Contents

Chapter 1

The geometry of Euclidean Space

We consider the basic operations of vectors in 3 and 3 dim. space: vector addition, scalar multiplication, dot product and cross product. In section 1.6 we generalize these notions to n dim'l space.

1.1 Vectors in 2, 3 dim space

Definition 1.1.1. A vector in $\mathbb{R}^n, n = 2, 3$ is an ordered pair(triple) of real numbers, such as

$$(a_1, a_2), \text{ or } (a_1, a_2, a_3).$$

Here a_1, a_2 are called **x-coordinate**, **y-coordinate** or **x-component**, **y-component** of (a_1, a_2). The point $(0, 0)$ is called the **origin** and denoted by O.

We use the boldface to denote vectors, e.g, $\mathbf{a} = (a_1, a_2)$ or $\mathbf{a} = (a_1, a_2, a_3)$ are standard notations for vectors. The notation $\vec{a}$ is also used. A point P in $\mathbb{R}^n$ can be represented by an ordered pair of real numbers (a_1, a_2) or (a_1, a_2, a_3) called **Cartesian coordinate)** of P. Thus, vectors are identified with points in the plane or space.

$$\mathbb{R}^2 = \{(a_1, a_2) \mid a_1 \in \mathbb{R}, a_2 \in \mathbb{R}\}.$$

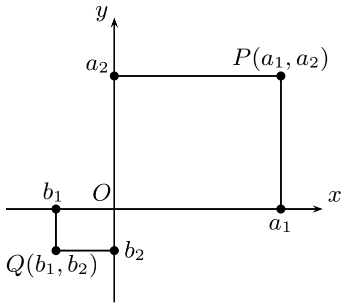

Figure 1.1: Coordinate plane

Vector addition and scalar multiplication-algebraic view

The operation of addition can be extended to $\mathbb{R}^3$. Given two triples, $\mathbf{a} = (a_1, a_2, a_3), \mathbf{b} = (b_1, b_2, b_3)$, we define

$$\mathbf{a} + \mathbf{b} = (a_1, a_2, a_3) + (b_1, b_2, b_3) = (a_1 + b_1, a_2 + b_2, a_3 + b_3)$$

to be the **sum** of (a_1, a_2, a_3) and (b_1, b_2, b_3). Thus we see that

$$\mathbf{a} + \mathbf{b} = \mathbf{b} + \mathbf{a}.$$

Two vectors $\mathbf{a} = (a_1, a_2, a_3)$ and $\mathbf{b} = (b_1, b_2, b_3)$ are equal if $a_1 = b_1, a_2 = b_2$ and $a_3 = b_3$. The vector $\mathbf{0} = (0, 0, 0)$ is the **zero element**. The vector $-(a_1, a_2, a_3) = (-a_1, -a_2, -a_3)$ is called the **additive inverse or negative** of (a_1, a_2, a_3).

Commutative law and associate law for additions:

(i) $(x, y, z) + (u, v, w) = (u, v, w) + (x, y, z)$ (commutative law)

(ii) $((x, y, z) + (u, v, w)) + (l, m, n)$
$= (x, y, z) + ((u, v, w) + (l, m, n))$ (associate law)

The **difference** is defined as

$$(a_1, a_2, a_3) - (b_1, b_2, b_3) = (a_1 - b_1, a_2 - b_2, a_3 - b_3).$$

Example 1.1.2.

$$(6,0,2)+(-10,3,2)=(-4,3,4)$$
$$(3,0,3)-(5,0,-2)=(-2,0,5)$$
$$(0,0,0)+(1,3,2)=(1,3,2)$$

For any real α, and (a_1, a_2, a_3) in $\mathbb{R}^3$, the **scalar multiple** $\alpha(a_1, a_2, a_3)$ is defined as

$$\alpha(a_1, a_2, a_3) = (\alpha a_1, \alpha a_2, \alpha a_3).$$

Additions and scalar multiplication has the following properties:

(i) $(\alpha\beta)(x, y, z) = \alpha(\beta(x, y, z))$ (associate law)

(ii) $(\alpha + \beta)(x, y, z) = \alpha(x, y, z) + \beta(x, y, z)$ (distributive law)

(iii) $\alpha((x, y, z) + (u, v, w)) = \alpha(x, y, z) + \alpha(u, v, w)$ (distributive law)

(iv) $\alpha(0, 0, 0) = (0, 0, 0)$ (property of 0)

(v) $0(x, y, z) = (0, 0, 0)$ (property of 0)

(vi) $1(x, y, z) = (x, y, z)$ (property of 1)

Example 1.1.3.

$$3\,(6,-3,2)=(18,-9,6)$$
$$1\,(3,5,-2)=(3,5,-2)$$
$$0\,(1,3,2)=(0,0,0)$$
$$(-2)\,(-2,1,3)=(4,-2,-6)$$

Example 1.1.4. Show

(1) $(\alpha + \beta)(x, y) = \alpha(x, y) + \beta(x, y)$

(2) $\alpha((x, y) + (u, v)) = \alpha(x, y) + \alpha(u, v)$

sol. (1) LHS is

$$\begin{aligned}(\alpha+\beta)(x,y) &= ((\alpha+\beta)x,(\alpha+\beta)y)\\ &= (\alpha x+\beta x,\alpha y+\beta y)\\ &= (\alpha x,\alpha y)+(\beta x,\beta y)\\ &= \alpha(x,y)+\beta(x,y)\end{aligned}$$

(2) LHS is

$$\begin{aligned}\alpha((x,y)+(u,v)) &= \alpha(x+u,y+v)\\ &= (\alpha(x+u),\alpha(y+v))\\ &= (\alpha x+\alpha u,\alpha y+\alpha v)\\ &= (\alpha x,\alpha y)+(\alpha u,\alpha v)\\ &= \alpha(x,y)+\alpha(u,v)\end{aligned}$$

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1.1.1 Lines, Planes and the Space

(1) The set of all real numbers is denoted by $\mathbb{R}$.

(2) The set of all ordered pairs of real numbers (x, y) is denoted by $\mathbb{R}^2$.

(3) The set of all ordered triples of real numbers (x, y, z) is denoted by $\mathbb{R}^3$.

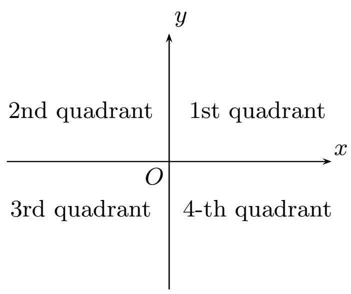

Figure 1.2: quadrant

The planes in $\mathbb{R}^3$determined by $z = 0$.(resp. $x = 0$ and $y = 0$) are called xy**-plane, (resp.** yz**-plane,** zx**-plane)** These planes divides the space into

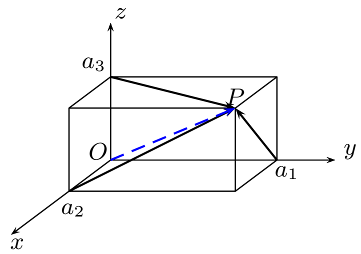

Figure 1.3: A point $P(a_1, a_2, a_3)$ as a vector

eight parts: Each of them is called **octant**. If every component is positive, it is called **the first octant**.

Example 1.1.5. (1) The xz-plane is the set of all points with $y = 0$:

$$\{(x, y) \mid y = 0\}.$$

(2) Similarly, the xy-plane is determined by $z = 0$:

$$\{(x, y, z) \mid z = 0\}.$$

(3) x-axis is determined by

$$\begin{cases} y = 0 \\ z = 0 \end{cases}$$

or

$$\{(x, y, z) \mid y = 0, z = 0\}.$$

Vectors-Geometric view

We can associate a vector **a** with a point (a_1, a_2, a_3) in the space. For example, we can visualize it with an arrow starting at the origin and ending at the point $\mathbf{a} = (a_1, a_2, a_3)$. One can also interpret a **vector** as a **directed line segment** i.e, a line segment with specified *magnitude* and *direction*.

