Chapter 12

The geometry of Euclidean Space

We consider the basic operations of vectors in 3 and 3 dim. space: vector addition, scalar multiplication, dot product and cross product. In section **??** we generalize these notions to n dim'l space.

12.1 3D coordinates

(1) The set of all real numbers is denoted by $\mathbb{R}$.

(2) The set of all ordered pairs of real numbers (x, y) is denoted by $\mathbb{R}^2$.

(3) The set of all ordered triples of real numbers (x, y, z) is denoted by $\mathbb{R}^3$.

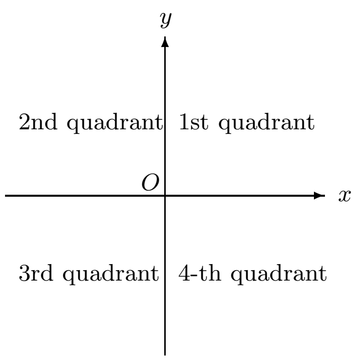


Figure 12.1: quadrant

The planes in $\mathbb{R}^3$determined by $z = 0$.(resp. $x = 0$ and $y = 0$) are called xy**-plane, (resp.** yz**-plane,** zx**-plane)** These planes divides the space into eight parts: Each of them is called **octant**. If every component is positive, it is called **the first octant**.

Example 12.1.1. (1) The xz-plane is the set of all points with $y = 0$:

$$\{(x, y) \mid y = 0\}.$$

(2) Similarly, the xy-plane is determined by $z = 0$:

$$\{(x, y, z) \mid z = 0\}.$$

(3) x-axis is determined by

$$\begin{cases} y = 0 \\ z = 0 \end{cases}$$

or

$$\{(x, y, z) \mid y = 0, z = 0\}.$$

12.2 Vectors in 2, 3 dim space

Definition 12.2.1. A vector in $\mathbb{R}^n, n = 2, 3$ is an ordered pair(triple) of real numbers, such as

$$(u_1, u_2), \text{ or } (v_1, v_2, v_3).$$

Here u_1, u_2 are called x**-coordinate**, y**-coordinate** or x**-component**, y**-component** of (u_1, u_2). The point $(0, 0)$ is called the **origin** and denoted by O.

We use the boldface to denote vectors, e.g, $\mathbf{u} = (u_1, u_2)$ or $\mathbf{v} = (v_1, v_2, v_3)$ are standard notations for vectors. The notation $\vec{u}$ is also used. A point P in $\mathbb{R}^n$ can be represented by an ordered pair of real numbers (u_1, u_2) or (v_1, v_2, v_3) called **Cartesian coordinate)** of P. Thus, vectors are identified with points in the plane or space.

$$\mathbb{R}^2 = \{(a_1, a_2) \mid a_1 \in \mathbb{R}, a_2 \in \mathbb{R}\}.$$

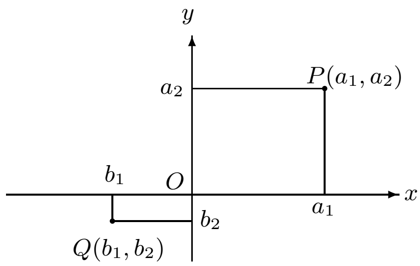


Figure 12.2: Coordinate plane

Vector addition and scalar multiplication-algebraic view

The operation of addition can be extended to $\mathbb{R}^3$. Given two triples, $\mathbf{u} = (u_1, u_2, u_3), \mathbf{v} = (v_1, v_2, v_3)$, we define

$$\mathbf{u} + \mathbf{v} = (u_1, u_2, u_3) + (v_1, v_2, v_3) = (u_1 + v_1, u_2 + v_2, u_3 + v_3)$$

to be the **sum** of (u_1, u_2, u_3) and (v_1, v_2, v_3). Thus we see that

$$\mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u}.$$

Two vectors $\mathbf{u} = (u_1, u_2, u_3)$ and $\mathbf{v} = (v_1, v_2, v_3)$ are equal if $u_1 = v_1, u_2 = v_2$ and $u_3 = v_3$. The vector $\mathbf{0} = (0, 0, 0)$ is the **zero element**. The vector $-(u_1, u_2, u_3) = (-u_1, -u_2, -u_3)$ is called the **additive inverse or negative** of (u_1, u_2, u_3).

Commutative law and associate law for additions:

(i) $(x, y, z) + (u, v, w) = (u, v, w) + (x, y, z)$ (commutative law)

(ii) $((x, y, z) + (u, v, w)) + (l, m, n)$
$= (x, y, z) + ((u, v, w) + (l, m, n))$ (associate law)

The **difference** is defined as

$$(u_1, u_2, u_3) - (v_1, v_2, v_3) = (u_1 - v_1, u_2 - v_2, u_3 - v_3).$$

Example 12.2.2.

$$(6,0,2)+(-10,3,2)=(-4,3,4)$$
$$(3,0,3)-(5,0,-2)=(-2,0,5)$$
$$(0,0,0)+(1,3,2)=(1,3,2)$$

For any real α, and (u_1, u_2, u_3) in $\mathbb{R}^3$, the **scalar multiple** $\alpha(u_1, u_2, u_3)$ is defined as

$$\alpha(u_1, u_2, u_3) = (\alpha u_1, \alpha u_2, \alpha u_3).$$

Additions and scalar multiplication has the following properties:

(i) $(\alpha\beta)(x, y, z) = \alpha(\beta(x, y, z))$ (associate law)

(ii) $(\alpha+\beta)(x, y, z) = \alpha(x, y, z) + \beta(x, y, z)$ (distributive law)

(iii) $\alpha((x, y, z) + (u, v, w)) = \alpha(x, y, z) + \alpha(u, v, w)$ (distributive law)

(iv) $\alpha(0,0,0) = (0,0,0)$ (property of 0)

(v) $0(x, y, z) = (0,0,0)$ (property of 0)

(vi) $1(x, y, z) = (x, y, z)$ (property of 1)

Example 12.2.3.

$$3\,(6,-3,2)=(18,-9,6)$$
$$1\,(3,5,-2)=(3,5,-2)$$
$$0\,(1,3,2)=(0,0,0)$$
$$(-2)\,(-2,1,3)=(4,-2,-6)$$

Example 12.2.4. Show

(1) $(\alpha+\beta)(x, y) = \alpha(x, y) + \beta(x, y)$

(2) $\alpha((x, y) + (u, v)) = \alpha(x, y) + \alpha(u, v)$

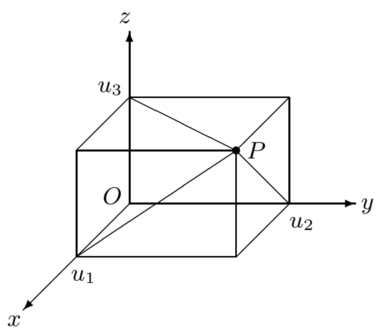


Figure 12.3: A point $P(u_1, u_2, u_3)$ as a vector

sol. (1) LHS is

$$\begin{aligned}(\alpha+\beta)(x,y) &= ((\alpha+\beta)x, (\alpha+\beta)y)\\ &= (\alpha x+\beta x, \alpha y+\beta y)\\ &= (\alpha x, \alpha y) + (\beta x, \beta y)\\ &= \alpha(x,y) + \beta(x,y)\end{aligned}$$

(2) LHS is

$$\begin{aligned}\alpha((x,y)+(u,v)) &= \alpha(x+u, y+v)\\ &= (\alpha(x+u), \alpha(y+v))\\ &= (\alpha x+\alpha u, \alpha y+\alpha v)\\ &= (\alpha x, \alpha y) + (\alpha u, \alpha v)\\ &= \alpha(x,y) + \alpha(u,v)\end{aligned}$$

□

12.2.1 Lines, Planes and the Space

Vectors-Geometric view

We can associate a vector $\mathbf{u}$ with a point (u_1, u_2, u_3) in the space. For example, we can visualize it with an arrow starting at the origin and ending at the point $\mathbf{u} = (u_1, u_2, u_3)$. One can also interpret a **vector** as a **directed line segment** i.e, a line segment with specified *magnitude* and *direction.*

Referring to the Figure 12.4, we denote the directed line segment PQ from

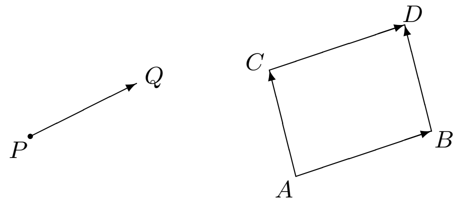


Figure 12.4: vector

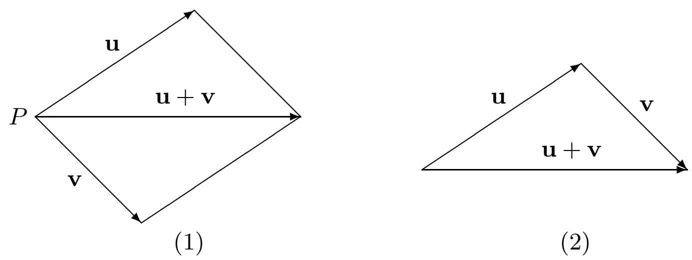


Figure 12.5: sum of two vectors

P to Q by $\overrightarrow{PQ}$. P and Q are called **tail** and **head** respectively. A vector with tail at the origin is called a **position vector**. If two vectors have the same magnitude direction, we regard it as the same vector. In this case two vector can overlap exactly when moved in parallel. Given two points $P_1(x_1, y_1, z_1)$ and $P_2(x_2, y_2, z_2)$, the **displacement vector** from P_1 to P_2 is

$$\overrightarrow{P_1P_2} = (x_2 - x_1, y_2 - y_1, z_2 - z_1).$$

Referring to the parallelogram $ABDC$ in Figure 12.4, we see $\overrightarrow{AB} = \overrightarrow{CD}$ and $\overrightarrow{AC} = \overrightarrow{BD}$.

See figure 12.5 (1). If two vectors $\mathbf{u}$, $\mathbf{v}$ have the same tail P, the sum $\mathbf{u}+\mathbf{v}$ is the vector ending at the opposite vertex of the parallelogram formed by $\mathbf{u}$ and $\mathbf{v}$.

(i) $\mathbf{u}+\mathbf{v} = \mathbf{v}+\mathbf{u}$ (commutative law)

(ii) $(\mathbf{u}+\mathbf{v})+\mathbf{w} = \mathbf{u}+(\mathbf{v}+\mathbf{w})$ (associate law)

Scalar multiple of a vector

For a real number(scalar) s and a vector $\mathbf{v}$, the scalar multiple $s\mathbf{v}$(see Fig 1.11) is the vector having magnitude $|s|$ times that of $\mathbf{v}$, having the same direction

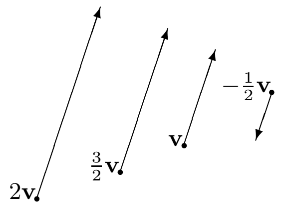


Figure 12.6: scalar multiple of $\mathbf{v}$

as $\mathbf{v}$ when $s > 0$, opposite direction when $s < 0$.

