Chapter 11

Parametric Equations and Plane Curves

11.1 Plane curves

Parameterized curve

Definition 11.1.1. Let $I = [a, b]$ be an interval. If $\gamma(t) = (f(t), g(t)) \colon I \to \mathbb{R}^2$ is a function defined on I, then the set of points $(x, y) = (f(t), g(t))$ is called a **parametric curve**. The relations $x = f(t),\ y = g(t)$ are called **parametric equations**.

The image of the curve is denoted by $C = \gamma(I)$. (Sometimes the function $\gamma(t)$ itself is called a parametric curve.) The variable t is called a **parameter**. When a parametric curve γ is given, the point $\gamma(a)$ is the **initial point** of γ, and $\gamma(b)$ is the **terminal point**(or **end point**) of γ.

Example 11.1.2. Find a parametric equation of the unit circle $x^2 + y^2 = 1$.

sol. We can represent it as

$$\gamma_1(t) = (x(t), y(t)), \quad x(t) = \cos(2\pi t),\ y(t) = \sin(2\pi t),\ t \in [0, 1].$$

Another parametrization is possible:

$$\gamma_2(t) = \left(\cos(-4\pi t + \frac{\pi}{2}), \sin(-4\pi t + \frac{\pi}{2})\right),\ t \in [0, \frac{1}{2}].$$

□

Drawing the graph

Example 11.1.3. Draw the graph of $\gamma(t) = (2t^2 - 1, \sin \pi t)$.

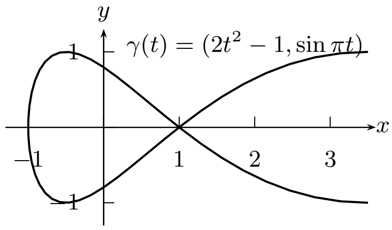


Figure 11.1: $\gamma(t) = (2t^2 - 1, \sin \pi t)$

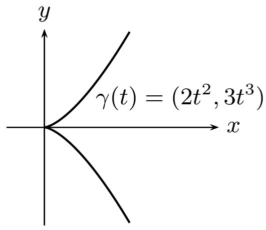


Figure 11.2: $\gamma(t) = (2t^2, 3t^3)$

Example 11.1.4. Find a parameterized representation of $y^2 = x^2 + x^3$.

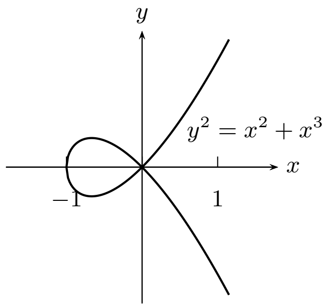


Figure 11.3: $y^2 = x^2 + x^3$

sol. Let $y = tx$. Then from $y^2 = x^2 + x^3$ obtain (See the graph in Figure 11.3)

$$x^2(t^2 - 1 - x) = 0.$$

Set $x = t^2 - 1$ so that $y = t(t^2 - 1)$. Hence $\gamma(t) = (t^2 - 1, t(t^2 - 1))$ is a parametrization. Other parametrizations are possible.

$\square$

Cycloid

Assume circle of radius a is rolling on the x-axis. Let P be a point on the circle starting to move from the origin. (Fig 11.4) If circle rolls by t radian,

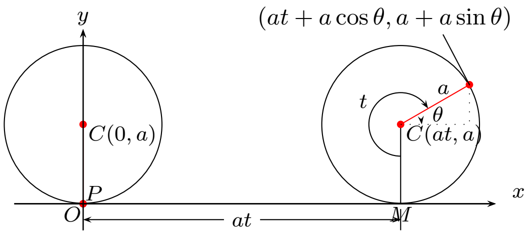


Figure 11.4: Cycloid

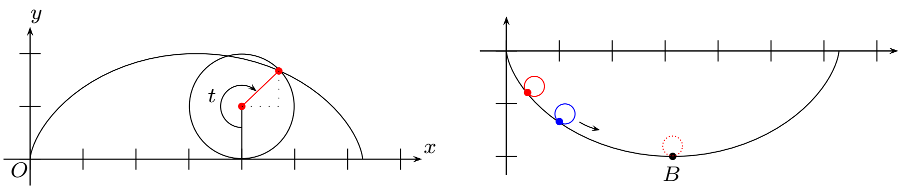


Figure 11.5: cycloid - brachistochrone

then the center point C moves to the right by at. If θ is the angle between the x-axis and the axis CP, then the position of $P(x, y)$ is

$$x = at + a\cos\theta, \qquad y = a + a\sin\theta. \tag{11.1}$$

Since $\theta = (3\pi)/2 - t$ we have

$$x = a(t - \sin t), \qquad y = a(1 - \cos t).$$

This is a **cycloid**.

Brachistochrone and Tautochrone

If we upside down the figure of cycloid (Fig. 11.5), the curve has two physical properties. Imagine we slide a bead along a frictionless wall from a point O and reach the point B.

(1) Shortest time path - Among all curves joining O and B, the cycloid

is the one along which a frictionless bead, subject only to gravitational force,will slide down the fastest. In this sense the curve is called "Brachistochrone"

(2) Same time path - Even if you start the bead anywhere on the curve, the time to reach the bottom B is the same. In this sense the curve is called "Tautochrone"

Brachistochrone: Imagine the bead is sliding along some path.

Recall : The work (kinetic energy) done to the bead along the any from $(0,0)$ to (x,y) is

$$mgy = \frac{1}{2}mv^2 - \frac{1}{2}m0^2 = \frac{1}{2}mv^2.$$

Thus the speed when the bead reaches $B = (x,y)$ is

$$v(= \frac{ds}{dt}) = \sqrt{2gy}$$

or

$$dt = \frac{ds}{\sqrt{2gy}} = \frac{\sqrt{1+(dy/dx)^2}}{\sqrt{2gy}}dx.$$

Thus the total time T_f it takes the bead to reach along the path $y = f(x)$ from O to $B(a\pi, 2a)$ is

$$T = \int_{x=0}^{x=\pi} \frac{\sqrt{1+(dy/dx)^2}}{\sqrt{2gy}}dx \tag{11.2}$$

One can use the thecnique of calculus of variation to show the solution to this DE is the cycloid. This is out of the scope of thei book.

Tautochrone: We see

$$\begin{aligned} T &= \int_{x=0}^{x=\pi} \frac{\sqrt{1+(dy/dx)^2}}{\sqrt{2gy}}dx \\ &= \int_{t=0}^{t=\pi} \frac{\sqrt{(dy/dt)^2+(dx/dt)^2}}{\sqrt{2gy}}dt \\ &= \int_{t=0}^{t=\pi} \sqrt{\frac{a}{g}}dt = \pi\sqrt{\frac{a}{g}} \end{aligned}$$

if the path is cycloid. Change the initial point and compute the arrival time. you will see it will be independent of initial point.

11.2 Calculus with Parametric Curves

Slopes of parametrized curves

If $f(t)$ and $g(t)$ are differentiable and $f'(t) \neq 0$. Then $t = f^{-1}(x)$ exists and $y(x) = (g \circ f^{-1})(x)$ is well defined. By implicit function theorem, the derivative of y w.r.t x is given by

$$\boxed{\frac{dy}{dx} = \frac{dy}{dt} \cdot \frac{dt}{dx} = \frac{dy/dt}{dx/dt}}.$$

Example 11.2.1. Find the tangent to the curve $x = t - t^2$, $y = t - t^3$ at the point $(-2, -6)$.

sol. Solving $t - t^2 = -2$, $t - t^3 = -6$, we see $t = 2$. Chain rule and implicit differentiation

$$\begin{aligned} \frac{dy}{dx} &= \frac{dy/dt}{dx/dt} = \frac{1-3t^2}{1-2t} \\ \left.\frac{dy}{dx}\right|_{t=2} &= \frac{1-12}{1-4} = \frac{11}{3}. \end{aligned}$$

Thus the equation of tangent is

$$y = \frac{11}{3}(x+2) - 6.$$

□

Second derivative for a parametric equation

If the relations $x = f(t)$, $y = g(t)$ define y as a twice differentiable function of x at the point where $dx/dt \neq 0$, then

$$\boxed{\frac{d^2y}{dx^2} = \frac{d}{dt}\left(\frac{dy}{dx}\right) \Big/ \frac{dx}{dt} = \frac{dy'/dt}{dx/dt}}.$$

Example 11.2.2. Compute dy/dx and d^2y/dx^2 when $x = t - t^2$, $y = t - t^3$.

sol. Implicit differentiation gives

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{1-3t^2}{1-2t},$$
$$\frac{d^2y}{dx^2} = \frac{d}{dt}\left(\frac{dy}{dx}\right) \Big/ \frac{dx}{dt}$$
$$= \frac{d}{dt}\left(\frac{1-3t^2}{1-2t}\right) \Big/ (1-2t)$$
$$= \frac{2-6t+6t^2}{(1-2t)^3}.$$

□

Example 11.2.3. Find the area enclosed by the asteroid (Fig. 11.2)

$$x = \cos^3 t, \ y = \sin^3 t, \ 0 \le t \le 2\pi.$$

$$A = 4\int_0^1 y dx$$
$$= 4\int_0^{\pi/2} \sin^3 t \cdot 3\cos^2 t \sin t dt$$
$$= 12\int_0^{\pi/2} \left(\frac{1-\cos 2t}{2}\right)^2 \left(\frac{1+\cos 2t}{2}\right) dt$$
$$\cdots$$
$$= \frac{3\pi}{8}.$$

11.2.1 Arc Length of a Parametric Curve

Definition 11.2.4. If C is given by $(x,y) = (f(t), g(t))$, $a \le t \le b$, and f', g' are continuous and not simultaneously zero and C is one-to-one. Then the **length of** C is given by

$$L = \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} dt. \tag{11.3}$$

If $P = \{t_0, t_1, \ldots, t_n\}$ is a partition of $[a,b]$, then the curve is approximated

