Chapter 7

Transcendental Functions

7.1 Inverse functions and their derivatives

One to one function

A function is one -to- one on a domain if $f(x_1) \neq f(x_2)$ whenever $x_1 \neq x_2$ in D.

(1) $f(x) = \sqrt{x}$ for $x > 0$

(2) $y = x^2$ is not one-to-one on $[-1, 1]$ but is one-to-one on $[0, m]$.

A test for one-one-one function: A function $y = f(x)$ is one-one-one if its graph intersects each *horizontal line* at most once.

Definition 7.1.1. Suppose f is one-one-one function on D with range R. Then the inverse function f^{-1} is defined by

$$f^{-1}(a) = b \text{ if } f(b) = a$$

The domain of f^{-1} is R and range is D.

Example 7.1.2. $y = x^2$

sol. Since $x = \sqrt{y}$. We interchange x, y obtain, $y = \sqrt{x}$.

□

$$f^{-1} \circ f(x) = x, \quad f \circ f^{-1}(y) = y$$

Definition 7.1.3. $y = \log_a x$ is the inverse of exponential function $y = a^x$

$\log_{10} x$ is written as $\log x$, $\log_e x$ is written as $\ln x$. $\ln x$ is called *natural logarithm* and

$$y = \ln x \text{ iff } e^y = x$$

In particular, if $x = e$ we get $\ln e = 1$.

Inverse trig functions

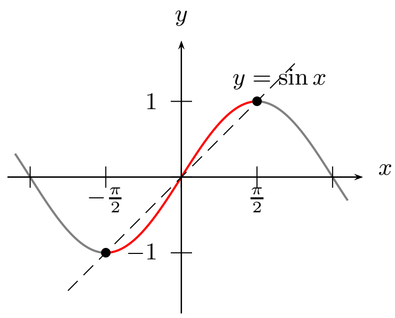

Figure 7.1: $y = \sin x$

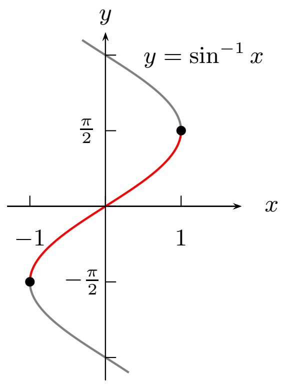

Figure 7.2: $y = \sin^{-1} x$

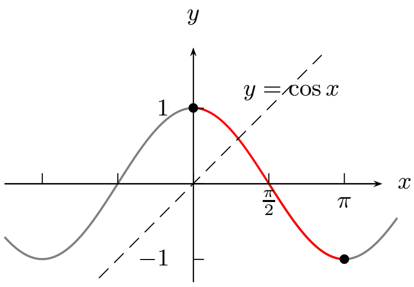

Figure 7.3: $y = \cos x$

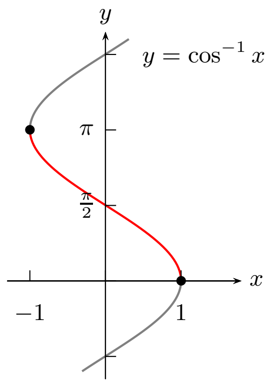

Figure 7.4: $y = \cos^{-1} x$

Example 7.1.4. Half life of Polonium 210. The time for radioactive substance required to decay by half is independent of the initial quantity. $y = y_0 e^{-kt}$. So $t = \ln 2/k$. For Polonium 210, $k = 5 \cdot 10^{-3}$.

In chapter 2 we introduced natural logarithmic function as the inverse of the exponential function e^x, where the number e was chosen to satisfy certain slope condition. In this chapter, we introduce an alternative theory for exponential and log. function.

Definition 7.1.5. A function f is one-to-one on a domain D if $f(x_1) \neq f(x_2)$ whenever $x_1 \neq x_2$.

Definition 7.1.6. Suppose a function f is one-to-one on a domain D with range R. The **inverse function** f^{-1} exists and is defined by

$$f^{-1}(b) = a \text{ if } f(a) = b.$$

The domain of f^{-1} is R and range is D.

We have

$$(f^{-1} \circ f)(x) = x, \quad x \in D, \quad (f \circ f^{-1})(y) = y, \quad y \in R.$$

Derivatives of inverse function

Theorem 7.1.7. *Suppose f is differentiable in I. If $f'(x)$ is never zero, then f^{-1} exists, differentiable. Furthermore for $a \in I$, $f(a) = b$,*

$$\boxed{(f^{-1})'(b) = \frac{1}{f'(f^{-1}(b))} = \frac{1}{f'(a)}.} \tag{7.1}$$

Set $y = f(x)$. Then the inverse function is $x = f^{-1}(y)$, and its derivative is

$$\left.\frac{dx}{dy}\right|_{y=f(a)} = \frac{1}{dy/dx|_{x=a}}, \quad a \in I$$

Proof. Differentiate $x = (f^{-1} \circ f)(x) = f^{-1}(f(x)) = f^{-1}(y)$ w.r.t x using the Chain rule, we have

$$1 = (f^{-1})'(f(x))f'(x).$$

Setting $x = a$, we see $1 = (f^{-1})'(f(a))f'(a)$. Thus

$$(f^{-1})'(b) = 1/f'(a).$$

□

Usually, we use the notation $y = f^{-1}(x)$. The graph of $y = f(x)$ and that of $y = f^{-1}(x)$ are symmetric w.r.t the line $y = x$.

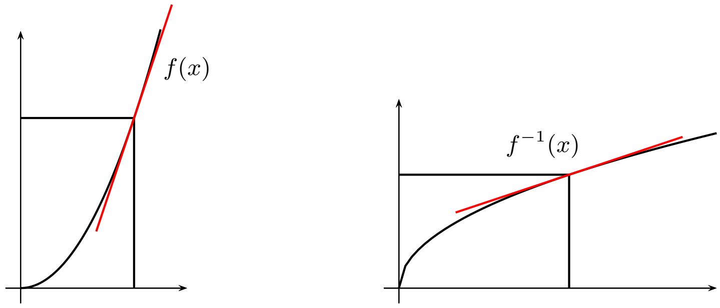

Figure 7.5: Slope of inverse function

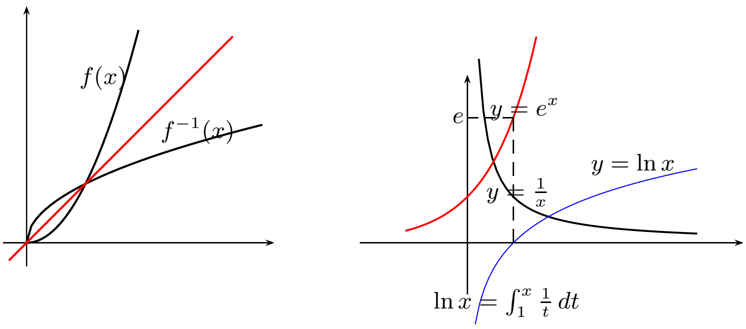

Figure 7.6: Graph of inverse functions, Graph of $\ln x$ and e^x

Example 7.1.8. (1) $f(x) = x^7 + 8x^3 + 4x - 2$. Find $(f^{-1})'(-2)$.

(2) $f(x) = \sin^{-1} x$. Find f'.

sol. (1) Since $f' = 7x^6 + 24x^2 + 4 \geq 4$ inverse f^{-1} exists. Since $f(0) = -2$ we have

$$(f^{-1})'(-2) = (f^{-1})'(f(0)) = \frac{1}{f'(0)} = \frac{1}{4}.$$

(2) $y = \sin^{-1} x$, $x = \sin y$. Hence

$$\begin{aligned} \frac{d}{dx}\sin^{-1} x = \frac{dy}{dx} &= \frac{1}{dx/dy} = \frac{1}{(d/dy)\sin y} \\ &= \frac{1}{\cos y} = \frac{1}{\sqrt{1-\sin^2 y}} = \frac{1}{\sqrt{1-x^2}}. \end{aligned}$$

□

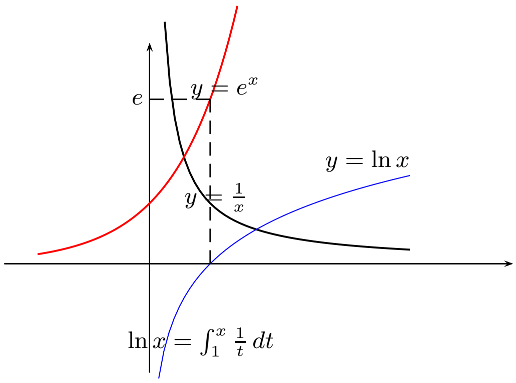

Figure 7.7: Graph of $\ln x$ and e^x

7.2 Natural logarithms (defined as integral)

Definition 7.2.1. For $x > 0$, the (natural) logarithmic function is defined by

$$\ln x = \int_1^x \frac{1}{t}\, dt.$$

Thus by fundamental theorem,

$$\frac{d}{dx} \ln x = \frac{1}{x}. \tag{7.2}$$

If $u(x)$ is any positive differentiable function,

$$\frac{d}{dx} \ln u(x) = \frac{1}{u} \frac{du}{dx}. \tag{7.3}$$

Definition 7.2.2. The **number** e is that number satisfying

$$\ln(e) = \int_1^e \frac{1}{t}\, dt = 1.$$

The derivative of the natural logarithmic function

The derivative of natural logarithmic function is by its definition and fundamental theorem of calculus, for $x > 0$

$$\frac{d}{dx} \int_1^x \frac{1}{t} dt = \frac{1}{x}.$$

One can show similar formula holds for $x < 0$. In fact, we have

$$\boxed{\frac{d}{dx}\ln|x| = \frac{1}{x}, \quad x \neq 0.} \tag{7.4}$$

Properties of logarithms

For $b > 0, x > 0$

(1) $\ln bx = \ln b + \ln x$

(2) $\ln \frac{b}{x} = \ln b - \ln x$

(3) $\ln \frac{1}{x} = -\ln x$

(4) $\ln x^r = r \ln x$ (For rational number r).