Referring to the Figure 1.4, we denote the directed line segment PQ from P to Q by $\overrightarrow{PQ}$. P and Q are called **tail** and **head** respectively. A vector with tail at the origin is called a **position vector**. If two vectors have the same magnitude direction, we regard it as the same vector. In this case two vector

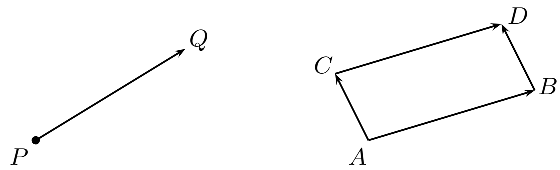

Figure 1.4: vectors

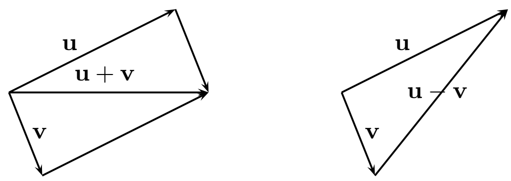

Figure 1.5: sum and difference of two vectors

can overlap exactly when moved in parallel. Given two points $P_1(x_1, y_1, z_1)$ and $P_2(x_2, y_2, z_2)$, the **displacement vector** from P_1 to P_2 is

$$\overrightarrow{P_1P_2} = (x_2 - x_1, y_2 - y_1, z_2 - z_1).$$

Referring to the parallelogram $ABDC$ in Figure 1.4, we see $\overrightarrow{AB} = \overrightarrow{CD}$ and $\overrightarrow{AC} = \overrightarrow{BD}$.

See figure 1.5 (1). If two vectors $\mathbf{u}$, $\mathbf{v}$ have the same tail P, the sum $\mathbf{u}+\mathbf{v}$ is the vector ending at the opposite vertex of the parallelogram formed by $\mathbf{u}$ and $\mathbf{v}$.

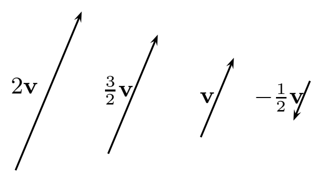

Figure 1.6: scalar multiples of $\mathbf{v}$

(i) $\mathbf{u}+\mathbf{v}=\mathbf{v}+\mathbf{u}$ (commutative law)

(ii) $(\mathbf{u}+\mathbf{v})+\mathbf{w}=\mathbf{u}+(\mathbf{v}+\mathbf{w})$ (associate law)

Scalar multiple of a vector

For a real number(scalar) s and a vector $\mathbf{v}$, the scalar multiple $s\mathbf{v}$(see Fig 1.11) is the vector having magnitude $|s|$ times that of $\mathbf{v}$, having the same direction as $\mathbf{v}$ when $s>0$, opposite direction when $s<0$.

The followings hold:

(iii) $(st)\mathbf{u}=s(t\mathbf{u})$ (associative law)

(iv) $(s+t)\mathbf{u}=s\mathbf{u}+t\mathbf{u}$ (distributive law)

(v) $s(\mathbf{u}+\mathbf{v})=s\mathbf{u}+s\mathbf{v}$ (distributive law)

(vi) $s\mathbf{0}=\mathbf{0}$ (0-vector)

(vii) $0\mathbf{u}=\mathbf{0}$

(viii) $1\mathbf{u}=\mathbf{u}$

Example 1.1.6 (3D vectors)**.** A 3D vector is denoted by, say

$$\mathbf{a}=(a_1,a_2,a_3).$$

Here $a_1,\ a_2,\ a_3$ are called **x-component, y- component, z -component** of $\mathbf{a}$. Let $A=(a_1,a_2,a_3)$. Shift the line segment OA by b_1 along x-axis, by b_2 along y-axis, b_3 along z-axis respectively. We obtain a vector denoted by BP. (See figure 1.7) Then the coordinate of B and P are (b_1,b_2,b_3) and $(a_1+b_1,a_2+b_2,a_3+b_3)$, and $OBPA$ form a parallelogram. Hence

$$\overrightarrow{OA}+\overrightarrow{OB}=\overrightarrow{OP}.$$

Standard basis vectors

Definition 1.1.7. The following vectors $\mathbf{i}$, $\mathbf{j}$, $\mathbf{k}$ are called **(standard basis vector)** of $\mathbb{R}^3$ (Figure 1.13).

$$\mathbf{i}=(1,0,0),\ \mathbf{j}=(0,1,0),\ \mathbf{k}=(0,0,1)$$

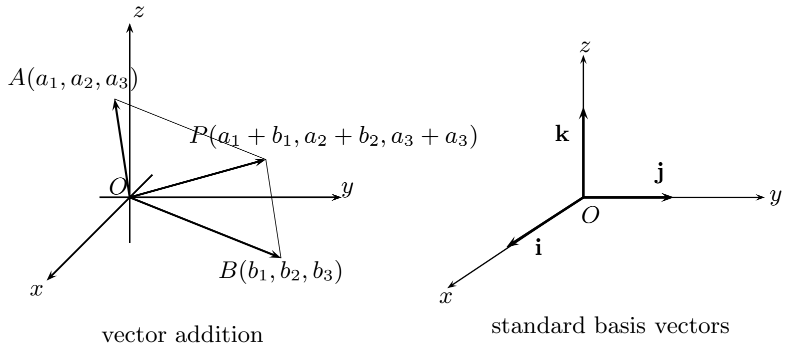

Figure 1.7: Vectors in 3D

Remark 1.1.8. (1) For a given $\mathbf{v} = (a_1, a_2, a_3)$

$$(a_1, a_2, a_3) = a_1(1, 0, 0) + a_2(0, 1, 0) + a_3(0, 0, 1) = a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k}$$

Example 1.1.9. Write the following using standard basis vectors.

(1) $\mathbf{v} = (-1/2, 3, 5)$.

(2) Express $3\mathbf{a} - 2\mathbf{b}$ when $\mathbf{a} = (3, 5, 0)$, $\mathbf{b} = (-4, 1, 1)$.

(3) Given two points $P(1, 4, 3)$ and $Q(4, 1, 2)$, express $\overrightarrow{PQ}$.

(4) Given three points $A(0, -1, 4)$, $B(2, 4, 1)$ and $C(3, 0, 2)$, express

$$\frac{1}{2}\overrightarrow{OA} + \frac{1}{3}\overrightarrow{OB} + \frac{1}{6}\overrightarrow{OC}.$$

sol.

(1) $\mathbf{v} = (-1/2)\mathbf{i} + 3\mathbf{j} + 5\mathbf{k}$

(2) $$\begin{aligned} 3\mathbf{a} - 2\mathbf{b} &= 3(3\mathbf{i} + 5\mathbf{j}) - 2(-4\mathbf{i} + \mathbf{j} + \mathbf{k}) \\ &= (9 + 8)\mathbf{i} + (15 - 2)\mathbf{j} + (-2)\mathbf{k} = 17\mathbf{i} + 13\mathbf{j} - 2\mathbf{k} \end{aligned}$$

(3) $\overrightarrow{PQ} = (4 - 1)\mathbf{i} + (1 - 4)\mathbf{j} + (2 - 3)\mathbf{k} = 3\mathbf{i} - 3\mathbf{j} - \mathbf{k}$

(4) $$\begin{aligned} &(1/2)\overrightarrow{OA} + (1/3)\overrightarrow{OB} + (1/6)\overrightarrow{OC} \\ &= (1/2)(-\mathbf{j} + 4\mathbf{k}) + (1/3)(2\mathbf{i} + 4\mathbf{j} + \mathbf{k}) + (1/6)(3\mathbf{i} + 2\mathbf{k}) \\ &= (7/6)\mathbf{i} + (5/6)\mathbf{j} + (8/3)\mathbf{k} \end{aligned}$$

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1.2 More about vectors

Parametric equation of lines(Point-direction form)

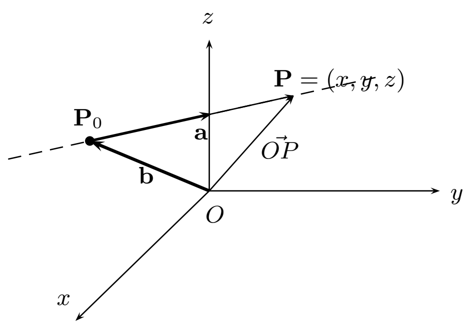

Figure 1.8: A line is determined by a point and a vector

The equation of the line ℓ through the point $P_0 = \mathbf{b}$ and pointing in the direction of $\vec{P_0P} = \mathbf{a}$ is given by

$$\ell(t) = \overrightarrow{OP_0} + t\overrightarrow{P_0P} = \mathbf{b} + t\mathbf{a},\ t \in \mathbb{R}$$

where $\mathbf{b} = (x_1, y_1, z_1)$ and $\mathbf{a} = (a, b, c)$. In coordinate form, we have

$$\begin{aligned} x &= x_1 + at, \\ y &= y_1 + bt, \\ z &= z_1 + ct, \end{aligned}$$

Example 1.2.1. (1) Find equation of line through $(2, 1, 5)$ in the direction of $4\mathbf{i} - 2\mathbf{j} + 5\mathbf{k}$.