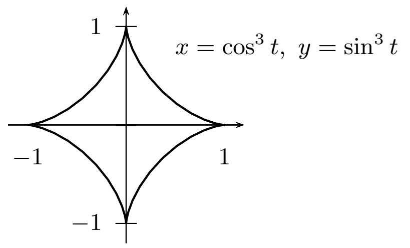


Figure 11.6: Astroid

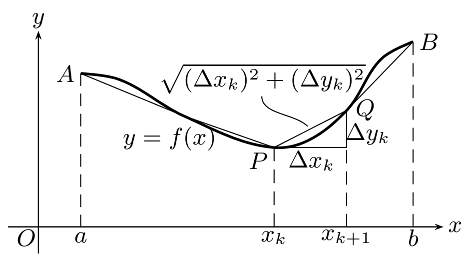


Figure 11.7: $PQ = \sqrt{(\Delta x_k)^2 + (\Delta y_k)^2}$, $\Delta y_k = f(x_{k+1}) - f(x_k)$

by the line segments joining the points $(x(t_i), y(t_i))$. Hence the length is

$$\sim \sum_{i=0}^{n-1} \sqrt{(x(t_{i+1}) - x(t_i))^2 + (y(t_{i+1}) - y(t_i))^2} \tag{11.4}$$

$$= \sum_{i=0}^{n-1} \sqrt{\left(\frac{x(t_{i+1}) - x(t_i)}{t_{i+1} - t_i}\right)^2 + \left(\frac{y(t_{i+1}) - y(t_i)}{t_{i+1} - t_i}\right)^2}\, \Delta t_{i+1}. \tag{11.5}$$

(Fig 11.7). Thus as $\|P\| \to 0$ we obtain

$$L = \int_a^b \sqrt{(x'(t))^2 + (y'(t))^2}\, dt.$$

Example 11.2.5. Find the arclength of the asteroid (Fig. 11.2)

$$x = \cos^3 t,\ y = \sin^3 t,\ 0 \le t \le 2\pi.$$

$$
\begin{aligned}
(\frac{dx}{dt})^2 &= (-3\cos^2 t \sin t)^2, \quad (\frac{dy}{dt})^2 = (3\sin^2 t \cos t)^2 \\
\sqrt{(\frac{dx}{dt})^2 + (\frac{dy}{dt})^2} &= \sqrt{9\cos^2 t \sin^2 t} \\
&= |3\cos t \sin t| \\
&= 3\cos t \sin t.
\end{aligned}
$$

So the length is

$$
\begin{aligned}
&= \int_0^{\pi/2} 3\cos t \sin t dt \\
&= \frac{3}{2}\int_0^{\pi/2} \sin 2t dt \\
&= \frac{3}{2}.
\end{aligned}
$$

Example 11.2.6. Find the perimeter of the ellipse $\frac{x^2}{a^2} + \frac{y^2}{a^2} = 1, (a > b)$. We can parameterize it by $x = a\cos t,\ y = b\sin t,\ 0 \le t \le 2\pi$.

$$
\begin{aligned}
(\frac{dx}{dt})^2 + (\frac{dy}{dt})^2 &= a^2\cos^2 t + b^2 \sin^2 t \\
&= a^2 - (a^2 - b^2)\sin^2 t \\
&= a^2(1 - e^2 \sin^2 t),\ e = 1 - \frac{b^2}{a^2}.
\end{aligned}
$$

So the perimeter is expressed as

$$
P = 4a\int_0^{\pi/2} \sqrt{(1 - e^2\sin^2 t)}dt.
$$

This is called elliptic integral of the second kind whose value can be found by say Taylor expansion:

$$
\begin{aligned}
P &= 4a\int_0^{\pi/2} \sqrt{(1 - e^2\sin^2 t)}dt \\
&= 4a\left[\frac{\pi}{2} - \left(\frac{1}{2}e^2\right)\left(\frac{1}{2}\frac{\pi}{2}\right) - \left(\frac{1}{2\cdot 4}e^4\right)\left(\frac{1\cdot 3}{2\cdot 4}\frac{\pi}{2}\right) - \left(\frac{1\cdot 3}{2\cdot 4\cdot 6}e^6\right)\left(\frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6}\right) - \cdots\right]
\end{aligned}
$$

Since $e < 1$, this series converges by comparison test (Compare with $\sum_{n=1}^{\infty}(e^2)^n$.)

Length of a curve $y = f(x)$

When $x = t$ in the parameterization, we obtain

$$L = \int_a^b \sqrt{(x'(t))^2 + (y'(t))^2}\, dt = \int_a^b \sqrt{1 + (f'(x))^2}\, dx$$

The arc length differential

We define the arc length function for the parameterized curve $x = f(t),\ y = g(t),\ a \le t \le b$ by

$$s(t) = \int_a^t \sqrt{(f'(t))^2 + (g'(t))^2}\, dt.$$

Then

$$\frac{ds}{dt} = \sqrt{(f'(t))^2 + (g'(t))^2} = \sqrt{(\frac{dx}{dt})^2 + (\frac{dy}{dt})^2}.$$

The differential of arc length is

$$ds = \sqrt{(f'(t))^2 + (g'(t))^2} = \sqrt{(\frac{dx}{dt})^2 + (\frac{dy}{dt})^2} dt = \sqrt{dx^2 + dy^2}. \tag{11.6}$$

Example 11.2.7. Find the centroid of the first quadrant of the asteroid (Fig. 11.2.9)

$$x = \cos^3 t,\ y = \sin^3 t,\ 0 \le t \le 2\pi.$$

We set the density $\delta = 1$. Then the typical segment of the curve has mass

$$dm = 1 \cdot ds = \sqrt{(\frac{dx}{dt})^2 + (\frac{dy}{dt})^2} dt = 3 \cos t \sin t\, dt,$$

Thus the curve's mass is

$$M = \int_0^{\pi/2} dm = \int_0^{\pi/2} 3 \cos t \sin t dt = \frac{3}{2}.$$

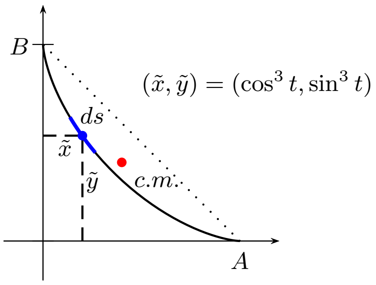


Figure 11.8: c.m.=Center of mass

Thus the curve's moment about the x-axis is

$$\begin{aligned} M_x &= \int_0^{\pi/2} \tilde{y} dm = \int_0^{\pi/2} \sin^3 t \cdot 3\cos t \sin t dt \\ &= \int_0^{\pi/2} \sin^4 t \cos t dt = \frac{3}{5}. \end{aligned}$$

Hence

$$\bar{y} = \frac{M_x}{M} = \frac{3/5}{3/2} = \frac{2}{5}.$$

By the symmetry of the point, c.m. is $(2/5, 2/5)$.

Example 11.2.8. Find the time T_c it takes for a frictionless bead to slide along the cycloid $x = a(t - \sin t),\ y = a(1 - \cos t)$ from $t = 0$ to $t = \pi$. From early section, we have seen the time is

$$T_c \quad = \quad \int_{x=0}^{x=\pi} \frac{\sqrt{1 + (dy/dx)^2}}{\sqrt{2gy}} dx = \int_{x=0}^{x=\pi} \frac{ds}{\sqrt{2gy}} \tag{11.7}$$

Use the arc length formula

$$\begin{aligned} ds &= \sqrt{(\frac{dx}{dt})^2 + (\frac{dy}{dt})^2} dt \\ &= \sqrt{a^2(1 - 2\cos t + \cos^2 t + \sin^2 t)} dt \\ &= \sqrt{a^2(2 - 2\cos t)} dt \end{aligned}$$

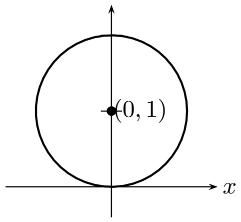


Figure 11.9: Unit circle at $(0,1)$ is rotated about x-axis

to see

$$\begin{aligned} T_c &= \int_{x=0}^{\pi} \sqrt{\frac{a^2(2-2\cos t)}{2ga(1-\cos t)}}dt \\ &= \int_0^{\pi} \sqrt{\frac{a}{g}}dt = \pi\sqrt{\frac{a}{g}}. \end{aligned}$$

11.2.2 Area of Surface of Revolution

The area of the surface generated by revolving the parametric curve $(x,y) = (f(t), g(t))$, $a \le t \le b$ about (either x or y axis) is given as follows:

$$S = \int_a^b 2\pi y\sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}dt \text{ if revolved about } x\text{-axis} \tag{11.8}$$

$$S = \int_a^b 2\pi x\sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}dt \text{ if revolved about } y\text{-axis} \tag{11.9}$$

Example 11.2.9. Find the surface area of revolution of the parameterized circle $x = \cos t,\ y = 1 + \sin t, 0 \le t \le 2\pi$ about the x-axis.

$$\begin{aligned} S &= \int_a^b 2\pi y\sqrt{(\frac{dx}{dt})^2 + (\frac{dy}{dt})^2}dt \\ &= \int_0^{2\pi} 2\pi(1+\sin t)\sqrt{(-2\sin t)^2 + (\cos t)^2}dt \\ &= 2\pi\int_0^{2\pi}(1+\sin t)dt = 4\pi^2. \end{aligned}$$

11.3 Polar coordinate

To define the polar coordinate, we fix the **origin** O (also called a **pole**) and an **initial ray** from O. Given a point P, let r be the distance from O to P, and θ be the angle between $\overrightarrow{OP}$ and the initial ray measured in **radian**. Then P is denoted by (r, θ). (figure 11.10) We allow r and θ to have negative value, i.e, if $r < 0$ the point (r, θ) represent the opposite point $(|r|, \theta)$. While if $\theta < 0$ the point (r, θ) represents $(r, |\theta|)$. (figure 11.10)

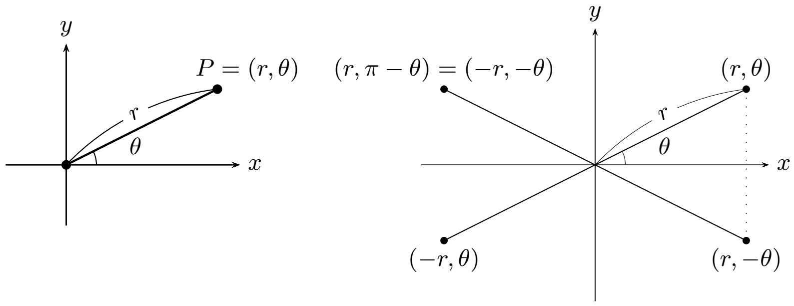


Figure 11.10:

Note the symmetry:

(1) The point $(r, -\theta)$ is symmetric to the point (r, θ) w.r.t. the x axis.

