# Chapter 5

# Stokes Equation

## 5.1 Mathematical Formulations

We introduce some notations: Let $\mathbf{v} = (v_1, v_2)^T$ or $\mathbf{v} = (v_1, v_2, v_3)^T$. When $n = 2$ we define

$$\operatorname{curl}\mathbf{v} = \frac{\partial v_2}{\partial x} - \frac{\partial v_1}{\partial y},$$
$$\mathbf{curl}\,\eta = \left(\tfrac{\partial \eta}{\partial y}, -\tfrac{\partial \eta}{\partial x}\right).$$

In fact, $\operatorname{curl}\mathbf{v}$ is obtained by imbedding $\mathbf{v}$ into $\mathbb{R}^3$, take 3D curl, then take the third component. For scalar function $\eta$, $\operatorname{curl}\eta$ the is same as imbedding $\eta$ into $\mathbb{R}^3$ as $(0, 0, \eta)$, take 3D curl, then take the first two components.

When $n = 3$ the curl of 3 -dim vector is defined as usual

$$\mathbf{curl}\,\mathbf{v} = \nabla \times \mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ v_1 & v_2 & v_3 \end{vmatrix} \tag{5.1}$$

$$\begin{aligned} \operatorname{curl}(\mathbf{curl}\,\phi) &= -\Delta\phi, \quad n = 2 \\ \left.\begin{aligned} \mathbf{curl}\,(\mathbf{curl}\,\mathbf{v}) \\ \mathbf{curl}\,(\operatorname{curl}\mathbf{v}) \end{aligned}\right\} &= -\Delta\mathbf{v} + \mathbf{grad}\,(\operatorname{div}\mathbf{v}) \quad \begin{aligned} n = 3 \\ n = 2 \end{aligned} \end{aligned}$$

We define

$$\mathbf{grad}\, p = \begin{pmatrix} \partial p/\partial x_1 \\ \partial p/\partial x_2 \end{pmatrix}, \quad \mathbf{div}\, \boldsymbol{\tau} = \begin{pmatrix} \partial \tau_{11}/\partial x_1 + \partial \tau_{12}/\partial x_2 \\ \partial \tau_{21}/\partial x_1 + \partial \tau_{22}/\partial x_2 \end{pmatrix},$$

$$\operatorname{div} \mathbf{v} = \partial v_1/\partial x_1 + \partial v_2/\partial x_2, \quad \mathbf{Grad}\, \mathbf{v} = \begin{pmatrix} \partial v_1/\partial x_1 & \partial v_1/\partial x_2 \\ \partial v_2/\partial x_1 & \partial v_2/\partial x_2 \end{pmatrix}.$$

We also define

$$\underline{\mathbf{curl}}\mathbf{v} = \begin{pmatrix} \mathbf{curl}\, v_1 \\ \mathbf{curl}\, v_2 \end{pmatrix}, \ \Delta \mathbf{u} = \begin{pmatrix} \Delta u_1 \\ \Delta u_2 \end{pmatrix}.$$

**Theorem 5.1.1.** *Let $\Omega$ be simply connected. Then $\mathbf{u} \in (L^2(\Omega))^n$ satisfies*

$$\mathbf{curl}\, \mathbf{u} = 0$$

*iff there exists $p \in H^1(\Omega)$ s.t $\mathbf{u} = \mathbf{grad}\, p$.*

For any two square matrices $A, B$ we write

$$A : B = \sum_{i,j} a_{ij} b_{ij}.$$

We also define special tensors

$$\boldsymbol{\delta} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \boldsymbol{\chi} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad \text{and} \quad \operatorname{tr}(\boldsymbol{\tau}) = \boldsymbol{\tau} : \boldsymbol{\delta} = \tau_{11} + \tau_{22}.$$

Let

$$\mathbf{Grad}\, \mathbf{u} = \nabla \mathbf{u} = \begin{pmatrix} \frac{\partial u_1}{\partial x_1} & \frac{\partial u_1}{\partial x_2} \\ \frac{\partial u_2}{\partial x_1} & \frac{\partial u_2}{\partial x_2} \end{pmatrix}. \tag{5.2}$$

Finally,

$$\boldsymbol{\epsilon}(\mathbf{v}) = \frac{1}{2}[\mathbf{Grad}\, \mathbf{v} + (\mathbf{Grad}\, \mathbf{v})^t], \quad \epsilon_{ij}(\mathbf{u}) = \frac{1}{2}(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i})$$

is called *deviatoric or deformation* tensor. Let $\Omega$ be a domain in $\mathbb{R}^n (n = 2, 3)$ with its boundary $\Gamma := \partial \Omega$.

The Navier-Stokes equations for a viscous fluid is are as follows:

$$
\begin{aligned}
\frac{\partial u_i}{\partial t} + \sum_{j=1}^{n} u_j \frac{\partial u_i}{\partial x_j} - 2\nu \sum_j \frac{\partial \epsilon_{ij}(\mathbf{u})}{\partial x_j} + \frac{\partial p}{\partial x_i} &= f_i(1 \le i \le n) \text{ in } \Omega, && (5.3)\\
\operatorname{div} \mathbf{u} &= 0 \text{ (incompressible)}, && (5.4)\\
\mathbf{u} &= \mathbf{g} \text{ on } \Gamma. && (5.5)
\end{aligned}
$$

Here $\mathbf{u}$ is the velocity of the fluid, $\nu > 0$ is the viscosity and $p$ is the pressure; (Here we assume $p$ and $\nu$ are normalized so we may assume $\rho = 1$) and the vector $\mathbf{f}$ represents body forces per unit mass. If we introduce the stress tensor $\sigma_{ij} := -p\delta_{ij} + 2\nu\boldsymbol{\epsilon}_{ij}(\mathbf{u})$ we have a simpler form :

$$
\begin{aligned}
\tfrac{\partial \mathbf{u}}{\partial t} + (\mathbf{u} \cdot \nabla)\mathbf{u} - \operatorname{div} \boldsymbol{\sigma} &= \mathbf{f} \text{ in } \Omega,\\
\operatorname{div} \mathbf{u} &= 0 \text{ in } \Omega,\\
\mathbf{u} &= \mathbf{g} \text{ on } \Gamma.
\end{aligned} \tag{5.6}
$$

