4.1 Discontinuous Galerkin Method

Consider

$$-\nabla \cdot (\beta(x)\nabla u) = f \quad \text{in } \Omega \tag{4.1}$$

$$u = g \quad \text{on } \partial\Omega \tag{4.2}$$

We define the discontinuous Galerkin methods which use completely discontinuous basis functions. Given any triangulation $\mathcal{T}_h$ of Ω, we consider X_h be the space of functions whose restriction to each $T \in \mathcal{T}_h$ are certain polynomials (not necessarily the same degree) and let the collection of all the edges of $T \in \mathcal{T}_h$ be denoted by $\mathcal{E}_h$ and we split $\mathcal{E}_h$ into two disjoint sets; $\mathcal{E}_h = \mathcal{E}_h^o \cup \mathcal{E}_h^b$, where $\mathcal{E}_h^o$ is the set of edges lying in the interior of Ω, and $\mathcal{E}_h^b$ is the set of edges on the boundary of Ω. For illustration, consider two adjacent element T_1, T_2. Let $\mathbf{n}_{T_i}, i = 1, 2$ be the unit outward normal vector to the boundary of T_i, but for the common edge $e = \partial T_1 \cap \partial T_2$, we choose a direction of the normal vector, say $\mathbf{n}_e = \mathbf{n}_{T_1}$ and fix it once and for all. We do this for every edges. For functions v defined on $T_1 \cup T_2$, we let $[\cdot]_e$ and $\{\cdot\}_e$ denote the jump and average across e respectively, i.e.

$$[v]_e = v^1 - v^2, \{v\}_e = \frac{1}{2}(v^1 + v^2).$$

We note the identity

$$a_1 b_1 - a_2 b_2 = \frac{1}{2}[(a_1 + a_2)(b_1 - b_2) + (a_1 - a_2)(b_1 + b_2)]. \tag{4.3}$$

Multiplying both sides of the equation (4.1) by $v \in H^1(T)$, applying Green's formula and adding, we get

$$\sum_{T \in \mathcal{T}_h} \left(\int_T \beta \nabla u \cdot \nabla v dx - \int_{\partial T} \beta \nabla u \cdot \mathbf{n}_T \, v ds \right) = \int_\Omega f v dx.$$

First assume $g = 0$. By using the identity (4.3) and the preassigned normal vectors $\mathbf{n}_e$ we see the above equation becomes

$$\sum_{T \in \mathcal{T}_h} \int_T \beta \nabla u \cdot \nabla v dx - \sum_{e \in \mathcal{E}_h^o} \int_e \{\beta \nabla u \cdot \mathbf{n}_e\}_e [v]_e - \sum_{e \in \mathcal{E}_h^b} \int_e \beta \nabla u \cdot \mathbf{n}_e \, v ds = \int_\Omega f v dx. \tag{4.4}$$

Since $[u] = 0$ on edges and $u = g = 0$ on $\partial\Omega$ we can add the unharmful term

$$-\epsilon \sum_{e \in \mathcal{E}_h} \int_e \{\beta \nabla v \cdot \mathbf{n}_e\}_e [u]_e ds \text{ and } j_\sigma(u, v) := \sum_{e \in \mathcal{E}_h^o} \int_e \frac{\sigma}{h} [u]_e [v]_e ds + \sum_{e \in \mathcal{E}_h^b} \frac{\sigma}{h} \int_e uv \, ds$$

for any ϵ, we get

$$\sum_{T\in\mathcal{T}_h}\int_T \beta\nabla u\cdot\nabla v dx - \sum_{e\in\mathcal{E}_h^o}\left(\int_e \{\beta\nabla u\cdot\mathbf{n}_e\}_e[v]_e ds + \epsilon\int_e\{\beta\nabla v\cdot\mathbf{n}_e\}_e[u]_e\right) ds$$
$$-\sum_{e\in\mathcal{E}_h^b}\left(\int_e \beta\nabla u\cdot\mathbf{n}_e\, v + \epsilon\beta\nabla v\cdot\mathbf{n}_e u\, ds\right) + j_\sigma(u,v) = \int_\Omega f v dx \quad (4.5)$$

Now if $u = g_1 \neq 0$ on $\partial\Omega$, we have to add the followings to rhs.

$$\sum_{e\in\mathcal{E}_h^b}\frac{\sigma}{h}\int_e g_1 v\, ds - \epsilon\sum_{e\in\mathcal{E}_h^b}\int_e \beta\nabla v\cdot\mathbf{n}_e g_1\, ds. \quad (4.6)$$