The followings hold:

(iii) $(st)\mathbf{u} = s(t\mathbf{u})$ (associative law)

(iv) $(s+t)\mathbf{u} = s\mathbf{u} + t\mathbf{u}$ (distributive law)

(v) $s(\mathbf{u}+\mathbf{v}) = s\mathbf{u} + s\mathbf{v}$ (distributive law)

(vi) $s\mathbf{0} = \mathbf{0}$ (0-vector)

(vii) $0\mathbf{u} = \mathbf{0}$

(viii) $1\mathbf{u} = \mathbf{u}$

Example 12.2.5 (3D vectors)**.** A 3D vector is denoted by, say

$$\mathbf{a} = (a_1, a_2, a_3).$$

Here a_1, a_2, a_3 are called x**-component,** y**- component,** z **-component** of $\mathbf{a}$. Let $A = (a_1, a_2, a_3)$. Shift the line segment OA by b_1 along x-axis, by b_2 along y-axis, b_3 along z-axis respectively. We obtain a vector denoted by BP. (See figure 12.7) Then the coordinate of B and P are (b_1, b_2, b_3) and $(a_1+b_1, a_2+b_2, a_3+b_3)$, and $OBPA$ form a parallelogram. Hence

$$\overrightarrow{OA} + \overrightarrow{OB} = \overrightarrow{OP}.$$

Standard basis vectors

Definition 12.2.6. The following vectors $\mathbf{i}$, $\mathbf{j}$, $\mathbf{k}$ are called **(standard basis vector)** of $\mathbb{R}^3$ (Figure 1.13).

$$\mathbf{i} = (1,0,0),\ \mathbf{j} = (0,1,0),\ \mathbf{k} = (0,0,1)$$

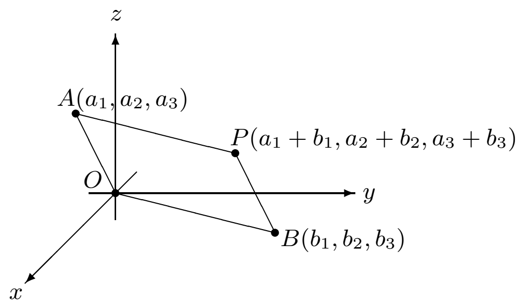


Figure 12.7: Addition

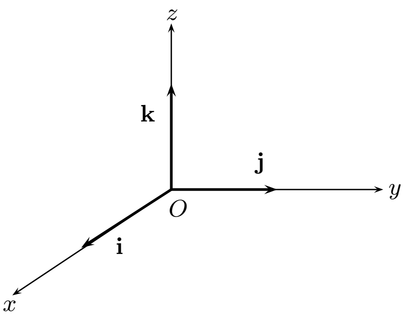


Figure 12.8: standard basis vector

Remark 12.2.7. (1) For a given $\mathbf{v} = (a_1, a_2, a_3)$

$$(a_1, a_2, a_3) = a_1(1, 0, 0) + a_2(0, 1, 0) + a_3(0, 0, 1) = a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k}$$

Example 12.2.8. Write the following using standard basis vectors.

(1) $\mathbf{v} = (-1/2, 3, 5)$.

(2) Express $3\mathbf{a} - 2\mathbf{b}$ when $\mathbf{a} = (3, 5, 0)$, $\mathbf{b} = (-4, 1, 1)$.

(3) Given two points $P(1, 4, 3)$ and $Q(4, 1, 2)$, express $\overrightarrow{PQ}$.

(4) Given three points $A(0, -1, 4)$, $B(2, 4, 1)$ and $C(3, 0, 2)$, express

$$\frac{1}{2}\overrightarrow{OA} + \frac{1}{3}\overrightarrow{OB} + \frac{1}{6}\overrightarrow{OC}.$$

sol.

(1) $\mathbf{v} = (-1/2)\mathbf{i} + 3\mathbf{j} + 5\mathbf{k}$

(2) $3\mathbf{a} - 2\mathbf{b} = 3(3\mathbf{i} + 5\mathbf{j}) - 2(-4\mathbf{i} + \mathbf{j} + \mathbf{k})$
$= (9+8)\mathbf{i} + (15-2)\mathbf{j} + (-2)\mathbf{k} = 17\mathbf{i} + 13\mathbf{j} - 2\mathbf{k}$

(3) $\overrightarrow{PQ} = (4-1)\mathbf{i} + (1-4)\mathbf{j} + (2-3)\mathbf{k} = 3\mathbf{i} - 3\mathbf{j} - \mathbf{k}$

(4) $(1/2)\overrightarrow{OA} + (1/3)\overrightarrow{OB} + (1/6)\overrightarrow{OC}$
$= (1/2)(-\mathbf{j} + 4\mathbf{k}) + (1/3)(2\mathbf{i} + 4\mathbf{j} + \mathbf{k}) + (1/6)(3\mathbf{i} + 2\mathbf{k})$
$= (7/6)\mathbf{i} + (5/6)\mathbf{j} + (8/3)\mathbf{k}$

□

12.3 Dot(Inner) product, length, distance

Dot product-Inner product

Definition 12.3.1. Given two vectors $\mathbf{a} = a_1\mathbf{i}+a_2\mathbf{j}+a_3\mathbf{k}$ and $\mathbf{b} = b_1\mathbf{i}+b_2\mathbf{j}+b_3\mathbf{k}$ we define

$$a_1b_1 + a_2b_2 + a_3b_3$$

to be the **dot product** or **(inner product)** of **a** and **b** and write $\mathbf{a} \cdot \mathbf{b}$.

Example 12.3.2. Let $\mathbf{a} = 2\mathbf{i} - 3\mathbf{j} + \mathbf{k}$, $\mathbf{b} = \mathbf{i} + 2\mathbf{j} - \mathbf{k}$. Find

(1) $\mathbf{a} \cdot \mathbf{a}$

(2) $\mathbf{a} \cdot \mathbf{b}$

(3) $\mathbf{a} \cdot (\mathbf{a} - 3\mathbf{b})$

(4) $(3\mathbf{a} + 2\mathbf{b}) \cdot (\mathbf{a} - \mathbf{b})$

sol. (1) $\mathbf{a} \cdot \mathbf{a} = 4 + 9 + 1 = 14$
(2) $\mathbf{a} \cdot \mathbf{b} = 2 - 6 - 1 = -5$
(3) $\mathbf{a} \cdot (\mathbf{a} - 3\mathbf{b}) = (2\mathbf{i} - 3\mathbf{j} + \mathbf{k}) \cdot (-\mathbf{i} - 9\mathbf{j} + 4\mathbf{k})$
$= -2 + 27 + 4 = 29$
(4) $(3\mathbf{a} + 2\mathbf{b}) \cdot (\mathbf{a} - \mathbf{b}) = (8\mathbf{i} - 5\mathbf{j} + \mathbf{k}) \cdot (\mathbf{i} - 5\mathbf{j} + 2\mathbf{k})$
$= 8 + 25 + 2 = 35$

□

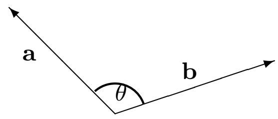


Figure 12.9: Angle between two vectors

Length of vectors

The **length, norm** of a vector $\mathbf{a} = (a_1, a_2, a_3)$ is

$$\sqrt{a_1^2 + a_2^2 + a_3^2},$$

denoted by $\|\mathbf{a}\|$. Also we note that

$$\|\mathbf{a}\| = (\mathbf{a} \cdot \mathbf{a})^{1/2}.$$

Example 12.3.3. Find the lengths of the following vectors.

(1) $\mathbf{a} = (3, 2, 1)$

(2) $3\mathbf{i} - 4\mathbf{j} + \mathbf{k}$

(3) $\overrightarrow{AB}$ when $A(2, -1/3, -1)$, $B(8/3, 0, 1)$.

sol. (1) $\|\mathbf{a}\| = \sqrt{9 + 4 + 1} = \sqrt{14}$
(2) $\|3\mathbf{i} - 4\mathbf{j} + \mathbf{k}\| = \sqrt{9 + 16 + 1} = \sqrt{26}$
(3) $\|\overrightarrow{AB}\| = \sqrt{(8/3 - 2)^2 + (0 - (-1/3))^2 + (1 - (-1))^2}$
$= \sqrt{4/9 + 1/9 + 4} = \sqrt{41}/3$

□

Definition 12.3.4. A vector with norm 1 is called a **unit vector**. Any nonzero vector $\mathbf{a}$ can be made into a unit vector by setting $\mathbf{a}/\|\mathbf{a}\|$. This process is called a **normalization**.

Example 12.3.5. Normalize the followings.

(1) $\mathbf{i} + \mathbf{j} + \mathbf{k}$

(2) $3\mathbf{i} + 4\mathbf{k}$

(3) $a\mathbf{i} - \mathbf{j} + c\mathbf{k}$

sol. (1) $(1/\sqrt{3})\,\mathbf{i} + (1/\sqrt{3})\,\mathbf{j} + (1/\sqrt{3})\,\mathbf{k}$

(2) $(3/5)\,\mathbf{i} + (4/5)\,\mathbf{k}$

(3) $(a/\sqrt{1+a^2+c^2})\,\mathbf{i} - (1/\sqrt{1+a^2+c^2})\,\mathbf{j} + (c/\sqrt{1+a^2+c^2})\,\mathbf{k}$

□

Angle between two vectors

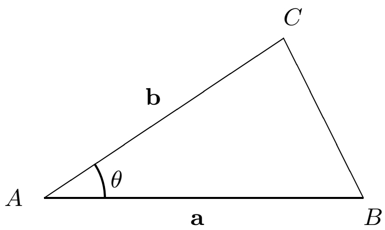
Proposition 12.3.6. *Let* $\mathbf{a}, \mathbf{b}$ *be two nonzero vectors in* $\mathbb{R}^2$ *or* $\mathbb{R}^3$ *and let* θ *be the angle between them. Then*

$$\mathbf{a}\cdot\mathbf{b} = \|\mathbf{a}\|\,\|\mathbf{b}\|\,\cos\theta$$

and hence

$$\boxed{\theta = \cos^{-1}\left(\frac{\mathbf{a}\cdot\mathbf{b}}{\|\mathbf{a}\|\,\|\mathbf{b}\|}\right)}$$

Proof. Let $\mathbf{a} = \overrightarrow{AB}$, $\mathbf{b} = \overrightarrow{AC}$. Then $\mathbf{a}-\mathbf{b} = \overrightarrow{CB}$. Let $\angle CAB = \theta$. Then by



$$|BC|^2 = |AB|^2 + |AC|^2 - 2|AB|\,|AC|\,\cos\theta$$

Figure 12.10: law of cosine

the law of cosine (figure 12.10) we have

$$\|\mathbf{b}-\mathbf{a}\|^2 = \|\mathbf{a}\|^2 + \|\mathbf{b}\|^2 - 2\|\mathbf{a}\|\,\|\mathbf{b}\|\,\cos\theta.$$

The left hand side is

$$\begin{aligned}\|\mathbf{a}-\mathbf{b}\|^2 &= (\mathbf{a}-\mathbf{b})\cdot(\mathbf{a}-\mathbf{b})\\ &= \mathbf{a}\cdot\mathbf{a} - \mathbf{a}\cdot\mathbf{b} - \mathbf{b}\cdot\mathbf{a} + \mathbf{b}\cdot\mathbf{b}\\ &= \|\mathbf{a}\|^2 - 2\,\mathbf{a}\cdot\mathbf{b} + \|\mathbf{b}\|^2.\end{aligned}$$

Hence we obtain

$$\|\mathbf{a}\|\,\|\mathbf{b}\|\,\cos\theta = \mathbf{a}\cdot\mathbf{b}.$$

□

Corollary 12.3.7. *Two nonzero vector* $\mathbf{a}$ *and* $\mathbf{b}$ *are* ***perpendicular, orthogonal*** *if and only if* $\mathbf{a}\cdot\mathbf{b}=0$.

Example 12.3.8. Find the angle between $\mathbf{i}+\mathbf{j}+2\mathbf{k}$ and $-\mathbf{i}+2\mathbf{j}+\mathbf{k}$.

sol. By proposition 1.2.10,

$$\frac{(\mathbf{i}+\mathbf{j}+2\mathbf{k})\cdot(-\mathbf{i}+2\mathbf{j}+\mathbf{k})}{\|\mathbf{i}+\mathbf{j}+2\mathbf{k}\|\,\|-\mathbf{i}+2\mathbf{j}+\mathbf{k}\|}=\frac{-1+2+2}{\sqrt{1+1+4}\sqrt{1+4+1}}=\frac{3}{6}=\frac{1}{2}.$$

Hence the angle is $\cos^{-1}(1/2)=\pi/3$.