Here the first term is interpreted as

$$
(\mathbf{u} \cdot \nabla)\mathbf{v} = \mathbf{e}_i \sum_j u_j \frac{\partial v_i}{\partial x_j} = \sum_j u_j \frac{\partial \mathbf{v}}{\partial x_j}.
$$

Note that if $\operatorname{div} \mathbf{u} = 0$, the following identity holds

$$
\sum_j \frac{\partial \boldsymbol{\epsilon}_{ij}(\mathbf{u})}{\partial x_j} = \frac{1}{2} \sum_j \left( \frac{\partial^2 u_i}{\partial x_j^2} + \frac{\partial^2 u_j}{\partial x_i \partial x_j} \right) = \frac{1}{2} \Delta u_i, \qquad \text{for each } i \tag{5.7}
$$

so that the equation can be written as

$$
\begin{cases}
\frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u} \cdot \nabla)\mathbf{u} - \nu \Delta \mathbf{u} + \mathbf{grad}\, p = \mathbf{f} \text{ in } \Omega,\\
\operatorname{div} \mathbf{u} = 0 \text{ in } \Omega,\\
\mathbf{u} = \mathbf{g} \text{ on } \Gamma.
\end{cases} \tag{5.8}
$$

**Remark 5.1.2.** If we define $\mathbf{uv} = \mathbf{u} \otimes \mathbf{v} = (u_i v_j)_{i,j}$, then when $\operatorname{div} \mathbf{u} = 0$, we see that the nonlinear term $(\mathbf{u} \cdot \nabla)\mathbf{u}$ can be written as $\nabla \cdot (\mathbf{u} \otimes \mathbf{u})$.

We only consider the steady-state case and assume that $\mathbf{u}$ is so small that we can ignore the non-linear convection term $u_j \frac{\partial u_i}{\partial x_j}$. Thus, we have the Stokes equation:

$$\begin{cases} -\nu\Delta \mathbf{u} + \mathbf{grad}\, p &= \mathbf{f} \text{ in } \Omega, \\ \operatorname{div} \mathbf{u} &= 0 \text{ in } \Omega, \\ \mathbf{u} &= \mathbf{g} \text{ on } \Gamma. \end{cases} \tag{5.9}$$

### 5.1.1 A weak formulation

Let

$$V = \{\mathbf{v} \in H_0^1(\Omega)^n, \operatorname{div} \mathbf{v} = 0\}$$

and $L_0^2(\Omega)$ be the space of all $L^2(\Omega)$ functions $q$ such that $\int_\Omega q\, dx = 0$.

Multiply (5.9) by $\mathbf{v} \in H_0^1(\Omega)^n$ and integrate by parts, we obtain

$$(\nu\nabla\mathbf{u}, \nabla\mathbf{v}) - (p, \operatorname{div}\mathbf{v}) = (\mathbf{f}, \mathbf{v}).$$

Define

$$a(\mathbf{u}, \mathbf{v}) := \nu \sum_{i,j=1}^{n} \left(\frac{\partial u_i}{\partial x_j}, \frac{\partial v_i}{\partial x_j}\right) = \nu \int_\Omega \mathbf{grad}\,\mathbf{u} : \mathbf{grad}\,\mathbf{v}\, d\mathbf{x} \tag{5.10}$$

$$b(\mathbf{v}, q) := -(q, \operatorname{div}\mathbf{v}). \tag{5.11}$$

Then we have the equivalent weak (abstract) form of (5.9) : Find $\mathbf{u} \in H^1(\Omega)^n$ s.t.

$$\begin{cases} a(\mathbf{u}, \mathbf{v}) + b(\mathbf{v}, p) &= \langle \mathbf{f}, \mathbf{v}\rangle \text{ for all } \mathbf{v} \in H_0^1(\Omega)^n, \\ b(\mathbf{u}, q) &= 0 \text{ for all } q \in L_0^2(\Omega), \\ \mathbf{u} &= \mathbf{g} \text{ on } \Gamma. \end{cases} \tag{5.12}$$

We can find a function $\mathbf{u}_g \in H^1(\Omega)^n$ such that

$$\operatorname{div}\mathbf{u}_g = 0 \text{ on } \Omega,\ \mathbf{u}_g = g \text{ on } \Gamma$$

so that $\mathbf{u}$ can be decomposed as $\mathbf{u} = \mathbf{w} + \mathbf{u}_g, \mathbf{w} \in H_0^1(\Omega)^n$. With

$$\langle \ell, \mathbf{v}\rangle := \langle \mathbf{f}, \mathbf{v}\rangle - a(\mathbf{u}_g, \mathbf{v})$$

the problem (5.12) is equivalent to : Find a unique pair of functions $(\mathbf{w}, p) \in$

$H_0^1(\Omega)^n \times L_0^2(\Omega)$ such that

$$\left\{ \begin{array}{rcl} a(\mathbf{w},\mathbf{v}) + b(\mathbf{v},p) & = & \langle \ell, \mathbf{v} \rangle \text{ for all } \mathbf{v} \in H_0^1(\Omega)^n \\ b(\mathbf{w},q) & = & 0 \text{ for all } q \in L_0^2(\Omega). \end{array} \right. \tag{5.13}$$

This problem actual has a unique solution by Corollary 5.1.5 below. Furthermore, we have the following :

$$\|\mathbf{u}\|_1 + \|p\|_0 \leq C(\|\mathbf{f}\|_{-1} + \|g\|_{1/2,\Gamma}). \tag{5.14}$$

### 5.1.2 A General result

Now let us put problem (5.13) into general framework of chap 4: We set

$$X = H_0^1(\Omega)^n, \quad M = L_0^2(\Omega).$$

Let $X$ and $M$ be two Hilbert spaces with norms $\|\cdot\|_X$ and $\|\cdot\|_M$ and let $X'$ and $M'$ be their dual spaces. As usual, we denote $\langle \cdot, \cdot \rangle$ be the duality pairing between $X$ and $X'$ or $M$ and $M'$.