(2) To what direction, does the line $x = 3t - 2, y = t - 1, z = 7t + 4$ points ?

sol. (1) $\mathbf{v} = (2, 1, 5) + t(4, -2, 5)$
(2) $(3, 1, 7) = 3\mathbf{i} + \mathbf{j} + 7\mathbf{k}$.

Two point form

We describe the equation of line through two points $P = (x_1, y_1, z_1), Q = (x_2, y_2, z_2)$. If we let $\mathbf{a} = (x_1, y_1, z_1), \mathbf{b} = (x_2, y_2, z_2)$.

The direction is given by $\mathbf{v} = \mathbf{b} - \mathbf{a} = (x_2 - x_1, y_2 - y_1, z_2 - z_1)$. So by the point-direction form we see the equation is

$$\ell(t) = \mathbf{a} + t(\mathbf{b} - \mathbf{a}).$$

In components, we see

$$\begin{aligned} x &= x_1 + (x_2 - x_1)t \\ y &= y_1 + (y_2 - y_1)t \\ z &= z_1 + (z_2 - z_1)t \end{aligned}$$

Solving these for t and equating, we see

$$\boxed{\frac{x - x_1}{x_2 - x_1} = \frac{y - y_1}{y_2 - y_1} = \frac{z - z_1}{z_2 - z_1}.} \tag{1.1}$$

This is another equation of the line **(symmetric form)**.

Example 1.2.2. Find the equation of a line through $(2, 1, -3)$ and $(6, -1, -5)$.

A direction vector is $(4, -2, 2)$. So

$$\ell(t) = (2, 1, -3) + t(4, -2, 2)$$

Example 1.2.3. Find the equation of the *line segment* between $(1, 1, -3)$ and $(2, -1, 0)$

sol. We get $\ell(t) = (1, 1, -3) + t(1, -2, 3)$ but note the domain $0 \leq t \leq 1$.

■

Example 1.2.4. Find the point where the the line given by the equations

$$\begin{cases} x &= t + 5 \\ y &= -2t - 4 \\ z &= 3t + 7 \end{cases}$$

intersect the plane $3x + 2y - 7z = 2$.

sol. We must find the value of t which gives the intersection point. Substituting the expression x, y, z into the equation of the plane, we see

$$3(t+5) + 2(-2t-4) - 7(3t+7) = 2.$$

Solving we get $t = -2$. Hence the point is $(3, 0, 1)$.

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Example 1.2.5. Does the two lines $(x, y, z) = (t, -6t+1, 2t-8)$ and $(3t+1, 2t, 0)$ intersect ?

sol. If two line intersect, we must have

$$(t_1, -6t_1, 2t_1 - 8) = (3t_2 + 1, 2t_2, 0)$$

for some numbers t_1, t_2.(Note: we have used two different parameters t_1 and t_2). Since the system of equation

$$\begin{aligned} t_1 &= 3t_2 + 1 \\ -6t_1 &= 2t_2 \\ 2t_1 - 8 &= 0 \end{aligned}$$

has no solution, the lines do not meet.

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Cycloid

Assume a circle of radius a is rolling on the x-axis. Let P be a point on the circle located at the origin in the beginning. As the circle rolls, P starts to move from the origin. (Fig 1.9) The trajectory of P is called a *cycloid*. If circle rotates by t radian, then $P = (x, y)$ is given by

$$x = at + a\cos\theta, \qquad y = a + a\sin\theta. \tag{1.2}$$

Since $\theta = \frac{3\pi}{2} - t$, $\cos\theta = -\sin t$, $\sin\theta = -\cos t$, we have

$$x = a(t - \sin t), \qquad y = a(1 - \cos t).$$

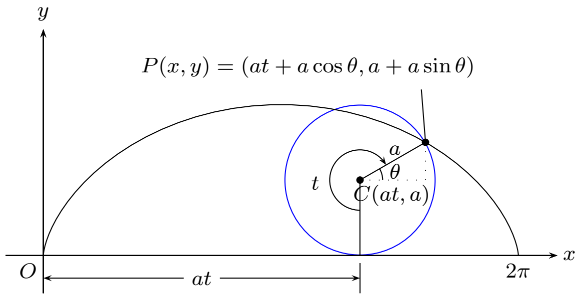


Figure 1.9: Cycloid

Involute

Imagine you unwind adhesive tape from a fixed circular tape dispenser(or a roll or wire). Assume the unwound tape is taut and tangent to the dispenser roll. The vector representing tip of the tape from the dispenser roll is

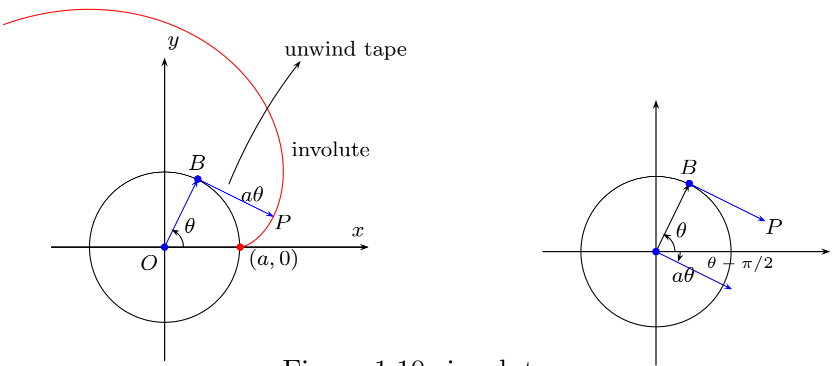


Figure 1.10: involute

$$\vec{BP} = a\theta\cos(\theta - \frac{\pi}{2})\mathbf{i} + a\theta\sin(\theta - \frac{\pi}{2})\mathbf{j}.$$

Hence

$$\vec{OP} = \vec{OB} + \vec{BP} = a(\cos\theta + \theta\sin\theta)\mathbf{i} + a(\sin\theta - \theta\cos\theta)\mathbf{j}.$$

So the coordinate of P is

$$\begin{cases} x = a(\cos\theta + \theta\sin\theta) \\ y = a(\sin\theta - \theta\cos\theta). \end{cases}$$

1.3 Inner product, length, distance

Dot product-Inner product

Definition 1.3.1. Given two vectors $\mathbf{a} = a_1\mathbf{i}+a_2\mathbf{j}+a_3\mathbf{k}$ and $\mathbf{b} = b_1\mathbf{i}+b_2\mathbf{j}+b_3\mathbf{k}$ we define

$$a_1b_1 + a_2b_2 + a_3b_3$$

to be the **dot product** or **(inner product)** of $\mathbf{a}$ and $\mathbf{b}$ and write $\mathbf{a}\cdot\mathbf{b}$.

