Chapter 8

Techniques of Integration

8.1 Basic integration formulas

(1) $\int x^n \, dx = \frac{x^{n+1}}{n+1} + C \quad (n \neq -1)$

(2) $\int \frac{1}{x} \, dx = \ln x + C$

(3) $\int \cos x \, dx = \sin x + C$

(4) $\int \sin x \, dx = -\cos x + C$

(5) $\int \tan x \, dx = -\ln |\cos x| + C$

(6) $\int \sec^2 x \, dx = \tan x + C$

(7) $\int \csc^2 x \, dx = -\cot x + C$

(8) $\int \sec x \tan x \, dx = \sec x + C$

(9) $\int \csc x \cot x \, dx = -\csc x + C$

(10) $\int \frac{1}{\sqrt{1-x^2}} \, dx = \sin^{-1} x + C$

(11) $\int \frac{1}{1+x^2} \, dx = \tan^{-1} x + C$

(12) $\int e^x \, dx = e^x + C$

(13) $\int a^x \, dx = \dfrac{a^x}{\ln a} + C \quad (a > 0, a \neq 1)$

(14) $\int \cosh x \, dx = \sinh x + C$

(15) $\int \sinh x \, dx = \cosh x + C$

(16) $\int \tanh x \, dx = \ln \cosh x + C$

(17) $\int \coth x \, dx = \ln |\sinh x| + C$

(18) $\int \operatorname{sech} x \, dx = \tan^{-1} \sinh x + C$

(19) $\int \operatorname{csch} x \, dx = \ln \left| \tanh \dfrac{x}{2} \right| + C$

Example 8.1.1.

$$\int_0^{\pi/4} \sqrt{1 + \cos 4x} \, dx.$$

Use

$$\cos^2 \theta = \frac{1 + \cos 2\theta}{2}.$$

Example 8.1.2. Find

$$\int \sec x \, dx.$$

The idea is to multiply $\sec x + \tan x$ both the numerator and denominator:

$$\begin{aligned} \int \sec x \, dx &= \int \sec x \cdot \frac{\sec x + \tan x}{\sec x + \tan x} \, dx \\ &= \int \frac{\sec^2 x + \sec x \tan x}{\sec x + \tan x} \, dx \\ &= \int \frac{du}{u} \\ &= \ln |\sec x + \tan x| + C. \end{aligned}$$

Similarly, we obtain

$$\int \csc x \, dx = -\ln |\csc x + \cot x| + C.$$

Example 8.1.3. Find
$$\int \frac{dx}{1-\sin x}.$$

$$\begin{aligned}
\int \frac{dx}{1-\sin x} &= \int \frac{1}{1-\sin x} \cdot \frac{1-\sin x}{1+\sin x} dx \\
&= \int \frac{1+\sin x}{1-\sin^2 x} dx \\
&= \int \frac{1+\sin x}{\cos^2 x} dx \\
&= \int (\sec^2 x + \sec x \tan x) dx \\
&= \tan x + \sec x + C.
\end{aligned}$$

Example 8.1.4. Find
$$\int \frac{dx}{(1+\sqrt{x})^3}.$$

Substitution $u = \sqrt{x}$ does not work! However, using $u = 1 + \sqrt{x}$, we see

$$\begin{aligned}
\int \frac{dx}{(1+\sqrt{x})^3} &= \int \frac{2(u-1)}{u^3} du \\
&= \int \left(\frac{2}{u^2} - \frac{2}{u^3} du \right) du \\
&= -\frac{2}{u} + \frac{1}{u^2} + C \\
&= \frac{1 - 2(1+\sqrt{x})}{(1+\sqrt{x})^2} + C \\
&= -\frac{1+2\sqrt{x}}{(1+\sqrt{x})^2} + C
\end{aligned}$$

Integral tables

(1) $\displaystyle\int \frac{1}{a^2+u^2} du = \frac{1}{a} \tan^{-1} \frac{u}{a} \quad (a > 0).$

(2) $\displaystyle\int \frac{1}{\sqrt{a^2-u^2}} du = \sin^{-1} \frac{u}{a} \quad (a > 0).$

Example 8.1.5. For $\int 1/(4+9x^2)\, dx$, use substitution first. Let $3x/2 = u$.

Then $3/2dx = du$ and

$$\begin{aligned}\int \frac{1}{4+9x^2}\,dx &= \frac{1}{4}\int \frac{1}{1+(\frac{3x}{2})^2}\,dx \\ &= \frac{1}{6}\int \frac{1}{1+u^2}\,du \\ &= \frac{1}{6}\tan^{-1}\frac{3}{2}x + C.\end{aligned}$$

8.2 Integration by Parts

Recall the Leibnitz formula:

$$\frac{d}{dx}[f(x)g(x)] = f(x)'g(x) + f(x)g'(x).$$

Integrating w.r.t x, we have

$$f(x)g(x) = \int f(x)g'(x)\,dx + \int f'(x)g(x)\,dx. \tag{8.1}$$

Thus

Proposition 8.2.1 (Integration by Parts I)**.**

$$\int f(x)g'(x)\,dx = f(x)g(x) - \int f'(x)g(x)\,dx. \tag{8.2}$$

Proposition 8.2.2 (Definite integral)**.**

$$\int_a^b f(x)g'(x)\,dx = [f(x)g(x)]_a^b - \int_a^b f'(x)g(x)\,dx.$$

Sometimes writing $u = f(x), v = g(x)$, the integration by parts can have the following form:

Proposition 8.2.3 (Integration by Parts II)**.**

$$\int u\,dv = uv - \int v\,du. \tag{8.3}$$

Example 8.2.4. Find the following

(1) $\int_0^\pi x\sin x\,dx$

(2) $\int \ln x \, dx$.

sol. (1) Let $u = x$, $dv = \sin x \, dx$. Then $du = dx$, $v = -\cos x$. (Fig 8.1)

$$\begin{aligned}\int_0^\pi x \sin x \, dx &= [x(-\cos x)]_0^\pi - \int_0^\pi (-\cos x) \, dx \\ &= \pi + [\sin x]_0^\pi \\ &= \pi.\end{aligned}$$

(2) Let $u = \ln x$, $dv = dx$. Then we have $du = (1/x)dx$, $v = x$.

$$\begin{aligned}\int \ln x \, dx &= (\ln x)x - \int x \cdot \frac{1}{x} \, dx \\ &= x \ln x - x + C.\end{aligned}$$

□

Example 8.2.5. Find $\int x^2 \sin x \, dx$.

sol. Let $u = x^2$, $dv = \sin x \, dx$. Then $du = 2xdx$, $v = -\cos x$ and hence

$$\begin{aligned}\int x^2 \sin x \, dx &= x^2(-\cos x) - \int (-\cos x) 2x \, dx \\ &= -x^2 \cos x + \int 2x \cos x \, dx.\end{aligned}$$

Again, set $u = 2x$, $dv = \cos x \, dx$. Then $du = 2 \, dx$, $v = \sin x$.

$$\begin{aligned}\int x^2 \sin x \, dx &= -x^2 \cos x + 2x \sin x - \int 2 \sin x \, dx \\ &= -x^2 \cos x + 2x \sin x + 2 \cos x + C.\end{aligned}$$

□

Repeated integration by parts - using table

Example 8.2.6. Find $\int x^2 e^x \, dx$.

sol. Referring to Table 8.1 with $f(x) = x^2$, $g(x) = e^x$, we have

$$\int x^2 e^x \, dx = x^2 e^x - 2xe^x + 2e^x + C.$$

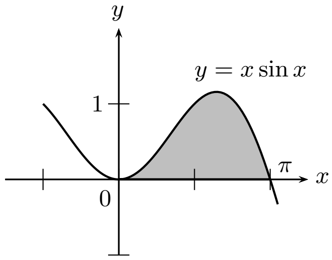

Figure 8.1:

f and its derivative		g and its integral
x^2	$(+)$	e^x
$2x$	$(-)$	e^x
2	$(+)$	e^x
0		e^x

Table 8.1: x^2e^x

□

f and its derivative		g and its integral
x^3	$(+)$	$\sin x$
$3x^2$	$(-)$	$-\cos x$
$6x$	$(+)$	$-\sin x$
6	$(-)$	$\cos x$
0		$\sin x$

Table 8.2: $x^3 \sin x$

Example 8.2.7. Find $\displaystyle\int x^3 \sin x\, dx$.