Consider

$$\frac{d}{dx}\ln(bx) = \frac{1}{x} = \frac{d}{dx}\ln x$$

So by above result,

$$\ln(bx) = \frac{1}{x} = \ln x + C$$

Place $x = 1$ to see $C = \ln b$.

Proof of Log rule $\ln x^r = r \ln x$

Consider (assuming r is rational)

$$\frac{d}{dx}\ln x^r = \frac{1}{x^r}\frac{d}{dx}(x^r) = \frac{1}{x^r} r x^{r-1} = \frac{r}{x} = \frac{d}{dx}(r \ln x)$$

Thus $\ln x^r$ and $r \ln x$ have same derivative and we have

$$\ln x^r = r \ln x + C,$$

for some constant C. To find the constant C, we let $x = 1$. Then $C = 0$.

The integral $\int (1/u) du$

To evaluate the following integral

$$\int \frac{f'(x)}{f(x)} dx \tag{7.5}$$

we use the substitution $u = f(x), du = f'(x)dx$ to see

$$\int \frac{f'(x)}{f(x)}dx = \int \frac{1}{u}du = \ln|u| + C = \ln|f(x)| + C. \tag{7.6}$$

Example 7.2.3.

$$\begin{aligned}
\int_{\pi/6}^{\pi/2} \frac{5\cos\theta}{2+\sin\theta}d\theta &= \int_{1/2}^{1} \frac{5}{2+u}du \\
&= 5\ln(2+u)|_{1/2}^{1} \\
&= 5(\ln 3 - \ln\frac{5}{2}) = 5\ln\frac{6}{5}.
\end{aligned}$$

Integral of $\tan x, \cot x, \sec x$ and $\csc x$

$$\begin{aligned}
\int \tan x\, dx &= \int \frac{\sin x}{\cos x}\, dx \\
&= -\int \frac{du}{u} \\
&= -\ln|u| + C \\
&= -\ln|\cos x| + C \\
&= \ln\frac{1}{|\cos x|} + C \\
&= \ln|\sec x| + C.
\end{aligned}$$

For $\sec x$ we need special trick:

$$\begin{aligned}
\int \sec x\, dx &= \int \sec x \frac{(\sec x + \tan x)}{(\sec x + \tan x)}\, dx \\
&= \int \frac{(\sec^2 x + \sec x \tan x)}{\sec x + \tan x}\, dx \\
&= \int \frac{du}{u} \\
&= \ln|u| + C \\
&= \ln|\sec x + \tan x| + C.
\end{aligned}$$

For $\csc x$ we do similarly. Thus we have

$$\boxed{\begin{array}{rcl} \int \sec x\,dx &=& \ln|\sec x + \tan x| + C \\ \int \csc x\,dx &=& -\ln|\csc x + \cot x| + C. \end{array}}$$

Logarithmic Differentiation

Example 7.2.4. Find dy/dx if $y = \frac{(x^2+1)^{1/3}(x-3)^{1/2}}{x+5}$.

sol. We take the logarithm and then take derivative to get

$$\begin{aligned} \ln y &= \ln\left(\frac{(x^2+1)^{1/3}(x-3)^{1/2}}{x+5}\right) \\ &= \frac{1}{3}\ln(x^2+1) + \frac{1}{2}\ln(x-3) - \ln(x+5) \\ \frac{y'}{y} &= \frac{1}{3}\frac{2x}{x^2+1} + \frac{1}{2}\frac{1}{x-3} - \frac{1}{x-5} \\ y' &= \frac{(x^2+1)^{1/3}(x-3)^{1/2}}{x+5}\left(\frac{1}{3}\frac{2x}{x^2+1} + \frac{1}{2}\frac{1}{x-3} - \frac{1}{x-5}\right) \end{aligned}$$

□

7.3 Exponential function e^x

Definition 7.3.1. Define the (natural) exponential function $e^x = exp(x) := \ln^{-1} x$ as the inverse function of $\ln x$. Thus

$$y = exp(x) \Leftrightarrow x = \ln y.$$

Thus

$$\begin{aligned} exp(\ln x) &= x, \ (x > 0) \\ \ln(exp(x)) &= x. \end{aligned} \tag{7.7}$$

It turns out that when x is a rational number m/n, $exp(m/n) = e^{m/n} = \sqrt[n]{e^m}$.

The derivative of e^x

$f(x) = \ln x$. The derivative of its inverse function $y = f^{-1}(x) = e^x$ is

$$\begin{aligned} (f^{-1})'(x) &= \frac{1}{f'(f^{-1}(x))} \\ &= \frac{1}{1/y} \\ &= y \\ &= e^x. \end{aligned}$$

Alterative way: Let $y = e^x$. Implicit differentiation w.r.t. x gives

$$\begin{aligned} \ln y &= x \\ \frac{1}{y}\frac{dy}{dx} &= 1 \\ \frac{dy}{dx} &= y = e^x. \end{aligned}$$

Antiderivative of e^x

$$\boxed{\int e^x dx = e^x + C.} \tag{7.8}$$

Theorem 7.3.2 (Laws of exponents)**.** *(1)* $e^x e^y = e^{x+y}$.

(2) $e^{-x} = \frac{1}{e^x}$.

(3) $\frac{e^x}{e^y} = e^{x-y}$.

(4) $(e^x)^r = e^{rx}$.

Example 7.3.3. Sketch the graph of $x^{1/x}$. To do this, we first investigate the behavior of the function as $x \to 0^+$ and $x \to \infty$. To study the limit, we take the logarithm:

$$\lim_{x\to\infty} \ln x^{1/x} = \lim_{x\to\infty} \frac{\ln x}{x} = 0.$$

Hence

$$\lim_{x\to\infty} x^{1/x} = \exp(\lim_{x\to\infty} \frac{\ln x}{x}) = e^0 = 1.$$

Meanwhile

$$\lim_{x\to 0^+} \ln x^{1/x} = \lim_{x\to 0^+} \frac{\ln x}{x} = -\infty.$$

Hence

$$\lim_{x\to 0+} x^{1/x} = \exp(\lim_{x\to 0+} \frac{\ln x}{x}) = e^{-\infty} = 0.$$

To see the local extrema, take the derivative and find the critical point. $f'(x) = (1-\ln x)/x^2 = 0$ for $x = e$. By checking the sign of $f'(x)$ near $x = e$, we conclude $x = e$ is a point of local maximum.

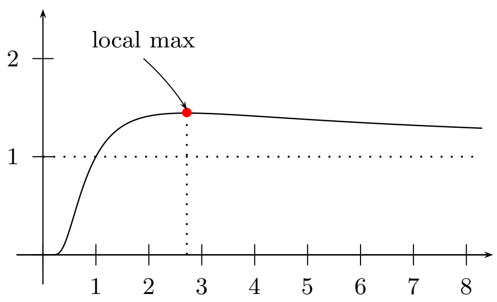

Figure 7.8: Graph of $y = x^{1/x}$

Derivatives of Power function

We prove the following theorem which was stated earlier.

Theorem 7.3.4. *For any real r, it holds that*

$$\frac{d}{dx}u^r = u^{r-1}\frac{du}{dx}.$$

Proof. Since $u^r = e^{r\ln u}$ we have

$$\frac{d}{dx}u^r = ru^r\frac{d\ln u}{dx} = ur^r\frac{1}{u}\frac{du}{dx} = ru^{r-1}\frac{du}{dx}.$$

□

Example 7.3.5. Differentiate $f(x) = x^x, x > 0$

sol. Write $f(x) = x^x = e^{x \ln x}$. So

$$\begin{aligned} f'(x) &= \frac{d}{dx}(e^{x \ln x}) \\ &= (e^{x \ln x})\frac{d}{dx}(x \ln x) \\ &= e^{x \ln x}(\ln x + x \cdot \frac{1}{x}) \\ &= x^x(\ln x + 1). \end{aligned}$$

□

The number e was defined a number satisfying $\ln e = 1$. Hence

$$e = \ln^{-1}(1) = exp(1) \tag{7.9}$$

it is known that $e = 2.718281828 \cdots$.