Example 1.3.2. Let $\mathbf{a} = 2\mathbf{i} - 3\mathbf{j} + \mathbf{k}$, $\mathbf{b} = \mathbf{i} + 2\mathbf{j} - \mathbf{k}$. Find

(1) $\mathbf{a}\cdot\mathbf{a}$

(2) $\mathbf{a}\cdot\mathbf{b}$

(3) $\mathbf{a}\cdot(\mathbf{a} - 3\mathbf{b})$

(4) $(3\mathbf{a} + 2\mathbf{b})\cdot(\mathbf{a} - \mathbf{b})$

sol. (1) $\mathbf{a}\cdot\mathbf{a} = 4 + 9 + 1 = 14$

(2) $\mathbf{a}\cdot\mathbf{b} = 2 - 6 - 1 = -5$

(3) $\mathbf{a}\cdot(\mathbf{a} - 3\mathbf{b}) = (2\mathbf{i} - 3\mathbf{j} + \mathbf{k})\cdot(-\mathbf{i} - 9\mathbf{j} + 4\mathbf{k})$
$= -2 + 27 + 4 = 29$

(4) $(3\mathbf{a} + 2\mathbf{b})\cdot(\mathbf{a} - \mathbf{b}) = (8\mathbf{i} - 5\mathbf{j} + \mathbf{k})\cdot(\mathbf{i} - 5\mathbf{j} + 2\mathbf{k})$
$= 8 + 25 + 2 = 35$

∎

Proposition 1.3.3 (Properties of Inner Product)**.** *For vectors* $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$ *and scalar* α, *the following hold:*

(1) $\mathbf{a}\cdot\mathbf{a} \geq 0$ *(equality holds only when* $\mathbf{a} = \mathbf{0}$*)*

(2) $\mathbf{a}\cdot\mathbf{b} = \mathbf{b}\cdot\mathbf{a}$

(3) $(\mathbf{a} + \mathbf{b})\cdot\mathbf{c} = \mathbf{a}\cdot\mathbf{c} + \mathbf{b}\cdot\mathbf{c}$

(4) $(\alpha\mathbf{a})\cdot\mathbf{b} = \alpha(\mathbf{a}\cdot\mathbf{b})$

(5) $\|\mathbf{a}\| = \sqrt{\mathbf{a}\cdot\mathbf{a}}$

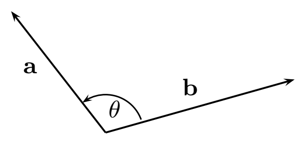


Figure 1.11: Angle between two vectors

Proof. These can be proved easily. □

Example 1.3.4. For **a**, **b**, **c** Show the following.

(1) $(\mathbf{a}-\mathbf{b})\cdot\mathbf{c}=\mathbf{a}\cdot\mathbf{c}-\mathbf{b}\cdot\mathbf{c}$

(2) $\mathbf{a}\cdot(\mathbf{b}+\mathbf{c})=\mathbf{a}\cdot\mathbf{b}+\mathbf{a}\cdot\mathbf{c}$

(3) $\mathbf{a}\cdot(\mathbf{b}-\mathbf{c})=\mathbf{a}\cdot\mathbf{b}-\mathbf{a}\cdot\mathbf{c}$

(4) $\mathbf{a}\cdot\mathbf{b}=\frac{1}{2}(\|\mathbf{a}\|^2+\|\mathbf{b}\|^2-\|\mathbf{a}-\mathbf{b}\|^2)$

sol. We see

(1) $(\mathbf{a}-\mathbf{b})\cdot\mathbf{c}=(\mathbf{a}+(-1)\mathbf{b}))\cdot\mathbf{c}=\mathbf{a}\cdot\mathbf{c}+((-1)\mathbf{b})\cdot\mathbf{w}$
$=\mathbf{a}\cdot\mathbf{c}+(-1)\mathbf{b}\cdot\mathbf{c}=\mathbf{a}\cdot\mathbf{c}-\mathbf{b}\cdot\mathbf{c}$

(2) $\mathbf{a}\cdot(\mathbf{b}+\mathbf{c})=(\mathbf{b}+\mathbf{c})\cdot\mathbf{a}=\mathbf{b}\cdot\mathbf{a}+\mathbf{c}\cdot\mathbf{a}=\mathbf{a}\cdot\mathbf{b}+\mathbf{a}\cdot\mathbf{c}$

(3) $\mathbf{a}\cdot(\mathbf{b}-\mathbf{c})=(\mathbf{b}-\mathbf{c})\cdot\mathbf{a}=\mathbf{b}\cdot\mathbf{a}-\mathbf{c}\cdot\mathbf{a}=\mathbf{a}\cdot\mathbf{b}-\mathbf{a}\cdot\mathbf{c}$

(4) $\|\mathbf{a}-\mathbf{b}\|^2=(\mathbf{a}-\mathbf{b})\cdot(\mathbf{a}-\mathbf{b})=\mathbf{a}\cdot(\mathbf{a}-\mathbf{b})-\mathbf{b}\cdot(\mathbf{a}-\mathbf{b})$
$=\mathbf{a}\cdot\mathbf{a}-\mathbf{a}\cdot\mathbf{b}-\mathbf{b}\cdot\mathbf{a}+\mathbf{b}\cdot\mathbf{b}=\|\mathbf{a}\|^2-2\,\mathbf{a}\cdot\mathbf{b}+\|\mathbf{b}\|^2$

■

Length of vectors

The **length, norm** of a vector $\mathbf{a}=(a_1,a_2,a_3)$ is

$$\sqrt{(a_1-0)^2+(a_2-0)^2+(a_3-0)^2}=\sqrt{a_1^2+a_2^2+a_3^2}$$

denoted by $\|\mathbf{a}\|$. Also we note that

$$\|\mathbf{a}\|=(\mathbf{a}\cdot\mathbf{a})^{1/2}.$$

Example 1.3.5. Find the lengths of the following vectors.

(1) $\mathbf{a} = (3, 2, 1)$

(2) $3\mathbf{i} - 4\mathbf{j} + \mathbf{k}$

(3) $\overrightarrow{AB}$ when $A(2, -1/3, -1)$, $B(8/3, 0, 1)$.

sol. (1) $\|\mathbf{a}\| = \sqrt{9+4+1} = \sqrt{14}$
(2) $\|3\mathbf{i} - 4\mathbf{j} + \mathbf{k}\| = \sqrt{9+16+1} = \sqrt{26}$
(3) $\|\overrightarrow{AB}\| = \sqrt{(8/3-2)^2 + (0-(-1/3))^2 + (1-(-1))^2}$
$= \sqrt{4/9 + 1/9 + 4} = \sqrt{41}/3$

■

Definition 1.3.6. A vector with norm 1 is called a **unit vector**. Any nonzero vector $\mathbf{a}$ can be made into a unit vector by setting $\mathbf{a}/\|\mathbf{a}\|$. This process is called a **normalization**.

Example 1.3.7. Normalize the followings.

(1) $\mathbf{i} + \mathbf{j} + \mathbf{k}$

(2) $3\mathbf{i} + 4\mathbf{k}$

(3) $a\mathbf{i} - \mathbf{j} + c\mathbf{k}$

sol. (1) $(1/\sqrt{3})\,\mathbf{i} + (1/\sqrt{3})\,\mathbf{j} + (1/\sqrt{3})\,\mathbf{k}$
(2) $(3/5)\,\mathbf{i} + (4/5)\,\mathbf{k}$
(3) $(a/\sqrt{1+a^2+c^2})\,\mathbf{i} - (1/\sqrt{1+a^2+c^2})\,\mathbf{j} + (c/\sqrt{1+a^2+c^2})\,\mathbf{k}$

■

Angle between two vectors

Proposition 1.3.8. *Let $\mathbf{a}, \mathbf{b}$ be two nonzero vectors in $\mathbb{R}^2$ or $\mathbb{R}^3$ and let θ be the angle between them. Then*

$$\mathbf{a} \cdot \mathbf{b} = \|\mathbf{a}\|\,\|\mathbf{b}\|\,\cos\theta$$

and hence

$$\theta = \cos^{-1}\left(\frac{\mathbf{a} \cdot \mathbf{b}}{\|\mathbf{a}\|\,\|\mathbf{b}\|}\right)$$

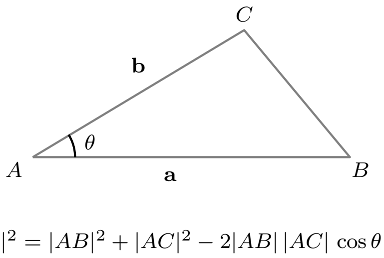


Figure 1.12: law of cosine

Proof. Let $\mathbf{a} = \overrightarrow{AB}$, $\mathbf{b} = \overrightarrow{AC}$. Then $\mathbf{a} - \mathbf{b} = \overrightarrow{CB}$.