(2) The point $(r, \pi - \theta)$ is symmetric to the point (r, θ) w.r.t. the y-axis.

(3) The point $(-r, \theta)$ is symmetric to the point (r, θ) about the origin.

Relation with Cartesian coordinate

Proposition 11.3.1 (Relations between polar and cartesian coordinate)**.**

$$\boxed{x = r\cos\theta,\ y = r\sin\theta,\ x^2 + y^2 = r^2,\ \tan\theta = \frac{y}{x}}$$

Example 11.3.2. (1) Line through the origin: $\theta = c$.

(2) Line : $r\cos(\alpha - \theta) = d$, where (d, α) is the point on the line closest to the origin.

(3) Line $x = 4$: $r\cos\theta = -4$.

(4) Circle : $r^2 = 4r\cos\theta$.

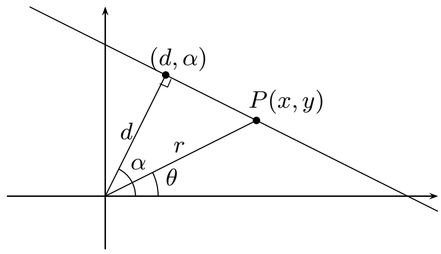


Figure 11.11: Polar form of lines, $r\cos(\alpha - \theta) = d$

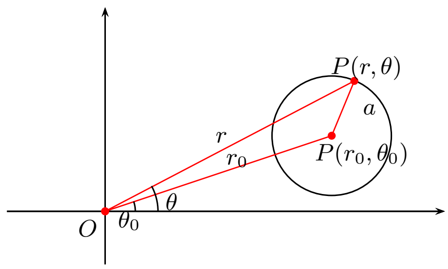


Figure 11.12: Polar equation of a circles, $a^2 = r_0^2 + r^2 - 2r_2 r\cos(\theta - \theta_0)$

(5) $r = \frac{4}{2\cos\theta - \sin\theta}$.

Polar Equation	Cartesian coord
$r\cos\theta = 2$	$x = 2$
$r^2\cos\theta\sin\theta = 4$	$xy = 4$
$r^2\cos^2\theta - r^2\sin^2\theta = 1$	$x^2 - y^2 = 1$
$r = 1 + 2r\cos\theta$	$y^2 - 3x^2 - 4x - 1 = 0$
$r = 1 - \cos\theta$	$x^4 + y^4 + 2x^2y^2 + 2x^3 + 2xy^2 - y^2 = 0$

Circles in polar coordinate

Use law of cosines to the triangle OP_0P, (fig. 11.12) to get

$$a^2 = r_0^2 + r^2 - 2r_2 r\cos(\theta - \theta_0).$$

If the circle passes the origin, then $r = a$ and this simplifies to

$$r = 2a\cos(\theta - \theta_0). \tag{11.10}$$

11.4 Drawing in Polar Coordinate

Example 11.4.1. Draw the graph of

$$r = 2\cos\theta.$$

sol. Multiplying r both sides, we have $r^2 = 2r\cos\theta$. Hence we obtain $x^2 + y^2 = 2x$, or $(x-1)^2 + y^2 = 1$.

□

Example 11.4.2. Draw the graph of $r = 1 - \sin\theta$.

sol. First draw the graph on the (θ, r) plane. Then translate it to cartesian coordinate.

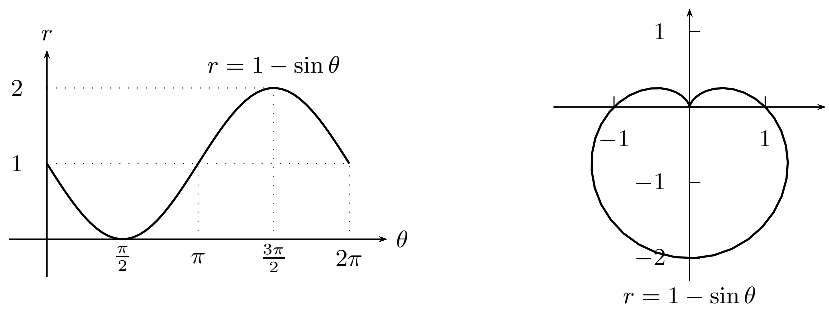


Figure 11.13: $r = 1 - \sin\theta$

□

Example 11.4.3. Draw the graph of $r^2 = 4\cos\theta$.

sol. We note $\cos\theta \geq 0$ so that $-\pi/2 \leq \theta \leq \pi/2$. Note the symmetry about x-axis and the origin. Graph the following, by filling in the Table. Figure 11.14.

$$r = \pm 2\sqrt{\cos\theta}.$$

□

θ	$\cos\theta$	r
0	1	± 2
$\pm\pi/6$	$\frac{\sqrt{3}}{2}$	$\approx \pm 1.9$
$\pm\pi/4$	$\frac{1}{\sqrt{2}}$	$\approx \pm 1.7$
$\pm\pi/3$	$\frac{1}{2}$	$\approx \pm 1.4$
$\pm\pi/2$	0	0

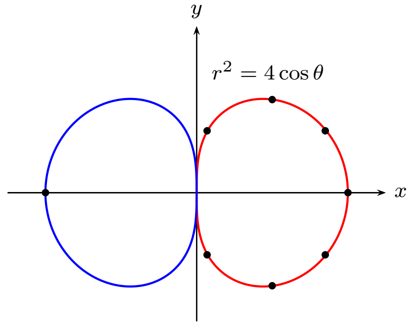


Figure 11.14: $r^2 = 4\cos\theta$

Example 11.4.4 (Limaçon)**.** Draw the graph of $r = 1 + 2\cos\theta$.

sol. Multiply r and changing to x, y coordinates, we get $r^2 = r + 2r\cos\theta$. Hence

$$\begin{aligned} x^2 + y^2 &= \sqrt{x^2 + y^2} + 2x \qquad (r \geq 0) \\ x^2 + y^2 &= -\sqrt{x^2 + y^2} + 2x \quad (r < 0). \end{aligned}$$

θ	r	θ	r
0	3	$\pm 2\pi/3$	0
$\pm\pi/6$	$1 + \sqrt{3}$	$\pm 3\pi/4$	$1 - \sqrt{2}$
$\pm\pi/4$	$1 + \sqrt{2}$	$\pm 5\pi/6$	$1 - \sqrt{3}$
$\pm\pi/3$	2	$\pm\pi$	-1
$\pm\pi/2$	1		

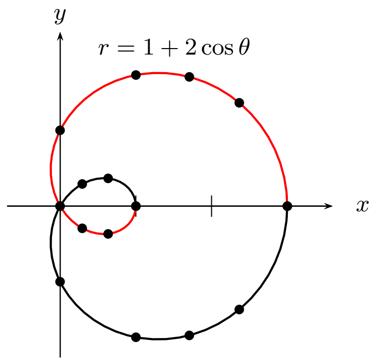


Figure 11.15: $r = 1 + 2\cos\theta$

□

Symmetry

Let $P = (r, \theta)$ be a given point. The point $(r, -\theta)$ is symmetric to the point (r, θ) w.r.t. the x axis, while the point $(r, \pi - \theta)$ is symmetric to the point (r, θ) w.r.t. the y-axis. Finally the point $(-r, \theta)$ (or $(r, \pi + \theta)$) is symmetric to the point (r, θ) about the origin. Hence we have the following result.

Proposition 11.4.5. *The graph of $f(r,\theta)=0$ is symmetric w.r.t.*

(1) x-axis if $f(r,-\theta)=f(r,\theta)$ or $f(-r,\pi-\theta)=f(r,\theta)$,

(2) y-axis if $f(r,\pi-\theta)=f(r,\theta)$ or $f(-r,-\theta)=f(r,\theta)$,

(3) the origin if $f(-r,\theta)=f(r,\theta)$ or $f(r,\pi+\theta)=f(r,\theta)$.

Example 11.4.6. Find the symmetry of $r^2=\sin 2\theta$.

sol. Set $f(r,\theta)=r^2-\sin 2\theta$. Then

$$f(-r,\theta)=(-r)^2-\sin 2\theta=f(r,\theta).$$

Hence it is symmetric about the origin. On the other hand,

$$\begin{aligned} f(r,-\theta)&=r^2-\sin(-2\theta)\neq f(r,\theta),\\ f(-r,\pi-\theta)&=r^2-\sin(2\pi-2\theta)\neq f(r,\theta). \end{aligned}$$

Hence it is not symmetric about the x-axis. Also, we see that

$$\begin{aligned} f(r,\pi-\theta)&=r^2-\sin(2\pi-2\theta)=r^2+\sin 2\theta\neq f(r,\theta),\\ f(-r,-\theta)&=r^2-\sin(-2\theta)=r^2+\sin 2\theta\neq f(r,\theta). \end{aligned}$$

Hence it is not symmetric about y-axis either.

□

Example 11.4.7. For the graph $r=2\cos 2\theta$, we let $f(r,\theta)=r-\cos 2\theta$ and we replace the x-axis symmetric point $(-r,\pi-\theta)$ for (r,θ) then

$$f(-r,\pi-\theta)=-r-\cos 2(\pi-\theta)=-r-\cos 2\theta\neq f(r,\theta).$$

This looks different from the given relation. However, if we replace another expression of the same x-axis symmetric point $(r,-\theta)$ for (r,θ), then

$$f(r,-\theta)=r-\cos(-2\theta)=r-\cos 2\theta=f(r,\theta).$$

Hence it is symmetric about x-axis.