Use the Table 8.2.

$$\int x^3 \sin x\, dx = -x^3 \cos x + 3x^2 \sin x + 6x \cos x - 6 \sin x + C.$$

Example 8.2.8. Find $\displaystyle\int e^x \sin x\, dx$.

sol. If $u = e^x$, $dv = \sin x\,dx$, then $du = e^x\,dx$, $v = -\cos x$.

$$\begin{aligned}\int e^x \sin x\,dx &= e^x(-\cos x) - \int e^x(-\cos x)\,dx \\ &= -e^x\cos x + \int e^x \cos x\,dx.\end{aligned}$$

Again let $u = e^x$, $dv = \cos x\,dx$ so that $du = e^x\,dx$, $v = \sin x$.

$$\begin{aligned}\int e^x \sin x\,dx &= -e^x\cos x + \int e^x \cos x\,dx \\ &= -e^x\cos x + e^x \sin x - \int e^x \sin x\,dx.\end{aligned}$$

Solving this for $\int e^x \sin x\,dx$ we obtain

$$\int e^x \sin x\,dx = \frac{1}{2}e^x(\sin x - \cos x) + C.$$

□

Reduction formula

Example 8.2.9. Express $\int \cos^n x\,dx$ in terms of low power of $\cos x$.

sol.

$$\begin{aligned}\int \cos^{n-1} x \cos x\,dx &= \cos^{n-1}\sin x + (n-1)\int \sin^2 x \cos^{n-2} x\,dx \\ &= \cos^{n-1}\sin x + (n-1)\int (1-\cos^2 x)\cos^{n-2} x\,dx \\ &= \cos^{n-1}\sin x + (n-1)\int \cos^{n-2} x\,dx - (n-1)\int \cos^n x\,dx.\end{aligned}$$

So

$$n\int \cos^n x\,dx = \cos^{n-1}\sin x + (n-1)\int \cos^{n-2} x\,dx.$$

□

Example 8.2.10. Prove

$$\int (a^2 \pm x^2)^n\,dx = \frac{x(a^2 \pm x^2)}{2n+1} + \frac{2na^2}{2n+1}\int (a^2 \pm x^2)^{n-1}\,dx, \quad (n \neq -\frac{1}{2}).$$

sol. Integration by parts

$$\begin{aligned}\int (a^2 \pm x^2)^n\,dx &= x(a^2 \pm x^2)^n - \int x \cdot n(a^2 \pm x^2)^{n-1}(\pm 2x)\,dx \\ &= x(a^2 \pm x^2)^n - \int 2n(a^2 \pm x^2)^{n-1}(a^2 \pm x^2 - a^2)\,dx \\ &= x(a^2 \pm x^2)^n - 2n\int (a^2 \pm x^2)^n\,dx + 2na^2\int (a^2 \pm x^2)^{n-1}\,dx.\end{aligned}$$

If $n \neq -1/2$,

$$\int (a^2 \pm x^2)^n\,dx = \frac{x(a^2 \pm x^2)^n}{2n+1} + \frac{2na^2}{2n+1}\int (a^2 \pm x^2)^{n-1}\,dx.$$

□

8.3 Integration of trigonometric function

Products of powers of sines and cosines

Integral of $\sin^m x \cos^n x$

(1) If m is odd, then set $m = 2k+1$ and use $\sin^2 x = 1 - \cos^2 x$, $\sin x\,dx = -d(\cos x)$ to transform it to

$$\int \sin^{2k+1} x \cos^n x\,dx = -\int (1 - \cos^2 x)^k \cos^n x\,d(\cos x).$$

(2) If n is odd $n = 2k+1$, use $\cos^2 x = 1 - \sin^2 x$ to obtain

$$\int \sin^m x \cos^{2k+1} x\,dx = \int \sin^m x(1 - \sin^2 x)^k\,d(\sin x).$$

(3) If both m, n are even, use $\sin^2 x = (1-\cos 2x)/2$, $\cos^2 x = (1+\cos 2x)/2$ to lower the degree and repeat the previous technique.

Example 8.3.1. Find $\int \sin^5 x\,dx$.

sol.
$$\begin{aligned}\int \sin^5 x\,dx &= -\int (1 - \cos^2 x)^2\,d(\cos x) \\ &= -\int (1 - 2\cos^2 x + \cos^4 x)\,d(\cos x) \\ &= -\frac{1}{5}\cos^5 x + \frac{2}{3}\cos^3 x - \cos x + C.\end{aligned}$$

□

Example 8.3.2. Find $\int \sin^2 x \cos^3 x \, dx$.

sol.
$$\begin{aligned}\int \sin^2 x \cos^3 x \, dx &= \int \sin^2 x (1-\sin^2 x) \, d(\sin x)\\ &= -\frac{1}{5}\sin^5 x + \frac{1}{3}\sin^3 x + C.\end{aligned}$$

□

Example 8.3.3. Find $\int \sin^4 x \cos^2 x \, dx$.

sol.
$$\begin{aligned}\int \sin^4 x \cos^2 x \, dx &= \int \left(\frac{1-\cos 2x}{2}\right)^2 \left(\frac{1+\cos 2x}{2}\right) dx\\ &= \frac{1}{8}\int \left(1 - 2\cos 2x + \cos^2 2x\right)(1+\cos 2x)\, dx\\ &= \frac{1}{8}\int \left(1 - \cos 2x - \cos^2 2x + \cos^3 2x\right) dx\\ &= \frac{1}{8}\int \left(1 - \cos 2x - \frac{1+\cos 4x}{2} + (1-\sin^2 2x)\cos 2x\right) dx\\ &= \frac{1}{16}\int \left(1 - \cos 4x - \sin^2 2x \cdot 2\cos 2x\right) dx\\ &= \frac{1}{16}\left(x - \frac{1}{4}\sin 4x - \frac{1}{3}\sin^3 2x\right) + C.\end{aligned}$$

□

Integral of $\sqrt{1 \pm \sin ax}$, $\sqrt{1 \pm \cos ax}$

Use the double angle formula.

$$\begin{aligned}\sin 2A &= 2\sin A \cos A\\ \cos 2A &= 2\cos^2 A - 1 = 1 - 2\sin^2 A\end{aligned}$$

to change the terms $1 \pm \sin ax$ or $1 \pm \cos ax$ to a complete square.

Example 8.3.4. Find $\int_0^{\pi} \sqrt{1 - \sin x}\, dx$.

sol. Use the identity:

$$1 - \sin x = 1 - 2\sin(\frac{x}{2})\cos(\frac{x}{2}) = \left(\sin(\frac{x}{2}) - \cos(\frac{x}{2})\right)^2.$$

$$\begin{aligned}
\int_0^{\pi} \sqrt{1-\sin x}\,dx &= \int_0^{\pi} \left|\sin\frac{x}{2} - \cos\frac{x}{2}\right|\,dx \\
&= \int_0^{\pi/2} \left(\cos\frac{x}{2} - \sin\frac{x}{2}\right) + \int_{\pi/2}^{\pi} \left(\sin\frac{x}{2} - \cos\frac{x}{2}\right)\,dx \\
&= \left[2\sin\frac{x}{2} + 2\cos\frac{x}{2}\right]_0^{\pi/2} + \left[-2\cos\frac{x}{2} - 2\sin\frac{x}{2}\right]_{\pi/2}^{\pi} \\
&= (\sqrt{2}+\sqrt{2}-2) + (-2+\sqrt{2}+\sqrt{2}) \\
&= 4(\sqrt{2}-1).
\end{aligned}$$