Let $\angle CAB = \theta$. Then by the law of cosine (figure 1.12) we have

$$\|\mathbf{b} - \mathbf{a}\|^2 = \|\mathbf{a}\|^2 + \|\mathbf{b}\|^2 - 2\|\mathbf{a}\|\,\|\mathbf{b}\|\,\cos\theta.$$

The left hand side is

$$\begin{aligned}\|\mathbf{a} - \mathbf{b}\|^2 &= (\mathbf{a} - \mathbf{b}) \cdot (\mathbf{a} - \mathbf{b}) \\ &= \mathbf{a} \cdot \mathbf{a} - \mathbf{a} \cdot \mathbf{b} - \mathbf{b} \cdot \mathbf{a} + \mathbf{b} \cdot \mathbf{b} \\ &= \|\mathbf{a}\|^2 - 2\,\mathbf{a} \cdot \mathbf{b} + \|\mathbf{b}\|^2.\end{aligned}$$

Hence we obtain

$$\|\mathbf{a}\|\,\|\mathbf{b}\|\,\cos\theta = \mathbf{a} \cdot \mathbf{b}.$$

□

Corollary 1.3.9. *Two nonzero vector* $\mathbf{a}$ *and* $\mathbf{b}$ *are* ***perpendicular, orthogonal*** *if and only if* $\mathbf{a} \cdot \mathbf{b} = 0$.

Example 1.3.10. Find the angle between $\mathbf{i} + \mathbf{j} + 2\mathbf{k}$ and $-\mathbf{i} + 2\mathbf{j} + \mathbf{k}$.

sol. By proposition 1.2.10,

$$\frac{(\mathbf{i} + \mathbf{j} + 2\mathbf{k}) \cdot (-\mathbf{i} + 2\mathbf{j} + \mathbf{k})}{\|\mathbf{i} + \mathbf{j} + 2\mathbf{k}\|\,\| - \mathbf{i} + 2\mathbf{j} + \mathbf{k}\|} = \frac{-1 + 2 + 2}{\sqrt{1 + 1 + 4}\sqrt{1 + 4 + 1}} = \frac{3}{6} = \frac{1}{2}.$$

Hence the angle is $\cos^{-1}(1/2) = \pi/3$.

■

Corollary 1.3.11. *Given two points* $A(a_1, a_2, a_3)$, $B(b_1, b_2, b_3)$, *the area of* *the triangle OAB is*

$$\frac{1}{2}\sqrt{(a_2b_3 - a_3b_2)^2 + (a_3b_1 - a_1b_3)^2 + (a_1b_2 - a_2b_1)^2}.$$

Proof. Let $\overrightarrow{OA} = \mathbf{a}$, $\overrightarrow{OB} = \mathbf{b}$, $\angle BOA = \theta$. Then the area of $\triangle OAB$ is

$$\begin{aligned}
&\frac{1}{2}|OA|\,|OB|\,\sin\theta \\
&= \frac{1}{2}\|\mathbf{a}\|\,\|\mathbf{b}\|\,\sqrt{1-\cos^2\theta} \\
&= \frac{1}{2}\sqrt{\|\mathbf{a}\|^2\,\|\mathbf{b}\|^2 - (\mathbf{a}\cdot\mathbf{b})^2} \\
&= \frac{1}{2}\sqrt{(a_1^2+a_2^2+a_3^2)(b_1^2+b_2^2+b_3^2) - (a_1b_1+a_2b_2+a_3b_3)^2} \\
&= \frac{1}{2}\sqrt{(a_2b_3 - a_3b_2)^2 + (a_3b_1 - a_1b_3)^2 + (a_1b_2 - a_2b_1)^2}.
\end{aligned}$$

□

Example 1.3.12. Find the area of the triangle with vertices $A(a,0,0), B(0,b,0), C(0,0,c)$.

sol. Shift(translate) A to the origin, then the points B, C are moved to the points $(-a,b,0)$ and $(-a,0,c)$. Hence

$$\frac{1}{2}\sqrt{(bc-0)^2 + (0+ac)^2 + (0+ab)^2} = \frac{1}{2}\sqrt{b^2c^2 + c^2a^2 + a^2b^2}.$$

■

Theorem 1.3.13 (Cauchy-Schwarz inequality)**.** *For any two vectors* **a**, **b**

$$|\mathbf{a}\cdot\mathbf{b}| \le \|\mathbf{a}\|\,\|\mathbf{b}\|$$

holds, and the equality holds iff **a** *and* **b** *are parallel.*

Proof. We may assume **a**, **b** are nonzero. Let θ be the angle between **a** and **b**. Then by prop 1.3.8

$$|\mathbf{a}\cdot\mathbf{b}| = \|\mathbf{a}\|\,\|\mathbf{b}\|\,|\cos\theta| \le \|\mathbf{a}\|\,\|\mathbf{b}\|$$

holds. Since $\|\mathbf{a}\|\,\|\mathbf{b}\| \neq 0$, if equality holds $|\cos\theta| = 1$ i.e, $\theta = 0$ or π. Hence **a** and **b** are parallel. □

Remark 1.3.14. The Cauchy-Schwarz inequality reads, componentwise, as

$$(ax+by+cz)^2 \le (a^2+b^2+c^2)(x^2+y^2+z^2).$$

Example 1.3.15. Show Cauchy-Schwarz inequality for $\mathbf{i}+3\mathbf{j}+2\mathbf{k}$, $-\mathbf{i}+\mathbf{j}$.

sol. Since the inner product and lengths are

$$\begin{aligned}&(\mathbf{i}+3\mathbf{j}+2\mathbf{k})\cdot(-\mathbf{i}+\mathbf{j}) = -1+3 = 2,\\ &\|\mathbf{i}+3\mathbf{j}+2\mathbf{k}\|\,\|-\mathbf{i}+\mathbf{j}\| = \sqrt{1+9+4}\,\sqrt{1+1} = \sqrt{28} = 2\sqrt{7}\end{aligned}$$

we have

$$|(\mathbf{i}+3\mathbf{j}+2\mathbf{k})\cdot(-\mathbf{i}+\mathbf{j})| \le \|\mathbf{i}+3\mathbf{j}+2\mathbf{k}\|\,\|-\mathbf{i}+\mathbf{j}\|.$$

■

Theorem 1.3.16 (Triangle inequality)**.** *For any two vector* $\mathbf{a}$, $\mathbf{b}$ *it holds that*

$$\|\mathbf{a}+\mathbf{b}\| \le \|\mathbf{a}\| + \|\mathbf{b}\|$$

and equality holds when $\mathbf{a}$, $\mathbf{b}$ *are parallel and having same direction.*

Proof. We have

$$\|\mathbf{a}+\mathbf{b}\|^2 = (\mathbf{a}+\mathbf{b})\cdot(\mathbf{a}+\mathbf{b}) = \|\mathbf{a}\|^2 + 2\mathbf{a}\cdot\mathbf{b} + \|\mathbf{b}\|^2.$$

By C-S

$$\|\mathbf{a}+\mathbf{b}\|^2 \le \|\mathbf{a}\|^2 + 2\|\mathbf{a}\|\,\|\mathbf{b}\| + \|\mathbf{b}\|^2 = (\|\mathbf{a}\|+\|\mathbf{b}\|)^2.$$

Equality holds iff

$$\mathbf{a}\cdot\mathbf{b} = \|\mathbf{a}\|\,\|\mathbf{b}\|,$$

i.e, the angle is 0. □

Example 1.3.17. Show triangle inequality for $-\mathbf{i}+\mathbf{j}$ and $\mathbf{i}+3\mathbf{j}+2\mathbf{k}$.