Slope of tangent

First we give a warning: The slope of the tangent at a point of polar curve $r = f(\theta)$ is *not* given by $r' = df/d\theta$, because the slope is measured as the ratio between the increase in y and increase in x(i.e, $\Delta y/\Delta x$). Let us consider the parametric expression

$$x = r\cos\theta = f(\theta)\cos\theta, \quad y = r\sin\theta = f(\theta)\sin\theta.$$

Using the parametric derivative, we have

$$\begin{aligned} \frac{dy}{dx} &= \frac{dy/d\theta}{dx/d\theta} \\ &= \frac{\frac{d}{d\theta}f(\theta)\sin\theta}{\frac{d}{d\theta}f(\theta)\cos\theta} \\ &= \frac{\frac{df}{d\theta}\sin\theta + f(\theta)\cos\theta}{\frac{df}{d\theta}\cos\theta - f(\theta)\sin\theta}. \end{aligned}$$

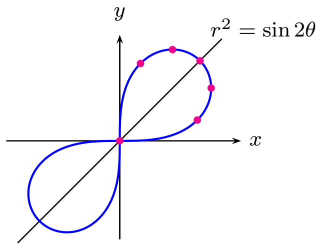
Hence

$$\boxed{\frac{dy}{dx} = \frac{f'(\theta)\sin\theta + f(\theta)\cos\theta}{f'(\theta)\cos\theta - f(\theta)\sin\theta}}$$

Example 11.4.8. Draw another form of Cardioid: $r = 1 - \cos\theta$, checking the the slope of the tangent at the origin.

Example 11.4.9. Draw the lemniscate $r^2 = \sin 2\theta$, noting the slope of tangent near the origin.

θ	r^2
0	0
$\pi/12$	$\frac{1}{2}$
$\pi/6$	$\frac{\sqrt{3}}{2}$
$\pi/4$	1
$\pi/3$	$\frac{\sqrt{3}}{2}$



Problems Caused by Polar Coordinates - skip

Example 11.4.10. Show the point $(2, \pi/2)$ lies on $r = 2\cos 2\theta$.

sol. Substitute $(r,\theta) = (2,\pi/2)$ into $r = 2\cos 2\theta$, we see

$$2 = 2\cos\pi = -2$$

does not holds. However, if we use alternative expression for the same point $(-2,-\pi/2)$, then

$$-2 = 2\cos 2(-\pi/2) = -2.$$

So the point $(2,\pi/2) = (-2,-\pi/2)$ line on the curve.

$\square$

Example 11.4.11. Find all the intersections of $r^2 = 4\cos\theta$ and $r = 1-\cos\theta$.

sol. First solve

$$\begin{aligned} r^2 &= 4\cos\theta \\ r &= 1-\cos\theta. \end{aligned}$$

Substitute $\cos\theta = r^2/4$ into $r = 1-\cos\theta$ to see

$$r = 1-\cos\theta = 1-r^2/4.$$

We get $r = -2\pm 2\sqrt{2}$. Among those $r = -2-2\sqrt{2}$ is too large, we only choose $r = -2+2\sqrt{2}$. So

$$\theta = \cos^{-1}(1-r) = \cos^{-1}(3-2\sqrt{2}) \approx 80^\circ.$$

But if we see the graph 11.16 there are four points A, B, C, D. These parameter θ in two equation is not necessarily the same (they run on different time) That is

The curve $r = 1-\cos\theta$ passes C when $\theta = \pi$, while the curve $r^2 = 4\cos\theta$ passed C when $\theta = 0$. The same phenomena arises with D.

$\square$

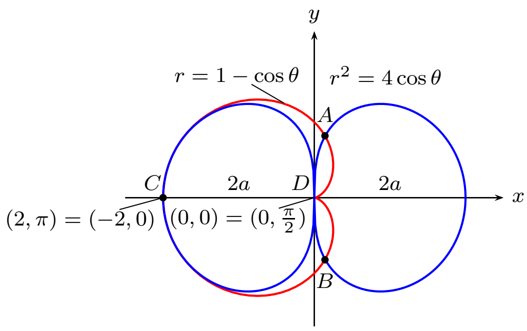


Figure 11.16: intersection of $r^2 = 4\cos\theta$ and $r = 1 - \cos\theta$

11.5 Areas and Lengths in Polar Coordinates

Areas

We want to find the area of the region bounded by the following curves.

$$r = f(\theta), \quad \theta = a, \quad \theta = b.$$

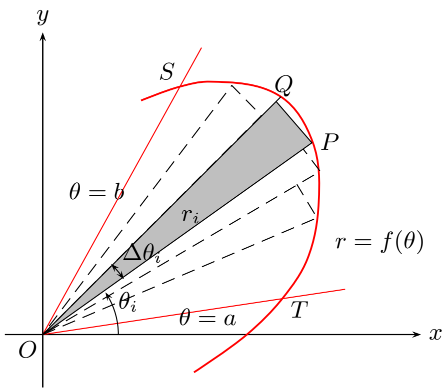


Figure 11.17: Area of the sector OST is approximated by sum of triangles

Let $P = \{\theta_0, \theta_1, \ldots, \theta_n\}$ be the partition of $[a, b]$(angle) and $r_i = r(\theta_i)$. Each region is approx'd by n sectors. The area of the sector determined by

$$r = f(\theta), \quad \theta_i \le \theta \le \theta_{i+1}$$

is approximated by circular sector whose area is $r_i^2(\theta_{i+1} - \theta_i)/2$. Hence the area will be obtained if we pass to the limit.

$$\lim_{n\to\infty}\sum_{i=0}^{n-1}\Delta A_i = \lim_{n\to\infty}\sum_{i=0}^{n-1}\frac{1}{2}{r_i}^2(\theta_{i+1}-\theta_i).$$

(See fig 11.17). Since $r_i \to r$, $\Delta\theta_i = \theta_{i+1}-\theta_i \to d\theta$, the area enclosed by the curve between $\theta = a, \theta = b$ is given by

$$\boxed{\int_a^b \frac{1}{2}r^2\,d\theta.} \tag{11.11}$$

Example 11.5.1. Find the area enclosed by the cardioid: $r = 2(1+\cos\theta)$.

sol.

$$\int_0^{2\pi}\frac{1}{2}(2+2\cos\theta)^2\,d\theta = 6\pi. \tag{11.12}$$

□

Arc Length

Find the arc-length of the curve $r = f(\theta), \quad \theta\in[a,b]$.

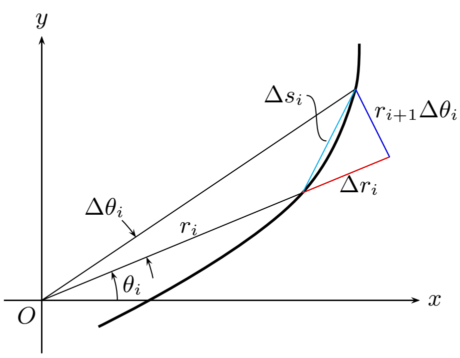


Figure 11.18: $r_i = r(\theta_i)$, $\Delta r_i = r_{i+1}-r_i$, $\Delta\theta_i = \theta_{i+1}-\theta_i$

Let $P = \{\theta_0, \theta_1, \ldots, \theta_n\}$ be the partition of $[a,b]$ and $r_i = r(\theta_i)$. The length of the line segment (see fig 11.18) connecting two points (r_i,θ_i) and (r_{i+1},θ_{i+1}) is

$$\Delta s_i = \sqrt{(r_{i+1}(\theta_{i+1}-\theta_i))^2 + (r_{i+1}-r_i)^2}.$$

Thus total curve length is approx'ed by

$$\begin{aligned}\lim_{n\to\infty}\sum_{i=0}^{n-1}\Delta s_i &= \sum_{i=0}^{n-1}\sqrt{(r_{i+1}(\theta_{i+1}-\theta_i))^2+(r_{i+1}-r_i)^2}\\ &= \sum_{i=0}^{n-1}\sqrt{r_{i+1}^2+\left(\frac{r_{i+1}-r_i}{\theta_{i+1}-\theta_i}\right)^2}\,(\theta_{i+1}-\theta_i).\end{aligned}$$

Since $\frac{r_{i+1}-r_i}{\theta_{i+1}-\theta_i}\to r'(\theta_i)$ as $\theta_{i+1}\to\theta_i$, the length of the curve between $\theta=a,\theta=b$ is

$$\boxed{\int_0^s ds = \int_a^b \sqrt{r^2+\left(\frac{dr}{d\theta}\right)^2}\,d\theta.} \tag{11.13}$$

Example 11.5.2. Find the length of the closed curve $r=2\cos\theta$.

sol. It can be changed to $x^2+y^2=2x$. Since θ ranges in the domain $[-\pi/2,\pi/2]$, the arclength is 2π.

$$\int_{-\pi/2}^{\pi/2}\sqrt{(2\cos\theta)^2+(-2\sin\theta)^2}\,d\theta = 2\pi. \tag{11.14}$$

□

Example 11.5.3. Find the length of the cardioid: $r=1-\cos\theta$.

$$\begin{aligned}L &= \int_0^{2\pi}\sqrt{r^2+\left(\frac{dr}{d\theta}\right)^2}\,d\theta\\ &= \int_0^{2\pi}\sqrt{2-2\cos\theta}\,d\theta\\ &= \int_0^{2\pi}\sqrt{4\sin^2\frac{\theta}{2}}\,d\theta\\ &= \int_0^{2\pi}2\sin\frac{\theta}{2}\,d\theta = 8.\end{aligned}$$