□

Example 8.3.5. Find $\displaystyle\int_0^{\pi/2} \sqrt{1+\cos 2x}\,dx$.

sol. $1+\cos 2x = 2\cos^2 x$,

$$\begin{aligned}
\int_0^{\pi/2} \sqrt{1+\cos 2x} &= \sqrt{2}\int_0^{\pi/2} |\cos x|\,dx \\
&= \sqrt{2}\,[\sin x]_0^{\pi/2} = \sqrt{2}.
\end{aligned}$$

□

Integrals of powers of $\tan x$ and $\sec x$

Recall

$$1+\tan^2 x = \sec^2 x,\ (\tan x)' = \sec^2 x,\ (\sec x)' = \sec x \tan x.$$

Example 8.3.6. $\displaystyle\int \sec x\,dx$.

sol. Multiply $\sec x + \tan x$.

$$\begin{aligned}
\int \sec x\,dx &= \int \frac{\sec x(\sec x + \tan x)}{\sec x + \tan x}\,dx \\
&= \int \frac{(\sec x + \tan x)'}{\sec x + \tan x}\,dx \\
&= \ln|\sec x + \tan x| + C.
\end{aligned}$$

□

Example 8.3.7. $\displaystyle\int \tan^2 x \sec x\,dx$.

sol. Since $\tan^2 x \sec x = (\sec^2 x - 1)\sec x = \sec^3 x - \sec x$, we can find $\int \sec^3 x\,dx$. Let $u = \sec x$, $dv = \sec^2 x\,dx$ then $v = \tan x$, $du = \sec x \tan x\,dx$, we have

$$\begin{aligned}\int \sec^3 x\,dx &= \sec x \tan x - \int (\tan x)\sec x \tan x\,dx \\ &= \sec x \tan x - \int (\sec^2 x - 1)\sec x\,dx \\ &= \sec x \tan x - \int \sec^3 x\,dx + \int \sec x\,dx.\end{aligned}$$

Hence we obtain

$$\int \sec^3 x\,dx = \frac{1}{2}\sec x \tan x + \frac{1}{2}\int \sec x\,dx.$$

Hence

$$\begin{aligned}\int \tan^2 x \sec x\,dx &= \int \sec^3 x\,dx - \int \sec x\,dx \\ &= \frac{1}{2}\sec x \tan x - \frac{1}{2}\int \sec x\,dx \\ &= \frac{1}{2}\sec x \tan x - \frac{1}{2}\ln|\sec x + \tan x| + C.\end{aligned}$$

□

Example 8.3.8. $\int \tan^6 x\,dx$.

sol. Since $\tan^2 x = \sec^2 x - 1$

$$\begin{aligned}\int \tan^6 x\,dx &= \int \tan^4 x(\sec^2 x - 1)\,dx \\ &= \int \tan^4 x \sec^2 x\,dx - \int \tan^4 x\,dx \\ &= \int \tan^4 x \sec^2 x\,dx - \int \tan^2 x(\sec^2 x - 1)\,dx \\ &= \int \tan^4 x \sec^2 x\,dx - \int \tan^2 x \sec^2 x\,dx + \int \tan^2 x\,dx \\ &= \int \tan^4 x \sec^2 x\,dx - \int \tan^2 x \sec^2 x\,dx + \int (\sec^2 x - 1)\,dx \\ &= \frac{1}{5}\tan^5 x - \frac{1}{3}\tan^3 x + \tan x - x + C.\end{aligned}$$

□

Remark 8.3.9. For $\cot x$ or $\csc x$, use

$$\begin{aligned} 1+\cot^2 x &= \csc^2 x, \\ (\cot x)' &= -\csc^2 x, \\ (\csc x)' &= -\csc x \cot x. \end{aligned}$$

Products such as $\sin mx \sin nx$, $\sin mx \cos nx$, $\cos mx \cos nx$

Recall addition formula:

$$\begin{aligned} \sin(A+B) &= \sin A \cos B + \cos A \sin B \\ \sin(A-B) &= \sin A \cos B - \cos A \sin B \\ \cos(A+B) &= \cos A \cos B - \sin A \sin B \\ \cos(A-B) &= \cos A \cos B + \sin A \sin B. \end{aligned}$$

From these we get(with $A = mx$, $B = nx$)

$$\begin{aligned} \sin mx \sin nx &= \frac{1}{2}[\cos(m-n)x - \cos(m+n)x] \\ \sin mx \cos nx &= \frac{1}{2}[\sin(m-n)x + \sin(m+n)x] \\ \cos mx \cos nx &= \frac{1}{2}[\cos(m-n)x + \cos(m+n)x]. \end{aligned}$$

Example 8.3.10. $\displaystyle\int_0^{\pi/6} \sin 4x \sin 3x \, dx.$

sol.

$$\begin{aligned} \int_0^{\pi/6} \sin 4x \sin 3x \, dx &= \frac{1}{2}\int_0^{\pi/6} (\cos x - \cos 7x)\, dx \\ &= \frac{1}{2}\left[\sin x - \frac{1}{7}\sin 7x\right]_0^{\pi/6} = \frac{2}{7}. \end{aligned}$$

□

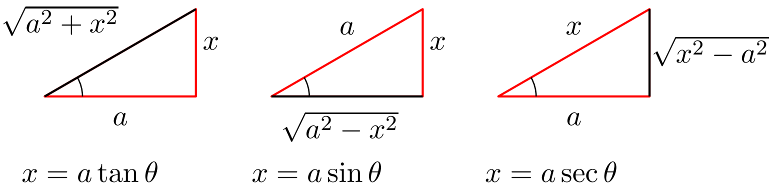

Figure 8.2: trig substitution

8.4 Trigonometric substitutions

Quadratic term

For the terms of the forms a^2-u^2, a^2+u^2 u^2-a^2, we can try to substitute $u = a\sin\theta$, $u = a\tan\theta$, $u = a\sec\theta$ resp.

$$a^2-u^2 = a^2 - a^2\sin^2\theta = a^2(1-\sin^2\theta) = a^2\cos^2\theta \tag{8.4}$$
$$a^2+u^2 = a^2 + a^2\tan^2\theta = a^2(1+\tan^2\theta) = a^2\sec^2\theta \tag{8.5}$$
$$u^2-a^2 = a^2\sec^2\theta - a^2 = a^2(\sec^2\theta - 1) = a^2\tan^2\theta. \tag{8.6}$$

Note the domain of definition

(1) $u = a\sin\theta$ is defined on $-\pi/2 \le \theta \le \pi/2$.

(2) $u = a\tan\theta$ $\theta = \tan^{-1}(u/a)$ on $-\pi/2 < \theta < \pi/2$.

(3) $u = a\sec\theta$ $\theta = \sec^{-1}(u/a)$ Since $|u| \ge a$ $0 \le \theta < \pi/2$ (if $u \ge a$), or $\pi/2 < \theta \le \pi$ (if $u \le -a$).