Introduce bilinear forms

$$a(\cdot,\cdot) : X \times X \to \mathbb{R}, \quad b(\cdot,\cdot) : X \times M \to \mathbb{R}$$

with norms

$$\|a\| = \sup_{u,v} \frac{a(u,v)}{\|u\|_X \|v\|_X}, \quad \|b\| = \sup_{v \in X, \mu \in M} \frac{b(v,\mu)}{\|v\|_X \|\mu\|_M}.$$

We consider the following two variational problem called problem (Q): Given $\ell \in X'$ and $\chi \in M'$, find a pair $(u,\lambda) \in X \times M$ such that

$$a(u,v) + b(v,\lambda) = \langle \ell, v \rangle \text{ for all } v \in X \tag{5.15}$$

$$b(u,\mu) = \langle \chi, \mu \rangle \text{ for all } \mu \in M. \tag{5.16}$$

In order to study (Q), we associate two linear operators $A \in \mathcal{L}(X; X')$ and $B \in \mathcal{L}(X; M')$ defined by

$$\langle Au, v \rangle = a(u,v) \text{ for all } u, v \in X \tag{5.17}$$

$$\langle Bv, \mu \rangle = b(v,\mu) \text{ for all } v \in X, \mu \in M. \tag{5.18}$$

Let $B' \in \mathcal{L}(M;X')$ be dual operators defined by

$$\left\langle B'\mu, v\right\rangle = \langle \mu, Bv\rangle = b(v,\mu), v \in X, \mu \in M. \tag{5.19}$$

With these, the problem can be written as:

Find $(u,\lambda) \in X \times M$ such that

$$\begin{aligned} Au + B'\lambda &= \ell \text{ in } X' && (5.20)\\ Bu &= \chi \text{ in } M'. && (5.21)\end{aligned}$$

We set $V = Ker(B)$ and more generally define

$$V(\chi) = \{v \in X; Bv = \chi\}.$$

Note that $V = V(0)$.

Now problem (Q) can be changed into equivalent form (P):

Find $u \in V(\chi)$ such that

$$a(u,v) = \langle \ell, v\rangle\,, \quad v \in V. \tag{5.22}$$

Define the polar set $V^0$ by

$$V^0 = \{g \in X'; < g, v >= 0,\ \forall v \in V\}.$$

**Lemma 5.1.3.** *The followings are equivalent: (i) There is a constant $\beta > 0$ such that*

$$\inf_{\mu\in M} \sup_{v\in X,} \frac{b(v,\mu)}{\|v\|_X\|\mu\|_M} \geq \beta > 0. \tag{5.23}$$

*(ii) The operator $B'$ is an isomorphism from $M$ onto $V^0$ and*

$$\|B'\mu\|_{X'} \geq \beta\|\mu\|_M. \tag{5.24}$$

*(ii) The operator $B$ is an isomorphism from $V^\perp$ onto $M'$ and*

$$\|Bv\|_{M'} \geq \beta\|v\|_X. \tag{5.25}$$

**Theorem 5.1.4.** *The problem (Q) has a unique solution if (i) $\pi A$ is an isomorphism from $V$ onto $V'$(p. 59) and (ii) there is a constant $\beta > 0$ such* *that*

$$\inf_{\mu\in M}\sup_{v\in X,}\frac{b(v,\mu)}{\|v\|_X\|\mu\|_M}\geq\beta>0. \tag{5.26}$$

**Corollary 5.1.5.** *Assume that $a(\cdot,\cdot)$ is coercive on $V$, i.e., there exists a constant $\alpha>0$ such that*

$$a(v,v)\geq\alpha\|v\|_X^2,\quad \forall v\in V. \tag{5.27}$$

*Then problem (Q) has unique solution if and only if $b(\cdot,\cdot)$ satisfies inf-sup condition.*

Now let us put problem (5.9) into general framework of chap 4.: We set

$$X=H_0^1(\Omega)^n,\quad M=L_0^2(\Omega).$$

The choice $M=L_0^2(\Omega)$ is a matter of convenience and we can just as well take $M=L^2(\Omega)/\mathbb{R}$.

## Finite dimensional problem

p 123. Girault - Raviart, 'Finite element approximation of the Navier-Stokes equations'.

Now change every space to finite dimensional one. Let

$$X_h\subset X,\quad M_h\subset M$$

be finite dimensional subspace with certain approximation properties.

As in the continuous case, we associate two linear operators $A_h\in\mathcal{L}(X;X_h')$, $B_h\in\mathcal{L}(X;M_h')$ and $B_h'\in\mathcal{L}(M;X_h')$ defined by

$$\begin{aligned}\langle A_hu,v_h\rangle &= a(u,v_h)\text{ for all }v_h\in X_h,u\in X, &(5.28)\\ \langle B_hv,\mu_h\rangle &= b(v,\mu_h)\text{ for all }\mu_h\in M_h,v\in X, &(5.29)\\ \left\langle v_h,B_h'\mu\right\rangle &= b(v_h,\mu)\text{ for all }v_h\in X_h,\mu\in M. &(5.30)\end{aligned}$$

We define the finite dimensional analogue of $V$:

$$V_h(\chi)=\{v_h\in X_h;b(v_h,\mu_h)=<\chi,\mu_h>,\mu_h\in M_h\}.$$

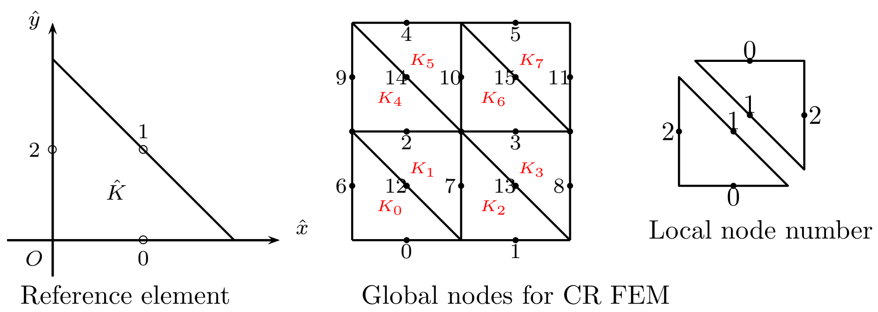



Figure 5.1: Nodes for CR nonconforming FEM

We set

$$V_h = V_h(0) = Ker(B_h) \cap X_h = \{v_h \in X_h; b(v_h, \mu_h) = 0, \mu_h \in M_h\}.$$

**Caution:** $V_h \not\subset V$ and $V_h(\chi) \not\subset V(\chi)$, since $M_h$ is a proper subspace of $M$.

We now define the approximate problem.