Area of a Surface of Revolution in Polar coordinate

Recall the formula: the surface area of Revolution

$$\text{revolved about } x\text{-axis } S = \int_a^b 2\pi y\sqrt{\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2}\,dt \tag{11.15}$$

$$\text{revolved about } y\text{-axis } S = \int_a^b 2\pi x\sqrt{\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2}\,dt. \tag{11.16}$$

Since $x = r\cos\theta$, $\quad y = r\sin\theta$, with $r = f(\theta)$, by changing it to polar coordinates; we have

$$\begin{aligned}\left(\frac{dx}{d\theta}\right)^2 + \left(\frac{dy}{d\theta}\right)^2 &= \big(f'(\theta)\cos\theta - f(\theta)\sin\theta\big)^2 + \big(f'(\theta)\sin(\theta) + f(\theta)\cos(\theta)\big)^2 \\ &= (f(\theta))^2 + \big(f'(\theta)\big)^2 \\ &= r^2 + \left(\frac{dr}{d\theta}\right)^2 = ds^2.\end{aligned}$$

If the graph is revolved, the area of the surface of the revolution is

$$\boxed{S = \begin{cases} \displaystyle\int_a^b 2\pi r\sin\theta\sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\,d\theta & \text{revolved about } x\text{-axis} \\ \displaystyle\int_a^b 2\pi r\cos\theta\sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\,d\theta & \text{revolved about } y\text{-axis}\end{cases}}. \tag{11.17}$$

Example 11.5.4. Find the area of the surface of the revolution when we revolve the right hand loop of lemniscate $r^2 = \cos 2\theta$ about y-axis. $(\pi^2/4)$

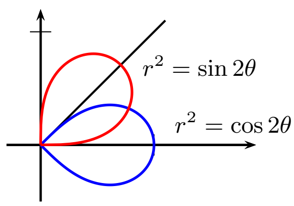


Figure 11.19: Lemniscates $r^2 = \sin 2\theta$ and $r^2 = \cos 2\theta$

11.6 Conic Sections and Quadratic Equations

Parabola

Definition 11.6.1. The set of all points in a plane equidistant from a *fixed point* and a *fixed line* is a **parabola**. The fixed point is called a **focus** and the line is called a **directrix**.

Find the equation of parabola whose focus is at $F = (p, 0)$ and directrix ℓ is $x = -p$. (Figure 11.21.) By definition it holds that $\overline{PQ} = \overline{PF}$. Thus

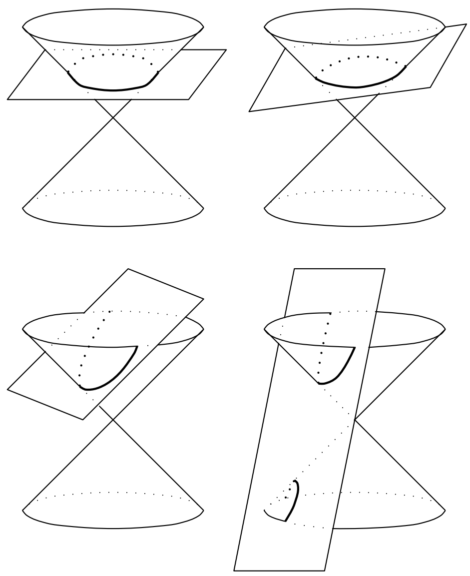


Figure 11.20: Conic sections

$$(x-p)^2 + y^2 = (x+p)^2$$

is the equation of parabola

$$y^2 = 4px. \tag{11.18}$$

The point on the curve closest to the directrix is called **vertex**, and the line connecting vertex and focus is the **axis**. For $y^2 = 4px$, vertex is $(0, 0)$ and x-axis is the axis of parabola.

If $F = (0, p)$ is the focus and the directrix ℓ is given by $y = -p$ then we get

$$x^2 = py.$$

Example 11.6.2. Find parabola whose directrix is $x = 1$, focus is at $(0, 3)$.

sol.

$$x^2 + (y-3)^2 = (x-1)^2.$$

So $y^2 - 6y + 2x + 8 = 0$.

□

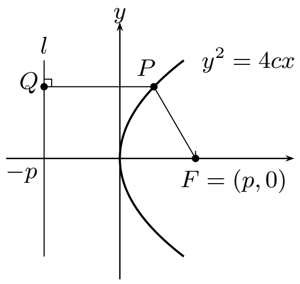


Figure 11.21: Parabola ($y^2 = 4cx$)

Ellipse

Definition 11.6.3. The set of all points in a plane whose sum of distances from two given focuses is a **ellipse**. If two points are identical, it becomes a **circle**.

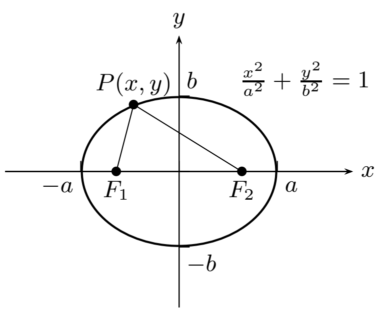


Figure 11.22: Ellipse ($x^2/a^2 + y^2/b^2 = 1$)

Now assume two points $F_1 = (-c, 0)$ and $F_2 = (c, 0)$ are given. Find the set of all points where the sum of distances from focuses are constant and let $P = (x, y)$. (Refer to fig 11.22)

$$PF_1 + PF_2 = 2a.$$

Since

$$\sqrt{(x+c)^2 + y^2} + \sqrt{(x-c)^2 + y^2} = 2a$$

we obtain

$$\frac{x^2}{a^2} + \frac{y^2}{a^2 - c^2} = 1. \tag{11.19}$$

Let let $b^2 = a^2 - c^2, b > 0$. Then $b \le a$ and hence from (11.19) we get

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1. \tag{11.20}$$

If $x = 0$ then $y = \pm b$ and if $y = 0$ we have $x = \pm a$. Two points $(\pm a, 0)$ are intersection of ellipse with x-axis $(0, \pm b)$ are intersection of ellipse with y-axis.

Foci are $F_1 = (0, -c)$ and $F_2 = (0, c)$. The set of all points whose sum of distance to these $2b$ is given by

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1.$$

The point $(0, \pm b)$ are vertices.

Example 11.6.4. Find the equation of an ellipse having foci $(\pm 1, 0)$ and sum of distance from the origin is 6.

sol. $c = 1$ and $a = 3$. Thus $b^2 = a^2 - c^2 = 9 - 1 = 8$. Hence

$$\frac{x^2}{9} + \frac{y^2}{8} = 1.$$

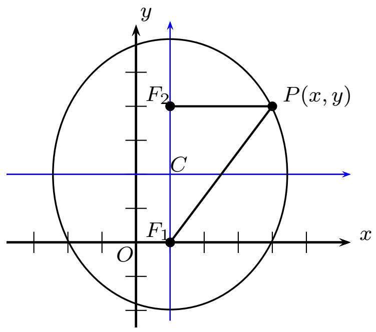
□

More generally, foci may not lie on the convenient axis.

Example 11.6.5. Find the equation of an ellipse whose foci are $(1, 0)$ and $(1, 4)$, and the sum of the distance is 8.

sol. $a = 4$. Noting that the foci lie on the line $x = 1$ and the center is at $(1, 2)$, the major axis is parallel to y axis. Use new coordinates $X = x - 1$, $Y = y - 2$. Then on the XY-plane the foci are $(0, \pm c) = (0, \pm 2)$. Thus $b^2 = a^2 - c^2 = 16 - 4 = 12$. Hence

$$\frac{X^2}{12} + \frac{Y^2}{16} = 1 \Rightarrow \frac{(x-1)^2}{12} + \frac{(y-2)^2}{16} = 1. \tag{11.21}$$



□

Hyperbola

Definition 11.6.6. If the difference of distances from given two foci are constant, we obtain **hyperbola**.

Find the equation of a hyperbola whose two foci are $F_1 = (-c, 0)$, $F_2 = (c, 0)$, and the difference of distance is $2a$. (Fig 11.23.) Then the point $P = (x, y)$ satisfies (since $|PF_1 - PF_2| = 2a$)

$$\sqrt{(x+c)^2 + y^2} - \sqrt{(x-c)^2 + y^2} = \pm 2a.$$

Or arranging terms, we get

$$\frac{x^2}{a^2} + \frac{y^2}{a^2 - c^2} = 1 \ (a^2 - c^2 < 0). \tag{11.22}$$

Let $b^2 = c^2 - a^2$. Then we obtain the following form.

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1. \tag{11.23}$$

The lines $\frac{x}{a} = \frac{y}{b}$ are two **asymptotes**.

On the other hand, if the distances from two foci $(0, \pm c)$ is $2b$, then the equation of hyperbola is

$$-\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1.$$

Example 11.6.7. Foci are $(\pm 2, 0)$ Find the locus whose difference is 2.

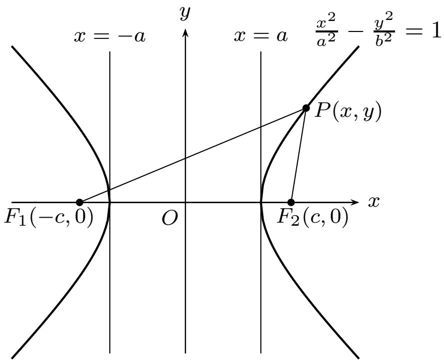


Figure 11.23: hyperbola $x^2/a^2 - y^2/b^2 = 1$

sol. Since $a = 1$, $c = 2$, $b = \sqrt{3}$

$$x^2 - \frac{y^2}{3} = 1.$$

Asymptote are $y = \pm\sqrt{3}x$, and vertices are $(\pm 1, 0)$.

□

11.7 Conic Sections in Polar Coordinate

eccentricity and directrix

From the definition of parabola we see that for any point P, the distance to focus F is the same as the distance to the directrix D. i.e,

$$PF = PD \text{ or } PF = e \cdot PD (e = 1).$$

Given an ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

define c by $c^2 = a^2 - b^2$ when $a \geq b$. Then $(\pm c, 0)$ are foci and $(\pm a, 0)$ are vertices.