□

Corollary 12.3.9. *Given two points* $A(a_1,a_2,a_3)$, $B(b_1,b_2,b_3)$, *the area of the triangle* OAB *is*

$$\frac{1}{2}\sqrt{(a_2b_3-a_3b_2)^2+(a_3b_1-a_1b_3)^2+(a_1b_2-a_2b_1)^2}$$

Proof. Let $\overrightarrow{OA}=\mathbf{a}$, $\overrightarrow{OB}=\mathbf{b}$, $\angle BOA=\theta$. Then the area of $\triangle OAB$ is

$$\begin{aligned}
&\frac{1}{2}|OA|\,|OB|\,\sin\theta\\
&=\frac{1}{2}\|\mathbf{a}\|\,\|\mathbf{b}\|\,\sqrt{1-\cos^2\theta}\\
&=\frac{1}{2}\sqrt{\|\mathbf{a}\|^2\,\|\mathbf{b}\|^2-(\mathbf{a}\cdot\mathbf{b})^2}\\
&=\frac{1}{2}\sqrt{(a_1^2+a_2^2+a_3^2)(b_1^2+b_2^2+b_3^2)-(a_1b_1+a_2b_2+a_3b_3)^2}\\
&=\frac{1}{2}\sqrt{(a_2b_3-a_3b_2)^2+(a_3b_1-a_1b_3)^2+(a_1b_2-a_2b_1)^2}.
\end{aligned}$$

□

Example 12.3.10. Find the area of the triangle with vertices $A(a,0,0), B(0,b,0), C(0,0,c)$.

sol. Shift(translate) A to the origin, then the points B,C are moved to the points $(-a,b,0)$ and $(-a,0,c)$. Hence

$$\frac{1}{2}\sqrt{(bc-0)^2+(0+ac)^2+(0+ab)^2}=\frac{1}{2}\sqrt{b^2c^2+c^2a^2+a^2b^2}.$$

□

Proposition 12.3.11 (Properties of Inner Product)**.** *For vectors* $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$ *and scalar* α, *the following hold:*

(1) $\mathbf{a} \cdot \mathbf{a} \geq 0$ *(equality holds only when* $\mathbf{a} = \mathbf{0}$*)*

(2) $\mathbf{a} \cdot \mathbf{b} = \mathbf{b} \cdot \mathbf{a}$

(3) $(\mathbf{a} + \mathbf{b}) \cdot \mathbf{c} = \mathbf{a} \cdot \mathbf{c} + \mathbf{b} \cdot \mathbf{c}$

(4) $(\alpha \mathbf{a}) \cdot \mathbf{b} = \alpha(\mathbf{a} \cdot \mathbf{b})$

(5) $\|\mathbf{a}\| = \sqrt{\mathbf{a} \cdot \mathbf{a}}$

Proof. These can be proved easily. □

Example 12.3.12. For $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$ Show the following.

(1) $(\mathbf{a} - \mathbf{b}) \cdot \mathbf{c} = \mathbf{a} \cdot \mathbf{c} - \mathbf{b} \cdot \mathbf{c}$

(2) $\mathbf{a} \cdot (\mathbf{b} + \mathbf{c}) = \mathbf{a} \cdot \mathbf{b} + \mathbf{a} \cdot \mathbf{c}$

(3) $\mathbf{a} \cdot (\mathbf{b} - \mathbf{c}) = \mathbf{a} \cdot \mathbf{b} - \mathbf{a} \cdot \mathbf{c}$

(4) $\mathbf{a} \cdot \mathbf{b} = \frac{1}{2}(\|\mathbf{a}\|^2 + \|\mathbf{b}\|^2 - \|\mathbf{a} - \mathbf{b}\|^2)$

sol. We see

(1) $(\mathbf{a} - \mathbf{b}) \cdot \mathbf{c} = (\mathbf{a} + (-1)\mathbf{b})) \cdot \mathbf{c} = \mathbf{a} \cdot \mathbf{c} + ((-1)\mathbf{b}) \cdot \mathbf{w}$
$= \mathbf{a} \cdot \mathbf{c} + (-1)\mathbf{b} \cdot \mathbf{c} = \mathbf{a} \cdot \mathbf{c} - \mathbf{b} \cdot \mathbf{c}$

(2) $\mathbf{a} \cdot (\mathbf{b} + \mathbf{c}) = (\mathbf{b} + \mathbf{c}) \cdot \mathbf{a} = \mathbf{b} \cdot \mathbf{a} + \mathbf{c} \cdot \mathbf{a} = \mathbf{a} \cdot \mathbf{b} + \mathbf{a} \cdot \mathbf{c}$

(3) $\mathbf{a} \cdot (\mathbf{b} - \mathbf{c}) = (\mathbf{b} - \mathbf{c}) \cdot \mathbf{a} = \mathbf{b} \cdot \mathbf{a} - \mathbf{c} \cdot \mathbf{a} = \mathbf{a} \cdot \mathbf{b} - \mathbf{a} \cdot \mathbf{c}$

(4) $\|\mathbf{a} - \mathbf{b}\|^2 = (\mathbf{a} - \mathbf{b}) \cdot (\mathbf{a} - \mathbf{b}) = \mathbf{a} \cdot (\mathbf{a} - \mathbf{b}) - \mathbf{b} \cdot (\mathbf{a} - \mathbf{b})$
$= \mathbf{a} \cdot \mathbf{a} - \mathbf{a} \cdot \mathbf{b} - \mathbf{b} \cdot \mathbf{a} + \mathbf{b} \cdot \mathbf{b} = \|\mathbf{a}\|^2 - 2\,\mathbf{a} \cdot \mathbf{b} + \|\mathbf{b}\|^2$

□

Theorem 12.3.13 (Cauchy-Schwarz inequality)**.** *For any two vectors* $\mathbf{a}$, $\mathbf{b}$

$$|\mathbf{a} \cdot \mathbf{b}| \leq \|\mathbf{a}\| \, \|\mathbf{b}\|$$

holds, and the equality holds iff $\mathbf{a}$ *and* $\mathbf{b}$ *are parallel.*

Proof. We may assume $\mathbf{a}$, $\mathbf{b}$ are nonzero. Let θ be the angle between $\mathbf{a}$ and $\mathbf{b}$. Then by prop 12.3.6

$$|\mathbf{a}\cdot\mathbf{b}| = \|\mathbf{a}\|\,\|\mathbf{b}\|\,|\cos\theta| \le \|\mathbf{a}\|\,\|\mathbf{b}\|$$

holds. Since $\|\mathbf{a}\|\,\|\mathbf{b}\| \neq 0$, if equality holds $|\cos\theta| = 1$ i.e, $\theta = 0$ or π. Hence $\mathbf{a}$ and $\mathbf{b}$ are parallel. □

Remark 12.3.14. The Cauchy-Schwarz inequality reads, componentwise, as

$$(ax+by+cz)^2 \le (a^2+b^2+c^2)(x^2+y^2+z^2).$$

Example 12.3.15. Show Cauchy-Schwarz inequality for $\mathbf{i}+3\mathbf{j}+2\mathbf{k}$, $-\mathbf{i}+\mathbf{j}$.

sol. Since the inner product and lengths are

$$(\mathbf{i}+3\mathbf{j}+2\mathbf{k})\cdot(-\mathbf{i}+\mathbf{j}) = -1+3 = 2,$$
$$\|\mathbf{i}+3\mathbf{j}+2\mathbf{k}\|\,\|-\mathbf{i}+\mathbf{j}\| = \sqrt{1+9+4}\,\sqrt{1+1} = \sqrt{28} = 2\sqrt{7}$$

we have

$$|(\mathbf{i}+3\mathbf{j}+2\mathbf{k})\cdot(-\mathbf{i}+\mathbf{j})| \le \|\mathbf{i}+3\mathbf{j}+2\mathbf{k}\|\,\|-\mathbf{i}+\mathbf{j}\|.$$

□

Theorem 12.3.16 (Triangle inequality)**.** *For any two vector* $\mathbf{a}$*,* $\mathbf{b}$ *it holds that*

$$\|\mathbf{a}+\mathbf{b}\| \le \|\mathbf{a}\| + \|\mathbf{b}\|$$

and equality holds when $\mathbf{a}$*,* $\mathbf{b}$ *are parallel and having same direction.*

Proof. We have

$$\|\mathbf{a}+\mathbf{b}\|^2 = (\mathbf{a}+\mathbf{b})\cdot(\mathbf{a}+\mathbf{b}) = \|\mathbf{a}\|^2 + 2\mathbf{a}\cdot\mathbf{b} + \|\mathbf{b}\|^2.$$

By C-S

$$\|\mathbf{a}+\mathbf{b}\|^2 \le \|\mathbf{a}\|^2 + 2\|\mathbf{a}\|\,\|\mathbf{b}\| + \|\mathbf{b}\|^2 = (\|\mathbf{a}\|+\|\mathbf{b}\|)^2.$$

Equality holds iff

$$\mathbf{a}\cdot\mathbf{b} = \|\mathbf{a}\|\,\|\mathbf{b}\|,$$

i.e, the angle is 0. □

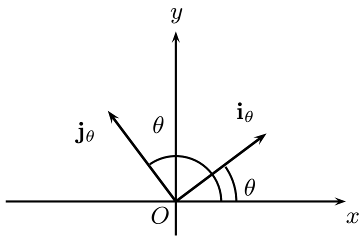


Figure 12.11: $\mathbf{i}_\theta$ and $\mathbf{j}_\theta$

Example 12.3.17. Show triangle inequality for $-\mathbf{i}+\mathbf{j}$ and $\mathbf{i}+3\mathbf{j}+2\mathbf{k}$.

sol. Sum and difference is

$$\begin{aligned} \|(\mathbf{i}+3\mathbf{j}+2\mathbf{k})+(-\mathbf{i}+\mathbf{j})\| &= \|4\mathbf{j}+2\mathbf{k}\| \;=\; \sqrt{16+4} \\ &= 2\sqrt{5} \;=\; 4.4721\ldots \\ \|\mathbf{i}+3\mathbf{j}+2\mathbf{k}\|+\|-\mathbf{i}+\mathbf{j}\| &= \sqrt{1+9+4}+\sqrt{1+1} \\ &= \sqrt{14}+\sqrt{2} \;=\; 5.1558\ldots \end{aligned}$$

Hence

$$\|(\mathbf{i}+3\mathbf{j}+2\mathbf{k})+(-\mathbf{i}+\mathbf{j})\| \le \|\mathbf{i}+3\mathbf{j}+2\mathbf{k}\|+\|-\mathbf{i}+\mathbf{j}\|.$$

□

Definition 12.3.18. If two vectors $\mathbf{a}$, $\mathbf{b}$ satisfy $\mathbf{a}\cdot\mathbf{b}=0$ then we say they are **orthogonal**(perpendicular).

Example 12.3.19. For any real θ, the two vectors $\mathbf{i}_\theta = (\cos\theta)\mathbf{i} + (\sin\theta)\mathbf{j}$, $\mathbf{j}_\theta = -(\sin\theta)\mathbf{i} + (\cos\theta)\mathbf{j}$ are orthogonal.

Example 12.3.20. Find a unit vector orthogonal to $2\mathbf{i}-\mathbf{j}+3\mathbf{k}$ and $\mathbf{i}+2\mathbf{j}+9\mathbf{k}$.

sol. Let $a\mathbf{i}+b\mathbf{j}+c\mathbf{k}$ be the desired vector. Then a, b, c are determined by

$$\begin{aligned} 2a-b+3c &= 0 \text{ (orthogonality)} \\ a+2b+9c &= 0 \text{ (orthogonality)} \\ a^2+b^2+c^2 &= 1 \text{ (unicity)}. \end{aligned}$$

Hence the desired vector is

$$\pm\frac{1}{\sqrt{19}}\left(3\mathbf{i}+3\mathbf{j}-\mathbf{k}\right).$$

□

Orthogonal projection

Given two nonzero vectors **a** and **b**, we may define the **orthogonal projection** of **b** onto **a** to be the vector **p** given in the figure 12.12. Since **p** is a scalar multiple of **a**, there is a constant c such that $\mathbf{p} = c\mathbf{a}$. We let

$$\mathbf{b} = c\mathbf{a} + \mathbf{q},$$

where **q** is a vector orthogonal to **a**. Taking inner product with **a**, we have

$$\mathbf{a} \cdot \mathbf{b} = c\mathbf{a} \cdot \mathbf{a}.$$

Hence we obtain $c = (\mathbf{a} \cdot \mathbf{b})/(\mathbf{a} \cdot \mathbf{a})$. Thus the orthogonal projection is

$$\mathbf{p} = \frac{\mathbf{a} \cdot \mathbf{b}}{\|\mathbf{a}\|^2}\mathbf{a}.$$

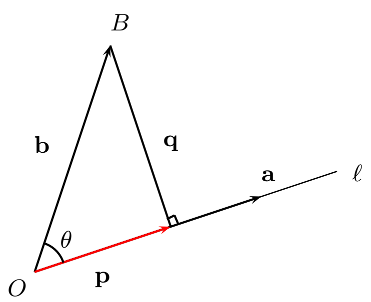


Figure 12.12: Projection of **b** onto **a**

The length of **p** is

$$\|\mathbf{p}\| = \frac{|\mathbf{a} \cdot \mathbf{b}|}{\|\mathbf{a}\|^2}\|\mathbf{a}\| = \frac{|\mathbf{a} \cdot \mathbf{b}|}{\|\mathbf{a}\|} = \|\mathbf{b}\| \cos\theta.$$

This agrees with the geometric interpretation.