$(Q_h)$: For $\ell_h$ given in $X'_h$ and $\chi_h \in M'_h$, find a pair $(u_h, \lambda_h)$ in $X_h \times M_h$ such that

$$a(u_h, v_h) + b(v_h, \lambda_h) = \langle \ell_h, v_h \rangle, \quad \forall v_h \in X_h \tag{5.31}$$

$$b(u_h, \mu_h) + c(\lambda_h, \mu_h) = \langle \chi_h, \mu_h \rangle, \quad \forall \mu_h \in M_h. \tag{5.32}$$

Now problem $(Q_h)$ can be changed into the equivalent problem.

$(P_h)$: Find $u_h \in V_h(\chi)$ such that

$$a(u_h, v_h) = \langle \ell, v_h \rangle, \quad v_h \in V_h. \tag{5.33}$$

### 5.1.3 Mapping to reference element

If $\hat{\phi}$ is any scalar function defined over $\hat{K}(\hat{K})$, we associate a function on $K$(pull back) by

$$\phi = \hat{\phi} \circ F_K^{-1}. \tag{5.34}$$

Hence

$$\nabla \phi = B_K^{-T} \nabla \hat{\phi} \circ F_K^{-1}, \quad \nabla \hat{\phi} = B_K^T \nabla \phi \circ F_K. \tag{5.35}$$

### 5.1.4 Matrix representation

A simplest choice $(X_h, M_h)$ is $(P_1^n, P_0)$. In this case, the matrix $A_h$ is just two copies of scalar Laplacian. For $B_h$, we test the following for $\mathbf{v}_h = (v_1, v_2)$ associated with edges $2, 3, 7, 10, 12, 13, 14, 15$ only. It suffices to test $\mathbf{v}_h = (v_1, 0)$ only.

We test the equation for $(0, \phi_2)$, where $\phi_2$ is the scalar basis function associated with the node 2. Since

$$\phi_2 = \begin{cases} \frac{2y-h}{h} & \text{in } K_1 \\ \frac{3h-2y}{h} & \text{in } K_4 \end{cases}$$

Or $\phi_2(x,y) = \hat{\phi}_2 \circ F_{K_1}^{-1}$ on $K_1$ and $\phi_2(x,y) = \hat{\phi}_2 \circ F_{K_4}^{-1}$ on $K_4$. Here

$$F_{K_1} = \begin{pmatrix} -h & 0 \\ 0 & -h \end{pmatrix} \begin{pmatrix} \hat{x} \\ \hat{y} \end{pmatrix} + \begin{pmatrix} h \\ h \end{pmatrix}, \quad F_{K_4} = \begin{pmatrix} h & 0 \\ 0 & h \end{pmatrix} \begin{pmatrix} \hat{x} \\ \hat{y} \end{pmatrix} + \begin{pmatrix} 0 \\ h \end{pmatrix}$$

We see

$$\begin{aligned} < B\mathbf{v}_h, p_h > &= b(\mathbf{v}_h, p_h) = -(\operatorname{div} \mathbf{v}_h, p_h)_{K_1 \cup K_4} \\ &= -\int_{K_1} \frac{2}{h} p_1 - \int_{K_4} -\frac{2}{h} p_4 = -h(p_1 - p_4) \end{aligned}$$

Similarly, since

$$\phi_{12} = \begin{cases} \frac{2x+2y-h}{h} & \text{in } K_0 \\ \frac{3h-2x-2y}{h} & \text{in } K_1 \end{cases}$$

if we test the equation for $\mathbf{v}_h = (\phi_{12}, 0)$,

$$\begin{aligned} < B\mathbf{v}_h, p_h > &= -(\operatorname{div} \mathbf{v}_h, p_h)_{K_0 \cup K_1} \\ &= -\int_{K_0} \frac{2}{h} p_0 - \int_{K_1} -\frac{2}{h} p_1 = -h(p_0 - p_1) \end{aligned}$$

Since

$$\phi_7 = \begin{cases} \frac{2x-h}{h} & \text{in } K_1 \\ \frac{3h-2x}{h} & \text{in } K_2 \end{cases}$$

$$< B\mathbf{v}_h, p_h > = -\int_{K_1} \frac{2}{h} p_1 - \int_{K_2} -\frac{2}{h} p_2 = -h(p_1 - p_2)$$

$B^t = (B_1, B_2)^t$ is $16 \times 8$ matrix, see (5.53), but if we only see the rows of $B^t$ corresp to $(\phi, 0)$ ($x$-component basis ftns) they are like $8 \times 8$ matrix

$$B_1^t = -h \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & -1 & 0 \\ 1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & -1 \end{pmatrix}$$

exer. Write down entries of $B$ corresponding to basis function of type $(0, \phi)$.