For the hyperbola

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$$

define c by $c^2 = a^2 + b^2$. In this case foci are $(\pm c, 0)$ and vertices are $(\pm a, 0)$.

Definition 11.7.1. For both cases (ellipse and hyperbola) we define **eccentricity** e by

$$e = \frac{\text{Distance between foci}}{\text{Distance between vertices}} = \frac{2c}{2a} = \frac{c}{a}.$$

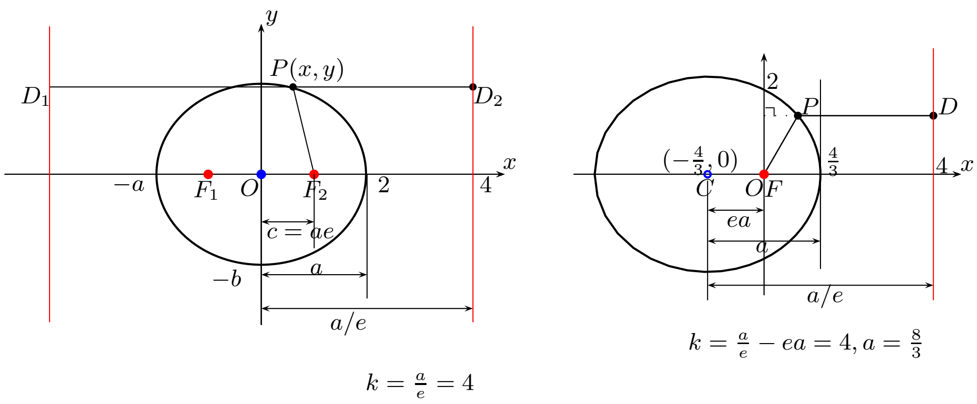


Figure 11.24: Ellipse $\frac{x^2}{4} + \frac{y^2}{3} = 1$ and $r = \frac{4}{2+\cos\theta}$ have same e and k, different center

For ellipse $x^2/a^2 + y^2/b^2 = 1$ $(a > b)$ the eccentricity is

$$e = \frac{c}{a} = \frac{\sqrt{a^2 - b^2}}{a}.$$

For hyperbola $x^2/a^2 - y^2/b^2 = 1$ the eccentricity is

$$e = \frac{c}{a} = \frac{\sqrt{a^2 + b^2}}{a}.$$

11.7.1 Classifying Conic sections by Eccentricity

The relation between the eccentricity and directrix also holds for other quadratic curves too! For the ellipse $x^2/a^2 + y^2/b^2 = 1$ $(a > b)$, the lines

$$x = \pm\frac{a}{e} = \pm\frac{a^2}{\sqrt{a^2 - b^2}}$$

are directrices. If $b > a$, the two lines

$$y = \pm\frac{b}{e} = \pm\frac{b^2}{\sqrt{b^2 - a^2}}$$

are directrices.

For hyperbola $x^2/a^2 - y^2/b^2 = 1$, the directrices are

$$x = \pm\frac{a}{e} = \pm\frac{a^2}{\sqrt{a^2 + b^2}}.$$

For hyperbola $-x^2/a^2 + y^2/b^2 = 1$, the directrix are

$$y = \pm\frac{b}{e} = \pm\frac{b^2}{\sqrt{b^2 + a^2}}.$$

We now classify conic sections using eccentricity and directrix

Definition 11.7.2. Suppose a point F and a line ℓ. If P satisfies

$$PF = e \cdot PD \tag{11.24}$$

then the conic section generated is

(1) an ellipse when $e < 1$

(2) a parabola when $e = 1$

(3) a hyperbola when $e > 1$.

The relation (11.24) is called the *focus-directrix* relation.

Example 11.7.3. Find the hyperbola with $e = \sqrt{3}$, foci $F = (\pm 3, 0)$, directrix is $x = 1$.

sol. We only see $x > 0$. Since $F = (3, 0)$, we have $c = 3$. From $PF = e \cdot PD$, we see

$$\sqrt{(x-3)^2 + y^2} = \sqrt{3}|x - 1| \Rightarrow \frac{x^2}{3} - \frac{y^2}{6} = 1.$$

□

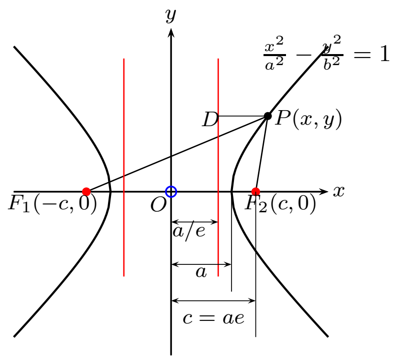


Figure 11.25: hyperbola $x^2/a^2 - y^2/b^2 = 1$

11.7.2 Polar Coordinates of Conic Sections

Refer to Fig 11.26. Assume the focus is F and directrix ℓ is $x = k > 0$. Let D be the foot of P to directrix ℓ while the foot on the x-axis is B. Then

$$PF = r, \qquad PD = k - FB = k - r\cos\theta.$$

So by the focus-directrix relation (11.24), we have

$$r = PF = e \cdot PD = e(k - r\cos\theta). \tag{11.25}$$

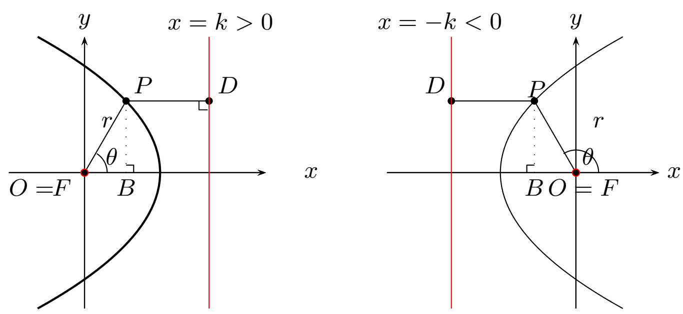


Figure 11.26: Conic sections in polar coordinate, focus is at the origin

Proposition 11.7.4. *The polar equation of a conic section with eccentricity*

e, directrix $x = k$, $k > 0$ having focus at the origin is

$$\boxed{r = \frac{ke}{1 + e\cos\theta}.} \tag{11.26}$$

Here $k = \frac{a}{e} - ea$ for ellipse, and $k = ea - \frac{a}{e}$ for hyperbola.

Remark 11.7.5. If $x = -k < 0$ is the directrix, we have the following.

$$r = \frac{ke}{1 - e\cos\theta}. \tag{11.27}$$

Example 11.7.6. Find the polar equation of a conic section with $e = 2$ directrix $x = -2$ and focus at origin.

sol. Since $k = 2$ and $e = 2$ we have from equation (11.27)

$$r = \frac{2(2)}{1 - 2\cos\theta} = \frac{4}{1 - 2\cos\theta}.$$

□

Example 11.7.7. Identify

$$r = \frac{3}{1 + 3\cos\theta}.$$

sol. Use (11.26). Since $ke = 3$, $e = 3$ we have $k = 1$. Hence directrix is $x = 1$ and $e = 3 > 1$. So a hyperbola.

□

Example 11.7.8. Find the polar equation of a conic section with directrix $x = 4$, eccentricity $e = 3/2$ and focus at the origin.

sol. Refer to Fig 11.27.

$$PF = r, \quad PD = 4 - r\cos\theta.$$

So $r = \frac{3}{2}(4 - r\cos\theta)$ and

$$r = \frac{6}{1 + 1.5\cos\theta}.$$

□

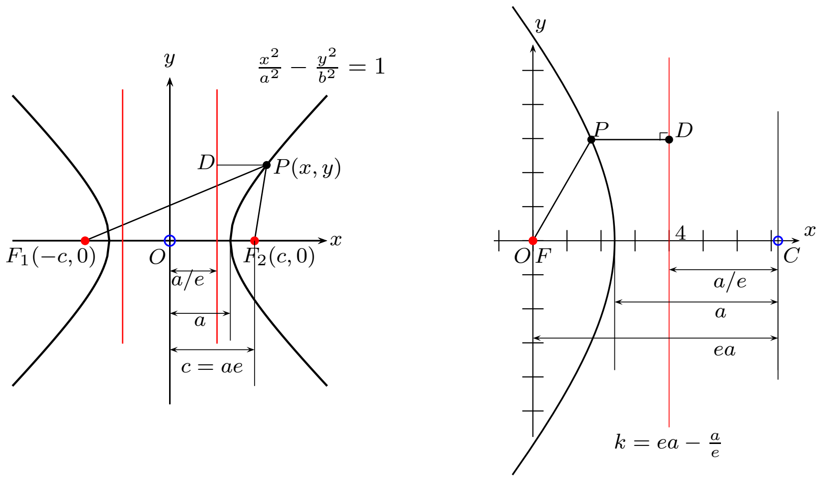


Figure 11.27: hyperbolas $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ and $r = \frac{6}{1+1.5\cos\theta}$, $k = 4, e = 3/2$