Definition 12.3.21. For nonzero vector **b** and any vector **a**, we define

$$proj_{\mathbf{a}}\mathbf{b} = \mathbf{p} = \frac{\mathbf{a} \cdot \mathbf{b}}{\|\mathbf{a}\|^2}\mathbf{a}.$$

We call it **orthogonal projection of b onto a**.

Example 12.3.22. $\mathbf{a} = 3\mathbf{i}+2\mathbf{j}-\mathbf{k}$, $\mathbf{b} = \mathbf{i}+\mathbf{j}+2\mathbf{k}$. Find orthogonal projection of $\mathbf{b}$ onto $\mathbf{a}$.

sol. The orthogonal projection is

$$\begin{aligned}\frac{\mathbf{a}\cdot\mathbf{b}}{\|\mathbf{a}\|^2}\mathbf{a} &= \frac{3\cdot 1+2\cdot 1+(-1)\cdot 2}{9+4+1}(3\mathbf{i}+2\mathbf{j}-\mathbf{k})\\ &= \frac{9}{14}\mathbf{i}+\frac{6}{14}\mathbf{j}-\frac{3}{14}\mathbf{k}.\end{aligned}$$

□

Theorem 12.3.23. *For any two nonzero* $\mathbf{u}$ *and* $\mathbf{v}$*, we can write* $\mathbf{v}$ *as the sum of two orthogonal vectors* $\mathbf{a}+\mathbf{b}$*, where* $\mathbf{a}$ *is the projection of* $\mathbf{v}$ *onto* $\mathbf{u}$ *and* $\mathbf{b}$ *is orthogonal to* $\mathbf{u}$*. This decomposition is unique.*

Proof. Denote by $\mathbf{a}$ the projection of $\mathbf{v}$ onto $\mathbf{u}$ and let $\mathbf{b} = \mathbf{v}-\mathbf{a}$. Then

$$\mathbf{v} = \frac{\mathbf{u}\cdot\mathbf{v}}{\|\mathbf{u}\|^2}\mathbf{u} + \mathbf{v} - \frac{\mathbf{u}\cdot\mathbf{v}}{\|\mathbf{u}\|^2}\mathbf{u} \equiv \mathbf{a}+\mathbf{b}.$$

We can check $\mathbf{b}$ is orthogonal to $\mathbf{u}$:

$$\begin{aligned}\mathbf{u}\cdot\mathbf{b} &= \mathbf{u}\cdot\left(\mathbf{v}-\frac{\mathbf{u}\cdot\mathbf{v}}{\|\mathbf{u}\|^2}\mathbf{u}\right)\\ &= \mathbf{u}\cdot\mathbf{v} - \frac{\mathbf{u}\cdot\mathbf{v}}{\|\mathbf{u}\|^2}\,\mathbf{u}\cdot\mathbf{u}\\ &= \mathbf{u}\cdot\mathbf{v}-\mathbf{u}\cdot\mathbf{v} = 0.\end{aligned}$$

This is an orthogonal decomposition. To see the uniqueness, assume there is real number α s.t. $\mathbf{v} = \alpha\mathbf{u}+\mathbf{c}$, with $\mathbf{u}\cdot\mathbf{c} = 0$. Then

$$\mathbf{u}\cdot\mathbf{v} = \mathbf{u}\cdot(\alpha\mathbf{u}+\mathbf{c}) = \alpha\mathbf{u}\cdot\mathbf{u}+\mathbf{u}\cdot\mathbf{c} = \alpha\|\mathbf{u}\|^2.$$

Hence we see

$$\begin{aligned}\alpha\mathbf{u} = \frac{\mathbf{u}\cdot\mathbf{v}}{\|\mathbf{u}\|^2}\mathbf{u} \;&=\; \mathbf{a}\\ \mathbf{c} = \mathbf{v}-\alpha\mathbf{u} \;&=\; \mathbf{v}-\mathbf{a} \;=\; \mathbf{b}.\end{aligned}$$

Thus the decomposition of $\mathbf{v}$ along $\mathbf{u}$ and its orthogonal component is unique.

□

Definition 12.3.24. The vector $\mathbf{a}$ is called the **component parallel to u** and $\mathbf{b}$ is the **component orthogonal to u**.(**orthogonal complement**).

Example 12.3.25. Find the orthogonal decomposition of $\mathbf{v} = 3\mathbf{i} + 5\mathbf{j} + \mathbf{k}$ w.r.t. $\mathbf{u} = \mathbf{i} + 2\mathbf{j} - \mathbf{k}$.

sol. Let $\mathbf{a}$ be the projection of $\mathbf{v}$ onto $\mathbf{u}$ and $\mathbf{b} = \mathbf{v} - \mathbf{a}$. Then

$$\begin{aligned}
\mathbf{a} &= \frac{\mathbf{u}\cdot\mathbf{v}}{\|\mathbf{u}\|^2}\mathbf{u} \\
&= \frac{1\cdot 3 + 2\cdot 5 + (-1)\cdot 1}{1+4+1}(\mathbf{i} + 2\mathbf{j} - \mathbf{k}) \\
&= 2\mathbf{i} + 4\mathbf{j} - 2\mathbf{k} \\
\mathbf{b} &= (3\mathbf{i} + 5\mathbf{j} + \mathbf{k}) - (2\mathbf{i} + 4\mathbf{j} - 2\mathbf{k}) \\
&= \mathbf{i} + \mathbf{j} + 3\mathbf{k}.
\end{aligned}$$

Here $\mathbf{a}$ is parallel to $\mathbf{u}$, $\mathbf{b}$ is orthogonal to $\mathbf{u}$ and $\mathbf{v} = \mathbf{a} + \mathbf{b}$.

□

Work

Displacement : If an object has moved from P to Q, then $\vec{PQ}$ is the displacement.

The **work** done by a constant force of magnitude F in moving an object along a straight line by D is $W = FD$. (Assume the force is directed along the line of motion) When the force is exerted in a different direction than the

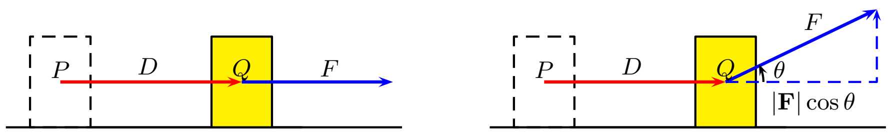


Figure 12.13: Work on a line

direction of the object, the work is defined as

$$\begin{aligned}
\text{Work} &= (\text{scalar compo. of } \mathbf{F} \text{ in the direction of } \mathbf{D}\ )\cdot(\text{length of } \mathbf{D}) \\
&= (|\mathbf{F}|\cos\theta)|\mathbf{D}| \\
&= \mathbf{F}\cdot\mathbf{D}.
\end{aligned}$$

12.4 Cross product

Definition 12.4.1. Let $\mathbf{u}, \mathbf{v}$ be two vectors in $\mathbb{R}^3$(not $\mathbb{R}^2$). The cross product of $\mathbf{u}, \mathbf{v}$, denoted by $\mathbf{u} \times \mathbf{v}$ is the vector whose length and direction are given as follows:

(1) The length is the area of the parallelogram spanned by $\mathbf{u}$ and $\mathbf{v}$.(zero if $\mathbf{u}, \mathbf{v}$ are parallel). Alternatively,

$$\|\mathbf{u} \times \mathbf{v}\| = \|\mathbf{u}\|\|\mathbf{v}\| \sin \theta,$$

where θ is the angle between $\mathbf{u}$ and $\mathbf{v}$.

(2) The direction of $\mathbf{u} \times \mathbf{v}$ is perpendicular to $\mathbf{u}$ and $\mathbf{v}$, and the triple $(\mathbf{u}, \mathbf{v}, \mathbf{u} \times \mathbf{v})$ form a right-handed set of vectors.

Hence we have

$$\boxed{\mathbf{u} \times \mathbf{v} = (\|\mathbf{u}\|\ \|\mathbf{v}\|\ \sin \theta)\mathbf{n}.}$$

Here $\mathbf{n}$ is the unit normal vector together with $\mathbf{u}, \mathbf{v}$ forming a right-handed set of vectors. Algebraic rules:

(1) $\mathbf{u} \times \mathbf{v} = 0$, if $\mathbf{u}$, $\mathbf{v}$ are parallel or one of them is zero.

(2) $\mathbf{u} \times \mathbf{v} = -(\mathbf{v} \times \mathbf{u})$

(3) $\mathbf{u} \times (\mathbf{v} + \mathbf{w}) = \mathbf{u} \times \mathbf{v} + \mathbf{u} \times \mathbf{w}$

(4) $(\mathbf{u} + \mathbf{v}) \times \mathbf{w} = \mathbf{u} \times \mathbf{w} + \mathbf{v} \times \mathbf{w}$

(5) $(\alpha \mathbf{u}) \times \mathbf{v} = \alpha(\mathbf{u} \times \mathbf{v})$ for scalar α.

Multiplication rules:

(1) $\mathbf{i} \times \mathbf{j} = \mathbf{k}, \quad \mathbf{j} \times \mathbf{k} = \mathbf{i}, \quad \mathbf{k} \times \mathbf{i} = \mathbf{j}.$

(2) $\mathbf{a} \times \mathbf{b} = -(\mathbf{b} \times \mathbf{a})$

(3) $\mathbf{i} \times \mathbf{i} = \mathbf{j} \times \mathbf{j} = \mathbf{k} \times \mathbf{k} = 0$

Note that

$$\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) \neq (\mathbf{a} \times \mathbf{b}) \times \mathbf{c}.$$

For example

$$\mathbf{i} \times (\mathbf{i} \times \mathbf{j}) = \mathbf{i} \times \mathbf{k} = -\mathbf{j} \neq (\mathbf{i} \times \mathbf{i}) \times \mathbf{j} = 0.$$

Proposition 12.4.2. *The area of parallelogram determined by the two vectors* $a\mathbf{i} + b\mathbf{j}$ *and* $c\mathbf{i} + d\mathbf{j}$ *is* $|ad - bc|$. *This is the absolute value of the determinant of the matrix determined by two two vectors:*

$$det \begin{bmatrix} a & b \\ c & d \end{bmatrix}.$$

Proof. Let $\mathbf{u} = a\mathbf{i} + b\mathbf{j}$, $\mathbf{v} = c\mathbf{i} + d\mathbf{j}$ and θ be the angle between them. Then the area of the parallelogram is

$$\begin{aligned} \|\mathbf{u}\| \, \|\mathbf{v}\| \, \sin\theta &= \|\mathbf{u}\| \, \|\mathbf{v}\| \sqrt{1 - \cos^2\theta} \\ &= \sqrt{\|\mathbf{u}\|^2 \, \|\mathbf{v}\|^2 - (\mathbf{u} \cdot \mathbf{v})^2} \\ &= \sqrt{(a^2 + b^2)(c^2 + d^2) - (ac + bd)^2} \\ &= \sqrt{a^2 d^2 + b^2 c^2 - 2abcd} \\ &= |ad - bc|. \end{aligned}$$

□

3×3 **matrix**

A typical 3×3 matrix is given by

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}.$$

The **determinant** is defined as

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = a_{11} \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - a_{12} \begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + a_{13} \begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}. \qquad (12.1)$$

The RHS of 12.1 is expansion w.r.t **first row.** By theorem **??**, (1), (2), we can expand w.r.t. any row or column, except we multiply $(-1)^{i+j}$.