### 5.1.5 Error estimate

**Theorem 5.1.6.** *(A) Assume $V_h(\chi)$ is nonempty and there exists a constant $\alpha > 0$ such that*

$$a(v_h, v_h) \geq \alpha \|v_h\|_X^2, \quad \forall v_h \in V_h. \tag{5.36}$$

*(B) the discrete inf-sup condition hold*

$$\sup_{v_h \in X_h} \frac{b(v_h, \mu_h)}{\|v_h\|_X} \geq \beta^* \|\mu_h\|_M > 0, \quad \forall \mu_h \in M_h. \tag{5.37}$$

*Then the problem $(P_h)$ has a unique solution $u_h \in V_h(\chi)$ and there is a constant $C > 0$ such that Then there is a unique solution $(u_h, \lambda_h)$ of the problem $(Q_h)$*

$$\|u - u_h\|_X + \|\lambda - \lambda_h\|_M \leq C_2 \left\{ \inf_{v_h \in X_h} \|u - v_h\|_X + \inf_{\mu_h \in M} \|\lambda - \mu_h\|_M \right\}. \tag{5.38}$$

### Checking the discrete inf-sup condition

**Lemma 5.1.7.** *The the discrete inf-sup condition (5.37) holds with a constant $\beta^* > 0$ independent of $h$ if and only if there exists an operator $\Pi_h \in \mathcal{L}(X; X_h)$ satisfying*

$$b(\mathbf{v} - \Pi_h \mathbf{v}, \mu_h) = 0, \ \forall \mu_h \in M_h, \mathbf{v} \in X \tag{5.39}$$

*and*

$$\|\Pi_h \mathbf{v}\|_X \leq C \|\mathbf{v}\|_X, \ \forall \mathbf{v} \in X. \tag{5.40}$$

*Proof.* Assume such $\Pi_h$ exists. Then by $\|\Pi_h \mathbf{v}\|_X \leq C\|\mathbf{v}\|_X$, we see

$$\begin{aligned} \sup_{\mathbf{v}_h \in X_h} \frac{b(\mathbf{v}_h, \mu_h)}{\|\mathbf{v}_h\|_X} &\geq \sup_{v \in X} \frac{b(\Pi_h \mathbf{v}, \mu_h)}{\|\Pi_h \mathbf{v}\|_X} \\ &= \sup_{\mathbf{v} \in X} \frac{b(\mathbf{v}, \mu_h)}{\|\Pi_h \mathbf{v}\|_X} \\ &\geq \frac{1}{C} \sup_{\mathbf{v} \in X} \frac{b(\mathbf{v}_h, \mu_h)}{\|\mathbf{v}\|_X} \\ &\geq \frac{\beta}{C} \|\mu_h\|_M. \end{aligned}$$

□

### 5.1.6 Error Estimate

#### Hypothesis

(1) There exists an operator $\mathbf{r}_h : H^{m+1}(\Omega)^n \cap H_0^1(\Omega)^n \to X_h$ such that

$$\|\mathbf{v} - \mathbf{r}_h \mathbf{v}\|_1 \leq Ch^m \|\mathbf{v}\|_{m+1}, \quad \forall \mathbf{v} \in H^{m+1}(\Omega)^n \quad 1 \leq m \leq l. \tag{5.41}$$

(2) There exists an operator $S_h : L^2(\Omega) \to M_h$ such that

$$\|q - S_h q\|_0 \leq Ch^m \|q\|_m, \quad \forall q \in H^m(\Omega)^n, \quad 0 \leq m \leq l. \tag{5.42}$$

(3) (Uniform inf-sup condition) For each $q_h \in M_h$ there exists $\mathbf{v}_h \in X_h$ such that

$$(q_h, \operatorname{div} \mathbf{v}_h) = \|q_h\|_0^2, \tag{5.43}$$

$$|\mathbf{v}_h|_1 \leq C\|q_h\|_0, \tag{5.44}$$

where $C > 0$ is independent of $h, q_h$ and $\mathbf{v}_h$.

**Theorem 5.1.8.** *Under the Hypothesis the solution of the problem(5.31) satisfies*

$$\|\mathbf{u} - \mathbf{u}_h\|_1 + \|p - p_h\|_0 \leq Ch^m \{\|\mathbf{u}\|_{m+1} + \|p\|_m\}. \tag{5.45}$$

**Remark 5.1.9.** One can expect one higher order for $L^2$ error estimate by duality technique.

$$\|\mathbf{u} - \mathbf{u}_h\|_0 \leq Ch\{|\mathbf{u} - \mathbf{u}_h|_1 + \inf \|p - p_h\|_0\}. \tag{5.46}$$

Stabilization. (Verfürth note) We may add the following term $\delta_K h^2(\operatorname{div}[\Delta\mathbf{u}+\nabla p]-\operatorname{div}\mathbf{f})$ to the second equation for each element $K$.

## 5.2 Solver

### Assembly of $A$ and $B$

One way to look at the eq. is

$$-\Delta u_1 + p_x = f_1 \tag{5.47}$$

$$-\Delta u_2 + p_y = f_2 \tag{5.48}$$

$$(u_1)_x + (u_2)_y = 0. \tag{5.49}$$

With $b_1(u_1, p) = -((u_1)_x, p)$, $b_2(u_2, p) = -((u_2)_y, p)$, its weak form is

$$a(u_1, v_1) + b_1(v_1, p) = f_1 \tag{5.50}$$

$$a(u_2, v_2) + b_2(v_2, p) = f_2 \tag{5.51}$$

$$b_1(u_1, q) + b_2(u_2, q) = 0 \tag{5.52}$$

In matrix, Let $2n_v = dim X_h$, $n_p = dim\, M_h$. Then for $n_v \times n_v$ matrix $A$ and $n_p \times n_v$ matrix $B$

$$\begin{bmatrix} A & 0 & B_1^t \\ 0 & A & B_2^t \\ B_1 & B_2 & 0 \end{bmatrix} \begin{bmatrix} U_x \\ U_y \\ P \end{bmatrix} = \begin{bmatrix} F_1 \\ F_2 \\ 0 \end{bmatrix} \tag{5.53}$$

If we use Uzawa we do not need explicit $B$. Instead only need action of $b(\mathbf{v}_h, p_h)$ for given $\mathbf{v}_h$ and $p_h$.