Example 8.4.1. $\displaystyle\int \frac{du}{a^2+u^2}$.

sol. Use substitution $u = a\tan\theta$, $du = a\sec^2\theta\, d\theta$ to get

$$\int \frac{du}{a^2+u^2} = \int \frac{a\sec^2\theta\, d\theta}{a^2\sec^2\theta} = \int \frac{d\theta}{a}$$
$$= \frac{\theta}{a} + C$$
$$= \frac{1}{a}\tan^{-1}\frac{u}{a} + C.$$

□

Example 8.4.2. Find $\int \sqrt{a^2 - u^2}\, du,\ (a > 0)$.

sol. Use $u = a\sin\theta$, $du = a\cos\theta\, d\theta$ to get

$$\begin{aligned}
\int \sqrt{a^2 - u^2}\, du &= \int a\cos\theta \cdot a\cos\theta\, d\theta \\
&= \frac{a^2}{2}\int (1 + \cos 2\theta)\, d\theta \\
&= \frac{a^2}{2}\left(\theta + \frac{\sin 2\theta}{2}\right) + C = \frac{a^2}{2}(\theta + \sin\theta\cos\theta) + C \\
&= \frac{a^2}{2}\left(\sin^{-1}\frac{u}{a} + \frac{u}{a}\sqrt{1 - \frac{u^2}{a^2}}\right) + C \\
&= \frac{a^2}{2}\sin^{-1}\frac{u}{a} + \frac{1}{2}u\sqrt{a^2 - u^2} + C.
\end{aligned}$$

□

Example 8.4.3. Find $\int \frac{du}{\sqrt{u^2 - a^2}},\ (|u| > a > 0)$.

sol. Let $u = a\sec\theta$

$$\begin{aligned}
u^2 - a^2 &= a^2(\sec^2\theta - 1) \\
&= a^2\tan^2\theta, \\
du &= a\sec\theta\tan\theta\, d\theta.
\end{aligned}$$

Then

$$\begin{aligned}
\int \frac{du}{\sqrt{u^2 - a^2}} &= \int \frac{a\sec\theta\tan\theta\, d\theta}{a|\tan\theta|} \\
&= \begin{cases} \int \sec\theta\, d\theta & (0 < \theta < \pi/2) \\ -\int \sec\theta\, d\theta & (\pi/2 < \theta < \pi) \end{cases} \\
&= \begin{cases} \ln|\sec\theta + \tan\theta| + C & (0 < \theta < \pi/2) \\ -\ln|\sec\theta + \tan\theta| + C & (\pi/2 < \theta < \pi) \end{cases}
\end{aligned}$$

$$= \begin{cases} \ln\left|\dfrac{u}{a} + \dfrac{\sqrt{u^2-a^2}}{a}\right| + C \quad (u > a) \\ \\ -\ln\left|\dfrac{u}{a} - \dfrac{\sqrt{u^2-a^2}}{a}\right| + C \quad (u < -a). \end{cases}$$

Last integrals can be simplified as follows:

$$\ln\left|\frac{u}{a} + \frac{\sqrt{u^2-a^2}}{a}\right| = \ln\left|u + \sqrt{u^2-a^2}\right| - \ln a.$$

$$\begin{aligned} -\ln\left|\frac{u}{a} - \frac{\sqrt{u^2-a^2}}{a}\right| &= \ln\left|\frac{a}{u-\sqrt{u^2-a^2}}\right| \\ &= \ln\left|\frac{a(u+\sqrt{u^2-a^2})}{(u-\sqrt{u^2-a^2})(u+\sqrt{u^2-a^2})}\right| \\ &= \ln\left|\frac{a(u+\sqrt{u^2-a^2})}{a^2}\right| = \ln\left|\frac{u+\sqrt{u^2-a^2}}{a}\right| \\ &= \ln\left|u+\sqrt{u^2-a^2}\right| - \ln a. \end{aligned}$$

Hence

$$\int \frac{du}{\sqrt{u^2-a^2}} = \ln\left|u+\sqrt{u^2-a^2}\right| + C'.$$

□

Example 8.4.4. $\displaystyle\int \frac{dx}{\sqrt{x^2+9}}$.

sol. Let $x = 3\tan\theta$ $(-\pi/2 < \theta < \pi/2)$, $dx = 3\sec^2\theta\, d\theta$,

$$\begin{aligned} \int \frac{dx}{\sqrt{x^2+9}} &= \int \frac{3\sec^2\theta}{3\sec\theta}\, d\theta \\ &= \int \sec\theta\, d\theta \\ &= \ln|\sec\theta + \tan\theta| + C \\ &= \ln\left|\sqrt{\left(\frac{x}{3}\right)^2 + 1} + \frac{x}{3}\right| + C \\ &= \ln\left|x + \sqrt{x^2+9}\right| + C. \end{aligned}$$

□

Involving $ax^2 + bx + c$ — Completing the square

For factors like ax^2+bx+c, $(a, b \neq 0)$, use $u = x+b/(2a)$ to get $ax^2+bx+c = a(u^2 \pm p^2)$.

Example 8.4.5. Find $\int \sqrt{2x - x^2}\, dx$.

sol. Since $2x - x^2 = 1 - (x-1)^2$ $u = x - 1$ we have as in example 8.4.2 with $a = 1$,

$$\begin{aligned}\int \sqrt{2x - x^2}\, dx &= \int \sqrt{1-u^2}\, du \\ &= \frac{1}{2}\sin^{-1} u + \frac{1}{2}u\sqrt{1-u^2} + C \\ &= \frac{1}{2}\sin^{-1}(x-1) + \frac{1}{2}(x-1)\sqrt{2x-x^2} + C.\end{aligned}$$

□

Example 8.4.6. $\int \frac{dx}{x^2+x+1}$.

sol. $x^2 + x + 1 = (x+1/2)^2 + 3/4$ $u = x + 1/2$ $a = \sqrt{3}/2$

$$\begin{aligned}\int \frac{dx}{x^2+x+1} &= \int \frac{du}{u^2+3/4} \\ &= \frac{2}{\sqrt{3}}\tan^{-1}\frac{2u}{\sqrt{3}} + C \\ &= \frac{2}{\sqrt{3}}\tan^{-1}\frac{2x+1}{\sqrt{3}} + C.\end{aligned}$$

□

8.5 Integration of rational functions

When $p(x)$, $q(x)$ are rational functions, we can always write it as

$$\frac{p(x)}{q(x)} = Q(x) + \frac{r(x)}{q(x)}$$

for some polynomial $Q(x), r(x)$ (degree of $r(x)$ is less than that of $q(x)$.) Here Q is zero if degree of $p(x)$ is less than that of $q(x)$.

Distinct linear factors

Suppose $\alpha_1, \dots, \alpha_r$ are distinct and $p(x)$ is polynomial of degree less than r. Then we can set

$$\frac{p(x)}{(x-\alpha_1)\cdots(x-\alpha_r)} = \frac{A_1}{x-\alpha_1} + \cdots + \frac{A_r}{x-\alpha_r}. \tag{8.7}$$

Here A_i's can be obtained by method of undetermined coefficients.(There is another method, called Heaviside cover up method, see below)

$$\int \frac{dx}{(x-\alpha_1)\cdots(x-\alpha_r)} = \sum_{i=1}^{r} A_i \ln|x-\alpha_i| + C.$$

Example 8.5.1. Find $\displaystyle\int \frac{x+1}{x(x+2)}\,dx$.

sol. One can find the following partial fraction

$$\frac{x+1}{x(x+2)} = \frac{1}{2x} + \frac{1}{2(x+2)}.$$

$$\begin{aligned}\int \frac{x+1}{x(x+2)}\,dx &= \frac{1}{2}\int \left(\frac{1}{x} + \frac{1}{x+2}\right) dx \\ &= \frac{1}{2}\ln|x(x+2)| + C.\end{aligned}$$

□

Example 8.5.2. Find $\displaystyle\int \frac{2x+1}{x^3-x}\,dx$.

sol. Since $x^3 - x = x(x-1)(x+1)$ we can set

$$\frac{2x+1}{x^3-x} = \frac{A}{x} + \frac{B}{x-1} + \frac{C}{x+1}.$$

Solving for A, B, C we get $A = -1,\ B = 3/2,\ C = -1/2$. Hence

$$\begin{aligned}\int \frac{2x+1}{x^3-x}\,dx &= \int \left(\frac{-1}{x} + \frac{3/2}{x-1} + \frac{-1/2}{x+1}\right) dx \\ &= -\ln|x| + \frac{3}{2}\ln|x-1| - \frac{1}{2}\ln|x+1| + C.\end{aligned}$$

□

Repeated linear factors

Assume the degree of $p(x)$ is less than that of $r(x)$. Then

$$\frac{p(x)}{(x-\alpha)^r} = \frac{A_1}{x-\alpha} + \frac{A_2}{(x-\alpha)^2} + \cdots + \frac{A_r}{(x-\alpha)^r}.$$