Example 7.3.6. Find the point where the line through of origin $y = mx$ is tangent to the graph of $y = \ln x$.

sol. We must have $m = \frac{1}{x}$ and $mx = \ln x$. Hence we get $m = \frac{1}{e}$ and $x = e$.

□

The number e as a limit

Theorem 7.3.7. *The number e satisfies*

$$e = \lim_{x \to 0}(1 + x)^{1/x}.$$

Proof. If $f(x) = \ln x$. Then $f'(1) = \frac{1}{x}|_{x=1} = 1$. By definition of derivative

$$\begin{aligned} 1 &= f'(1) = \lim_{x \to 0^+} \frac{\ln(1 + x) - \ln 1}{x} = \lim_{x \to 0^+} \ln[(1 + x)^{\frac{1}{x}}] \\ &= \ln[\lim_{x \to 0^+}(1 + x)^{\frac{1}{x}}]. \end{aligned}$$

Now exponentiate. □

The general exponential function a^x

Since $a = e^{\ln a}$ for any positive number a, we can define a^x by

$$\begin{aligned} a^x &= e^{\ln a^x} \\ &= e^{x \ln a} \\ &= e^{(\ln a)x}. \end{aligned}$$

Definition 7.3.8. If a is a positive number and x is any number, we define

$$\boxed{a^x = e^{x \ln a}}. \tag{7.10}$$

Since $\ln e^x = x$ for all real x, we have

$$\ln x^n = \ln(e^{n \ln x}) = n \ln x, x > 0.$$

One can also use the definition of $\ln x = \int_1^x dt$ to prove it.

Example 7.3.9. [Power rule] The derivative of x^n for any number n:

$$\begin{aligned} \frac{d}{dx} x^n &= \frac{d}{dx} e^{n \ln x} \ (x > 0) \\ &= e^{n \ln x} \cdot \frac{d}{dx}(n \ln x) \\ &= x^n \cdot \frac{n}{x} \\ &= nx^{n-1}. \end{aligned}$$

Derivative of a^x

By definition, $a^x = e^{x \ln a}$. Thus

$$\frac{d}{dx} a^x = \frac{d}{dx} e^{x \ln a} = \ln a (e^{x \ln a}) = a^x \ln a$$

and

$$\boxed{\frac{d}{dx} a^u = a^u \ln a \frac{du}{dx}} \tag{7.11}$$

General logarithmic function $\log_a x$

$y = \log_a x$ is defined as the inverse function of $y = a^x (a > 0, a \neq 1)$. Thus

$$\log_a x = y \Leftrightarrow a^y = x$$

$$\log_a(a^x) = x, \text{ for all } x, \text{ and } a^{(\log_a x)} = x, (x > 0)$$

$\log_{10} x$ is written as $\log x$ and called *common logarithmic function*

Properties

(1) Product rule: $\log_a xy = \log_a x + \log_a y$.

(2) Quotient rule: $\log_a \frac{x}{y} = \log_a x - \log_a y$.

(3) Product rule: $\log_a \frac{1}{y} = -\log_a y$.

(4) Power rule: $\log_a x^y = y \log_a x$.

Inverse properties

(1) Base a: $a^{\log_a x} = x$, $\log_a(a^x) = x (a > 0, a \neq 1, x > 0)$.

(2) Base e: $e^{\ln x} = x$, $\ln(e^x) = x (x > 0)$.

Derivative of $\log_a x$

We have

$$\log_a x = \frac{\ln x}{\ln a}. \tag{7.12}$$

So

$$\boxed{\begin{array}{rcl} \frac{d}{dx} \log_a x & = & \frac{1}{x \ln a} \\ \frac{d}{dx} \log_a u & = & \frac{1}{u \ln a} \frac{du}{dx} \end{array}} \tag{7.13}$$

Logarithmic Differentiation

Find dy/dx if $y = \frac{(x^2+1)^{1/3}(x-3)^{1/2}}{x+5}$.

Example 7.3.10. (1) $\displaystyle\int_0^2 \frac{2x}{x^2-5} dx = \ln |u|_{-5}^{-1}$.

(2) $\displaystyle\int_{-\pi/2}^{\pi/2} \frac{4\cos\theta}{3 + 2\sin\theta} d\theta = \int_1^5 \frac{2}{u} du$.

7.4 Exponential change and separable differentiable equations

The decay of radioactive material or money earning interests in bank account, temperature between a cup of hot water and room air it sits, etc follows the law of *exponential change* Suppose $y(t)$ denotes some quantity which changes according to the exponential law: The rate of change of y is proportional to y.

$$\frac{dy}{dt} = ky$$

with I.C. Then $y = Ae^{kt}$.

Example 7.4.1. Assume a disease is spreading "Entero virus", "A.I" Let y be the number of people infected by disease. Assume we cure people as much as possible. Then dy/dt is proportional to y.(The more people, the more infected, the more cured) Suppose for each year the number is reduced by 20% and 10,000 people infected today, how many years will it take to reduce to $1,000$?

sol. $y = Ae^{kt}$, $A = 10,000$ Since it is reduced by 0.2 each year, we see

$$0.8 = e^{k\cdot 1} \to k = \ln 0.8 < 0$$

So we have $y = 10,000e^{(\ln 0.8)t}$ we want $10,000e^{(\ln 0.8)t} = 1,000$. So $e^{(\ln 0.8)t} = \frac{1}{10}$. $\ln(0.8)t = \ln(0.1)$. $t = \frac{\ln(0.1)}{\ln(0.8)} \approx 10.32$ yrs.

□

Example 7.4.2 (Half life of a radioactive material)**.** $y_0e^{-kt} = \frac{1}{2}y_0$. so $t = \ln 2/k$.

Example 7.4.3 (Carbon 14)**.** It is estimated the half life of Carbon 14 is 5700 yrs. AS wooden artifact was found from an ancient site. This contains carbon 14 about 10% less than the living tree. How old is the site? $k = \ln 2$ Half life is $\ln 2/5700$. $y = y_0e^{-kt} = 0.9y_0$ So $e^{-kt} = 0.9$ or $t = -5700\frac{\ln 0.9}{\ln 2} = 866$ yrs.

Example 7.4.4 (Law of Cooling)**.** If H is the temperature of an object and H_s the surrounding temperature. Then the rate of change(cooling) is proportional to the temperature difference. Thus

$$\frac{dH}{dt} = -k(H - H_s).$$

Solving

$$H - H_s = (H_0 - H_s)e^{-kt}.$$

A boiled egg at 98^o is put in the sink of 18^o to cool down. In 5 min, the egg was 38^o. how much longer will it take to reach 20^o?

sol.

$$H - 18 = (98 - 18)e^{-kt}, \quad H = 18 + 80e^{-kt}.$$

Set $H = 38, t = 5$. Then $e^{-5k} = 1/4$ and

$$k = -\frac{\ln 1/4}{5} = 0.2 \ln 4 \approx 0.28.$$

$$H = 18 + 80e^{-(0.2 \ln 4)t}.$$

Solving $t \approx 13$ min.

□

Separable Differential Equations

A general differential equation is given in the form

$$\frac{dy}{dx} = f(x, y) \tag{7.14}$$

with certain initial condition such as $y(x_0) = y_0$. Such equation is called **separable** if f is expressed as a product of a function of x and a function of y, i.e,

$$\frac{dy}{dx} = g(x)H(y).$$

We rewrite it in the form

$$\frac{dy}{dx} = \frac{g(x)}{h(y)}$$

and obtain

$$\int h(y)\,dy = \int g(x)\,dx. \tag{7.15}$$

Example 7.4.5. Solve

$$\frac{dy}{dx} = (1 + y)e^x, \; y > -1. \tag{7.16}$$

Separate variables,

$$\begin{aligned}\frac{dy}{1+y} &= e^x dx \\ \int \frac{dy}{1+y} &= \int e^x dx \\ \ln(1+y) &= e^x + C.\end{aligned}$$

7.5 Intermediate form aand L'Hopital's Rule

L'Hopital's Rule

When $f(a) = g(a) = 0$ or $f(a) = g(a) = \infty$, the limit

$$\lim_{x\to a} \frac{f(x)}{g(x)}$$

cannot be found by directly substituting a. In this case we can use L'Hopital's Rule.