Refer to Fig 11.28. Assume the focus is F and directrix ℓ is $y = k > 0$. Let D be the foot of P to directrix ℓ while the foot on the y-axis is B. Then

$$PF = r, \quad PD = k - FB = k - r\sin\theta.$$

So

$$r = PF = e \cdot PD = e(k - r\sin\theta). \tag{11.28}$$

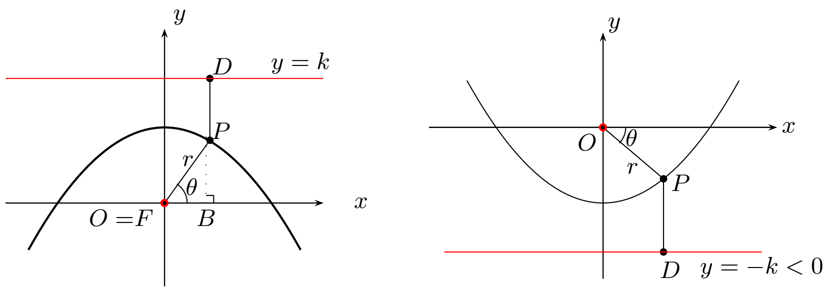


Figure 11.28: conic sections, focus at the origin, directrix $y = \pm k$

Proposition 11.7.9. *The polar equation of a conic section with eccentricity e, directrix $y = k$, $(k > 0)$ having focus at the origin is*

$$r = \frac{ke}{1 + e\sin\theta}. \tag{11.29}$$

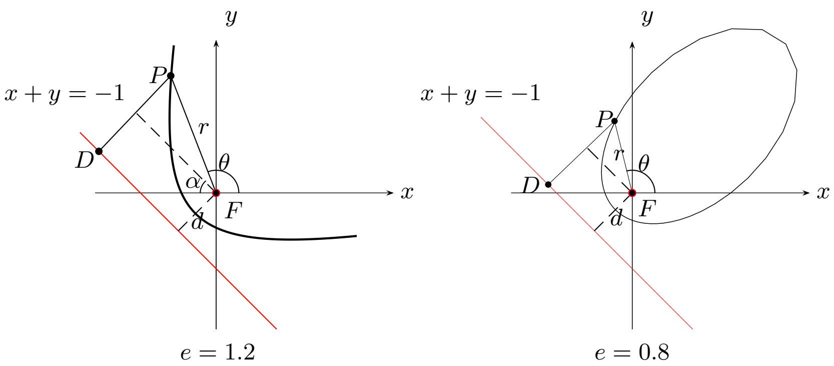


Figure 11.29: $r = e(r\sin(\pi - \alpha - \theta) + \frac{1}{\sqrt{2}})$

Remark 11.7.10. If the directrix is $y = -k < 0$ in the above proposition, we obtain the following equation.

$$\boxed{r = \frac{ke}{1 - e\sin\theta}}. \tag{11.30}$$

Example 11.7.11. 1. Find the polar equation of conic section with $e = 1.2$, directrix $\ell : x + y = -1$.

2. Repeat the problem with $e = 0.8$.

sol. Refer to figure 11.29. Since $\alpha = \pi/4$, we see that $PD = r\sin(\frac{3\pi}{4} - \theta) + \frac{1}{\sqrt{2}}$. Hence $r = e(r\sin(\frac{3\pi}{4} - \theta) + \frac{1}{\sqrt{2}})$ from which we get

$$r = \frac{e/\sqrt{2}}{1 - e\sin(\frac{3\pi}{4} - \theta)}.$$

□

Example 11.7.12 (Epicycloids)**.** A circle of radius b is rolling along the circle of radius $a (a > b)$, centered at the origin. Find the locus of a point P on the smaller circle, initially located at $(a, 0)$.

Sol. Since the position of $C = (a + b)(\cos\theta, \sin\theta)$, the coordinate of P is

$$(a + b)(\cos\theta, \sin\theta) + (0.5\cos(\phi + \theta - \pi), 0.5\sin(\phi + \theta - \pi)).$$

Example 11.7.13 (Scarabaeus)**.**

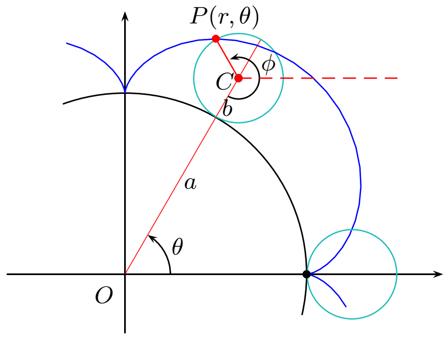


Figure 11.30: Epicycloids

11.7.3 Slope of a tangent line to the curve given by polar equation

Since $r = f(\theta)$ and $x = r\cos\theta$, $y = r\sin\theta$, the slope of the tangent line can be computed as

$$\boxed{\frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta} = \frac{f'(\theta)\sin\theta + f(\theta)\cos\theta}{f'(\theta)\cos\theta - f(\theta)\sin\theta}}.$$

Example 11.7.14. Find the equation of the tangent line to the curve $r = 1 + 2\cos(2\theta)$ at the point when $\theta = \pi/4$.

Sol. First we compute the slope of the tangent. Since $f(\theta) = 1+2\cos(2\theta)$, we have $f'(\theta) = -4\sin(2\theta)$ and

$$\frac{dy}{dx} = \frac{-4\sin(2\theta)\sin\theta + (1+2\cos(2\theta))\cos\theta}{-4\sin(2\theta)\cos\theta - (1+2\cos(2\theta))\sin\theta}.$$

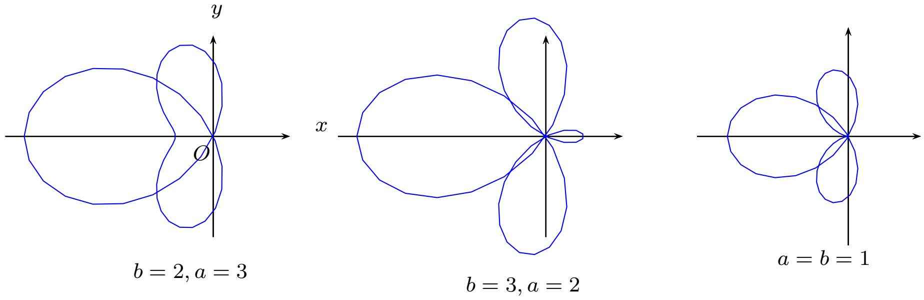


Figure 11.31: Scarabaeus $r = b\cos(2\theta) - a\cos\theta$

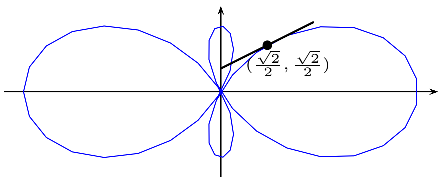


Figure 11.32: $r = 1 + 2\cos(2\theta)$

When $\theta = \pi/4$ the point is $(\sqrt{2}/2, \sqrt{2}/2)$. Hence the slope and the equation of tangent line is

$$\frac{-4\frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}}{-4\frac{\sqrt{2}}{2} - 2\frac{\sqrt{2}}{2}} = \frac{1}{2}, \quad y = \frac{1}{2}(x - \sqrt{2}/2) + \sqrt{2}/2.$$

11.7.4 Meaning of $\frac{dr}{d\theta}$

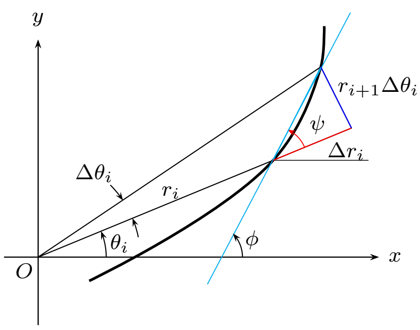


Figure 11.33: meaning of $\frac{dr}{d\theta}$

From figure 11.33, we see that

$$\lim_{\Delta\theta \to 0} r\frac{\Delta\theta}{\Delta r} = \tan\psi = r\frac{d\theta}{dr} = \frac{r}{r'}.$$

Since $\psi = \phi - \theta$ we have

$$\tan\psi = \tan(\phi - \theta) = \frac{\tan\phi - \tan\theta}{1 + \tan\phi\tan\theta}. \tag{11.31}$$

In the limit as $\Delta\theta \to 0$,

$$\tan\phi \to \frac{r'\sin\theta + r\cos\theta}{r'\cos\theta - r\sin\theta} \equiv \frac{r's + rc}{r'c - rs}.$$

Substituting into (11.31)

$$\begin{aligned}\tan\psi &= \frac{\tan\phi - \tan\theta}{1 + \tan\phi\tan\theta} \\ &= \frac{\frac{r's+rc}{r'c-rs} - \frac{s}{c}}{1 + \frac{r's+rc}{r'c-rs}\frac{s}{c}} \\ &= \frac{c(r's + rc) - s(r'c - rs)}{c(r'c - rs) + s(r's + rc)} \\ &= \frac{r}{r'} = \frac{r}{dr/d\theta}.\end{aligned}$$

Mysubcycloid

A small circle of radius b is rolling inside a cycloid. We will find the locus of a point P on the circle of radius b, originally located at the origin, tangent inside cycloid at the point T. The contact point is $T = (t - \sin t, 1 - \cos t), 0 \le t \le 2\pi$. The arclength along the cycloid is $s = 4(1 - \cos\frac{t}{2})$, hence the angle along which the smaller circle rolled is

$$\phi = \frac{s}{b} = \frac{4(1 - \cos\frac{t}{2})}{b}.$$

Tangent line at t_0 is

$$y = \frac{\sin t_0}{1 - \cos t_0}(x - t_0 + \sin t_0) + 1 - \cos t_0.$$

Let $t_0 = 2\pi/3$. Then $T = (2\pi/3 - \frac{\sqrt{3}}{2}, 1 + \frac{1}{2})$=$(1.2283679, 1.5)$. Then

$$y = \frac{1}{\sqrt{3}}(x - 1.2283679) + 1.5 = 0.57735(x - 1.2283679) + 1.5.$$

Two points are $(0.5, 1.0794768), (2, 1.9455018)$. Center C is. add $(b\sin\psi, -b\cos\psi)$, where $\tan\psi = \frac{\sin t}{1-\cos t}$. In this case, $\tan\psi = \frac{\pi}{6}$, so

$$(b\sin\psi, -b\cos\psi) = (\frac{1}{4}, -\frac{\sqrt{3}}{4}) = (0.25, -0.4330127),$$

$b = 1/2, C = (1.2283679, 1.5) + (0.25, -0.4330127) = (1.4783679, 1.0669873)$.