To find the coefficients $A_1, A_2, \dots, A_r$, multiply $(x-\alpha)^r$. Then

$$p(x) = A_1(x-\alpha)^{r-1} + A_2(x-\alpha)^{r-1} + \cdots + A_r.$$

Now we can use the method of undetermined coefficients to find A_i's. (Another way of finding $A_i's$ by derivative will be introduced below). Once $A_i's$ are known, we can find the integral:

$$\begin{aligned}\int \frac{p(x)}{(x-\alpha)^r}\,dx &= \int \left(\frac{A_1}{x-\alpha} + \frac{A_2}{(x-\alpha)^2} + \cdots + \frac{A_r}{(x-\alpha)^r}\right) dx \\ &= A_1 \ln|x-\alpha| - \frac{A_2}{x-\alpha} - \cdots - \frac{(r-1)A_r}{(x-\alpha)^{r-1}} + C.\end{aligned}$$

Example 8.5.3. Find $\int \frac{x^2}{(x-2)^3}\,dx$.

sol. Since $x^2 = (x-2)^2 + 4(x-2) + 4$, we have

$$\frac{x^2}{(x-2)^3} = \frac{1}{x-2} + \frac{4}{(x-2)^2} + \frac{4}{(x-2)^3}.$$

Hence

$$\begin{aligned}\int \frac{x^2}{(x-2)^3}\,dx &= \int \left(\frac{1}{x-2} + \frac{4}{(x-2)^2} + \frac{4}{(x-2)^3}\right) dx \\ &= \ln|x-2| - \frac{4}{x-2} - \frac{8}{(x-2)^2} + C.\end{aligned}$$

□

*Irreducible quadratic factor

Suppose $x^2 + \beta_1 x + \gamma_1, \dots, x^2 + \beta_r x + \gamma_r$ are distinct quadratic factor without having real roots(we say irreducible quadratic factor). Suppose $p(x)$ is polynomial of degree less than $2r$. So we have

$$\frac{p(x)}{(x^2+\beta_1 x+\gamma_1)\cdots(x^2+\beta_r x+\gamma_r)} = \sum_{i=1}^{r} \frac{B_i x + C_i}{x^2+\beta_i x+\gamma_i}$$

for some $B_1, \dots, B_r$ and $C_1, \dots, C_r$. Hence

$$\int \frac{p(x)}{(x^2+\beta_1 x+\gamma_1)\cdots(x^2+\beta_r x+\gamma_r)}\, dx = \sum_{i=1}^{r} \int \frac{B_i x + C_i}{x^2+\beta_i x+\gamma_i}\, dx.$$

Again we can find the coefficients by method of undetermined coefficients. Now since

$$\begin{aligned} B_i x + C_i &= \frac{B_i}{2}(2x+\beta_i) + D_i, \ \ (D_i = C_i - B_i\beta_i/2) \\ &= \frac{B_i}{2}(x^2+\beta_i x+\gamma_i)' + D_i, \end{aligned}$$

we have

$$\begin{aligned} \int \frac{B_i x + C_i}{x^2+\beta_i x+\gamma_i}\, dx &= \int \left(\frac{B_i}{2} \frac{(x^2+\beta_i x+\gamma_i)'}{x^2+\beta_i x+\gamma_i} + \frac{D_i}{x^2+\beta_i x+\gamma_i} \right) dx \\ &= \frac{B_i}{2} \ln(x^2+\beta_i x+\gamma_i) + \int \frac{D_i}{x^2+\beta_i x+\gamma_i}\, dx. \end{aligned}$$

For $D_i/(x^2+\beta_i x+\gamma_i)$ use the formula:

$$\int \frac{du}{u^2+a^2} = \frac{1}{a}\tan^{-1}\frac{u}{a} + C.$$

Example 8.5.4. Find $\displaystyle\int \frac{2x}{x^4+x^2+1}\, dx$.

sol. Since $x^4+x^2+1 = (x^2-x+1)(x^2+x+1)$, we set

$$\frac{2x}{x^4+x^2+1} = \frac{B_1 x + C_1}{x^2-x+1} + \frac{B_2 x + C_2}{x^2+x+1}.$$

By comparing, we obtain $B_1 = B_2 = 0$, $C_1 = 1$, $C_2 = -1$. Since

$$x^2 \pm x + 1 = (x \pm 1/2)^2 + (\sqrt{3}/2)^2,$$

we see

$$\begin{aligned}
&\int \frac{2x}{x^4+x^2+1}\,dx \\
&\quad = \int \left(\frac{1}{(x-1/2)^2+(\sqrt{3}/2)^2} - \frac{1}{(x+1/2)^2+(\sqrt{3}/2)^2} \right) dx \\
&\quad = \frac{2}{\sqrt{3}} \left(\tan^{-1} \frac{2x-1}{\sqrt{3}} - \tan^{-1} \frac{2x+1}{\sqrt{3}} \right) + C.
\end{aligned}$$

□

*Repeated irreducible quadratic factor

Suppose $p(x)$ is polynomial of degree less than $2r$, and $x^2+\beta x+\gamma$ does not have real roots. Then we can set

$$\frac{p(x)}{(x^2+\beta x+\gamma)^r} = \frac{B_1x+C_1}{x^2+\beta x+\gamma} + \frac{B_2x+C_2}{(x^2+\beta x+\gamma)^2} + \cdots + \frac{B_rx+C_r}{(x^2+\beta x+\gamma)^r}$$

for some $B_1, B_2, \ldots, B_r, C_1, C_2, \ldots, C_r$. Then

$$\begin{aligned}
&\int \frac{p(x)}{(x^2+\beta x+\gamma)^r}\,dx \\
&\quad = \int \left(\frac{B_1x+C_1}{x^2+\beta x+\gamma} + \frac{B_2x+C_2}{(x^2+\beta x+\gamma)^2} + \cdots + \frac{B_rx+C_r}{(x^2+\beta x+\gamma)^r} \right) dx.
\end{aligned}$$

By the same way as before we see, with $D_i = C_i - B_i\beta/2$

$$\begin{aligned}
\int \frac{B_ix+C_i}{(x^2+\beta x+\gamma)^i}\,dx &= \int \left(\frac{B_i}{2} \frac{(x^2+\beta x+\gamma)'}{(x^2+\beta x+\gamma)^i} + \frac{D_i}{(x^2+\beta x+\gamma)^i} \right) dx \\
&= -\frac{B_i}{2(i-1)(x^2+\beta x+\gamma)^{i-1}} + \int \frac{D_i}{(x^2+\beta x+\gamma)^i}\,dx.
\end{aligned}$$

For the integral of $D_i/(x^2+\beta x+\gamma)^i$ $(i \ge 2)$, use the recurrence relation

$$\int \frac{du}{(u^2+a^2)^i} = \frac{u}{a^2(2i-2)(u^2+a^2)^{i-1}} + \frac{2i-3}{a^2(2i-2)} \int \frac{du}{(u^2+a^2)^{i-1}}.$$

Example 8.5.5. Find $\displaystyle\int \frac{x^4+2x^3+5x^2+6}{(x^2+2)^3}\,dx.$

sol.

$$\frac{x^4+2x^3+5x^2+6}{(x^2+2)^3} = \frac{A_1x+B_1}{x^2+2} + \frac{A_2x+B_2}{(x^2+2)^2} + \frac{A_3x+B_3}{(x^2+2)^3}.$$

Multiply $(x^2+2)^3$ to see

$$x^4+2x^3+5x^2+6 = A_1x^5+B_1x^4+(4A_1+A_2)x^3+(4B_1+B_2)x^2 + (4A_1+2A_2+A_3)x+4B_1+2B_2+B_3.$$

Comparing, we get $A_1=0$, $A_2=2$, $A_3=-2$, $B_1=1$, $B_2=1$, $B_3=0$. Hence the integrand is

$$\frac{x^4+2x^3+5x^2+6}{(x^2+2)^3} = \frac{1}{x^2+2}+\frac{2x+1}{(x^2+2)^2}+\frac{-4x}{(x^2+2)^3}.$$