Theorem 7.5.1 (L'Hopital's Rule: First form)**.** *Suppose $f(a) = g(a) = 0$ that $f'(a)$, $g'(a)$ exist and $g'(a) \neq 0$ then*

$$\lim_{x\to a} \frac{f(x)}{g(x)} = \frac{f'(a)}{g'(a)}.$$

Proof. $\lim_{x\to a} \frac{f(x)}{g(x)} = \frac{\lim_{x\to a}(f(x) - f(a))/(x-a)}{\lim_{x\to a}(g(x) - g(a))/(x-a)} = \frac{f'(a)}{g'(a)}.$ □

Example 7.5.2. (1) $\lim_{x\to 0} \frac{\sqrt{1+x} - 1}{x} \left(\frac{0}{0}\right) = \left.\frac{1/2\sqrt{1+x}}{1}\right|_{x=0} = \frac{1}{2}.$

(2) $\lim_{x\to 1} \frac{x^2 - 1}{x-1} \left(\frac{0}{0}\right) = \left.\frac{2x}{1}\right|_{x=1} = 2.$

Example 7.5.3. (1) $\lim_{x\to(\pi/2)^-} \frac{\cos x}{\sin x - 1} \left(\frac{0}{0}\right) = \lim_{x\to(\pi/2)^-} \frac{-\sin x}{\cos x} = -\infty.$

(2) $\lim_{x\to 0} \left(\frac{1}{\sin x} - \frac{1}{x}\right) (\infty - \infty) = \lim_{x\to\infty} \frac{x - \sin x}{x \sin x} \left(\frac{0}{0}\right).$

$$= \lim_{x\to 0} \frac{1 - \cos x}{\sin x + x\cos x} \left(\frac{0}{0}\right) = \lim_{x\to 0} \frac{\sin x}{2\cos x - x\sin x} = \frac{0}{2} = 0.$$

Theorem 7.5.4 (L'Hopital's Rule:Stronger form). *Suppose that $f(a) = g(a) = 0$ and f, g are differentiable on (a, b). (The case $f'(c) = g'(c) = 0$ is allowed) and that $g'(x) =\neq 0$ for $x \neq a$. Then*

$$\lim_{x\to a} \frac{f(x)}{g(x)} = \lim_{x\to a} \frac{f'(x)}{g'(x)}$$

as long as the rhs limit exists.

The proof is based on the following result.

Theorem 7.5.5 (Cauchy's Mean value theorem). *Suppose f and g are conti in $[a, b]$, diff'ble in (a, b). If $g' \neq 0$ on (a, b) then $g(b) \neq g(a)$ and there exist $c \in (a, b)$ such that*

$$\frac{f(b) - f(a)}{g(b) - g(a)} = \frac{f'(c)}{g'(c)}$$

Proof. Suppose $g(b) = g(a)$ then by Mean value theorem

$$g'(c) = \frac{g(b) - g(a)}{b - a} = 0$$

for some $c \in (a, b)$. This contradict to $g' \neq 0$. So, $g(b) \neq g(a)$. Next consider the function F defined by

$$F(x) = f(x) - f(a) - \left(\frac{f(b) - f(a)}{g(b) - g(a)}\right)(g(x) - g(a)).$$

We apply Rolle's theorem to F. Hence there exist $c \in (a, b)$ such that $F'(c) = 0$. Since

$$F'(c) = f'(c) - \left(\frac{f(b) - f(a)}{g(b) - g(a)}\right) g'(c) = 0$$

we have

$$\frac{f'(c)}{g'(c)} = \frac{f(b) - f(a)}{g(b) - g(a)}.$$

□

Proof. First show

$$\lim_{x\to c^+} \frac{f(x)}{g(x)} = \lim_{x\to c^+} \frac{f'(x)}{g'(x)}.$$

When $c < x < b$ use thm 7.5.5(Cauchy' MVT) on $[c, x]$. Then there is $d \in (c, x)$ s.t.

$$\frac{f'(d)}{g'(d)} = \frac{f(x) - f(c)}{g(x) - g(c)} = \frac{f(x)}{g(x)}$$

and $d \to c^+$ as as $x \to c^+$

$$\lim_{x\to c^+} \frac{f(x)}{g(x)} = \lim_{d\to c^+} \frac{f'(d)}{g'(d)}$$
$$= \lim_{x\to c^+} \frac{f'(x)}{g'(x)}.$$

The following can be shown the same way.

$$\lim_{x\to c^-} \frac{f(x)}{g(x)} = \lim_{x\to c^-} \frac{f'(x)}{g'(x)}.$$

□

One sided intermediate form

Example 7.5.6. (1) $\lim_{x\to 0^+} \frac{\sin x}{x^2} = \lim_{x\to 0^+} \frac{\cos x}{2x} = \infty.$

(2) $\lim_{x\to 0^-} \frac{\sin x}{x^2} = \lim_{x\to 0^-} \frac{\cos x}{2x} = -\infty$

Intermediate form ∞/∞, $\infty \cdot 0$, $\infty - \infty$

Example 7.5.7.

(1) $\lim_{x\to 0} x \sin \frac{1}{x}$

(2) $\lim_{x\to 0^+} \sqrt{x} \ln x$

(3) $\lim_{x\to (\pi/2)^-} \frac{\tan x}{1+\tan x} \left(\frac{\infty}{\infty}\right) = \lim_{x\to (\pi/2)^-} \frac{\sec^2 x}{\sec^2 x} = 1.$

(4) $\lim_{x\to\infty} \frac{\pi/2 - \tan^{-1} x}{1/x} \left(\frac{0}{0}\right) = \lim_{x\to\infty} \frac{-1/(1+x^2)}{-1/x^2}$

$$= \lim_{x\to\infty} \frac{x^2}{1+x^2} \left(\frac{\infty}{\infty}\right) = \lim_{x\to\infty} \frac{2x}{2x} = 1.$$

(5) $\lim_{x\to \pi/2} \frac{\sec x}{1+\tan x}$

(6) $\lim_{x\to\infty} \frac{\ln x}{2\sqrt{x}}$

Wrong use of L'Hopital's rule

$$\lim_{x\to\infty} \frac{\pi/2 - \tan^{-1} x}{1/x} \left(\frac{0}{0}\right) = \lim_{x\to\infty} \frac{-1/(1+x^2)}{-1/x^2}$$
$$= \lim_{x\to\infty} \frac{x^2}{1+x^2} \left(\frac{\infty}{\infty}\right) = \lim_{x\to\infty} \frac{2x}{2x} = 1.$$

In this case we can find limit as follows:

$$\lim_{x\to\infty} \frac{\sqrt{9x+1}}{\sqrt{4x+1}} = \sqrt{\lim_{x\to\infty} \frac{9x+1}{4x+1}} = \sqrt{\frac{9}{4}} = \frac{3}{2}.$$

$$\lim_{x\to 0} \frac{1-\cos x}{x+x^2} = \lim_{x\to 0} \frac{\sin x}{1+2x} = 0 \neq \lim_{x\to 0} \frac{\cos x}{2}.$$

As the last equation shows, we cannot use L'Hopital's rule when the quotient has a limit.

Intermediate form 0^∞, ∞^0, $\infty - \infty$

Example 7.5.8. Use continuity

If $\lim \ln f(x) = L$ then $f(x) = \lim e^{\ln f(x)} = e^L$.

(1) $\lim_{x\to 0^+} (1+x)^{1/x}$

(2) $\lim_{x\to\infty} x^{1/x}$

(3) $\lim_{x\to 0} \left(\frac{1}{\sin x} - \frac{1}{x}\right)$

7.6 Inverses trigonometric functions

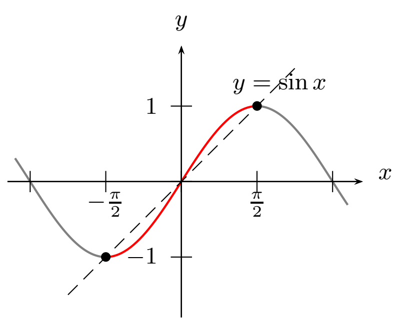

Figure 7.9: $y = \sin x$

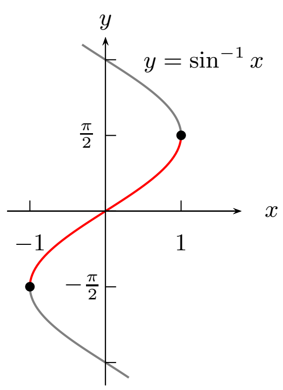

Figure 7.10: $y = \sin^{-1} x$

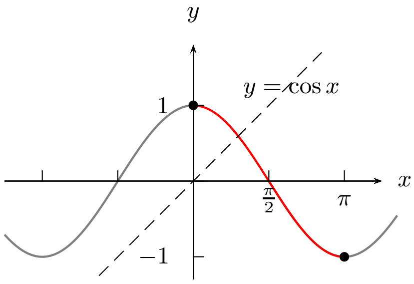

Figure 7.11: $y = \cos x$

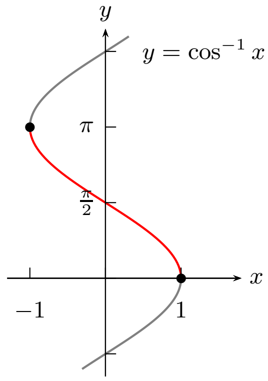

Figure 7.12: $y = \cos^{-1} x$

Arcsine and Arccosine functions

Since the trig functions are not one-to-one in general, the inverse functions do not exist. However, if we restrict the domain properly so that the functions are one-to-one, we can **define** the inverses.

First consider the function $y = \sin x$. The function $\sin x\colon [-\pi/2, \pi/2] \to [-1, 1]$ is one-to-one on $[-\pi/2, \pi/2]$. We choose this interval to define its inverse function. Define

$$y = \sin^{-1} x\colon [-1, 1] \longrightarrow [-\pi/2, \pi/2].$$

whenever $x = \sin y$ for $y \in [-\pi/2, \pi/2]$. Its graph is given in Figure 7.10. The inverse sine function $\sin^{-1} x$ is sometimes written as $\arcsin x$.