sol. Sum and difference is

$$\begin{aligned}\|(\mathbf{i}+3\mathbf{j}+2\mathbf{k})+(-\mathbf{i}+\mathbf{j})\| &= \|4\mathbf{j}+2\mathbf{k}\| \;=\; \sqrt{16+4}\\ &= 2\sqrt{5} \;=\; 4.4721\ldots\\ \|\mathbf{i}+3\mathbf{j}+2\mathbf{k}\| + \|-\mathbf{i}+\mathbf{j}\| &= \sqrt{1+9+4}+\sqrt{1+1}\\ &= \sqrt{14}+\sqrt{2} \;=\; 5.1558\ldots\end{aligned}$$

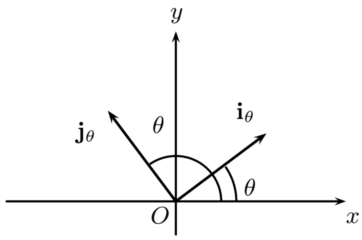


Figure 1.13: $\mathbf{i}_\theta$ and $\mathbf{j}_\theta$

Hence

$$\|(\mathbf{i}+3\mathbf{j}+2\mathbf{k})+(-\mathbf{i}+\mathbf{j})\| \le \|\mathbf{i}+3\mathbf{j}+2\mathbf{k}\| + \|-\mathbf{i}+\mathbf{j}\|.$$

■

Definition 1.3.18. If two vectors **a**, **b** satisfy $\mathbf{a}\cdot\mathbf{b} = 0$ then we say they are **orthogonal**(perpendicular).

Example 1.3.19. For any real θ, the two vectors $\mathbf{i}_\theta = (\cos\theta)\mathbf{i} + (\sin\theta)\mathbf{j}$, $\mathbf{j}_\theta = -(\sin\theta)\mathbf{i} + (\cos\theta)\mathbf{j}$ are orthogonal.

Example 1.3.20. Find a unit vector orthogonal to $2\mathbf{i}-\mathbf{j}+3\mathbf{k}$ and $\mathbf{i}+2\mathbf{j}+9\mathbf{k}$.

sol. Let $a\mathbf{i}+b\mathbf{j}+c\mathbf{k}$ be the desired vector. Then a, b, c are determined by

$$\begin{aligned} 2a-b+3c &= 0 \text{ (orthogonality)} \\ a+2b+9c &= 0 \text{ (orthogonality)} \\ a^2+b^2+c^2 &= 1 \text{ (unicity).} \end{aligned}$$

Hence the desired vector is

$$\pm\frac{1}{\sqrt{19}}(3\mathbf{i}+3\mathbf{j}-\mathbf{k}).$$

■

Orthogonal projection

Given two nonzero vectors **a** and **b**, we may define the **orthogonal projection** of **b** onto **a** to be the vector **p** given in the figure 1.14. Since **p** is a scalar multiple of **a**, there is a constant c such that $\mathbf{p} = c\mathbf{a}$. We let

$$\mathbf{b} = c\mathbf{a} + \mathbf{q},$$

where $\mathbf{q}$ is a vector orthogonal to $\mathbf{a}$. Taking inner product with $\mathbf{a}$, we have

$$\mathbf{a}\cdot\mathbf{b} = c\mathbf{a}\cdot\mathbf{a}.$$

Hence we obtain $c = (\mathbf{a}\cdot\mathbf{b})/(\mathbf{a}\cdot\mathbf{a})$. Thus the orthogonal projection is

$$\mathbf{p} = \frac{\mathbf{a}\cdot\mathbf{b}}{\|\mathbf{a}\|^2}\mathbf{a}.$$

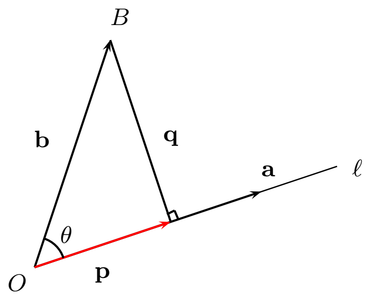


Figure 1.14: Projection of **b** onto **a**

The length of $\mathbf{p}$ is

$$\|\mathbf{p}\| = \frac{|\mathbf{a}\cdot\mathbf{b}|}{\|\mathbf{a}\|^2}\|\mathbf{a}\| = \frac{|\mathbf{a}\cdot\mathbf{b}|}{\|\mathbf{a}\|} = \|\mathbf{b}\|\cos\theta.$$

This agrees with the geometric interpretation.

Definition 1.3.21. For nonzero vector $\mathbf{b}$ and any vector $\mathbf{a}$, we define

$$proj_{\mathbf{a}}\mathbf{b} = \mathbf{p} = \frac{\mathbf{a}\cdot\mathbf{b}}{\|\mathbf{a}\|^2}\mathbf{a}.$$

We call it **orthogonal projection of b onto a**.

Example 1.3.22. $\mathbf{a} = 3\mathbf{i}+2\mathbf{j}-\mathbf{k}$, $\mathbf{b} = \mathbf{i}+\mathbf{j}+2\mathbf{k}$. Find orthogonal projection of $\mathbf{b}$ onto $\mathbf{a}$.

sol. The orthogonal projection is

$$\begin{aligned}\frac{\mathbf{a}\cdot\mathbf{b}}{\|\mathbf{a}\|^2}\mathbf{a} &= \frac{3\cdot 1 + 2\cdot 1 + (-1)\cdot 2}{9+4+1}(3\mathbf{i}+2\mathbf{j}-\mathbf{k})\\ &= \frac{9}{14}\mathbf{i} + \frac{6}{14}\mathbf{j} - \frac{3}{14}\mathbf{k}.\end{aligned}$$

■

Theorem 1.3.23. *For any two nonzero* $\mathbf{u}$ *and* $\mathbf{v}$, *we can write* $\mathbf{v}$ *as the sum of two orthogonal vectors* $\mathbf{a}+\mathbf{b}$, *where* $\mathbf{a}$ *is the projection of* $\mathbf{v}$ *onto* $\mathbf{u}$ *and* $\mathbf{b}$ *is orthogonal to* $\mathbf{u}$. *This decomposition is unique.*

Proof. Denote by $\mathbf{a}$ the projection of $\mathbf{v}$ onto $\mathbf{u}$ and let $\mathbf{b}=\mathbf{v}-\mathbf{a}$. Then

$$\mathbf{v}=\frac{\mathbf{u}\cdot\mathbf{v}}{\|\mathbf{u}\|^2}\mathbf{u}+\mathbf{v}-\frac{\mathbf{u}\cdot\mathbf{v}}{\|\mathbf{u}\|^2}\mathbf{u}\equiv\mathbf{a}+\mathbf{b}.$$

We can check $\mathbf{b}$ is orthogonal to $\mathbf{u}$:

$$\begin{aligned}\mathbf{u}\cdot\mathbf{b}&=\mathbf{u}\cdot\left(\mathbf{v}-\frac{\mathbf{u}\cdot\mathbf{v}}{\|\mathbf{u}\|^2}\mathbf{u}\right)\\&=\mathbf{u}\cdot\mathbf{v}-\frac{\mathbf{u}\cdot\mathbf{v}}{\|\mathbf{u}\|^2}\mathbf{u}\cdot\mathbf{u}\\&=\mathbf{u}\cdot\mathbf{v}-\mathbf{u}\cdot\mathbf{v}=0.\end{aligned}$$

This is an orthogonal decomposition. To see the uniqueness, assume there is real number α s.t. $\mathbf{v}=\alpha\mathbf{u}+\mathbf{c}$, with $\mathbf{u}\cdot\mathbf{c}=0$. Then

$$\mathbf{u}\cdot\mathbf{v}=\mathbf{u}\cdot(\alpha\mathbf{u}+\mathbf{c})=\alpha\mathbf{u}\cdot\mathbf{u}+\mathbf{u}\cdot\mathbf{c}=\alpha\|\mathbf{u}\|^2.$$

Hence we see

$$\begin{aligned}\alpha\mathbf{u}=\frac{\mathbf{u}\cdot\mathbf{v}}{\|\mathbf{u}\|^2}\mathbf{u}&=\mathbf{a}\\ \mathbf{c}=\mathbf{v}-\alpha\mathbf{u}&=\mathbf{v}-\mathbf{a}=\mathbf{b}.\end{aligned}$$

Thus the decomposition of $\mathbf{v}$ along $\mathbf{u}$ and its orthogonal component is unique.