Cross product-using determinant

In the previous section, we have defined the cross product using geometry, but did not show how to compute it. Now we can give a formula for the cross product using the determinant:

Definition 12.4.3 (Alternative definition)**.** For $\mathbf{u} = u_1\mathbf{i} + u_2\mathbf{j} + u_3\mathbf{k}$ and $\mathbf{v} = v_1\mathbf{i} + v_2\mathbf{j} + v_3\mathbf{k}$, the **cross product** $\mathbf{u} \times \mathbf{v}$ is defined by

$$\mathbf{u} \times \mathbf{v} = \begin{vmatrix} u_2 & u_3 \\ v_2 & v_3 \end{vmatrix} \mathbf{i} - \begin{vmatrix} u_1 & u_3 \\ v_1 & v_3 \end{vmatrix} \mathbf{j} + \begin{vmatrix} u_1 & u_2 \\ v_1 & v_2 \end{vmatrix} \mathbf{k}. \tag{12.2}$$

Using the definition of determinant (12.1) symbolically, we have

$$\mathbf{u} \times \mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix}.$$

Example 12.4.4. $\mathbf{i} \times \mathbf{i} = \mathbf{0}, \quad \mathbf{j} \times \mathbf{j} = \mathbf{0}, \quad \mathbf{k} \times \mathbf{k} = \mathbf{0}.$

Example 12.4.5. Compute $(2\mathbf{i} - \mathbf{j} + 3\mathbf{k}) \times (\mathbf{i} + \mathbf{j} + 2\mathbf{k})$.

sol. By the definition of cross product, we see

$$(2\mathbf{i} - \mathbf{j} + 3\mathbf{k}) \times (\mathbf{i} + \mathbf{j} + 2\mathbf{k}) = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & -1 & 3 \\ 1 & 1 & 2 \end{vmatrix} = -5\mathbf{i} - \mathbf{j} + 3\mathbf{k}.$$

□

A geometric meaning of the cross product

To see the relation with the geometric definition of the cross product, we define the triple product of three vectors: Let

$$\mathbf{a} = a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k}, \mathbf{b} = b_1\mathbf{i} + b_2\mathbf{j} + b_3\mathbf{k}, \mathbf{c} = c_1\mathbf{i} + c_2\mathbf{j} + c_3\mathbf{k}.$$

The dot product between $(\mathbf{a}\times\mathbf{b})$ and $\mathbf{c}$ is $(\mathbf{a}\times\mathbf{b})\cdot\mathbf{c}$, called the **triple product** $(\mathbf{a}\times\mathbf{b})\cdot\mathbf{c}$ of three vectors, $\mathbf{a}, \mathbf{b}$ and $\mathbf{c}$. We see by definition

$$\begin{aligned}
(\mathbf{a}\times\mathbf{b})\cdot\mathbf{c} &= \left(\begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix}\mathbf{i} - \begin{vmatrix} a_1 & a_3 \\ b_1 & b_3 \end{vmatrix}\mathbf{j} + \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix}\mathbf{k}\right)\cdot(c_1\mathbf{i}+c_2\mathbf{j}+c_3\mathbf{k}) \\
&= \begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix}c_1 - \begin{vmatrix} a_1 & a_3 \\ b_1 & b_3 \end{vmatrix}c_2 + \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix}c_3 \\
&= \begin{vmatrix} c_1 & c_2 & c_3 \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}.
\end{aligned}$$

We observe the following properties of $(\mathbf{a}\times\mathbf{b})$:

(1) If $\mathbf{c}$ is a vector in the plane spanned by $\mathbf{a}, \mathbf{b}$, then the third row in the determinant is a linear combination of the first and second row, and hence $(\mathbf{a}\times\mathbf{b})\cdot\mathbf{c} = 0$. In other words, *the vector* $\mathbf{a}\times\mathbf{b}$ *is orthogonal to any vector in the plane spanned by* $\mathbf{a}$ *and* $\mathbf{b}$.

(2) We compute length of $\mathbf{a}\times\mathbf{b}$.

$$\begin{aligned}
\|\mathbf{a}\times\mathbf{b}\|^2 &= \begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix}^2 + \begin{vmatrix} a_1 & a_3 \\ b_1 & b_3 \end{vmatrix}^2 + \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix}^2 \\
&= (a_2b_3 - a_2b_2)^2 + (a_1b_3 - b_1a_3)^2 + (a_1b_2 - b_1a_2)^2 \\
&= (a_1^2 + a_2^2 + a_3^2)(b_1^2 + b_2^2 + b_3^2) - (a_1b_1 + a_2b_2 + a_3b_3)^2.
\end{aligned}$$

Hence

$$\|\mathbf{a}\times\mathbf{b}\|^2 = \|\mathbf{a}\|^2\|\mathbf{b}\|^2 - (\mathbf{a}\cdot\mathbf{b})^2 = \|\mathbf{a}\|^2\|\mathbf{b}\|^2(1-\cos^2\theta) = \|\mathbf{a}\|^2\|\mathbf{b}\|^2\sin^2\theta.$$

So we conclude that $\mathbf{a}\times\mathbf{b}$ is a vector perpendicular to the plane $\mathcal{P}$ spanned by $\mathbf{a}$ and $\mathbf{b}$ with length $\|\mathbf{a}\|\|\mathbf{b}\||\sin\theta|$.

(3) Finally, the right handed rule can be checked with $\mathbf{i}\times\mathbf{j} = \mathbf{k}$.

Hence this alternative definition is the same as the geometric definition of the cross product given earlier.

Theorem 12.4.6 (Alternative cross Product)**.** *For* $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$, *it holds that*

(1) $\|\mathbf{a}\times\mathbf{b}\| = \|\mathbf{a}\|\|\mathbf{b}\|\sin\theta$, *the area of the parallelogram spanned by* $\mathbf{a}$ *and* $\mathbf{b}$.

(2) $\mathbf{a} \times \mathbf{b}$ *is perpendicular to* $\mathbf{a}$ *and* $\mathbf{b}$*, and the triple* $(\mathbf{a}, \mathbf{b}, \mathbf{a} \times \mathbf{b})$ *form a right-handed rule.*

Component formula using determinant

$$(a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k}) \times (b_1\mathbf{i} + b_2\mathbf{j} + b_3\mathbf{k}) = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$$
$$= (a_2b_3 - a_3b_2)\mathbf{i} - (a_1b_3 - a_3b_1)\mathbf{j} + (a_1b_2 - a_2b_1)\mathbf{k}.$$

Example 12.4.7. Find $(\mathbf{i} + \mathbf{j}) \times (\mathbf{j} - 2\mathbf{k})$.

sol. $(\mathbf{i} + \mathbf{j}) \times (\mathbf{j} - 2\mathbf{k}) = \mathbf{i} \times j - 2\mathbf{i} \times k + \mathbf{j} \times j - 2\mathbf{j} \times k = -2\mathbf{i} + 2\mathbf{j} + \mathbf{k}$.

□

Theorem 12.4.8 (Cross product II)**.**

(1) $\|\mathbf{u} \times \mathbf{v}\|^2 = \|\mathbf{u}\|^2\|\mathbf{v}\|^2 - (\mathbf{u} \cdot \mathbf{v})^2$. *In particular,* $\mathbf{u} \times \mathbf{u} = \mathbf{0}$.

(2) If is θ *the angle between* $\mathbf{u}$ *and* $\mathbf{v}$*,* $\|\mathbf{u} \times \mathbf{v}\| = \|\mathbf{u}\|\,\|\mathbf{v}\|\,\sin\theta$. *Hence nec. and suff. condition for* $\mathbf{u}$ *and* $\mathbf{v}$ *are parallel is* $\mathbf{u} \times \mathbf{v} = \mathbf{0}$.

(3) $(\mathbf{u} \times \mathbf{v}) \cdot \mathbf{w} = \mathbf{u} \cdot (\mathbf{v} \times \mathbf{w})$.

(4) $(\mathbf{u} \times \mathbf{v}) \cdot \mathbf{u} = (\mathbf{u} \times \mathbf{v}) \cdot \mathbf{v} = 0$*, i.e,* $\mathbf{u} \times \mathbf{v}$ *is orthogonal to* $\mathbf{u}$ *and* $\mathbf{v}$.

(5) $|(\mathbf{a} \times \mathbf{b}) \cdot \mathbf{c}|$ *is the volume of parallelepiped formed by three vectors* $\mathbf{a}, \mathbf{b}$ *and* $\mathbf{c}$.*(See below)*

Proof. Let $\mathbf{u} = a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k}$, $\mathbf{v} = b_1\mathbf{i} + b_2\mathbf{j} + b_3\mathbf{k}$, $\mathbf{w} = c_1\mathbf{i} + c_2\mathbf{j} + c_3\mathbf{k}$.

(1) $\|\mathbf{u} \times \mathbf{v}\|^2 = \|\mathbf{u}\|^2\|\mathbf{v}\|^2 - (\mathbf{u} \cdot \mathbf{v})^2$ as shown before.
So $\mathbf{u} \times \mathbf{u} = \mathbf{0}$.

(2) Since $\mathbf{u} \cdot \mathbf{v} = \|\mathbf{u}\|\,\|\mathbf{v}\|\,\cos\theta$, we have by (1)

$$\begin{aligned} \|\mathbf{u} \times \mathbf{v}\| &= \sqrt{\|\mathbf{u}\|^2\|\mathbf{v}\|^2 - (\mathbf{u} \cdot \mathbf{v})^2} \\ &= \|\mathbf{u}\|\,\|\mathbf{v}\|\,\sqrt{1 - \cos^2\theta} \\ &= \|\mathbf{u}\|\,\|\mathbf{v}\|\,\sin\theta. \end{aligned}$$

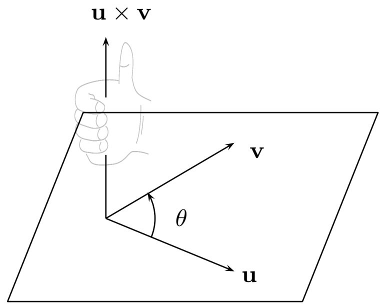


Figure 12.14: right handed rule

(3)

$$
\begin{aligned}
(\mathbf{u}\times\mathbf{v})\cdot\mathbf{w} &= \left(\begin{vmatrix} a_2 & a_3\\ b_2 & b_3\end{vmatrix}\mathbf{i} - \begin{vmatrix} a_1 & a_3\\ b_1 & b_3\end{vmatrix}\mathbf{j} + \begin{vmatrix} a_1 & a_2\\ b_1 & b_2\end{vmatrix}\mathbf{k}\right)\cdot(c_1\mathbf{i}+c_2\mathbf{j}+c_3\mathbf{k})\\
&= \begin{vmatrix} a_2 & a_3\\ b_2 & b_3\end{vmatrix}c_1 - \begin{vmatrix} a_1 & a_3\\ b_1 & b_3\end{vmatrix}c_2 + \begin{vmatrix} a_1 & a_2\\ b_1 & b_2\end{vmatrix}c_3\\
&= \begin{vmatrix} a_1 & a_2 & a_3\\ b_1 & b_2 & b_3\\ c_1 & c_2 & c_3\end{vmatrix}
\end{aligned}
$$

exapnding w.r.t first row, this is

$$
\begin{aligned}
&= a_1\begin{vmatrix} b_2 & b_3\\ c_2 & c_3\end{vmatrix} - a_2\begin{vmatrix} b_1 & b_3\\ c_1 & c_3\end{vmatrix} + a_3\begin{vmatrix} b_1 & b_2\\ c_1 & c_2\end{vmatrix}\\
&= (a_1\mathbf{i}+a_2\mathbf{j}+a_3\mathbf{k})\cdot\left(\begin{vmatrix} b_2 & b_3\\ c_2 & c_3\end{vmatrix}\mathbf{i} - \begin{vmatrix} b_1 & b_3\\ c_1 & c_3\end{vmatrix}\mathbf{j} + \begin{vmatrix} b_1 & b_2\\ c_1 & c_2\end{vmatrix}\mathbf{k}\right)\\
&= \mathbf{u}\cdot(\mathbf{v}\times\mathbf{w}).
\end{aligned}
$$

□

Geometry of Determinant

2×2 matrix: If $\mathbf{a} = (a_1, a_2)$ and $\mathbf{b} = (b_1, b_2)$ then we can view them as vectors in $\mathbb{R}^3$ and define

$$
\mathbf{a}\times\mathbf{b} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k}\\ a_1 & a_2 & 0\\ b_1 & b_2 & 0\end{vmatrix} = \begin{vmatrix} a_1 & a_2\\ b_1 & b_2\end{vmatrix}\mathbf{k} = (a_1b_2 - a_2b_1)\mathbf{k}.
$$

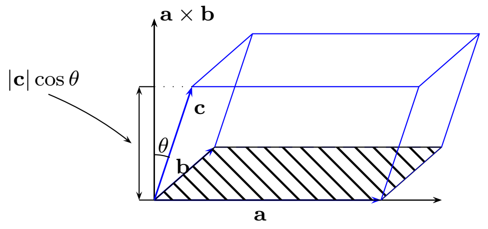


Figure 12.15: Meaning of triple product: $(\mathbf{a} \times \mathbf{b}) \cdot \mathbf{c}$

Hence $\|\mathbf{a} \times \mathbf{b}\|$ is the area of the parallelogram formed by the two vectors.