In order to define inverse cosine function, we restrict the domain of $y = \cos x$ to $[0, \pi]$. Then we define $\cos^{-1} x$ as

$$y = \cos^{-1} x\colon [-1, 1] \longrightarrow [0, \pi].$$

whenever $\cos y = x$ for any $x \in [0, \pi]$. The graph of $\cos^{-1} x$ is as figure 7.12. It is also written as $\arccos x$.

Example 7.6.1. (1) $\sin^{-1}(1/2) = \pi/6$

(2) $\sin^{-1}(1) = \pi/2$

Example 7.6.2. (1) $\cos^{-1}(1/2) = \pi/3$

(2) $\cos^{-1} 0 = \pi/2$

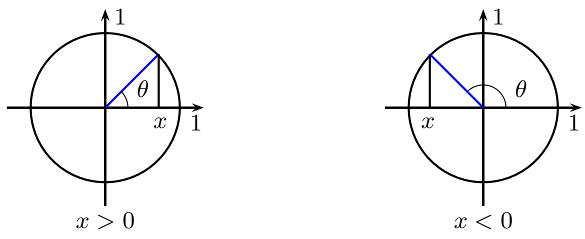

Figure 7.13: $\theta = \cos^{-1} x$

Inverse of $\tan x$

The function $\tan x$ is one to one on $(-\pi/2, \pi/2)$, thus we define its inverse function.

$$y = \tan^{-1} x \colon \mathbb{R} \longrightarrow (-\pi/2, \pi/2)$$

iff $\tan y = x$. See Figure 7.15. It is als written as $\arctan x$.

Example 7.6.3. (1) $\tan^{-1}(1) = \pi/4$

(2) $\tan^{-1}(0) = 0$.

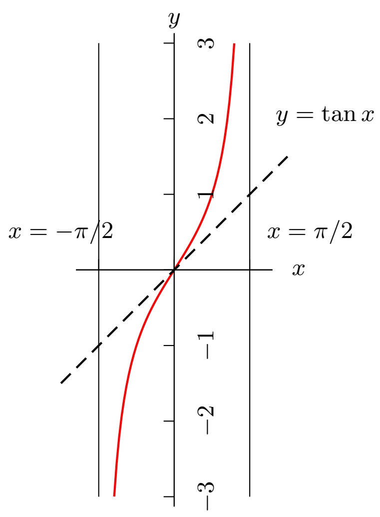

Figure 7.14: $y = \tan x$

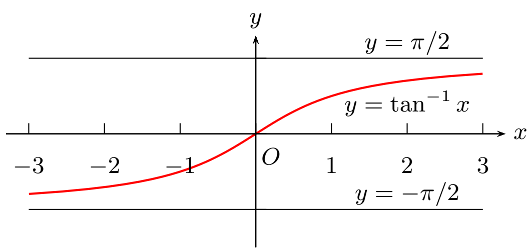

Figure 7.15: $y = \tan^{-1} x$

Inverses of $\sec x$**,** $\cot x$**,** $\csc x$

Let us look at the inverse of $\sec x$ first:

Inverses of $\cot x$, $\csc x$ are similarly defined.

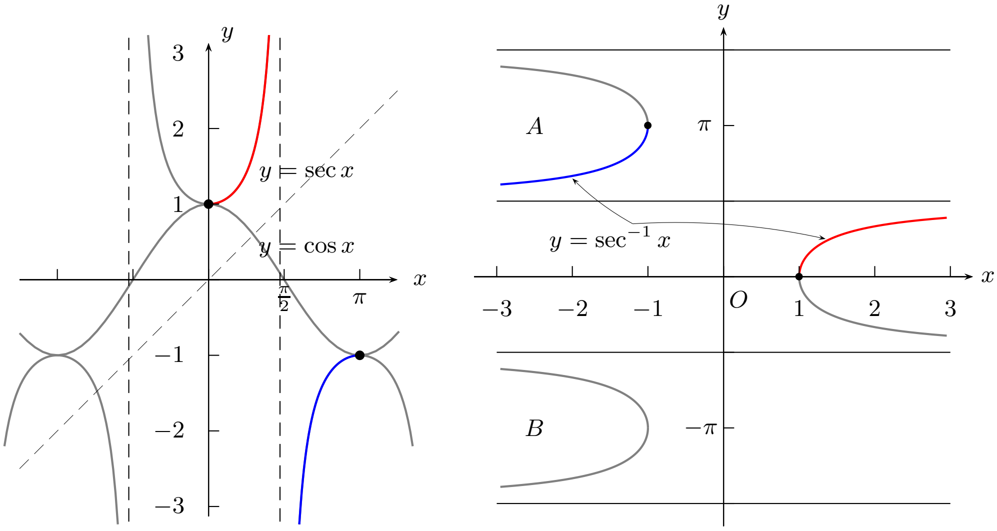

Figure 7.16: $y = \sec x$ and $y = \sec^{-1} x$

$\sec^{-1} x : \mathbb{R} - (-1, 1) \to [0, \pi] - \{\pi/2\}$

$\cot^{-1} x : \mathbb{R} \to (0, \pi)$.

$\csc^{-1} x : \mathbb{R} - (-1, 1) \to [-\pi/2, \pi/2] - \{0\}$

Identities involving arcsine and arccosine

Example 7.6.4.

$$\sin^{-1} x + \cos^{-1} x = \frac{\pi}{2}, \quad \cos^{-1} x + \cos^{-1}(-x) = \pi.$$

Proposition 7.6.5. *The following relations hold.*

$$\begin{array}{|lcl|}\hline \cos^{-1} x & = & (\pi/2) - \sin^{-1} x \\ \cot^{-1} x & = & (\pi/2) - \tan^{-1} x \\ \csc^{-1} x & = & (\pi/2) - \sec^{-1} x \\ \cot^{-1} x & = & \tan^{-1}(1/x) \\ \sec^{-1} x & = & \cos^{-1}(1/x) \\ \csc^{-1} x & = & \sin^{-1}(1/x) \\ \hline \end{array}$$

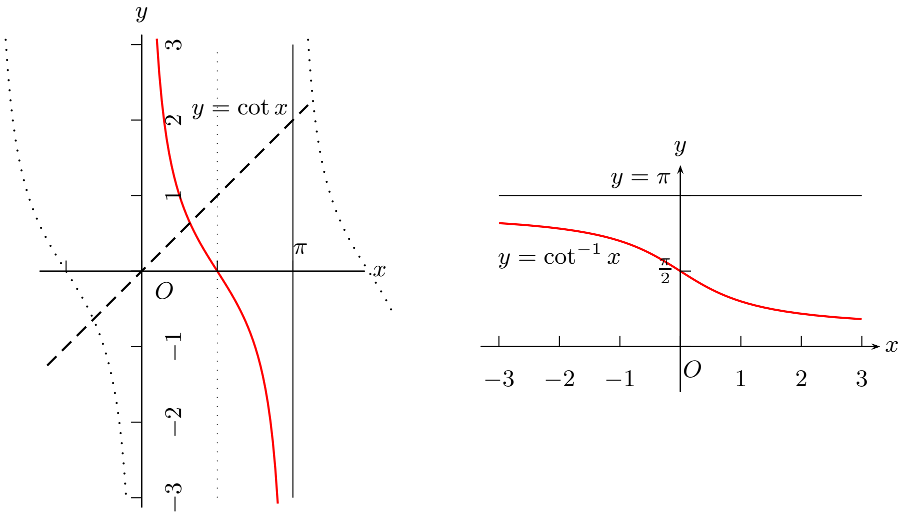

Figure 7.17: $y = \cot x$ and $y = \cot^{-1} x$

Example 7.6.6. (1) Find $\sin(\cos^{-1}(3/5))$

(2) Simplify $\tan(\sin^{-1} a)$

sol. (1) Let $\theta = \cos^{-1}(3/5)$. Then $\cos\theta = 3/5$ and $0 \le \theta \le \pi$. Hence

$$\sin\theta = \sqrt{1 - \frac{9}{25}} = \frac{4}{5}.$$

(2) Let $\theta = \sin^{-1} a$. Then $\sin\theta = a$ and $-\pi/2 \le \theta \le \pi/2$.

$$\cos\theta = \sqrt{1 - a^2}.$$

Hence

$$\tan\theta = \sin\theta / \cos\theta = a/\sqrt{1 - a^2}. \tag{7.17}$$

□

Derivative of inverse functions

Example 7.6.7. Find the derivative of $\sin^{-1} x$ and $\sin^{-1} u$, where $u = u(x)$.