### Standard Uzawa

Let $p_h^0$ given. Let small $\epsilon > 0$ be fixed ($\epsilon = 1.5$ in Verfüth note). Solve for $m = 0, 1, \cdots$, until $\|p_h^{m+1} - p_h^m\|$ is sufficiently small.

$$a(\mathbf{u}_h^{m+1}, \mathbf{v}_h) + b(\mathbf{v}_h, p_h^m) = (\mathbf{f}, \mathbf{v}_h) - a(\mathbf{u}_g, \mathbf{v}_h), \quad \mathbf{v}_h \in X_h$$

$$b(\mathbf{u}_h^{m+1}, q) + \delta c(p_h^m, q) = \frac{1}{\epsilon}(p_h^{m+1} - p_h^m, q) + \delta\chi_h(q), \quad q \in M_h$$

Normalize $p_h^{m+1}$ each step so that it belongs to $M = L_0^2(\Omega)$.

$$A\mathbf{u}_h + B^T p_h = \ell_h \tag{5.54}$$
$$B\mathbf{u}_h - \delta C p_h = -\delta\chi_h. \tag{5.55}$$

Thus in matrix form we have

$$\begin{pmatrix} A & B^T \\ B & -\delta C \end{pmatrix} \begin{pmatrix} \mathbf{u} \\ p \end{pmatrix} = \begin{pmatrix} \ell_h \\ -\delta\chi_h \end{pmatrix} \tag{5.56}$$

### Standard Uzawa-Again

(1) Given: an initial guess $p^0$ for the pressure, a tolerance $Tol > 0$ and a relaxation parameter $\epsilon > 0$.

(2) Apply a few Gauss-Seidel iterations (fix $p_h^m$) to the linear system

$$A\mathbf{u}_h^m = \ell_h - B^T p_h^m$$

and denote the result by $\mathbf{u}_h^{m+1}$. Compute

$$p_h^{m+1} = p_h^m + \epsilon(B\mathbf{u}_h^{m+1} - \delta C p_h^m + \delta\chi_h)$$

(3) Stop if

$$\|A\mathbf{u}_h^{m+1} + B^T p_h^{m+1} - \ell_h\| + \|B\mathbf{u}_h^{m+1} - \delta C p_h^{m+1} + \delta\chi_h\| \leq Tol$$

### 5.2.1 Improved Uzawa-cg MG

Solve the first equation for $\mathbf{u}$ (you may use multigrid) as

$$\mathbf{u}_h = A^{-1}(\ell_h - B^T p_h)$$

and insert into the second eq.

$$BA^{-1}(\ell_h - B^T p_h) - \delta C p_h = -\delta\chi_h$$

This gives

$$\mathcal{A} p_h := [BA^{-1}B^T + \delta C] p_h = BA^{-1}\ell_h + \delta\chi_h := \tilde{\mathbf{f}}. \tag{5.57}$$

One can show that $\mathcal{A}$ is SPD and the condition number is $O(1)$, hence we can Apply conjugate gradient method to this system $\mathcal{A}p_h = \tilde{\mathbf{f}}$. This algorithm requires evaluation of $A^{-1}\tilde{\mathbf{f}}$ where one can use a fast algorithm such as mutligrid method.

The next task is how to construct spaces $X_h$ and $M_h$ which satisfy the hypotheses. The CG-algorithm in general breaks down for non-symmetric or indefinite systems. However, there are various variants of the CG-algorithm which can be applied to these problems. A naive approach consists in applying the CG-algorithm to the squared system $L_k^T L_k x_k = L_k^T b_k$. This approach cannot be recommended since squaring the systems squares its condition number. A more efficient algorithm is the stabilized bi-conjugate gradient algorithm, shortly Bi-CG-stab. The underlying idea roughly is to solve simultaneously the original problem $L_k x_k = b_k$ and its adjoint $L_k^T y_k = b_k^T$.

### Stable pair for Stokes equation

For Stokes equation, we need to choose pair of spaces so that inf-sup condition holds. Assume $\mathcal{T}_h$ consists of triangles. Typically we use $P_2$ for the velocity and $P_1$ for the pressure. Another choice is $P_1$-nonconforming for velocity and $P_0$ for pressure(Called C-R(Crouzeix-Raviart-1973) element).

### $(P_2, P_1)$ pair -Taylor Hood

We can show the inf-sup condition and we have

$$\|\mathbf{u} - \mathbf{u}_h\|_h + \|p - p_h\|_0 \le Ch^2\{\|\mathbf{u}\|_2 + \|p\|_1\}. \tag{5.58}$$

### $(P_1^n, P_0)$ pair- Crouzeix- Raviart

Let $P_0$ be the space of all functions which are piecewise constant on each $T$. We have

$$\|\mathbf{u} - \mathbf{u}_h\|_h + \|p - p_h\|_0 \le Ch\{\|\mathbf{u}\|_2 + \|p\|_1\}. \tag{5.59}$$

### 5.2.2 II.3. Petrov-Galerkin stabilization

The mini element revisited.

Table 5.1: Summary of 2D triangular elements

● velocity
○ pressure

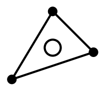
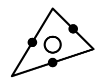
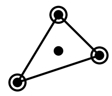
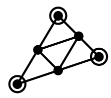
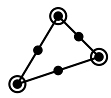
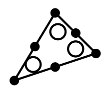
| Name | Sketch | LBB | Order | Remarks |
|---|---|---|---|---|
| $P_1P_0$ |  | N | 1 | Rarely used |
| $P_1^nP_0$ |  | Y | 1 | C-R, not for natural BC. |
| $P_1^+P_1$ (mini) |  | Y | 1 | cubic bubble |
| $P_1P_1$ (4 patch macro) |  | Y | 1 | iso $P_2$-$P_1$ |
| $P_kP_{k-1}$ (Taylor-Hood) |  | Y | 2 | $P_2P_1$ engineer's favor |
| $P_k \oplus B_{k+1}P_{k-1}^{-1}$ |  | Y | 2 | C-R $P_2^+P_1^{-1}$ |

Table 5.2: Pressure given by circle in the interior means discontinuous.