Example 12.4.9. Find the area of triangle with vertices at $(1,1), (0,2)$ and $(3,2)$.

sol. Two sides are $(0,2)-(1,1) = (-1,1)$ and $(3,2)-(1,1) = (2,1)$. Thus the area is the absolute value of $\frac{1}{2}\begin{vmatrix} -1 & 1 \\ 2 & 1 \end{vmatrix} = -\frac{3}{2}$.

□

Proposition 12.4.10. *The volume of parallelepiped with sides* $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$ *is give by the absolute value of triple product* $(\mathbf{a} \times \mathbf{b}) \cdot \mathbf{c}$ *which is the determinant*

$$|(\mathbf{a} \times \mathbf{b}) \cdot \mathbf{c}| = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}.$$

Proof. Consider a parallelogram with two sides $\mathbf{a}$, $\mathbf{b}$ as bottom of the parallelepiped. Then the height is length of the orthogonal projection of $\mathbf{c}$ onto $\mathbf{a} \times \mathbf{b}$ which is $\left\| \frac{(\mathbf{a}\times\mathbf{b})\cdot\mathbf{c}}{\|\mathbf{a}\times\mathbf{b}\|^2} \mathbf{a} \times \mathbf{b} \right\|$. Hence the volume is

$$Area(bottom) \times height = \|\mathbf{a} \times \mathbf{b}\| \left\| \frac{(\mathbf{a} \times \mathbf{b}) \cdot \mathbf{c}}{\|\mathbf{a} \times \mathbf{b}\|^2} \mathbf{a} \times \mathbf{b} \right\| = |(\mathbf{a} \times \mathbf{b}) \cdot \mathbf{c}|.$$

□

Example 12.4.11. Three points $A(1,2,3)$, $B(0,1,2)$, $C(0,3,2)$ are given. Find the volume of hexahedron having three vectors OA, OB, OC as sides.

sol. By proposition 12.4.10, we have

$$\begin{vmatrix} 1 & 2 & 3 \\ 0 & 1 & 2 \\ 0 & 3 & 2 \end{vmatrix} = 1 \begin{vmatrix} 1 & 2 \\ 3 & 2 \end{vmatrix} - 0 \begin{vmatrix} 2 & 3 \\ 3 & 2 \end{vmatrix} + 0 \begin{vmatrix} 2 & 3 \\ 1 & 2 \end{vmatrix} = -4.$$

□

Torque

Imagine we are trying to fasten a bolt with a wrench. If one apply the force $\mathbf{F}$ as the figure, we see the amount force acting to the action of bolt is $\|\mathbf{r}\|\|\mathbf{F}\|\sin\theta$.

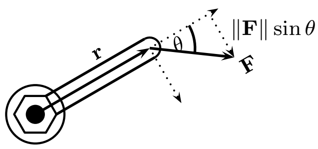


Figure 12.16: Turning a hexagonal bolt with a wrench with force $\mathbf{F}$. Torque vector is $\mathbf{r} \times \mathbf{F}$.

Then

$$\begin{aligned} \text{Amount of Torque } &= (\text{length of wrench}) \cdot (\text{component of } \mathbf{F} \perp \text{wrench}) \\ &= \|\mathbf{r}\|\|\mathbf{F}\|\sin\theta = \|\mathbf{r} \times \mathbf{F}\|. \end{aligned}$$

Also, the direction of the vector $\mathbf{r} \times \mathbf{F}$ is the same direction as the bolt moves. Hence it is natural to define $\mathbf{r} \times \mathbf{F}$ to be the torque vector.

12.5 Lines and planes

Parametric equation of lines(Point-direction form)

The equation of the line ℓ through the point P_0 and pointing in the direction of $\vec{P_0P} = \mathbf{v}$ is given by

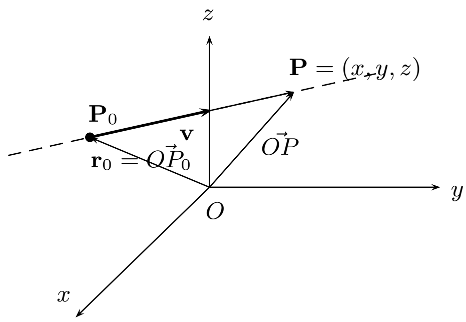


Figure 12.17: A line is determined by a point and a vector

$$\mathbf{r}(t) = \overrightarrow{OP_0} + t\overrightarrow{P_0P} = \mathbf{r}_0 + t\mathbf{v},\ t \in \mathbb{R}$$

where $\mathbf{r}_0 = (x_0, y_0, z_0)$ and $\mathbf{v} = (v_1, v_2, v_3)$. In coordinate form, we have

$$\begin{aligned} x &= x_0 + v_1 t, \\ y &= y_0 + v_2 t, \\ z &= z_0 + v_3 t, \end{aligned}$$

Example 12.5.1. (1) Find equation of line through $(2, 1, 5)$ in the direction of $4\mathbf{i} - 2\mathbf{j} + 5\mathbf{k}$.

(2) In what direction, the the line $x = 3t - 2, y = t - 1, z = 7t + 4$ points ?

sol. (1) $\mathbf{v} = (2, 1, 5) + t(4, -2, 5)$
(2) $(3, 1, 7) = 3\mathbf{i} + \mathbf{j} + 7\mathbf{k}$.

□

Two point form

We describe the equation of line through two points $P = (x_1, y_1, z_1), Q = (x_2, y_2, z_2)$. If we let $\mathbf{a} = (x_1, y_1, z_1), \mathbf{b} = (x_2, y_2, z_2)$.

The direction is given by $\mathbf{v} = \mathbf{b} - \mathbf{a} = (x_2 - x_1, y_2 - y_1, z_2 - z_1)$. So by the point-direction form we see the equation is

$$\mathbf{r}(t) = \mathbf{a} + t(\mathbf{b} - \mathbf{a}).$$

In components, we see

$$\begin{aligned} x &= x_1 + (x_2 - x_1)t \\ y &= y_1 + (y_2 - y_1)t \\ z &= z_1 + (z_2 - z_1)t \end{aligned}$$

Solving these for t and equating, we see

$$\boxed{\frac{x - x_1}{x_2 - x_1} = \frac{y - y_1}{y_2 - y_1} = \frac{z - z_1}{z_2 - z_1}.} \tag{12.3}$$

This is another equation of the line (called a **symmetric form**).

Example 12.5.2. Find the equation of a line through $(2, 1, -3)$ and $(6, -1, -5)$.

Example 12.5.3. Find the equation of the line segment between $(1, 1, -3)$ and $(2, -1, 0)$

sol. $0 \leq t \leq 1$

□

Example 12.5.4. Find where the the line given by the equations

$$\begin{cases} x &= t + 5 \\ y &= -2t - 4 \\ z &= 3t + 7 \end{cases}$$

intersect the plane $3x + 2y - 7z = 2$.

sol. We must find the value of t which gives the intersection point. Substituting the expression x, y, z into the equation of the plane, we see

$$3(t + 5) + 2(-2t - 4) - 7(3t + 7) = 2$$

Solving we get $t = -2$.

□

Example 12.5.5. Does the two lines $(x, y, z) = (t, -6t+1, 2t-8)$ and $(3t+1, 2t, 0)$ intersect ?

sol. If two line intersect, we must have

$$(t_1, -6t_1, 2t_1 - 8) = (3t_2 + 1, 2t_2, 0)$$

for some numbers t_1, t_2.(Note: we have used two different parameters t_1 and t_2). But since the system of equation

$$\begin{aligned} t_1 &= 3t_2 + 1 \\ -6t_1 &= 2t_2 \\ 2t_1 - 8 &= 0 \end{aligned}$$

has no solution, the lines do not meet.

□

Distance between a point and a line

Example 12.5.6. Find the distance from the point $P_0(2, 1, 3)$ to the line $\ell(t) = t(-1, 1, -2) + (2, 3, -2)$.

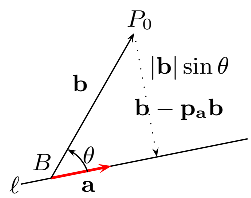


Figure 12.18: Distance from a point to a line

sol. Choose any point B on the line and find an orthogonal decomposition of $\overrightarrow{BP_0}$ onto the direction vector $\mathbf{a} = (-1, 1, -2)$ of the line. Then the length of the orthogonal complement is the distance. Choose $B = (2, 3, -2)$. Then

$$\begin{aligned} \overrightarrow{BP_0} := \mathbf{b} &= (2, 1, 3) - (2, 3, -2) \\ &= (0, -2, 5). \end{aligned}$$

Hence the orthogonal projection onto $\mathbf{a}$ is

$$\begin{aligned} \mathbf{p_a b} &= \frac{\mathbf{a}\cdot\mathbf{b}}{\|\mathbf{a}\|^2}\mathbf{a} \\ &= (2,-2,4). \end{aligned}$$

Thus the distance is

$$\|\mathbf{b}-\mathbf{p_a b}\| = \|(0,-2,5)-(2,-2,4)\| = \sqrt{5}.$$

Method 2: it is nothing but(by definition of cross product)
$\|\mathbf{b}\|\sin\theta = \frac{\|\mathbf{b}\times\mathbf{a}\|}{\|\mathbf{a}\|}$

□

12.5.1 Equation of a plane in space

Let $\mathcal{P}$ be a plane and $P_0 = (x_0, y_0, z_0)$ a point on that plane, and suppose that $\mathbf{n} = A\mathbf{i} + B\mathbf{j} + C\mathbf{k}$ is a normal vector. Let $P = (x, y, z)$ be any point in $\mathbb{R}^3$. Then P lies in the plane iff the vector $\overrightarrow{P_0P} = (x - x_0, y - y_0, z - z_0)$ is perpendicular to $\mathbf{n}$, that is, $\overrightarrow{P_0P} \cdot \mathbf{n} = 0$. In other words,

$$(A\mathbf{i} + B\mathbf{j} + C\mathbf{k}) \cdot [(x - x_0)\mathbf{i} + (y - y_0)\mathbf{j} + (z - z_0)\mathbf{k}] = 0.$$

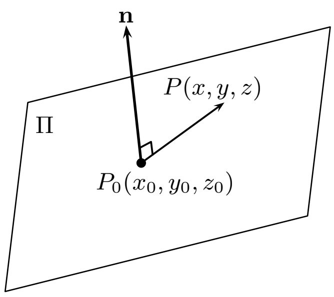


Figure 12.19: A plane is det'd by a point and normal vector

Proposition 12.5.7. *Equation of plane through (x_0, y_0, z_0) that has normal* *vector* $\mathbf{n}$ *is given by three forms:*

$$\begin{aligned} \textit{vector eq.} &= \overrightarrow{P_0P}\cdot\mathbf{n}=0 \\ \textit{component eq.} &= (A\mathbf{i}+B\mathbf{j}+C\mathbf{k})\cdot[(x-x_0)\mathbf{i}+(y-y_0)\mathbf{j}+(z-z_0)\mathbf{k}]=0 \\ \textit{component eq. 2} &= Ax+By+Cz=D(=Ax_0+By_0+Cz_0) \end{aligned}$$

Example 12.5.8. Find the equation of plane through the points $A(-3,0,-1)$, $B(-2,3,2)$, $C(1,1,3)$.

sol. Draw some graph describing the normal vector.