Method 1. Use Theorem 7.1.7. Let $f(x) = \sin x$. Its inverse function is

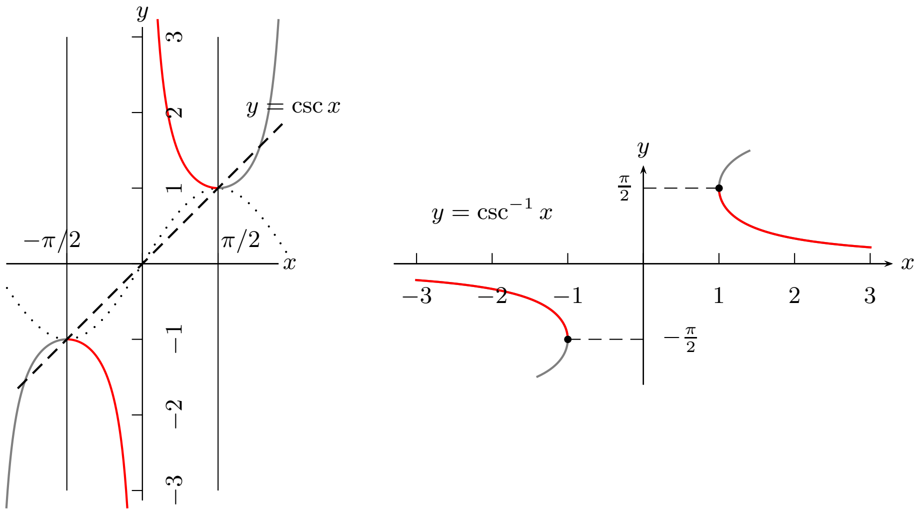

Figure 7.18: $y = \csc x$ and $y = \csc^{-1} x$

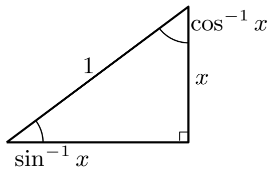

Figure 7.19: relation between $\sin^{-1} x$ and $\cos^{-1} x$

$y = f^{-1}(x) = \sin^{-1} x$. Hence we see

$$\begin{aligned}
(f^{-1})'(x) &= \frac{1}{f'(f^{-1}(x))} = \frac{1}{f'(y)} \\
&= \frac{1}{\cos y} \\
&= \frac{1}{\sqrt{1-\sin^2 y}} \\
&= \frac{1}{\sqrt{1-x^2}}.
\end{aligned}$$

Thus $\frac{d}{dx} \sin^{-1} x = \frac{1}{\sqrt{1-x^2}}$ and by Chain rule, $\frac{d}{dx} \sin^{-1} u(x) = \frac{1}{\sqrt{1-u^2}} \frac{du}{dx}$.

Method 2. Note that

$$y = \sin^{-1} x \Leftrightarrow \sin y = x.$$

Take derivative of this function w.r.t x (assuming y is a function of x). Thus

$$\begin{aligned} \sin y &= x \\ \left(\frac{d}{dy}\sin y\right)\frac{dy}{dx} &= \frac{d}{dx}(x) \\ \cos y\frac{dy}{dx} &= 1. \end{aligned}$$

Hence

$$\begin{aligned} \frac{dy}{dx} &= \frac{1}{\cos y} \\ &= \frac{1}{\sqrt{1-\sin^2 y}} \\ &= \frac{1}{\sqrt{1-x^2}}. \end{aligned}$$

Example 7.6.8. Find the derivative of $\tan^{-1} x$.

From $y = \tan^{-1} x$, we see by Theorem 7.1.7

$$\begin{aligned} (f^{-1})'(x) &= \frac{1}{f'(y)} \\ &= \frac{1}{1+\tan^2 y} \\ &= \frac{1}{1+x^2}. \end{aligned}$$

Thus $(f^{-1})'(x) = \frac{1}{1+x^2}$.

Example 7.6.9. Find the derivative of $y = \sec^{-1} x, \quad |x| \geq 1$.

sol. Let $y = \sec^{-1} x$. Then $x = \sec y$. (Refer to 7.16). Taking derivative w.r.t x, we get $1 = \sec y \tan y (dy/dx)$. Thus

$$\frac{dy}{dx} = \frac{1}{\sec y \tan y}.$$

We need to change it to expression in x. From, trig. identity we have

$$x^2 = \sec^2 y = \tan^2 y + 1, \text{ hence } \tan y = \pm\sqrt{x^2-1}.$$

For $x > 1$, we choose positive sign, $\tan y = \sqrt{x^2 - 1}$. Hence, we have

$$\frac{dy}{dx} = \frac{1}{x\sqrt{x^2-1}}, \quad x > 1.$$

For $x < -1$, we choose negative sign $\tan y = -\sqrt{x^2 - 1}$ to get

$$\frac{dy}{dx} = \frac{1}{-x\sqrt{x^2-1}}, \quad x < -1.$$

Hence

$$\boxed{\frac{d}{dx}\sec^{-1} x = \frac{1}{|x|\sqrt{x^2-1}}, \quad |x| > 1.}$$

□

Proposition 7.6.10. *The derivatives of inverse trig. functions :*

(1) $\frac{d}{dx}\sin^{-1} x = \frac{1}{\sqrt{1-x^2}}$

(2) $\frac{d}{dx}\cos^{-1} x = -\frac{1}{\sqrt{1-x^2}}$

(3) $\frac{d}{dx}\tan^{-1} x = \frac{1}{1+x^2}$

(4) $\frac{d}{dx}\csc^{-1} x = -\frac{1}{|x|\sqrt{x^2-1}}, \ |x| > 1$

(5) $\frac{d}{dx}\sec^{-1} x = \frac{1}{|x|\sqrt{x^2-1}}, \ |x| > 1$

(6) $\frac{d}{dx}\cot^{-1} x = -\frac{1}{1+x^2}.$

Integrals related to inverse trigonometric functions

Proposition 7.6.11. *The following integral formulas hold:*

(1) $\int \frac{dx}{\sqrt{1-x^2}} = \sin^{-1} x + C$

(2) $\int \frac{dx}{1+x^2} = \tan^{-1} x + C$

(3) $\int \frac{dx}{x\sqrt{x^2-1}} = \sec^{-1} |x| + C$

Example 7.6.12. $\displaystyle\int \frac{du}{a^2+u^2}.$

sol. Use substitution $u = a\tan\theta$, $du = a\sec^2\theta\,d\theta$ to get

$$\begin{aligned}\int \frac{du}{a^2+u^2} &= \int \frac{a\sec^2\theta\,d\theta}{a^2\sec^2\theta} = \int \frac{d\theta}{a}\\ &= \frac{\theta}{a} + C\\ &= \frac{1}{a}\tan^{-1}\frac{u}{a} + C.\end{aligned}$$

□

Example 7.6.13. Find $\int \sqrt{a^2-u^2}\,du$, $(a>0)$.

sol. Use $u = a\sin\theta$, $du = a\cos\theta\,d\theta$ to get

$$\begin{aligned}\int \sqrt{a^2-u^2}\,du &= \int a\cos\theta\cdot a\cos\theta\,d\theta\\ &= \frac{a^2}{2}\int (1+\cos 2\theta)\,d\theta\\ &= \frac{a^2}{2}\left(\theta + \frac{\sin 2\theta}{2}\right) + C = \frac{a^2}{2}(\theta + \sin\theta\cos\theta) + C\\ &= \frac{a^2}{2}\left(\sin^{-1}\frac{u}{a} + \frac{u}{a}\sqrt{1-\frac{u^2}{a^2}}\right) + C\\ &= \frac{a^2}{2}\sin^{-1}\frac{u}{a} + \frac{1}{2}u\sqrt{a^2-u^2} + C.\end{aligned}$$

□

Example 7.6.14. Find $\int \frac{du}{\sqrt{u^2-a^2}}$, $(|u|>a>0)$.

sol. Let $u = a\sec\theta$

$$\begin{aligned}u^2-a^2 &= a^2(\sec^2\theta - 1)\\ &= a^2\tan^2\theta,\\ du &= a\sec\theta\tan\theta\,d\theta.\end{aligned}$$

Then

$$\begin{aligned}
\int \frac{du}{\sqrt{u^2-a^2}} &= \int \frac{a\sec\theta\tan\theta\,d\theta}{a|\tan\theta|} \\
&= \begin{cases} \int \sec\theta\,d\theta & (0<\theta<\pi/2) \\ -\int \sec\theta\,d\theta & (\pi/2<\theta<\pi) \end{cases} \\
&= \begin{cases} \ln|\sec\theta+\tan\theta|+C & (0<\theta<\pi/2) \\ -\ln|\sec\theta+\tan\theta|+C & (\pi/2<\theta<\pi) \end{cases} \\
&= \begin{cases} \ln\left|\frac{u}{a}+\frac{\sqrt{u^2-a^2}}{a}\right|+C & (u>a) \\ \\ -\ln\left|\frac{u}{a}-\frac{\sqrt{u^2-a^2}}{a}\right|+C & (u<-a). \end{cases}
\end{aligned}$$