### 5.2.3 Stabilization

Motivated by mini element, we can solve the following eq. with $P_1/P_1$ pair.

$$\begin{cases} -\nu\Delta\mathbf{u} + \mathbf{grad}\, p &= \mathbf{f} \text{ in } \Omega, \\ \operatorname{div}\mathbf{u} - \alpha\Delta p &= -\alpha \operatorname{div}\mathbf{f} \text{ in } \Omega, \\ \mathbf{u} &= \mathbf{g} \text{ on } \Gamma. \end{cases} \tag{5.60}$$

Taking into account that $\Delta\mathbf{u}_T$ vanishes elementwise (for mini), the discrete problem does not change if we also add the term $\alpha\Delta \operatorname{div}\mathbf{u}$ to the left-hand side of the second equation. This shows that in total we may add the divergence of the momentum equation as a penalty. For the general form see, Verfurth note. (also my paper with Kwon)

## 5.3 Numerical method for Navier Stokes equation

### 5.3.1 Picard's iteration

$$
\begin{aligned}
-\Delta \mathbf{u}^{m+1} + \nabla p^{m+1} &= \mathbf{f} - (\mathbf{u}^m \cdot \nabla)\mathbf{u}^m \text{ in } \Omega, \\
\operatorname{div} \mathbf{u}^{m+1} &= 0 \text{ in } \Omega, \\
\mathbf{u}^{m+1} &= \mathbf{g} \text{ on } \Gamma.
\end{aligned} \tag{5.61}
$$

### 5.3.2 Newton's method

Consider

$$
\begin{aligned}
-\Delta \mathbf{u} + (\mathbf{u} \cdot \nabla)\mathbf{u} + \nabla p &= \mathbf{f} \text{ in } \Omega, \\
\operatorname{div} \mathbf{u} &= 0 \text{ in } \Omega, \\
\mathbf{u} &= \mathbf{g} \text{ on } \Gamma.
\end{aligned} \tag{5.62}
$$

Linearize (or correct with) $\mathbf{u}^{m+1} = \mathbf{u}^m + \delta\mathbf{u}$ to see

$$
\begin{aligned}
-\Delta \mathbf{u}^{m+1} + (\mathbf{u}^{m+1} \cdot \nabla)(\mathbf{u}^m + \delta\mathbf{u}) + \nabla p^{m+1} &= \mathbf{f} \text{ in } \Omega, \\
-\Delta \mathbf{u}^{m+1} + (\mathbf{u}^{m+1} \cdot \nabla)\mathbf{u}^m + (\mathbf{u}^m \cdot \nabla)\delta\mathbf{u} + \nabla p^{m+1} + (\delta\mathbf{u})^2 &= \mathbf{f} \text{ in } \Omega, \\
-\Delta \mathbf{u}^{m+1} + (\mathbf{u}^{m+1} \cdot \nabla)\mathbf{u}^m + (\mathbf{u}^m \cdot \nabla)(\mathbf{u}^{m+1} - \mathbf{u}^m) + \nabla p^{m+1} &\doteq \mathbf{f} \text{ in } \Omega.
\end{aligned} \tag{5.63}
$$

Thus we define the Newton iteration as : Given initial guess $\mathbf{u}^0, p^0$ solve the following for $m = 0, 1, \cdots$ ,, say with Uzawa for each $m$ until convergence.

$$
\begin{aligned}
-\Delta \mathbf{u}^{m+1} + \nabla p^{m+1} \quad &+(\mathbf{u}^{m+1} \cdot \nabla)\mathbf{u}^m \\
&+(\mathbf{u}^m \cdot \nabla)\mathbf{u}^{m+1} = \mathbf{f} + (\mathbf{u}^m \cdot \nabla)\mathbf{u}^m \text{ in } \Omega \\
&\operatorname{div} \mathbf{u}^{m+1} = 0 \text{ in } \Omega, \\
&\mathbf{u}^{m+1} = \mathbf{g} \text{ on } \Gamma.
\end{aligned} \tag{5.64}
$$

The Newton iteration converges quadratically. However, the initial guess must be close to the sought solution, otherwise the iteration may diverge. To avoid this, one can use a damped Newton iteration.

### 5.3.3 Projection scheme for Navier-Stokes eq... Chorin 68

See Jie Shen note IMS NUS.pdf. Consider the Full Navier Stokes problem:

$$
\begin{aligned}
\tfrac{\partial \mathbf{u}}{\partial t} + (\mathbf{u} \cdot \nabla)\mathbf{u} - \operatorname{div} \boldsymbol{\sigma} &= \mathbf{f} \text{ in } \Omega, \\
\operatorname{div} \mathbf{u} &= 0 \text{ in } \Omega, \\
\mathbf{u} &= \mathbf{g} \text{ on } \Gamma.
\end{aligned} \tag{5.65}
$$

The original projection method, proposed by Chorin [12] and Temam [69], was motivated by the idea of operator splitting, its semi-discrete version.

**Remark 5.3.1.** The above scheme has only 1/2 -order for velocity in $L^2(0,T;H^1)$ due to the nonphysical boundary condition $\frac{\partial p^{k+1}}{\partial n}|_\Gamma = 0$.

The improved projection type scheme appears to be, the so called pressure-correction scheme introduced in [26] (K. Goda. A multistep ...cavity flows. J. C. P., 1979.). Its first-order version reads :

Find $\tilde{\mathbf{u}}^{k+1}$ by solving

$$\begin{array}{rcl} \frac{\tilde{\mathbf{u}}^{k+1}-\mathbf{u}^k}{\delta t} - \nu\Delta\tilde{\mathbf{u}}^{k+1} + (\mathbf{u}^k\cdot\nabla)\tilde{\mathbf{u}}^{k+1} + \nabla p^k & = & \mathbf{f}(t_{k+1}) \text{ in } \Omega, \\ \tilde{\mathbf{u}}^{k+1}|_\Gamma & = & 0. \end{array} \tag{5.66}$$