Let $X_h(\Omega)$ be any space of piecewise polynomials on Ω. We define the following bilinear forms

$$\begin{aligned} a_\epsilon(u,v) &:= a_h(u,v) + b_\epsilon(u,v) + j_\sigma(u,v) \\ b_\epsilon(u,v) &:= -\sum_{e\in\mathcal{E}_h^o}\left(\int_e\{\beta\nabla u\cdot\mathbf{n}\}_e[v]_e ds + \epsilon\int_e\{\beta\nabla v\cdot\mathbf{n}\}_e[u]_e ds\right) \\ &\quad -\sum_{e\in\mathcal{E}_h^b}\left(\int_e \beta\nabla u\cdot\mathbf{n}_e\, v + \epsilon\beta\nabla v\cdot\mathbf{n}_e u\, ds\right) \end{aligned}$$

Now, for each $\epsilon = 0$, $\epsilon = \pm 1$, we obtain the DG for the problem (4.1): Find $u \in X_h$ such that for all $v \in X_h$

$$a_\epsilon(u,v) = (f,v) + \sum_{e\in\mathcal{E}_h^b}\frac{\sigma}{h}\int_e g_1 v\, ds - \epsilon\sum_{e\in\mathcal{E}_h^b}\int_e \beta\nabla v\cdot\mathbf{n}_e g_1\, ds. \quad (4.7)$$

This corresponds IP, SIPG, NIPG, if $\epsilon = 0$, $\epsilon = -1$, $\epsilon = 1$. If some part of the boundary is Neumann type, say, $\beta\nabla u\cdot\mathbf{n}_e = g_2$ on $\Gamma_2 := \partial\Omega\setminus\Gamma_1$, then change the corresponding term of $b_\epsilon(u,v)$ in (4.6) to

$$\sum_e\int_{e\cap\Gamma_2} g_2 v\, ds - \epsilon\sum_e\int_{e\cap\Gamma_2} g_1\nabla v\cdot\mathbf{n}_e v\, ds + \sum_{e\cap\Gamma_1}\frac{\sigma}{h}\int_e g_1 v\, ds$$

similarly for j_σ, i.e., $j_\sigma(u,v) = \sum_{e\cap\Gamma_1}\frac{\sigma}{h}\int_e uv\, ds$.

Remark 4.1.1. *Advantage of DG method:*

(1) Label the nodes element wise.

(2) We can use nonmatching grids, variable degree of basis functions.

(3) Capture shocks well.

(4) Mass matrix for parabolic problem is block diagonal.

(5) Many more dof than continuous basis.

4.1.1 The general idea for the Stokes problem

Consider the incompressible Stokes problem in with data $\mathbf{f} \in L^2(\Omega)$

$$\begin{cases} -\nu\Delta\mathbf{u} + \mathbf{grad}\, p &= \mathbf{f} \text{ in } \Omega, \\ \operatorname{div}\mathbf{u} &= 0 \text{ in } \Omega, \\ \mathbf{u} &= 0 \text{ on } \Gamma. \end{cases} \tag{4.8}$$

We take a test function $\mathbf{v} \in H^1(K)$, and assuming the p is smooth, we obtain

$$\int_\Omega \nabla p \cdot \mathbf{v}\, dx = \sum_T \left(-\int_T p \operatorname{div}\mathbf{v}\, dx \right) + \sum_e \int_e \{p\}[\mathbf{v}] \cdot \mathbf{n}\, ds \tag{4.9}$$

Thus combining this with (4.5), we have the following IIPG, SIPG, NIPG formulations for the Stokes problem (11):

$$\begin{aligned} &\sum_{T\in\mathcal{T}_h} \mu \int_T \nabla\mathbf{u} : \nabla\mathbf{v} dx - \mu \left(\sum_{e\in\mathcal{E}_h^o} \int_e \{\nabla\mathbf{u}\cdot\mathbf{n}_e\}_e[\mathbf{v}]_e ds + \epsilon \int_e \{\nabla\mathbf{v}\cdot\mathbf{n}_e\}_e[\mathbf{u}]_e ds \right) \\ &\qquad - \mu \sum_{e\in\mathcal{E}_h^b} \int_e (\nabla\mathbf{u}\cdot\mathbf{n}_e\,\mathbf{v} + \epsilon\nabla\mathbf{v}\cdot\mathbf{n}_e\,\mathbf{u}) ds + \mu j_\sigma(\mathbf{u},\mathbf{v}) \\ &+ \sum_{T\in\mathcal{T}_h} \left(-\int_T p \operatorname{div}\mathbf{v}\, dx \right) + \sum_{e\in\mathcal{E}_h^o} \int_e \{p\}[\mathbf{v}]\cdot\mathbf{n}\, ds + \sum_{e\in\mathcal{E}_h^b} \int_e p\cdot\mathbf{v}\cdot\mathbf{n}_e\, v ds = \int_\Omega f v dx \\ &\sum_{T\in\mathcal{T}_h} \left(\int_K q \operatorname{div}\mathbf{u}\, dx \right) - \sum_{e\in\mathcal{E}_h^o} \int_e \{q\}[\mathbf{u}]\cdot\mathbf{n}\, ds - \sum_{e\in\mathcal{E}_h^b} \int_e q\mathbf{u}\cdot\mathbf{n}\, ds = 0. \end{aligned} \tag{4.10}$$