Last integrals can be simplified as follows:

$$\ln\left|\frac{u}{a}+\frac{\sqrt{u^2-a^2}}{a}\right| = \ln\left|u+\sqrt{u^2-a^2}\right| - \ln a.$$

$$\begin{aligned}
-\ln\left|\frac{u}{a}-\frac{\sqrt{u^2-a^2}}{a}\right| &= \ln\left|\frac{a}{u-\sqrt{u^2-a^2}}\right| \\
&= \ln\left|\frac{a(u+\sqrt{u^2-a^2})}{(u-\sqrt{u^2-a^2})(u+\sqrt{u^2-a^2})}\right| \\
&= \ln\left|\frac{a(u+\sqrt{u^2-a^2})}{a^2}\right| = \ln\left|\frac{u+\sqrt{u^2-a^2}}{a}\right| \\
&= \ln\left|u+\sqrt{u^2-a^2}\right| - \ln a.
\end{aligned}$$

Hence

$$\int \frac{du}{\sqrt{u^2-a^2}} = \ln\left|u+\sqrt{u^2-a^2}\right| + C'.$$

□

7.7 Hyperbolic function

Any function $f(x)$ can be written as even part and odd part:

$$f(x) = \underbrace{\frac{f(x)+f(-x)}{2}}_{\text{even part}} + \underbrace{\frac{f(x)-f(-x)}{2}}_{\text{odd part}}.$$

In particular, e^x can be written as

$$e^x = \frac{e^x + e^{-x}}{2} + \frac{e^x - e^{-x}}{2}. \tag{7.18}$$

Each of the functions on the right hand side is useful and thus has a name:

Definition 7.7.1 (hyperbolic function)**.** [1]

$$\begin{aligned}
\cosh x &= \frac{e^x + e^{-x}}{2}, \textbf{ hyperbolic cosine}\\
\sinh x &= \frac{e^x - e^{-x}}{2}, \textbf{ hyperbolic sine}\\
\tanh x = \frac{\sinh x}{\cosh x} &= \frac{e^x - e^{-x}}{e^x + e^{-x}}, \textbf{ hyperbolic tangent}\\
\coth x = \frac{1}{\tanh x} &= \frac{e^x + e^{-x}}{e^x - e^{-x}}, \textbf{ hyperbolic cotangent}\\
\operatorname{sech} x = \frac{1}{\cosh x} &= \frac{2}{e^x + e^{-x}}, \textbf{ hyperbolic secant}\\
\operatorname{csch} x = \frac{1}{\sinh x} &= \frac{2}{e^x - e^{-x}}. \textbf{ hyperbolic cosecant}.
\end{aligned}$$

Some identities of hyperbolic functions:

Proposition 7.7.2.

(1) $\sinh 2x = 2\sinh x \cosh x$

(2) $\cosh 2x = \cosh^2 x + \sinh^2 x$

[1] hyperbolic functions have many things in common with trig. functions. We can define trig. functions $\sin x$ and $\cos x$ using complex numbers. We define $e^{i\theta} = \cos\theta + i\sin\theta$. Then $e^{-i\theta} = \cos\theta - i\sin\theta$ and hence

$$\cos\theta = \frac{e^{i\theta} + e^{-i\theta}}{2}, \quad \sin\theta = \frac{e^{i\theta} - e^{-i\theta}}{2i}$$

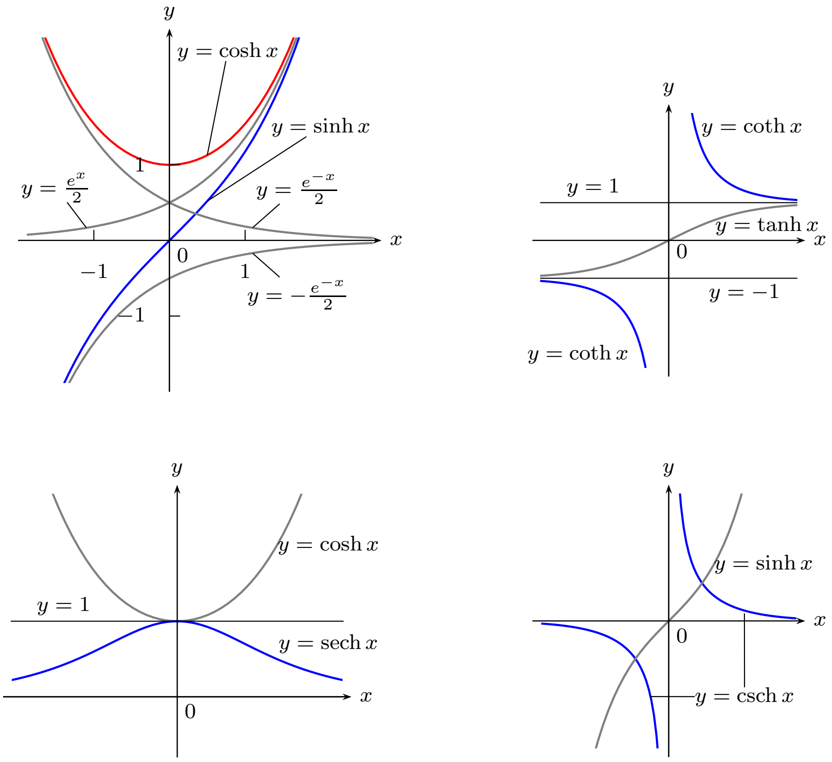

Figure 7.20: hyperbolic functions

(3) $\sinh^2 x = \dfrac{\cosh 2x - 1}{2}$

(4) $\cosh^2 x = \dfrac{\cosh 2x + 1}{2}$

(5) $\cosh^2 x - \sinh^2 x = 1$

(6) $\tanh^2 x = 1 - \operatorname{sech}^2 x$

(7) $\coth^2 x = 1 + \operatorname{csch}^2 x$

Derivatives and integrals of hyperbolic functions

Proposition 7.7.3.

(1) $\dfrac{d}{dx}(\sinh u) = \cosh u \dfrac{du}{dx}$

(2) $\dfrac{d}{dx}(\cosh u) = \sinh u \dfrac{du}{dx}$

(3) $\dfrac{d}{dx}(\tanh u) = \operatorname{sech}^2 u \dfrac{du}{dx}$

(4) $\frac{d}{dx}(\coth u) = -\operatorname{csch}^2 u \frac{du}{dx}$

(5) $\frac{d}{dx}(\operatorname{sech} u) = -\operatorname{sech} u \tanh u \frac{du}{dx}$

(6) $\frac{d}{dx}(\operatorname{csch} u) = -\operatorname{csch} u \coth u \frac{du}{dx}$

Proposition 7.7.4.

(1) $\int \sinh u\, du = \cosh u + C$

(2) $\int \cosh u\, du = \sinh u + C$

(3) $\int \operatorname{sech}^2 u\, du = \tanh u + C$

(4) $\int \operatorname{csch}^2 u\, du = -\coth u + C$

(5) $\int \operatorname{sech} u \tanh u\, du = -\operatorname{sech} u + C$

(6) $\int \operatorname{csch}\ u \coth u du = -\operatorname{csch} u + C$

Example 7.7.5. (1) The indefinite integral of $\sinh^2 x$ can be computed just as that of $\sin^2 x$.

$$\begin{aligned}\int_0^1 \sinh^2 x\, dx &= \int_0^1 \frac{\cosh 2x - 1}{2}\, dx \\ &= \frac{1}{2}\left[\frac{\sinh 2x}{2} - x\right]_0^1 \\ &= \frac{\sinh 2}{4} - \frac{1}{2}.\end{aligned}$$

(2) Using the definition of $\sinh x$

$$\begin{aligned}\int_0^{\ln 2} 4e^x \sinh x\, dx &= \int_0^{\ln 2} 4e^x \frac{e^x - e^{-x}}{2}\, dx = \int_0^{\ln 2} (2e^{2x} - 2)\, dx \\ &= \left[e^{2x} - 2x\right]_0^{\ln 2} \\ &= 3 - 2\ln 2.\end{aligned}$$

Inverse hyperbolic functions

The function $y = \sinh x$ is 1-1 and defined on $(-\infty, \infty)$ having values in $(-\infty, \infty)$. So (the inverse hyperbolic sine) $y = \sinh^{-1} x$ is defined on $(-\infty, \infty)$.

The function $y = \cosh x$ restricted to $[0, \infty)$ is 1-1 and its image is $[1, \infty)$. Hence (the inverse hyperbolic cosine) $y = \cosh^{-1}$ is defined on $[1, \infty)$ having values in $[0, \infty)$.

The function $y = \operatorname{sech} x$ restricted to $[0, \infty)$ is one-to-one, having values in $(0, 1]$. Hence its inverse function $y = \operatorname{sech}^{-1} x$ is defined on $(0, 1]$. Meanwhile $y = \tanh x$, $y = \coth x$, $y = \operatorname{csch} x$ are one-to-one on $(-\infty, \infty)$. Hence their inverses $y = \tanh^{-1} x$, $y = \coth^{-1} x$, $y = \operatorname{csch}^{-1} x$ are defined accordingly. The graphs are as in figure 7.21.