□

Definition 1.3.24. The vector $\mathbf{a}$ is called the **component parallel to u** and $\mathbf{b}$ is the **component orthogonal to u**.(**orthogonal complement**).

Example 1.3.25. Find the orthogonal decomposition of $\mathbf{v}=3\mathbf{i}+5\mathbf{j}+\mathbf{k}$ w.r.t. $\mathbf{u}=\mathbf{i}+2\mathbf{j}-\mathbf{k}$.

sol. Let $\mathbf{a}$ be the projection of $\mathbf{v}$ onto $\mathbf{u}$ and $\mathbf{b} = \mathbf{v} - \mathbf{a}$. Then

$$\begin{aligned}
\mathbf{a} &= \frac{\mathbf{u}\cdot\mathbf{v}}{\|\mathbf{u}\|^2}\mathbf{u} \\
&= \frac{1\cdot 3 + 2\cdot 5 + (-1)\cdot 1}{1+4+1}(\mathbf{i} + 2\mathbf{j} - \mathbf{k}) \\
&= 2\mathbf{i} + 4\mathbf{j} - 2\mathbf{k} \\
\mathbf{b} &= (3\mathbf{i} + 5\mathbf{j} + \mathbf{k}) - (2\mathbf{i} + 4\mathbf{j} - 2\mathbf{k}) \\
&= \mathbf{i} + \mathbf{j} + 3\mathbf{k}.
\end{aligned}$$

Here $\mathbf{a}$ is parallel to $\mathbf{u}$, $\mathbf{b}$ is orthogonal to $\mathbf{u}$ and $\mathbf{v} = \mathbf{a} + \mathbf{b}$.

□

Do examples 4,5 in p.22, 23.

Triangle inequality

Theorem 1.3.26. *For any vectors* $\mathbf{a}, \mathbf{b}$*, we have*

$$\|\mathbf{a} + \mathbf{b}\| \le \|\mathbf{a}\| + \|\mathbf{b}\|.$$

Use C-S.

Physical applications

Displacement : If an object has moved from P to Q, then $\vec{PQ}$ is the displacement.

Example 1.3.27. A ship is running on the sea at the speed of $20km$ to north. but the current is flowing at the speed of $20km$ to the east, then in one hr, the displacement of the ship is $(20\sqrt{2}, 20\sqrt{2})$.

1.4 Matrices and Cross product

Cross product

Definition 1.4.1. Let $\mathbf{a}, \mathbf{b}$ be two vectors in $\mathbb{R}^3$(not $\mathbb{R}^2$). The cross product of $\mathbf{a}, \mathbf{b}$, denoted by $\mathbf{a} \times \mathbf{b}$ is the vector whose length and direction are given as follows:

(1) The length is the area of the parallelogram spanned by $\mathbf{a}$ and $\mathbf{b}$.(zero if $\mathbf{a}, \mathbf{b}$ are parallel). Alternatively,

$$\|\mathbf{a} \times \mathbf{b}\| = \|\mathbf{a}\|\|\mathbf{b}\| \sin\theta,$$

where θ is the angle between $\mathbf{a}$ and $\mathbf{b}$.

(2) The direction of $\mathbf{a}\times\mathbf{b}$ is perpendicular to $\mathbf{a}$ and $\mathbf{b}$, and the triple $(\mathbf{a}, \mathbf{b}, \mathbf{a}\times\mathbf{b})$ form a right-handed set of vectors.

Algebraic rules:

(1) $\mathbf{a} \times \mathbf{b} = 0$, if $\mathbf{a}$, $\mathbf{b}$ are parallel or one of them is zero.

(2) $\mathbf{a} \times \mathbf{b} = -(\mathbf{b} \times \mathbf{a})$

(3) $\mathbf{a} \times (\mathbf{b} + \mathbf{c}) = \mathbf{a} \times \mathbf{b} + \mathbf{a} \times \mathbf{c}$

(4) $(\mathbf{a} + \mathbf{b}) \times \mathbf{c} = \mathbf{a} \times \mathbf{c} + \mathbf{b} \times \mathbf{c}$

(5) $(\alpha\mathbf{a}) \times \mathbf{b} = \alpha(\mathbf{a} \times \mathbf{b})$ for scalar α.

Multiplication rules:

(1) $\mathbf{i} \times \mathbf{j} = \mathbf{k}, \quad \mathbf{j} \times \mathbf{k} = \mathbf{i}, \quad \mathbf{k} \times \mathbf{i} = \mathbf{j}.$

(2) $\mathbf{a} \times \mathbf{b} = -(\mathbf{b} \times \mathbf{a})$

(3) $\mathbf{i} \times \mathbf{i} = \mathbf{j} \times \mathbf{j} = \mathbf{k} \times \mathbf{k} = 0$

Note that

$$\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) \neq (\mathbf{a} \times \mathbf{b}) \times \mathbf{c}.$$

For example

$$\mathbf{i} \times (\mathbf{i} \times \mathbf{j}) = \mathbf{i} \times \mathbf{k} = -\mathbf{j} \neq (\mathbf{i} \times \mathbf{i}) \times \mathbf{j} = 0.$$

2×2 matrix

The array of numbers $a_{11},\ a_{12},\ a_{21},\ a_{22}$ in the form

$$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$$

is called 2×2 **matrix** and

$$[a_{11}\ a_{12}], \qquad \begin{bmatrix} a_{12} \\ a_{22} \end{bmatrix}$$

are the first row and second column. The real number $a_{11}a_{22} - a_{12}a_{21}$ is **determinant** and denoted by

$$det(A) = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}$$

Example 1.4.2. Find determinant of 2×2 matrices.

$$\begin{vmatrix} 3 & 4 \\ 2 & 1 \end{vmatrix} = 3 - 8 = -5, \quad \begin{vmatrix} 0 & 3 \\ -1 & 1 \end{vmatrix} = 0 - (-3) = 3, \quad \begin{vmatrix} 1 & 2 \\ -2 & 1 \end{vmatrix} = 1 - (-4) = 5$$

Proposition 1.4.3. *The area of parallelogram determined by the two vectors* $a\mathbf{i} + b\mathbf{j}$ *and* $c\mathbf{i} + d\mathbf{j}$ *is* $|ad - bc|$. *This is the absolute value of the determinant of the matrix determined by two two vectors:*

$$det \begin{bmatrix} a & b \\ c & d \end{bmatrix}.$$

Proof. Let $\mathbf{u} = a\mathbf{i} + b\mathbf{j}$, $\mathbf{v} = c\mathbf{i} + d\mathbf{j}$ and θ be the angle between them. Then the area of the parallelogram is

$$\begin{aligned} \|\mathbf{u}\|\,\|\mathbf{v}\|\,\sin\theta &= \|\mathbf{u}\|\,\|\mathbf{v}\|\,\sqrt{1 - \cos^2\theta} \\ &= \sqrt{\|\mathbf{u}\|^2\,\|\mathbf{v}\|^2 - (\mathbf{u}\cdot\mathbf{v})^2} \\ &= \sqrt{(a^2 + b^2)(c^2 + d^2) - (ac + bd)^2} \\ &= \sqrt{a^2d^2 + b^2c^2 - 2abcd} \\ &= |ad - bc|. \end{aligned}$$

□

3×3 matrix

A typical 3×3 matrix is given by

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}.$$

Here

$$[a_{31}\ a_{32}\ a_{33}], \qquad \begin{bmatrix} a_{12} \\ a_{22} \\ a_{32} \end{bmatrix}$$

are third row and second column. The **determinant** is defined as

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = a_{11} \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - a_{12} \begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + a_{13} \begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}. \tag{1.3}$$