4.2 Elasticity equation

The pure traction problem is

$$\begin{aligned} -\mathbf{div}\,\{2\mu\boldsymbol{\epsilon}(\mathbf{u}) + \lambda\, tr(\boldsymbol{\epsilon}(\mathbf{u}))\boldsymbol{\delta}\} = \mathbf{f}, &\quad \text{in } \Omega, \\ \boldsymbol{\sigma}(\mathbf{u})\cdot\mathbf{n} = \mathbf{g}, &\quad \text{on } \partial\Omega, \end{aligned} \tag{4.11}$$

with compatibility condition:

$$\int_\Omega \mathbf{f}\cdot\mathbf{v}\, d\mathbf{x} + \int_{\partial\Omega} \mathbf{g}\cdot\mathbf{v} ds = 0, \text{ for } \mathbf{v} \in RM := \{(a+by, c-bx)\}.$$

Multiply $\mathbf{v} \in (H_0^1(T))^2$ and integrate by part of first term in (4.11), we see

$$\begin{aligned}
& -2\mu \int_T \sum_{i,j} \frac{\partial \boldsymbol{\epsilon}_{ij}(\mathbf{u})}{\partial x_j} v_i \, d\mathbf{x} \\
= & -2\mu \int_{\partial T} \sum_{i,j} \boldsymbol{\epsilon}_{ij}(\mathbf{u}) n_j v_i ds + \mu \int_T \sum_{i,j} \boldsymbol{\epsilon}_{ij}(\mathbf{u}) (\frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i}) \, d\mathbf{x} \\
= & -2\mu \int_{\partial T} \sum_{i,j} \boldsymbol{\epsilon}_{ij}(\mathbf{u}) n_j v_i ds + 2\mu \int_T \sum_{i,j} \boldsymbol{\epsilon}_{ij}(\mathbf{u}) \boldsymbol{\epsilon}_{ij}(\mathbf{v}) \, d\mathbf{x} \\
= & -2\mu \int_{\partial T} \boldsymbol{\epsilon}(\mathbf{u}) \mathbf{n} \cdot \mathbf{v} \, ds + 2\mu \int_T \boldsymbol{\epsilon}(\mathbf{u}) : \boldsymbol{\epsilon}(\mathbf{v}) \, d\mathbf{x}.
\end{aligned}$$

Here we have use the symmetry of $\boldsymbol{\epsilon}_{ij}(\mathbf{u})$. The second term of (4.11) is

$$\lambda \int_T \operatorname{div} \mathbf{u} \operatorname{div} \mathbf{v} \, d\mathbf{x} - \lambda \int_{\partial T} tr(\boldsymbol{\epsilon}(\mathbf{u})) \mathbf{v} \cdot \mathbf{n} \, ds.$$

Hence we get

$$\begin{aligned}
a(\mathbf{u}, \mathbf{v}) = & \sum_T 2\mu \int_T \boldsymbol{\epsilon}(\mathbf{u}) : \boldsymbol{\epsilon}(\mathbf{v}) \, d\mathbf{x} - 2 \sum_{e \in \mathcal{E}_h^o} \mu \int_e \{\boldsymbol{\epsilon}(\mathbf{u}) \mathbf{n}\} \cdot [\mathbf{v}] ds \\
& + \sum_T \lambda \int_T \operatorname{div} \mathbf{u} \operatorname{div} \mathbf{v} \, d\mathbf{x} - \sum_{e \in \mathcal{E}_h^o} \lambda \int_e \{tr(\boldsymbol{\epsilon}(\mathbf{u}))\} [\mathbf{v} \cdot \mathbf{n}] \, ds.
\end{aligned}$$

Here we assumed the homogeneous BC for simplicity. Add the symmetrizing term

$$2\epsilon \sum_{e \in \mathcal{E}_h^o} \mu \int_e \{\boldsymbol{\epsilon}(\mathbf{v}) \mathbf{n}\} \cdot [\mathbf{u}] ds + \epsilon \sum_{e \in \mathcal{E}_h^o} \lambda \int_e \{tr(\boldsymbol{\epsilon}(\mathbf{v}))\} [\mathbf{u} \cdot \mathbf{n}] \, ds.$$