Location of P is obtained by adding $(b\cos\xi, b\sin\xi)$ to the coordinate of center of smaller circle

$$P(x,y) = (t - \sin t, 1 - \cos t) + (b\sin\psi, -b\cos\psi) + (b\cos\xi, b\sin\xi), \quad (11.32)$$

where

$$\begin{aligned}
\xi &= \phi - (\frac{\pi}{2} - \psi) - \pi = \frac{4(1-\cos\frac{t}{2})}{b} - \frac{3}{2}\pi + \arctan\left(\frac{\sin t}{1-\cos t}\right) \\
&= A - \frac{3}{2}\pi + \psi \\
\sin\psi &= \frac{\tan\psi}{1+\tan^2\psi} = \frac{\sin t}{\sqrt{2(1-\cos t)}} \\
\cos\psi &= \frac{1}{1+\tan^2\psi} = \frac{\sqrt{1-\cos t}}{\sqrt{2}} \\
\sin\xi &= \cos(A+\psi) = \cos A\cos\psi - \sin A\sin\psi \\
\cos\xi &= -\sin(A+\psi) = -\sin A\cos\psi - \cos A\sin\psi.
\end{aligned}$$

Substituting $\sin\psi, \cos\psi$, etc. to the equation of $P(x,y)$ in (11.32) ($b = \frac{1}{2}$), we get

$$\begin{aligned}
x &= t - \sin t + \frac{b\sin t}{\sqrt{2(1-\cos t)}} \\
&\quad -b\left(\sin 8(1-\cos\frac{t}{2})\frac{\sqrt{1-\cos t}}{\sqrt{2}} + \cos 8(1-\cos\frac{t}{2})\frac{\sin t}{\sqrt{2(1-\cos t)}}\right) \\
y &= 1 - \cos t - \frac{b\sqrt{1-\cos t}}{\sqrt{2}} \\
&\quad +b\left(\cos 8(1-\cos\frac{t}{2})\frac{\sqrt{1-\cos t}}{\sqrt{2}} - \sin 8(1-\cos\frac{t}{2})\frac{\sin t}{\sqrt{2(1-\cos t)}}\right)
\end{aligned}$$

11.8 Quadratic Equations and Rotations

General quadratic curves are give by

$$Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0. \quad (11.33)$$

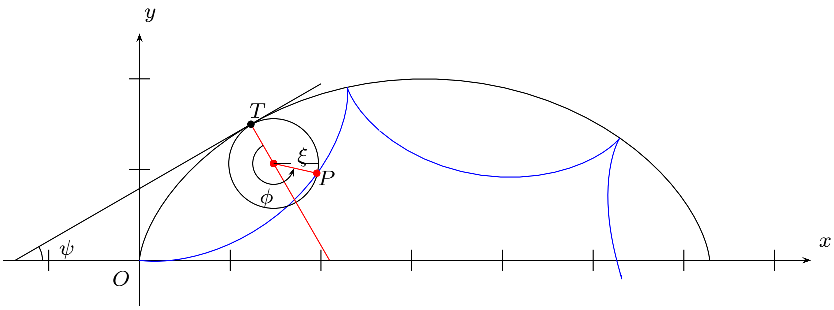


Figure 11.34: cycloid - mysubcycloid $b = 1/2$

The case $B = 0$, i.e, no xy-term

In this case the equation (11.33) is

$$Ax^2 + Cy^2 + Dx + Ey + F = 0. \tag{11.34}$$

If $AC \neq 0$ then these are classified into three classes:

(1) If $AC = 0$, but $A^2 + C^2 \neq 0$, we have a parabola:

$$A(x-\alpha)^2 + Ey = \delta.$$

(2) If $AC > 0$, we have an ellipse (Assume $A > 0$).

$$\frac{(x-\alpha)^2}{C\gamma^2} + \frac{(y-\beta)^2}{A\gamma^2} = \frac{1}{AC\gamma}. \tag{11.35}$$

(3) If $AC < 0$, we have a hyperbola (Assume $A > 0$)

$$\frac{(x-\alpha)^2}{|C|\gamma^2} - \frac{(y-\beta)^2}{A\gamma^2} = \frac{\gamma}{|AC\gamma^2|}.$$

Theorem 11.8.1. *Assume a quadratic curve of the following form is given.*

$$Ax^2 + Cy^2 + Dy^2 + Ey + F = 0.$$

(1) If $A = C = 0$ and one of D E is nonzero, then it is a line.

(2) If one of A or C is zero, it is parabola.

(3) If $AC > 0$, it is an ellipse.

(4) If $AC < 0$, it is a hyperbola.

The case $B \neq 0$, i.e the presence of xy-term

Example 11.8.2. Find the equation of hyperbola whose two foci are $F_1 = (-3,-3)$, $F_2 = (3,3)$ and difference of the distances are 6.

sol. From $|PF_1 - PF_2| = 6$, we get

$$\sqrt{(x+3)^2 + (y+3)^2} - \sqrt{(x-3)^2 + (y-3)^2} = \pm 6 \Rightarrow 2xy = 9.$$

$\square$

Rotation

Rotate xy-coordinate by an angle α and call new coordinate $x'y'$. Then a point $P(x,y)$ is represented by $P(x',y')$ in $x'y'$-coordinate. We will see the relation between the two expressions.

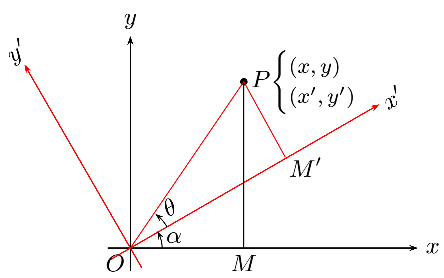


Figure 11.35: Rotation of axis

From fig 11.35, we see

$$x = OM = OP\cos(\theta+\alpha) = OP\cos\theta\cos\alpha - OP\sin\theta\sin\alpha$$
$$y = MP = OP\sin(\theta+\alpha) = OP\cos\theta\sin\alpha + OP\sin\theta\cos\alpha.$$

On the other hand,

$$OP\cos\theta = OM' = x', \qquad OP\sin\theta = M'P' = y'.$$

Proposition 11.8.3. *The point $P = (x, y)$ is related to the $x'y'$-coordinate by*

$$\begin{aligned} x &= x'\cos\alpha - y'\sin\alpha \\ y &= x'\sin\alpha + y'\cos\alpha. \end{aligned}$$

We see by direct computation that

$$A'x'^2 + B'x'y' + C'y'^2 + D'x' + E'y' + F' = 0, \tag{11.36}$$

where

$$\begin{aligned} A' &= A\cos^2\alpha + B\cos\alpha\sin\alpha + C\sin^2\alpha \\ B' &= B\cos 2\alpha + (C - A)\sin 2\alpha \\ C' &= A\sin^2\alpha - B\sin\alpha\cos\alpha + C\cos^2\alpha \\ D' &= D\cos\alpha + E\sin\alpha \\ E' &= -D\sin\alpha + E\cos\alpha \\ F' &= F. \end{aligned}$$

We will choose α so that the coefficient $B' = B\cos 2\alpha + (C - A)\sin 2\alpha$ vanishes.

Theorem 11.8.4. *Given the curve*

$$Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0.$$

If we choose

$$\tan 2\alpha = \frac{B}{A - C},$$

then cross product term disappears in the $x'y'$-coordinate system.

Example 11.8.5. Classify the quadratic curve

$$x^2 + xy + y^2 - 6 = 0.$$

$\boxed{\textbf{sol.}}$ From $\tan 2\alpha = B/(A - C)$ we have $2\alpha = \frac{\pi}{2}$ i.e, $\alpha = \frac{\pi}{4}$. Hence

$$\begin{aligned} x &= x'\cos\alpha - y'\sin\alpha = \frac{\sqrt{2}}{2}x' - \frac{\sqrt{2}}{2}y' \\ y &= x'\sin\alpha + y'\cos\alpha = \frac{\sqrt{2}}{2}x' + \frac{\sqrt{2}}{2}y'. \end{aligned}$$

Substitute into $x^2 + xy + y^2 - 6 = 0$ to get

$$\frac{x'^2}{4} + \frac{y'^2}{12} = 1.$$

See Fig 11.36.

□

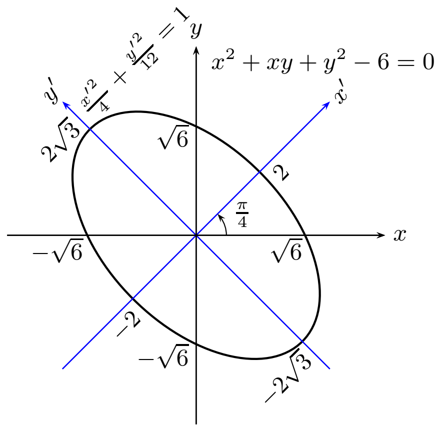


Figure 11.36: $x^2 + xy + y^2 - 6 = 0$

Invariance of Discriminant

Given a quadratic curve in xy-coordinate, we rotated the axis and obtain new equation in $x'y'$-coordinate. In this case, one can choose the angle so that no $x'y'$ term exists. However, if we are only interested in classification, there is a simple way. Given a quadratic curve

$$Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0,$$

the new equation after certain rotation of axis has the form

$$A'x'^2 + B'x'y' + C'x'^2 + D'x' + E'y' + F' = 0.$$

After some computation we can verify that

$$B^2 - 4AC = B'^2 - 4A'C'. \tag{11.37}$$

Theorem 11.8.6. *For the quadratic curves given in x, y*

$$Ax^2 + Bxy + Cx^2 + Dx + Ey + F = 0$$

we have the following classification:

(1) $B^2 - 4AC = 0$ *parabola*

(2) $B^2 - 4AC < 0$ *ellipse*

(3) $B^2 - 4AC > 0$ *hyperbola.*

This follows from (11.37) and Theorem 11.8.1 if we choose rotation so that $B' = 0$ in the new equation.

Example 11.8.7. (1) $3x^2 - 5xy + y^2 - 2x + 3y - 5 = 0$ has $B^2 - 4AC = 25 - 12 > 0$. Thus a hyperbola.

(2) $x^2 + xy + y^2 - 5 = 0$ has $B^2 - 4AC = -3 < 0$. Thus ellipse.

(3) $x^2 - 2xy + y^2 - 5x - 3 = 0$ satisfies $B^2 - 4AC = 0$, a parabola.