Then find $(p^{k+1}, \mathbf{u}^{k+1})$ from

$$\begin{array}{rcl} \frac{\mathbf{u}^{k+1}-\tilde{\mathbf{u}}^{k+1}}{\delta t} + \nabla(p^{k+1}-p^k) & = & 0, \\ \nabla\cdot\mathbf{u}^{k+1} & = & 0, \\ \mathbf{u}^{k+1}\cdot\mathbf{n}|_\Gamma & = & 0. \end{array} \tag{5.67}$$

By taking the divergence of the first equation in (5.67), we find that the second step is equivalent to

$$\begin{array}{rcl} \Delta(p^{k+1}-p^k) = \frac{1}{\delta t}\nabla\cdot\tilde{\mathbf{u}}^{k+1}, \ \frac{\partial(p^{k+1}-p^k)}{\partial n}|_\Gamma & = & 0, \\ \mathbf{u}^{k+1} = \tilde{\mathbf{u}}^{k+1} - \delta t\nabla(p^{k+1}-p^k). & & \end{array} \tag{5.68}$$

..... It is also shown that the above scheme is first-order accurate for the velocity in $L^\infty(0,T;L^2)\cap L^2(0,T;H^1)$ and pressure in $L^2(0,T;L^2)$. A popular second-order version reads

$$\begin{cases} \frac{3\tilde{\mathbf{u}}^{k+1}-4\mathbf{u}^k+\mathbf{u}^{k-1}}{2\delta t} + (2\mathbf{u}^k - \mathbf{u}^{k-1})\cdot\nabla\tilde{\mathbf{u}}^{k+1} - \nu\Delta\tilde{\mathbf{u}}^{k+1} + \nabla p^k = \mathbf{f}(t^{k+1}), \\ \tilde{\mathbf{u}}^{k+1}|_{\partial\Omega} = 0. \end{cases} \tag{5.69}$$

then find $(p^{k+1}, \mathbf{u}^{k+1})$ by

$$\begin{cases} \frac{3\mathbf{u}^{k+1}-3\tilde{\mathbf{u}}^{k+1}}{2\delta t} + \nabla(p^{k+1}-p^k) = 0, \\ \operatorname{div}\mathbf{u}^{k+1} = 0 \\ \mathbf{u}^{k+1}\cdot\mathbf{n}|_{\partial\Omega} = 0. \end{cases} \tag{5.70}$$

(To solve again take divergence as before) The above scheme is second-order for the velocity in the $L^2(0,T;L^2(\Omega))$-norm.

## 5.4 Stream-function formulation

The discrete velocity fields computed by the methods of the previous sections in general are not exactly incompressible. It is only weakly incompressible. In this section we will consider a formulation of the Stokes equations which leads to conforming solenoidal discretizations. This advantage, of course, has to be paid for by other drawbacks. Throughout this section we assume that $\Omega$ is a two dimensional, simply connected polygonal domain.

### 5.4.1 The curl operators

We need two curl-operators:

$$\mathbf{curl}\,\phi = (-\frac{\partial\phi}{\partial y},\frac{\partial\phi}{\partial x}),\ \mathrm{curl}\mathbf{v} = (\frac{\partial v_1}{\partial y} - \frac{\partial v_2}{\partial x})\ \text{(sgn is diffent from usual)}$$

(Stokes-2D)

$$\int_\Omega \mathrm{curl}\mathbf{u}_h \cdot \xi \, d\mathbf{x} = \int_\Omega \mathbf{u}_h \mathbf{curl}\,\xi \, d\mathbf{x} - \int_{\partial\Omega} \mathbf{u}\cdot\tau\xi \, ds \tag{5.71}$$

The following deep mathematical result is fundamental: A vector-field $\mathbf{v} : \Omega \to \mathbb{R}^2$ is solenoidal, i.e. $div\mathbf{v} = 0$, if and only if there is a unique stream-function $\phi\Omega \to \mathbb{R} :$ such that $\mathbf{v} = \mathbf{curl}\,\phi$ in $\Omega$ and $\phi = 0$ on $\Gamma$.

### 5.4.2 Stream-function formulation of the Stokes equations

Let $(\mathbf{u}, p)$ be the solution of the Stokes equations with force $\mathbf{f}$ and homogeneous boundary conditions and denote by $\psi$ the stream function corresponding to $\mathbf{u}$. Since

$$\mathbf{u}\cdot\mathbf{t} = 0 \text{ on } \Gamma,$$

we conclude that in addition

$$\frac{\partial\psi}{\partial\mathbf{n}} = \mathbf{t}\cdot\mathbf{curl}\,\psi = 0 \text{ on } \Gamma.$$

Inserting this representation of $\mathbf{u}$ in the momentum equation and applying the operator curl we obtain

$$\begin{aligned} \operatorname{curl}\mathbf{f} &= \operatorname{curl}(-\Delta\mathbf{u}+\nabla p) \quad (5.72)\\ &= -\Delta(\operatorname{curl}\mathbf{u})+\Delta(\nabla p) \quad (5.73)\\ &= -\Delta(\operatorname{curl}(\mathbf{curl}\,\psi))=\Delta^2\psi \quad (5.74) \end{aligned}$$

This proves that the stream function solves the biharmonic equation with homo. BC.

$$\begin{aligned} \Delta^2\psi &= \operatorname{curl}\mathbf{f} \text{ in } \Omega\\ \phi &= 0 \text{ on } \partial\Omega\\ \tfrac{\partial\phi}{\partial\mathbf{n}} &= 0 \text{ on } \partial\Omega \end{aligned}$$

How about two phase in this form? Change this to MFVM Conversely, one can prove: If solves the above biharmonic equation, there is a unique pressure p with mean-value 0 such that $\mathbf{u}=\mathbf{curl}\,\psi$ and $p$ solve the Stokes equations. In this sense, the Stokes equations and the biharmonic equation are equivalent. Remark II.5.1. Given a solution $\psi$ of the biharmonic equation and the corresponding velocity $\mathbf{u}=\mathbf{curl}\,\psi$ the pressure is determined by the equation $\mathbf{f}+\Delta\mathbf{u}=\nabla p$. But there is no constructive way to solve this problem. Hence, the biharmonic equation is only capable to yield the velocity field of the Stokes equations.