Find a vector $\mathbf{n}$ orthogonal to plane.

$$\begin{aligned} \mathbf{n} &= \overrightarrow{AB}\times\overrightarrow{AC} \\ &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -2-(-3) & 3-0 & 2-(-1) \\ 1-(-3) & 1-0 & 3-(-1) \end{vmatrix} \\ &= \begin{vmatrix} 3 & 3 \\ 1 & 4 \end{vmatrix}\mathbf{i} - \begin{vmatrix} 1 & 3 \\ 4 & 4 \end{vmatrix}\mathbf{j} + \begin{vmatrix} 1 & 3 \\ 4 & 1 \end{vmatrix}\mathbf{k} \\ &= 9\mathbf{i}+8\mathbf{j}-11\mathbf{k}. \end{aligned}$$

By proposition 12.5.7, the equation is

$$9(x+3)+8(y-0)-11(z+1)=0$$

or $9x+8y-11z+16=0$.

□

Lines of intersection

Example 12.5.9. Find a vector parallel to the line of intersection of two planes $2x-y+z-4=0$ and $3x-5y+z-1=0$.

sol. it is determined by two normals to the planes. The normals are $\mathbf{n}_1 = (2,-1,1)$ and $\mathbf{n}_2=(3,-5,1)$. Thus the direction is

$$\mathbf{n}_1\times\mathbf{n}_2 = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & -1 & 1 \\ 3 & -5 & 1 \end{vmatrix} = 4\mathbf{i}+\mathbf{j}-7\mathbf{k}.$$

□

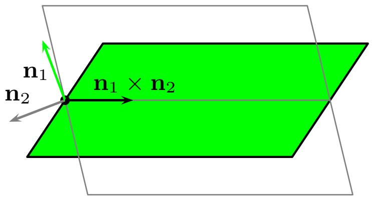


Figure 12.20: intersection of planes

Distance from a point to plane

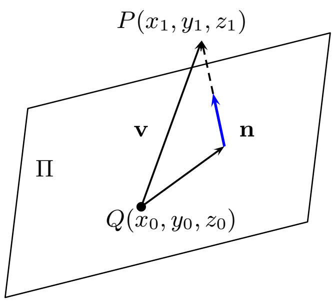


Figure 12.21: Distance from a point to plane

Proposition 12.5.10. *The distance from* $P(x_1, y_1, z_1)$ *to the plane* $Ax+By+Cz+D=0$ *is*

$$\frac{|Ax_1 + By_1 + Cz_1 + D|}{\sqrt{A^2+B^2+C^2}}.$$

Proof. Let $\mathbf{n}$ be a normal vector to the plane. If $Q(x_0, y_0, z_0)$ lies in the plane, the distance from P to the plane is the orthogonal projection of $\overrightarrow{PQ}$ along $\mathbf{n}$. Note that from $A(x-x_0)+B(y-y_0)+C(z-z_0)=0$, we see $\mathbf{n}//A\mathbf{i}+B\mathbf{j}+C\mathbf{k}$. Hence length of the orthogonal projection of $\overrightarrow{PQ}$ along $\mathbf{n}$ is

$$\begin{aligned}\left\|\frac{\mathbf{n}\cdot\overrightarrow{PQ}}{\|\mathbf{n}\|^2}\mathbf{n}\right\| &= \frac{|\mathbf{n}\cdot\overrightarrow{PQ}|}{\|\mathbf{n}\|} \\ &= \frac{|A(x_0-x_1)+B(y_0-y_1)+C(z_0-z_1)|}{\sqrt{A^2+B^2+C^2}}\end{aligned}$$

$$= \frac{|Ax_0 + By_0 + Cz_0 - Ax_1 - By_1 - Cz_1|}{\sqrt{A^2+B^2+C^2}}$$
$$= \frac{|-D - Ax_1 - By_1 - Cz_1|}{\sqrt{A^2+B^2+C^2}} = \frac{|Ax_1 + By_1 + Cz_1 + D|}{\sqrt{A^2+B^2+C^2}}.$$

□

Example 12.5.11. Find the distance from $(3, 4, -2)$ to the plane $2x - y + z - 4 = 0$.

sol. Using above proposition, distance is

$$\frac{|2 \cdot 3 - 1 \cdot 4 + 1 \cdot (-2) - 4|}{\sqrt{4+1+1}} = \frac{|-4|}{\sqrt{6}} = \frac{2\sqrt{6}}{3}.$$

□

Example 12.5.12. Find a unit vector perpendicular to the plane $4x - 3y + z - 4 = 0$ and express it as a cross product of two unit orthogonal vectors lying in the plane.

sol. Let $\mathcal{S}$ be the given plane. By proposition 12.5.7 we see $4\mathbf{i} - 3\mathbf{j} + \mathbf{k}$ is orthogonal to $\mathcal{S}$. Hence a unit normal vector is

$$\mathbf{n} = \pm \frac{4\mathbf{i} - 3\mathbf{j} + \mathbf{k}}{\sqrt{4^2 + (-3)^2 + 1^2}} = \pm \frac{1}{\sqrt{26}}(4\mathbf{i} - 3\mathbf{j} + \mathbf{k}).$$

Now in order to express this as a cross product of two vectors lying in the plane, we choose three arbitrary points in $\mathcal{S}$. For example, we choose $(1, 0, 0), (0, 0, 4), (2, 1, -1)$. Then we obtain two vectors

$$\mathbf{u} = (1, 0, 0) - (2, 1, -1) = -\mathbf{i} - \mathbf{j} + \mathbf{k}$$
$$\mathbf{v} = (0, 0, 4) - (2, 1, -1) = -2\mathbf{i} - \mathbf{j} + 5\mathbf{k}$$

which lie in the plane $\mathcal{S}$. Now we orthogonalize them.

Let $\mathbf{a}$ be the orthogonal projection of $\mathbf{v}$ onto $\mathbf{u}$. Then let $\mathbf{b} = \mathbf{v} - \mathbf{a}$.

$$\mathbf{a} = \frac{\mathbf{u} \cdot \mathbf{v}}{\|\mathbf{u}\|^2}\mathbf{u} = \frac{8}{3}(-\mathbf{i} - \mathbf{j} + \mathbf{k})$$
$$\mathbf{b} = \mathbf{v} - \frac{\mathbf{u} \cdot \mathbf{v}}{\|\mathbf{u}\|^2}\mathbf{u} = (-2\mathbf{i} - \mathbf{j} + 5\mathbf{k}) - \frac{8}{3}(-\mathbf{i} - \mathbf{j} + \mathbf{k})$$
$$= \frac{1}{3}(2\mathbf{i} + 5\mathbf{j} + 7\mathbf{k}).$$

Now normalize them.

$$\mathbf{a}_1 = \frac{\mathbf{a}}{\|\mathbf{a}\|} = \frac{1}{\sqrt{3}}(-\mathbf{i}-\mathbf{j}+\mathbf{k}), \quad \mathbf{b}_1 = \frac{\mathbf{b}}{\|\mathbf{b}\|} = \frac{1}{\sqrt{78}}(2\mathbf{i}+5\mathbf{j}+7\mathbf{k}).$$

We can check that

$$\begin{aligned}
\mathbf{a}_1 \cdot \mathbf{b}_1 &= \frac{(-1)\cdot 2 + (-1)\cdot 5 + 1 \cdot 7}{\sqrt{3}\cdot\sqrt{78}} = 0(\text{orthogonal}) \\
\mathbf{a}_1 \times \mathbf{b}_1 &= \frac{1}{3\sqrt{26}} \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -1 & -1 & 1 \\ 2 & 5 & 7 \end{vmatrix} \\
&= \frac{1}{3\sqrt{26}} \left(\begin{vmatrix} -1 & 1 \\ 5 & 7 \end{vmatrix} \mathbf{i} - \begin{vmatrix} -1 & 1 \\ 2 & 7 \end{vmatrix} \mathbf{j} + \begin{vmatrix} -1 & -1 \\ 2 & 5 \end{vmatrix} \mathbf{k} \right) \\
&= -\frac{1}{\sqrt{26}}(4\mathbf{i} - 3\mathbf{j} + \mathbf{k}).
\end{aligned}$$

□

Angles between two planes

The angle between two planes whose normal vectors are $\mathbf{n}_1$ and $\mathbf{n}_2$ (See figure 12.5.1) is given by

$$\theta = \cos^{-1}\left(\frac{\mathbf{n}_1 \cdot \mathbf{n}_2}{|\mathbf{n}_1|\,|\mathbf{n}_2|}\right).$$

*Distance between two skewed lines

Two lines are said to be **skewed** if they are neither intersecting nor parallel. It follows that they must *lie in two parallel planes* and the distance between the lines is equal to the distance between the planes. Let us describe how to find the distance between them.

Assume we have two parallel planes Π_1 and Π_2(resp.) containing the line ℓ_1 and ℓ_2(resp.). They share a common normal vector $\mathbf{n}$. Assume $\mathbf{a}_1$ and $\mathbf{a}_2$ are two direction vectors of each line. Then the normal vector $\mathbf{n}$ is obtained by taking cross product of $\mathbf{a}_1$ and $\mathbf{a}_2$. Let $P_1 \in \ell_1$, $P_2 \in \ell_2$ be any two points on each line. Then we compute the projection of $\overrightarrow{P_1P_2}$ onto $\mathbf{n}$. Moving the projection along the line ℓ_1 so that the head ends at P_2, we see its length is the desired distance.

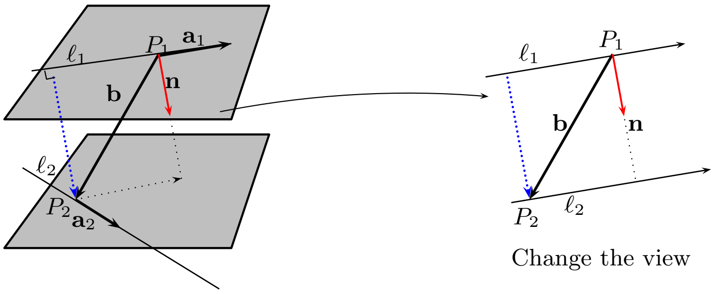


Figure 12.22: Distance between two lines is the length of $proj_{\mathbf{n}}\mathbf{b}$

Example 12.5.13. Find the distance between the two lines

$$\ell_1(t) = (0, 5, -1) + t(2, 1, 3), \text{ and } \ell_2(t) = (-1, 2, 0) + t(1, -1, 0).$$

sol. We have $\mathbf{a}_1 = (2, 1, 3)$ and $\mathbf{a}_2 = (1, -1, 0)$. Choose $P_1 = (2, 6, 2)$ and $P_2 = (0, 1, 0)$. Then $\mathbf{b} = (2, 6, 2) - (0, 1, 0) = (2, 5, 2)$. While

$$\mathbf{n} = \mathbf{a}_1 \times \mathbf{a}_2 = (2, 1, 3) \times (1, -1, 0) = (3, 3, -3).$$

Normalizing, we let $\mathbf{n} = (1, 1, -1)/\sqrt{3}$. Now the projection of $\mathbf{b}$ onto $\mathbf{n}$ is

$$proj_{\mathbf{n}}\mathbf{b} = (\mathbf{b} \cdot \mathbf{n})\mathbf{n} = \frac{(2 + 5 - 2)}{\sqrt{3}} \frac{(1, 1, -1)}{\sqrt{3}} = \frac{5}{3}(1, 1, -1).$$

Hence the distance is

$$\left\| \frac{5}{3}(1, 1, -1) \right\| = \frac{5}{\sqrt{3}}.$$

□

12.6 Quadric Surfaces

Visualizing functions

Definition 12.6.1. The **graph** of a functions of several variables $f : A \subset \mathbb{R}^n \to \mathbb{R}$ is (**graph**) the following set

$$graph(f) = \{(\mathbf{x}, f(\mathbf{x})) \in \mathbb{R}^{n+1} \mid \mathbf{x} \in A \subset \mathbb{R}^n\}.$$

Componentwise,

$$graph(f) = \{(x_1, \cdots, x_n, f(x_1, \cdots, x_n)) \in \mathbb{R}^{n+1} \mid \mathbf{x} \in A \subset \mathbb{R}^n\}.$$

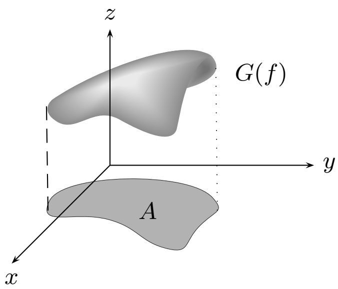


Figure 12.23: Graph of two variable function

Level sets, curves, surfaces

Definition 12.6.2. The **level set** of $f\colon \mathbb{R}^n \to \mathbb{R}$ is the set of all $\mathbf{x}$ where the function f has constant value:

$$S_c = \{\mathbf{x} \in \mathbb{R}^n \mid f(\mathbf{x}) = c, c \in \mathbb{R}\}.$$

If $n = 2$, it is **level curve** and if $n = 3$, **level surface**.

