\Omega v\phi \, dx = -\int_\Omega \mathbf{q} \cdot \nabla \phi \, dx \tag{1.9}$$

for all $\phi \in C_0^\infty(\Omega)$. In this case we let $\nabla \cdot \mathbf{q} = v$.

1.5.1 Sobolev Spaces

Let $m \in \mathbb{Z}^+$. The *Sobolev space* $H^m(\Omega)$ is the set of all functions u in $L^2(\Omega)$ which possess weak-derivatives $\partial^\alpha u \in L^2(\Omega)$ for all $|\alpha| \le m$. On $H^m(\Omega)$, we define the inner product

$$(u, v)_m = \sum_{|\alpha| \le m} (\partial^\alpha u, \partial^\alpha v) \quad \text{for } u, v \in H^m(\Omega) \tag{1.10}$$

with the corresponding norm

$$\|u\|_m = \|u\|_{H^m(\Omega)} = \big(\sum_{|\alpha|\le m} \int_\Omega |\partial^\alpha u|^2 dx\big)^{1/2} := (u,u)_m^{1/2}. \tag{1.11}$$

The norm $\|\cdot\|_m$ is called a *Sobolev norm.*

Theorem 1.5.1. *The space $H^m(\Omega)$ equipped with the inner product $(\cdot,\cdot)_m$ and the norm $\|\cdot\|_m$ is a Hilbert space.*

Exercise 1.5.2. *(1) Define the weak gradient of a function $u \in L^2(\Omega)$.*

(2) Show that there exist a sequence $u_n \in C(0,1)$, $\|u_n\|_0 \le 1$ but $\|u_n\|_1 \to \infty$ as $\to \infty$

(3) Let $V = C^1(\Omega)$, $\Omega = (0,1)$. Show that there is a Cauchy sequence $\{u_n\}$ in V such that $u_n \to u = x^\alpha(1-x)^\alpha$, $\frac{1}{2} < \alpha < 1$, but $u \notin V$. Thus V is not complete with respect to the norm $\|\cdot\|_1$.

Theorem 1.5.3. *(Sobolev imbedding Lemma) Let $m,k \in \mathbb{Z}$. If $m > k + \frac{n}{2}$ and the domain has sufficiently smooth boundary $\partial\Omega$, then every function $u \in H^m(\Omega)$ is equivalent to a function in $C^k(\Omega)$. Furthermore, there is a constant C independent of u such that*

$$\sup_{|\alpha|\le k} \|\partial^\alpha u\|_\infty \le C\|u\|_m. \tag{1.12}$$

There is an alternative definition of Sobolev spaces. Let V be the space of function $u \in C^m(\Omega)$ such that $\|u\|_m < \infty$. Then V is a normed linear space and its completion under the norm $\|\cdot\|$ is just $H^m(\Omega)$.
Let $H_0^1(\Omega)$ be the completion of $C_0^\infty(\Omega)$ under the same norm. Then we have

Theorem 1.5.4. *The space $H_0^1(\Omega)$ consists of the functions $u \in H^1(\Omega)$ which satisfy $u = 0$ (a.e.) on the boundary of Ω.*

We have the obvious inclusion:

$$L^2(\Omega) = H^0(\Omega) \supset H^1(\Omega) \supset H^2(\Omega) \supset \cdots$$

Analogous Sobolev spaces can be defined with $L^p(\Omega)$ norms with $p \ne 2$. These are denoted by $W^{m,p}$ and $W_0^{m,p}$. We need the following results which we assume without proof.

Theorem 1.5.5. *The spaces $C^1(\bar\Omega)$ and $C_0^1(\bar\Omega)$ are dense in $H^1(\Omega)$ and $H_0^1(\Omega)$ resp.*

Theorem 1.5.6. *(Poincaré inequality)*

(1) Let Ω be the square the domain $[0,d] \times [0,d] \subset \mathbb{R}^2$. Then we have

$$\|u\|_0^2 \le d^2 |u|_1^2 + d^{-2} \left(\int_\Omega u\,dx \right)^2, \quad \forall u \in H^1(\Omega) \tag{1.13}$$

(2) Let Ω be a bounded domain in $\mathbb{R}^2$ whose diameter is d. Then we have

$$\|u\|_0 \le d|u|_1, \quad \forall u \in H_0^1(\Omega), \tag{1.14}$$

where $|u|_m$ is the semi norm $\left(\sum_{|\alpha|=m} \int_\Omega |\partial^\alpha u|^2 dx\right)^{1/2}$.

Proof. HW. Hint: prove them for $u \in C^1(\bar{\Omega})$ (resp. $C_0^1(\bar{\Omega})$) and use Theorem 1.5.5. □