Hence

$$\begin{aligned}
&\int \frac{x^4+2x^3+5x^2+6}{(x^2+2)^3}\,dx \\
&= \int \frac{dx}{x^2+2}+\int \frac{2x}{(x^2+2)^2}\,dx+\int \frac{1}{(x^2+2)^2}\,dx+\int \frac{-4x}{(x^2+2)^3}\,dx \\
&= \frac{1}{\sqrt{2}}\tan^{-1}\frac{x}{\sqrt{2}}-\frac{1}{x^2+2}+\frac{x}{4(x^2+2)}+\frac{1}{4}\int \frac{1}{x^2+2}\,dx+\frac{1}{(x^2+2)^2} \\
&= \frac{5}{4\sqrt{2}}\tan^{-1}\frac{x}{\sqrt{2}}+\frac{x-4}{4(x^2+2)}+\frac{1}{(x^2+2)^2}+C.
\end{aligned}$$

□

Heaviside cover up method for linear factors

Example 8.5.6.

$$\frac{x^2+1}{(x-1)(x-2)(x-3)} = \frac{A}{x-1}+\frac{B}{x-2}+\frac{C}{x-3}.$$

Here

$$A = \frac{(1)^2+1}{\boxed{(x-1)}_{cover}(1-2)(1-3)}$$

$$B = \frac{(2)^2+1}{(2-1)\boxed{(x-2)}_{cover}(2-3)} = \frac{5}{(1)(-1)} = -5$$

$$C = \frac{(3)^2+1}{(3-1)(3-2)\boxed{(x-3)}_{cover}} = \frac{10}{(2)(1)} = 5.$$

Example 8.5.7. Do the same with

$$\int \frac{x+4}{x(x-2)(x+5)}.$$

sol. Note

$$\begin{aligned}
\frac{x+4}{x(x-2)(x+5)} &= \frac{A}{x} + \frac{B}{x-2} + \frac{C}{x+5} \\
A &= \frac{0+4}{\boxed{x}(0-2)(0+5)} = -\frac{2}{5} \\
B &= \frac{2+4}{2\boxed{(x-2)}(2+5)} = \frac{3}{7} \\
C &= \frac{-5+4}{(-5)(-5-2)\boxed{(x+5)}} = -\frac{1}{35}.
\end{aligned}$$

□

Using differentiation-repeated factors

Example 8.5.8.

$$\frac{x-1}{(x+1)^3} = \frac{A}{x+1} + \frac{B}{(x+1)^2} + \frac{C}{(x+1)^3}.$$

Write

$$x-1 = A(x+1)^2 + B(x+1) + C.$$

Substitute $x = -1$ to get $C = -2$. Then take derivative

$$1 = 2A(x+1) + B$$

and substitute $x = -1$ to get $B = 1$. Finally, taking derivative again, we see $A = 0$.

8.6 Integral Tables and CAS

Example 8.6.1. Find $\int x \sin^{-1} x \, dx$.

sol. We use the formula (derive it ?)

$$\int x^n \sin^{-1} ax\, dx = \frac{x^{n+1}}{n+1}\sin^{-1} ax - \frac{a}{n+1}\int \frac{x^{n+1}}{\sqrt{1-a^2x^2}}, n \neq -1.$$

□

Integration with Maple

For the indefinite integral of $f(x) = x^2\sqrt{a^2+x^2}$ in Maple, type

```
> f:=x^2*sqrt(a^2 +x^2)
> int(f,x)
```

Then you get the answer.

8.7 Numerical Integration

Trapezoidal Rule

To evaluate the definite integral $\int_a^b f(x)\,dx$ we divide the interval by n (uniform) subinterval and set

$$a = x_0 < x_1 < \cdots < x_n = b.$$

Here $\Delta x = h = \frac{b-a}{n}$ and

$$x_{i-1} = a + (i-1)\Delta x, i = 1, \cdots, n.$$

With $y_{x_i} = f(x_i)$ we use trapezoidal rule on each subinterval to get

$$\int_a^b f(x)\,dx \approx \frac{h}{2}(y_0 + 2y_1 + \cdots + 2y_{n-1} + y_n).$$

$$(Error) = |E_T| \leq \frac{M(b-a)^3}{12n^2}.$$

Here $M = \max|f''(x)|$.

Simpson's Rule

Replace the definite integral by an integral of quadratic interpolation. Exact for poly. of degree three. Assume $y = Ax^2 + Bx + C$ is an interpolating

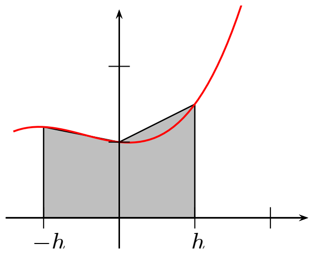

Figure 8.3: Trapezoidal Rule

polynomial of f in the sense that $y(x_i) = f(x_i)$ for $x_0 = -h, x_1 = 0, x_2 = h$

$$\begin{aligned}\int_{-h}^{h} f(x)\,dx &\approx \int_{-h}^{h} (Ax^2 + Bx + C)\,dx \\ &= \left.\frac{Ax^3}{3} + \frac{Bx^2}{2} + Cx\right]_{-h}^{h} \\ &= \frac{2Ah^3}{3} + 2Ch = \frac{h}{3}(2Ah^2 + 6C).\end{aligned}$$

Since

$$y_0 = Ah^2 - Bh + C,\ \ y_1 = C,\ \ y_2 = Ah^2 + Bh + C$$

we see

$$A = \frac{y_0 - 2y_1 + y_2}{2h^2},\ \ B = \frac{y_2 - y_0}{2h},\ \ C = y_1$$

and the the integral is

$$\frac{h}{3}(y_0 - 2y_1 + y_2 + 6y_1) = \frac{h}{3}(y_0 + 4y_1 + y_2).$$

Since this formula is exact for x^3, it is in general third order formula. When the general interval $[a, b]$ is divided by an even number of subintervals, we can apply it repeatedly to get

$$\int_a^b f(x)\,dx \approx \frac{h}{3}(y_0 + 4y_1 + 2y_2 + 4y_3 + \cdots + 2y_{n-2} + 4y_{n-1} + y_n).$$

$$|E_S| \leq \frac{M(b-a)^5}{180n^4}, \qquad M = \max|f^{(4)}(x)|.$$

Example 8.7.1. Find an upper bound for the error in estimating $\int_0^2 5x^4\,dx$ using Simpson's rule with $n = 4$.

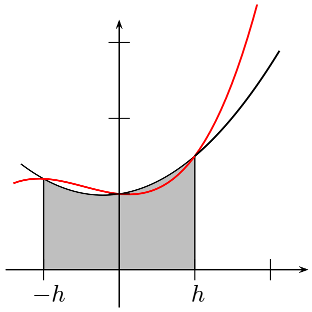

Figure 8.4: Simpson's Rule

sol. Let $f(x) = 5x^4$. Then $f^{(4)} = 120$. So $M = 120$. $b - a = 2$ and $n = 4$. The error bound is

$$|E_S| \le \frac{M(b-a)^5}{180n^4} = \frac{120(2)^5}{1804^4} = \frac{1}{12}.$$

□

Example 8.7.2. What is the minimum number of intervals needed to approximate above example using the Simpson's rule with an error less than 10^{-4}.

sol. We set

$$\frac{M(b-a)^5}{180n^4} < 10^{-4}.$$

Then

$$\begin{aligned} \frac{120(2)^5}{180n^4} &< 10^{-4} \\ n^4 &> \frac{64(10)^4}{3} \\ n &> 10\left(\frac{64}{3}\right)^{1/4} \approx 21.5. \end{aligned}$$

□

8.8 Improper Integral

So far the integral was defined only when

(1) The domain is finite like $[a, b]$.

(2) The range of the function is finite

In practice, there are cases when either one or both of these conditions violates.

Improper Integral

Definition 8.8.1 (Improper integral)**.** (1) When $a = -\infty$ or $b = \infty$,

(2) or f is undefined (infinite value) at either a or b

the integral $\int_a^b f(x)\,dx$ is called an **improper integral**.