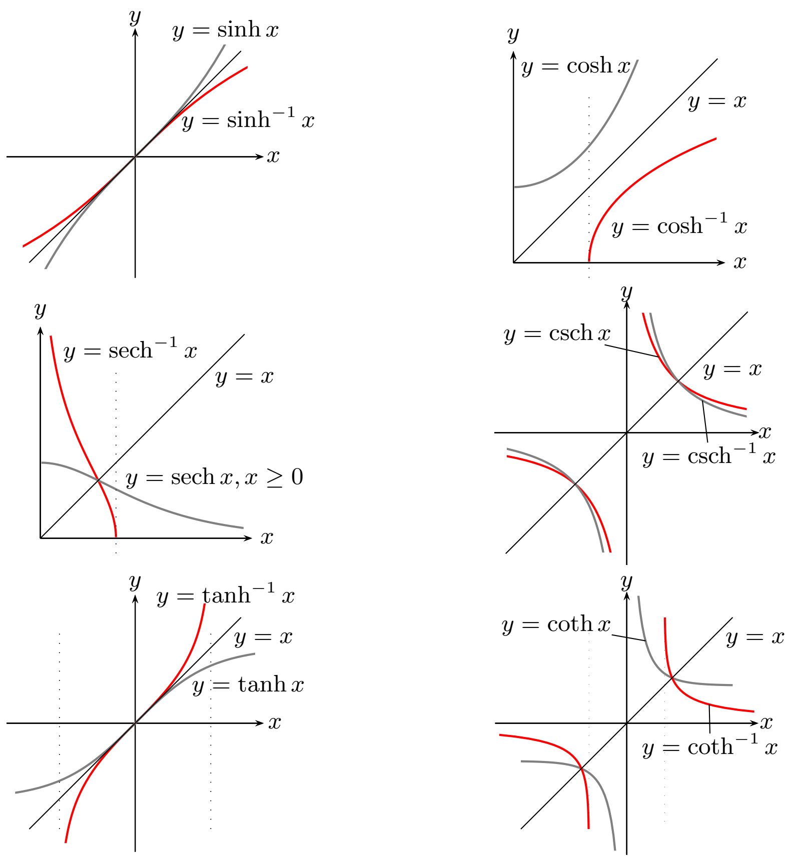

Figure 7.21: Inverse hyperbolic functions

Proposition 7.7.6. *Inverse hyperbolic functions can be represented by log functions.*

(1) $\sinh^{-1} x = \ln\big(x + \sqrt{x^2+1}\big), \quad -\infty < x < \infty$

(2) $\cosh^{-1} x = \ln\big(x + \sqrt{x^2-1}\big), \quad x \geq 1$

(3) $\tanh^{-1} x = \dfrac{1}{2}\ln\dfrac{1+x}{1-x}, \quad |x| < 1$

(4) $\operatorname{sech}^{-1} x = \ln\left(\dfrac{1+\sqrt{1-x^2}}{x}\right), \quad 0 < x \leq 1$

(5) $\operatorname{csch}^{-1} x = \ln\left(\dfrac{1}{x} + \dfrac{\sqrt{1+x^2}}{|x|}\right), \quad x \neq 0$

(6) $\coth^{-1} x = \dfrac{1}{2}\ln\dfrac{x+1}{x-1}, \quad |x| > 1.$

Proof. (1) Let $y = \sinh^{-1} x$.

$$\begin{aligned} x &= \sinh y = \frac{e^y - e^{-y}}{2}, \\ e^y - e^{-y} &= 2x, \\ e^{2y} - 2xe^y - 1 &= 0, \\ e^y &= x + \sqrt{x^2+1}. \quad \text{(Since } x - \sqrt{x^2+1} \text{ is negative, we drop it.)} \end{aligned}$$

Hence $y = \ln(x + \sqrt{x^2+1})$.

(4) The formula for $\operatorname{sech}^{-1} x \geq 0$.

$$\begin{aligned} y = \operatorname{sech}^{-1} x &\Rightarrow \operatorname{sech} y = x \Rightarrow x = \frac{2}{e^y + e^{-y}}, \\ e^y + e^{-y} &= \frac{2}{x}, \\ e^{2y} - \frac{2}{x}e^y + 1 &= 0, \\ e^y &= \frac{1}{x} \pm \sqrt{\frac{1}{x^2} - 1} = \frac{1 \pm \sqrt{1-x^2}}{x}. \end{aligned}$$

We choose positive sign and set $y := \ln\left(\frac{1+\sqrt{1-x^2}}{x}\right), \quad 0 < x \leq 1.$ ☐

Useful identities of inverse hyperbolic functions

Proposition 7.7.7.

(1) $\operatorname{sech}^{-1} x = \cosh^{-1} \frac{1}{x}$

(2) $\operatorname{csch}^{-1} x = \sinh^{-1} \frac{1}{x}$

(3) $\coth^{-1} x = \tanh^{-1} \frac{1}{x}$

This holds from definition. For example, if $0 < x \leq 1$, then

$$\operatorname{sech}\left(\cosh^{-1} \frac{1}{x}\right) = \frac{1}{\cosh\left(\cosh^{-1} \frac{1}{x}\right)} = \frac{1}{\left(\frac{1}{x}\right)} = x.$$

Derivatives of inverse hyperbolic functions

Proposition 7.7.8.

(1) $\dfrac{d(\sinh^{-1} u)}{dx} = \dfrac{1}{\sqrt{1+u^2}} \dfrac{du}{dx}$

(2) $\dfrac{d(\cosh^{-1} u)}{dx} = \dfrac{1}{\sqrt{u^2-1}} \dfrac{du}{dx}, \quad u > 1$

(3) $\dfrac{d(\tanh^{-1} u)}{dx} = \dfrac{1}{1-u^2} \dfrac{du}{dx}, \quad |u| < 1$

(4) $\dfrac{d(\coth^{-1} u)}{dx} = \dfrac{1}{1-u^2} \dfrac{du}{dx}, \quad |u| > 1$

(5) $\dfrac{d(\operatorname{sech}^{-1} u)}{dx} = \dfrac{-du/dx}{u\sqrt{1-u^2}}, \quad 0 < u < 1$

(6) $\dfrac{d(\operatorname{csch}^{-1} u)}{dx} = \dfrac{-du/dx}{|u|\sqrt{1+u^2}}, \quad u \neq 0$

Proof. (5) We verify the formula for $\operatorname{sech}^{-1} x$.

$$\begin{aligned} y = \operatorname{sech}^{-1} x \quad &\Rightarrow \quad \operatorname{sech} y = x, \\ -\operatorname{sech} y \tanh y \frac{dy}{dx} \quad &= \quad 1, \\ \frac{dy}{dx} \quad &= \quad -\frac{1}{\operatorname{sech} y \tanh y}, \\ &= \quad -\frac{1}{x\sqrt{1-x^2}}, \quad 0 < x \leq 1. \end{aligned}$$

□

Integrals of inverse hyperbolic functions

Proposition 7.7.9.

(1) $\displaystyle\int \frac{du}{\sqrt{1+u^2}} = \sinh^{-1} u + C$

(2) $\displaystyle\int \frac{du}{\sqrt{u^2-1}} = \cosh^{-1} u + C, \quad u > 1$

(3) $\displaystyle\int \frac{du}{1-u^2} = \begin{cases} \tanh^{-1} u + C, & \textit{if } |u| < 1, \\ \coth^{-1} u + C, & \textit{if } |u| > 1 \end{cases}$

(4) $\displaystyle\int \frac{du}{u\sqrt{1-u^2}} = -\operatorname{sech}^{-1} |u| + C = -\cosh^{-1}\left(\frac{1}{|u|}\right) + C$

(5) $\displaystyle\int \frac{du}{u\sqrt{1+u^2}} = -\operatorname{csch}^{-1} |u| + C = -\sinh^{-1}\left(\frac{1}{|u|}\right) + C$

7.8 Relative Rate of Growth

Definition 7.8.1. Suppose $f(x), g(x)$ are positive for sufficiently large x.

(1) f **grows faster than** g as $x \to \infty$ if

$$\lim_{x\to\infty} \frac{f(x)}{g(x)} = \infty.$$

(2) f **grows at the same rate** as g as $x \to \infty$ if

$$\lim_{x\to\infty} \frac{f(x)}{g(x)} = L, \text{ for some postive finite number } L.$$

Example 7.8.2. (1) e^x grows faster than x^3 as $x \to \infty$

(2) 3^x grows faster than 2^x as $x \to \infty$

(3) x grows faster than $\ln x$ as $x \to \infty$.

Order and Oh-notation

Definition 7.8.3. A function $f(x)$ is of smaller order than $g(x)$ as $x \to \infty$ if

$$\lim_{x\to\infty} \frac{f(x)}{g(x)} = 0.$$

In this case we write $f = o(g)$.

Definition 7.8.4. Suppose $f(x), g(x)$ are positive for sufficiently large x. Then a function $f(x)$ is a most the order of $g(x)$ as $x \to \infty$ if there is a positive number (not necessarily integer) M for which

$$\lim_{x\to\infty} \frac{f(x)}{g(x)} \leq M$$

for sufficiently large x. In this case we write $f = O(g)$.

Example 7.8.5. (1) $\ln x = o(x)$ as $x \to \infty$

(2) $x^2 = o(x^3)$ as $x \to \infty$

(3) $x + \sin x = O(x)$