Definition 12.6.3. Let $f\colon \mathbb{R}^2 \to \mathbb{R}$. The **section of the graph of** f **by the plane** $x = c$ is the set of all points (x, y, z) where $z = f(c, y)$. In symbol(notation),

$$\text{section by } x = c \text{ is } \{(x, y, x) \in \mathbb{R}^3 \mid z = f(c, y)\}.$$

Similarly, y-**section** of the graph of f is the set

$$\{(x, y, z) \in \mathbb{R}^3 | z = f(x, c), y = c\}.$$

Example 12.6.4. The graph of $f(x, y) = x^2 + y^2$ is called **paraboloid** or **paraboloid of revolution**. See figure 12.5 for the graph. Study the level sets.

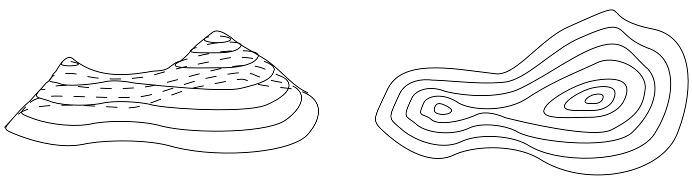


Figure 12.24: Contour curves and Level curves(set)

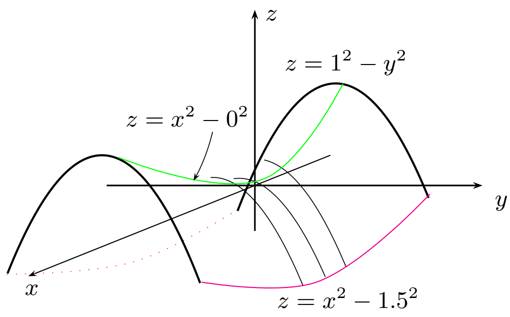


Figure 12.25: hyperbolic paraboloid $z = x^2 - y^2$

sol. The level set of $x^2 + y^2 = c$ is 0 if $c = 0$. For $c > 0$ it is a circle of radius $\sqrt{c}$. If $c < 0$, the level set is empty.

□

Example 12.6.5. Draw level sets of $f(x, y) = x^2 - y^2$. The graph is called **hyperbolic paraboloid** or **saddle**.

sol. **Detail view of the level set.** Consider the set $\{(x, y)|x^2 - y^2 = c\}$. If $c = 0$, then it is $y = \pm x$, two lines through origin. If $c > 0$, the level set is a hyperbola meeting with x-axis, and if $c < 0$ level set is a hyperbola meeting with y-axis. The intersection with xz-plane is the parabola $z = x^2$, and the intersection with yz-plane is the parabola $z = -y^2$. The graph of f is given in Figure 12.25.

□

Level surface of function of three variables

Example 12.6.6. Study the level surface of $f(x, y, z) = x^2 + y^2 + z^2$.

sol. The set $x^2 + y^2 + z^2 = c$ becomes

$$\begin{cases} \text{origin} & \text{if } c = 0 \\ \text{circle of radius } \sqrt{c} & \text{if } c > 0 \\ \text{empty if} & \text{if } c < 0. \end{cases}$$

To imagine the graph in $\mathbb{R}^4$, consider intersection with $\mathbb{R}^3_{z=0} = \{(x, y, z, w) \mid z = 0\}$. It is

$$\{(x, y, z, w) \mid w = x^2 + y^2, z = 0\}$$

□

Example 12.6.7. Describe the graph of $f(x, y, z) = x^2 + y^2 - z^2$.

sol. The graph of $f = x^2 + y^2 - z^2$ is a subset of 4-dimensional space. If we denote the points in this space by (x, y, z, w), then the graph is given by

$$\{(x, y, z, w)|w = x^2 + y^2 - z^2\}.$$

The level surface is

$$L_c = \{(x, y, z)|x^2 + y^2 - z^2 = c\}.$$

We have three cases:

(1) For $c = 0$, we have $z = \pm\sqrt{x^2 + y^2}$. This is a cone.

(2) If $c = -a^2$, we obtain $z = \pm\sqrt{x^2 + y^2 + a^2}$. This is a **hyperboloid of two sheets**.

(3) If $c = a^2 > 0$, we obtain $z = \pm\sqrt{x^2 + y^2 - a^2}$. This is **hyperboloid of single sheet**.

On the other hand, if we consider intersection with $y = 0$; $S_{y=0} = \{(x, y, z, w) \mid y = 0\}$, the intersection with the graph of f is

$$S_{y=0} \cap \text{ graph of } f = \{(x, y, z, w) \mid y = 0, w = x^2 - z^2\}.$$

By changing the role of y and z we have

$$\{(x, y, z, w) \mid w = x^2 - y^2, z = 0\}.$$

This set is considered to belong to (x, y, w)-space and is a hyperbolic paraboloid(saddle).

□

Example 12.6.8 (Hypersurface)**.** In general the graph of $w = F(x, y, z)$ is the set

$$\{(x, y, z, w)\}$$

which is a subset of $\mathbb{R}^3$ (called hypersurface) which one cannot draw. But we guess the shape by looking at the level surfaces of F.

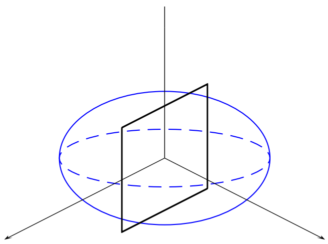
In $\mathbb{R}^3$ the analog of conic section is the surfaces defined by

$$Ax^2 + By^2 + Cz^2 + Dxy + Eyz + Fxz + \text{linear terms} = 0.$$

These are classified into the following typical surfaces.

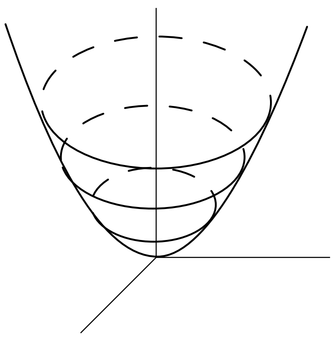
Ellipsoid

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1 \tag{12.4}$$



Elliptic paraboloid

$$z = \frac{x^2}{a^2} + \frac{y^2}{b^2} \tag{12.5}$$



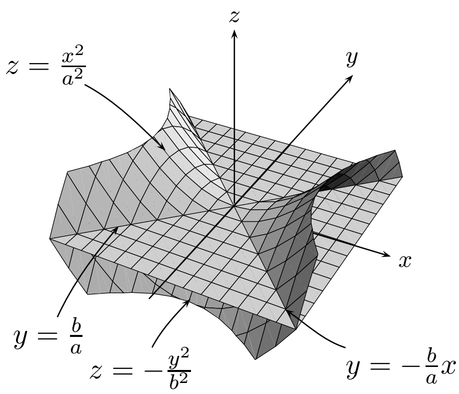


Figure 12.26: Saddle $z = \frac{x^2}{a^2} - \frac{y^2}{b^2} + k$

Hyperbolic paraboloid

$$z = \frac{x^2}{a^2} - \frac{y^2}{b^2}$$

The intersection with the plane $z = c$ is hyperbola except when $c = 0$, in which case the intersection is the two lines $y = \pm(b/a)x$. See figure 12.26.

Hyperboloid

Consider the surface **hyperboloid**:

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = k.$$

If $k > 0$, the intersection with $z = z_0$ is always an ellipse. This is called a **hyperboloid of one sheet**. Figure 12.28. On the other hand, The intersection of the set given by

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = -k(k > 0)$$

with $z = z_0$ is nonempty only when $|z_0| \geq 1$. For this reason it is called **hyperboloid of two sheets**.

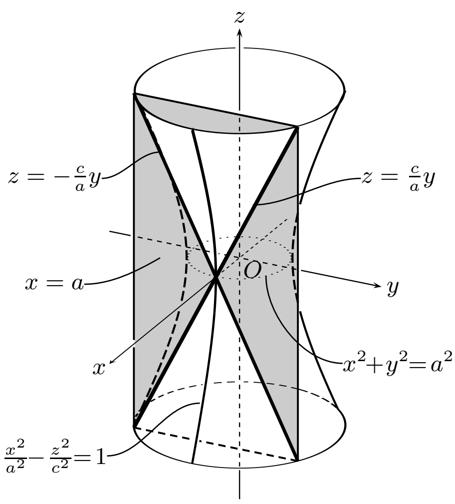


Figure 12.27: hyperboloid

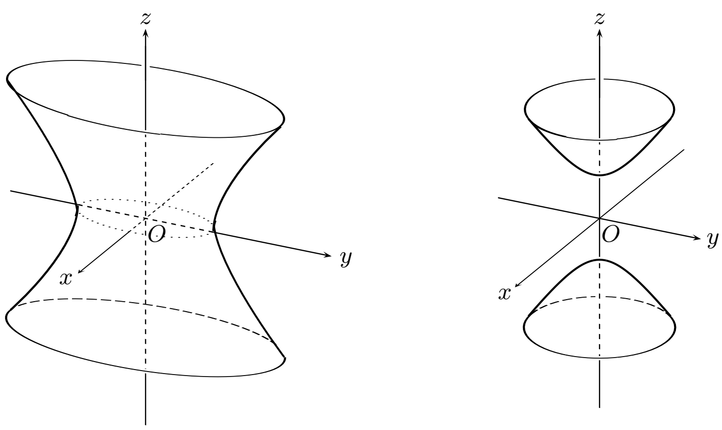


Figure 12.28: hyperboloid of one (two) sheet

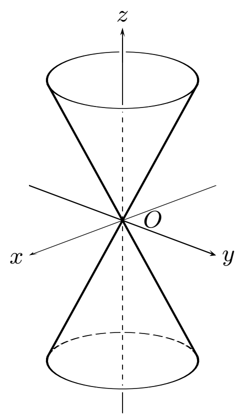


Figure 12.29: Elliptic cone

Elliptic cone

As a special case when $k = 0$ we obtain

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = \frac{z^2}{c^2}$$

called a **cone**

Example 12.6.9. Express the common part

$$\begin{cases} \sqrt{x^2+y^2} \le z \le \sqrt{1-x^2-y^2} \\ x \ge 0 \\ y \ge 0 \end{cases}$$

using spherical coordinate.

sol. Inside of $x^2 + y^2 = z^2$ and $x^2 + y^2 + z^2 = 1$.

$$\{(\rho, \theta, \phi) \mid 0 \le \rho \le 1, 0 \le \theta \le \frac{\pi}{2}, 0 \le \phi \le \frac{\pi}{4}\}$$

□