The Case when a or b is ∞

Definition 8.8.2 (Convergence of improper integral)**.**

(1) Suppose $f(x)$ is continuous on $[a, \infty)$. We set

$$\int_a^\infty f(x)\,dx = \lim_{b\to\infty} \int_a^b f(x)\,dx \tag{8.8}$$

provided the limit exists.

(2) Similarly, if $f(x)$ is continuous on $(-\infty, b]$, we set

$$\int_{-\infty}^b f(x)\,dx = \lim_{a\to-\infty} \int_a^b f(x)\,dx \tag{8.9}$$

provided the limit exists.

(3) If $f(x)$ is continuous on $(-\infty, \infty)$ then we set

$$\int_{-\infty}^\infty f(x)\,dx = \lim_{a\to-\infty} \int_a^0 f(x)\,dx + \lim_{b\to\infty} \int_0^b f(x)\,dx \tag{8.10}$$

provided both limits exist. In these cases, we say the **improper integral converges**. Otherwise, we say the integral **diverges**.

Example 8.8.3.

$$\int_1^\infty \frac{\ln x}{x^2}\,dx$$

Example 8.8.4.

$$\int_0^\infty \frac{1}{1+x^2}\,dx$$

The case when f is undefined (infinite value) at either a or b

Definition 8.8.5 (Convergence of Improper integral)**.**

(1) Suppose $f(x)$ is integrable on all closed subinterval of $[a, b)$ and we have either $\lim_{x\to b^-} f(x) = \pm\infty$. If the limit

$$L = \lim_{u\to b^-} \int_a^u f(x)\,dx \tag{8.11}$$

exists, then we say the **improper integral converges** and write its limit

$$\int_a^b f(x)\,dx = \lim_{u\to b^-} \int_a^u f(x)\,dx.$$

(2) The same definition holds when $\lim_{x\to a^+} f(x) = \pm\infty$. We write

$$\int_a^b f(x)\,dx = \lim_{\ell\to a^+} \int_\ell^b f(x)\,dx \tag{8.12}$$

if the latter limit exists. Otherwise, we say the integral **diverges**.

(3) The discontinuity can happen at an interior point. In this case, we break the integral at a point where f is discontinuous and apply the above definition to each of the integrals.

Computation of improper integral

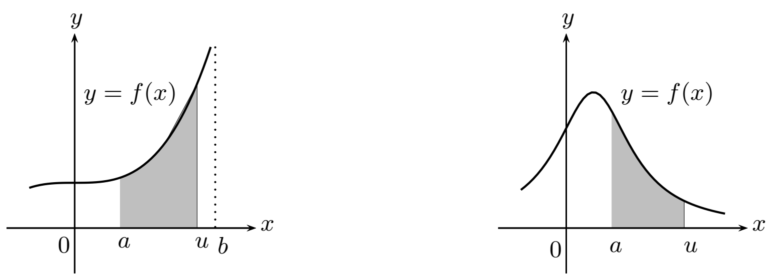

Figure 8.5: Improper integral on $[a, b)$

Example 8.8.6. Find the area surrounded by $y = 1/\sqrt{x}$, x-axis, y-axis, $x = 1$(fig 8.7).

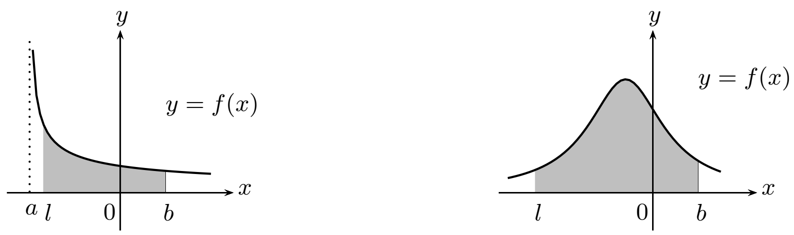

Figure 8.6: Improper integral on $(a, b]$

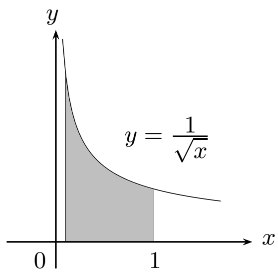

Figure 8.7: Improper Integral

sol. The function $1/\sqrt{x}$ is not defined at $x = 0$. But we can use the limit as

$$\begin{aligned}
(\text{Area}) &= \lim_{\varepsilon\to 0^+} \int_\varepsilon^1 \frac{dx}{\sqrt{x}} \\
&= \lim_{\varepsilon\to 0^+} \left[2x^{1/2}\right]_\varepsilon^1 \\
&= \lim_{\varepsilon\to 0^+} \left(2 - 2\varepsilon^{1/2}\right) = 2.
\end{aligned}$$

□

Example 8.8.7. $\displaystyle\int_{-1}^{1} \frac{1}{\sqrt{1-x^2}}\, dx.$

sol. We divide the integral on two intervals: $(-1, 0]$ and $[0, 1)$.

$$\begin{aligned}
\int_{-1}^{0} \frac{1}{\sqrt{1-x^2}}\, dx &= \lim_{\ell\to -1^+} \int_\ell^0 \frac{1}{\sqrt{1-x^2}}\, dx \\
&= \lim_{\ell\to -1^+} \left[\sin^{-1} x\right]_\ell^0 \\
&= -\sin^{-1}(-1) = \frac{\pi}{2}.
\end{aligned}$$

$$\int_0^1 \frac{1}{\sqrt{1-x^2}}\,dx = \lim_{u\to 1^-}\int_0^u \frac{1}{\sqrt{1-x^2}}\,dx$$
$$= \lim_{u\to 1^-}\left[\sin^{-1} x\right]_0^u$$
$$= \sin^{-1}(1) = \frac{\pi}{2}.$$

Hence

$$\int_{-1}^1 \frac{1}{\sqrt{1-x^2}}\,dx = \int_{-1}^0 \frac{1}{\sqrt{1-x^2}}\,dx + \int_0^1 \frac{1}{\sqrt{1-x^2}}\,dx = \pi.$$

□

Example 8.8.8. $\displaystyle\int_0^2 \frac{dx}{(x-1)^{4/3}}$.

sol. The function $1/(x-1)^{4/3}$ is not defined at $x = 1$. Hence we separate

$$\int_0^2 \frac{dx}{(x-1)^{4/3}} = \int_0^1 \frac{dx}{(x-1)^{4/3}} + \int_1^2 \frac{dx}{(x-1)^{4/3}}.$$

$$\int_0^1 \frac{dx}{(x-1)^{4/3}} = \lim_{u\to 1^-}\int_0^u \frac{dx}{(x-1)^{4/3}}$$
$$= \lim_{u\to 1^-}\left[-3(x-1)^{-1/3}\right]_0^u$$
$$= \lim_{u\to 1^-} -\frac{3}{(u-1)^{1/3}} - 3$$
$$= \infty.$$

Since $\displaystyle\int_0^1 \frac{dx}{(x-1)^{4/3}}$ diverges the integral diverges regardless of $\displaystyle\int_1^2 \frac{dx}{(x-1)^{4/3}}$.

□

The function $1/x^p$

The integral of $1/x^p$ on $(0,1]$ or $[1,\infty)$ depends on the value of p. In particular, the integral on $[1,\infty)$ is used to decide the convergence of the series $\sum 1/n^p$.

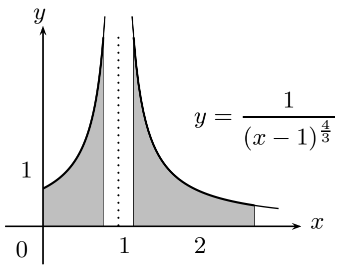

Figure 8.8:

Integral on $(0, 1]$

Example 8.8.9. Find $\displaystyle\int_0^1 \frac{dx}{x^p}$ $(p > 0)$.

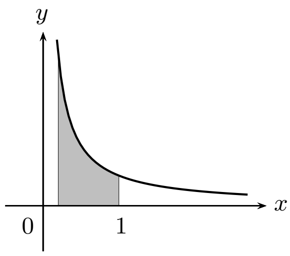

Figure 8.9: On $(0, 1]$

sol.

(1) For $0 < p < 1$

$$\int_0^1 \frac{dx}{x^p} = \lim_{\ell\to 0^+} \int_\ell^1 \frac{dx}{x^p} = \lim_{\ell\to 0^+} \left[\frac{x^{1-p}}{1-p}\right]_\ell^1 = \lim_{\ell\to 0^+} \frac{1-\ell^{1-p}}{1-p} = \frac{1}{1-p}.$$

(2) For $p = 1$

$$\int_0^1 \frac{dx}{x^p} = \lim_{\ell\to 0^+} \int_1^u \frac{dx}{x} = \lim_{\ell\to 0^+} [\ln x]_\ell^1 = \lim_{\ell\to 0^+} (-\ln \ell) = \infty.$$

(3) For $p > 1$

$$\int_0^1 \frac{dx}{x^p} = \lim_{\ell\to 0^+} \int_\ell^1 \frac{dx}{x^p} = \lim_{\ell\to 0^+} \left[\frac{x^{1-p}}{1-p}\right]_\ell^1 = \lim_{\ell\to 0^+} \frac{1-\ell^{1-p}}{1-p} = \infty.$$

□

Integral on $[1, \infty)$

Example 8.8.10. Find $\displaystyle\int_1^\infty \frac{dx}{x^p}$ $(p > 0)$.

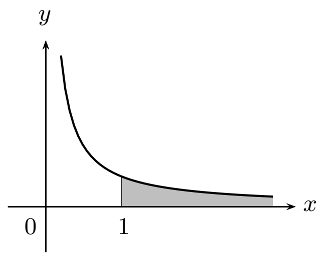

Figure 8.10: Improper integral on $[1, \infty)$

sol.

(1) For $0 < p < 1$,

$$\int_1^\infty \frac{dx}{x^p} = \lim_{u\to\infty} \int_1^u \frac{dx}{x^p} = \lim_{u\to\infty} \left[\frac{x^{1-p}}{1-p}\right]_1^u = \lim_{u\to\infty} \frac{u^{1-p}-1}{1-p} = \infty.$$

(2) For $p = 1$

$$\int_1^\infty \frac{dx}{x^p} = \lim_{u\to\infty} \int_1^u \frac{dx}{x} = \lim_{u\to\infty} [\ln x]_1^u = \lim_{u\to\infty} \ln u = \infty.$$

(3) For $p > 1$

$$\int_1^\infty \frac{dx}{x^p} = \lim_{u\to\infty} \int_1^u \frac{dx}{x^p} = \lim_{u\to\infty} \left[\frac{x^{1-p}}{1-p}\right]_1^u = \lim_{u\to\infty} \frac{u^{1-p}-1}{1-p} = \frac{1}{p-1}.$$

□

Test for convergence

Theorem 8.8.11 (Comparison test)**.** *Let $0 \leq f(x) \leq g(x)$ for all $x > a$. Then*

(1) If $\int_a^\infty g(x)\,dx$ converges, then $\int_a^\infty f(x)\,dx$ also converges.

(2) If $\int_a^\infty f(x)\,dx$ diverges, then $\int_a^\infty g(x)\,dx$ also diverges.

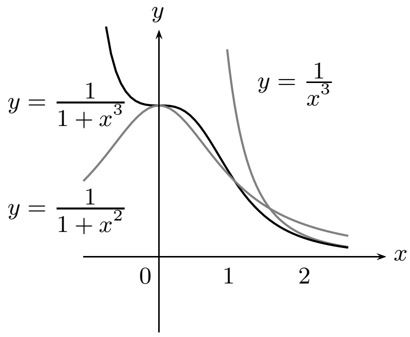

Figure 8.11:

Example 8.8.12. Test whether $\int_0^\infty \frac{dx}{1+x^3}$ converges or not?

sol. We see, for all $x \geq 1$, $1/(1+x^3) \leq 1/x^3$ holds. By example 8.8.10 we see $\int_1^\infty 1/x^3\,dx = 1/2$. Hence by Comparison test $\int_1^\infty 1/(1+x^3)\,dx$ converges. On the other hand, the integra $\int_0^1 1/(1+x^3)\,dx$ is well defined on $[0,1]$. Hence $\int_0^\infty 1/(1+x^3)\,dx$ converges and the value is $\int_0^1 1/(1+x^3)\,dx + \int_1^\infty 1/(1+x^3)\,dx$. (See Fig 8.11)

□

Theorem 8.8.13 (Limit Comparison Test)**.** *Assume $f(x)$, $g(x)$ are positive on $[a,\infty)$ and suppose*

$$\lim_{x\to\infty} \frac{f(x)}{g(x)} = L(0 < L < \infty).$$

Then the two integral $\int_a^\infty f(x)\,dx$ and $\int_a^\infty g(x)\,dx$ both converge or both diverge.

Proof. (1) Suppose $\int_a^\infty g(x)\,dx$ converges: Then there is $N > a$ such that $f(x)/g(x) \le L{+}1$ holds for all $x \ge N$. So we have $0 \le f(x) \le (L{+}1)g(x)$ and by Limit Comparison Test, $\int_N^\infty f(x)\,dx$ converge. Hence $\int_a^\infty f(x)\,dx$ converges to $\int_a^N f(x)\,dx + \int_N^\infty f(x)\,dx$.

(2) Suppose $\int_a^\infty g(x)\,dx$ diverges:There exists $N > a$ s.t. for all $x \ge N$, $f(x)/g(x) \ge L - L/2 = L/2$ holds. Hence $f(x) \ge (L/2)g(x) \ge 0$ and by Limit Comparison Test $\int_N^\infty f(x)\,dx$ diverges. So does $\int_a^\infty f(x)\,dx$.

□

Example 8.8.14. Test whether $\displaystyle\int_0^\infty \frac{dx}{1+e^x}$ converges or not?

sol. Let $f(x) = 1/(1+e^x)$, $g(x) = 1/e^x$. Then

$$\lim_{x\to\infty} \frac{f(x)}{g(x)} = \lim_{x\to\infty} \frac{e^x}{1+e^x} = 1$$

and

$$\int_0^\infty \frac{dx}{e^x} = \lim_{u\to\infty} \int_0^u \frac{dx}{e^x} = \lim_{u\to\infty} \left[-e^{-x}\right]_0^u = \lim_{u\to\infty} \left(-e^{-u}+1\right) = 1.$$

Hence by Limit Comparison Test, $\int_0^\infty 1/(1+e^x)\,dx$ converges.

□

Example 8.8.15. Test for convergence $\displaystyle\int_2^\infty \sqrt{\frac{x}{x^2-1}}\,dx$.

sol. Set $f(x) = \sqrt{\dfrac{x}{x^2-1}}$ and $g(x) = \dfrac{1}{\sqrt{x}}$. Then

$$\lim_{x\to\infty} \frac{f(x)}{g(x)} = \lim_{x\to\infty} \sqrt{\frac{x^2}{x^2-1}} = 1.$$

We see

$$\int_2^\infty \frac{dx}{\sqrt{x}} = \lim_{u\to\infty} \left[2\sqrt{x}\right]_2^u = \lim_{u\to\infty} \left(2\sqrt{u} - 2\sqrt{2}\right) = \infty.$$

By Limit Comparison Test, $\displaystyle\int_2^\infty \sqrt{\frac{x}{x^2-1}}\,dx$ diverges.

□

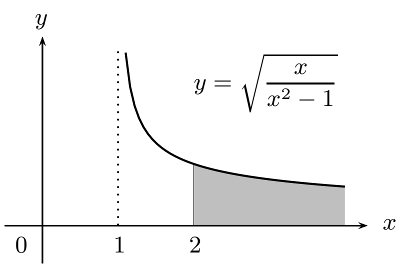

Figure 8.12: $\sqrt{\frac{x}{x^2-1}}$