Example 1.4.4.

$$\begin{vmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{vmatrix} = 1 \begin{vmatrix} 5 & 6 \\ 8 & 9 \end{vmatrix} - 2 \begin{vmatrix} 4 & 6 \\ 7 & 9 \end{vmatrix} + 3 \begin{vmatrix} 4 & 5 \\ 7 & 8 \end{vmatrix} = -3 + 12 - 9 = 0.$$

$$\begin{vmatrix} 1 & 1 & 1 \\ 2 & 4 & 8 \\ 3 & 9 & 27 \end{vmatrix} = 1 \begin{vmatrix} 4 & 8 \\ 9 & 27 \end{vmatrix} - 1 \begin{vmatrix} 2 & 8 \\ 3 & 27 \end{vmatrix} + 1 \begin{vmatrix} 2 & 4 \\ 3 & 9 \end{vmatrix} = 36 - 30 + 6 = 12.$$

Definition 1.4.5. If we exchange rows and columns of the following matrices

$$\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}, \qquad \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

to get

$$\begin{bmatrix} a_{11} & a_{21} \\ a_{12} & a_{22} \end{bmatrix}, \qquad \begin{bmatrix} a_{11} & a_{21} & a_{31} \\ a_{12} & a_{22} & a_{32} \\ a_{13} & a_{23} & a_{33} \end{bmatrix}$$

then resulting matrices are called the **transpose**.

Properties of determinant-skip for the time being

Theorem 1.4.6. *(1) Determinant of transposed matrix is the same the Determinant of original matrix.*

(2) If we exchange any two rows(columns), then determinant changes signs.

(3) $|det(\alpha A)| = \alpha^n |det(A)|$

(4) Adding a scalar multiple of row (column) to another row (column) does not change determinant.

Proof. (1) For 2×2

$$\begin{vmatrix} a_{11} & a_{21} \\ a_{12} & a_{22} \end{vmatrix} = a_{11}a_{22} - a_{21}a_{12} = a_{11}a_{22} - a_{12}a_{21} = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}.$$

For 3×3

$$\begin{aligned}
\begin{vmatrix} a_{11} & a_{21} & a_{31} \\ a_{12} & a_{22} & a_{32} \\ a_{13} & a_{23} & a_{33} \end{vmatrix} &= a_{11} \begin{vmatrix} a_{22} & a_{32} \\ a_{23} & a_{33} \end{vmatrix} - a_{21} \begin{vmatrix} a_{12} & a_{32} \\ a_{13} & a_{33} \end{vmatrix} + a_{31} \begin{vmatrix} a_{12} & a_{22} \\ a_{13} & a_{23} \end{vmatrix} \\
&= a_{11}(a_{22}a_{33} - a_{32}a_{23}) - a_{21}(a_{12}a_{33} - a_{32}a_{13}) \\
&\qquad + a_{31}(a_{12}a_{23} - a_{22}a_{13}) \\
&= a_{11}(a_{22}a_{33} - a_{32}a_{23}) - a_{21}a_{12}a_{33} + a_{21}a_{32}a_{13} \\
&\qquad + a_{31}a_{12}a_{23} - a_{31}a_{22}a_{13} \\
&= a_{11}(a_{22}a_{33} - a_{23}a_{32}) - a_{12}a_{21}a_{33} + a_{12}a_{23}a_{31} \\
&\qquad + a_{13}a_{21}a_{32} - a_{13}a_{22}a_{31} \\
&= a_{11}(a_{22}a_{33} - a_{23}a_{32}) - a_{12}(a_{21}a_{33} - a_{23}a_{31}) \\
&\qquad + a_{13}(a_{21}a_{32} - a_{22}a_{31}) \\
&= a_{11} \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - a_{12} \begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + a_{13} \begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix} \\
&= \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}.
\end{aligned}$$

(4) 2×2 case is easy.

For 3×3, we see by expanding w.r.t. first row

$$\begin{vmatrix} a_{11}+ta_{21} & a_{12}+ta_{22} & a_{13}+ta_{23} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$$

$$\begin{aligned} &= (a_{11}+ta_{21}) \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - (a_{12}+ta_{22}) \begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} \\ &\qquad + (a_{13}+ta_{23}) \begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix} \\ &= a_{11} \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - a_{12} \begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + a_{13} \begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix} \\ &\quad + t \left(a_{21} \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - a_{22} \begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + a_{23} \begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix} \right) \\ &= \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} + t \begin{vmatrix} a_{21} & a_{22} & a_{23} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}. \end{aligned}$$

Exchange second and third rows, do not change the value. By (2) there must be a sign change. Hence it is 0.

$$\begin{vmatrix} a_{11}+ta_{21} & a_{12}+ta_{22} & a_{13}+ta_{23} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}.$$

Hence (4) holds. □

The RHS of 1.3 is expansion w.r.t **first row.** By theorem 1.4.6, (1), (2), we can expand w.r.t. any row or column, except we multiply $(-1)^{i+j}$. So if we expand w.r.t 2nd row

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = -a_{21} \begin{vmatrix} a_{12} & a_{13} \\ a_{32} & a_{33} \end{vmatrix} + a_{22} \begin{vmatrix} a_{11} & a_{13} \\ a_{31} & a_{33} \end{vmatrix} - a_{23} \begin{vmatrix} a_{11} & a_{12} \\ a_{31} & a_{32} \end{vmatrix}.$$

If we expand w.r.t 3rd column

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = a_{13} \begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix} - a_{23} \begin{vmatrix} a_{11} & a_{12} \\ a_{31} & a_{32} \end{vmatrix} + a_{33} \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}.$$

Corollary 1.4.7. *(1) Determinant of a matrix one of whose row is zero is zero.*
(2) If any two rows (columns) are equal, the determinant is zero.

Example 1.4.8. The followings are expanded w.r.t 2nd, 3rd row.

$$\begin{vmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{vmatrix} = -2 \begin{vmatrix} 4 & 6 \\ 7 & 9 \end{vmatrix} + 5 \begin{vmatrix} 1 & 3 \\ 7 & 9 \end{vmatrix} - 8 \begin{vmatrix} 1 & 3 \\ 4 & 6 \end{vmatrix} = 12 - 60 + 48 = 0.$$

$$\begin{vmatrix} 1 & 2 & 3 \\ 4 & 0 & 6 \\ 0 & 8 & 0 \end{vmatrix} = 0 \begin{vmatrix} 2 & 3 \\ 0 & 6 \end{vmatrix} - 8 \begin{vmatrix} 1 & 3 \\ 4 & 6 \end{vmatrix} + 0 \begin{vmatrix} 1 & 2 \\ 4 & 0 \end{vmatrix} = 0 + 48 + 0 = 48.$$

Cross product-using determinant

In the previous section, we have defined the cross product using the geometric properties, but did not show how to compute it. Now we can give a formula for the cross product using the determinant:

Definition 1.4.9 (Alternative definition)**.** For $\mathbf{a} = a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k}$ and $\mathbf{b} = b_1\mathbf{i} + b_2\mathbf{j} + b_3\mathbf{k}$, the **cross product** $\mathbf{a} \times \mathbf{b}$ is defined by

$$\mathbf{a} \times \mathbf{b} = \begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix} \mathbf{i} - \begin{vmatrix} a_1 & a_3 \\ b_1 & b_3 \end{vmatrix} \mathbf{j} + \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix} \mathbf{k}. \tag{1.4}$$

Using the definition of determinant (1.3) symbolically, we have

$$\mathbf{a} \times \mathbf{b} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}.$$

Example 1.4.10. $\mathbf{i} \times \mathbf{i} = \mathbf{0}, \quad \mathbf{j} \times \mathbf{j} = \mathbf{0}, \quad \mathbf{k} \times \mathbf{k} = \mathbf{0}.$

Example 1.4.11. Compute $(2\mathbf{i} - \mathbf{j} + 3\mathbf{k}) \times (\mathbf{i} + \mathbf{j} + 2\mathbf{k})$.

sol. By the definition of cross product, we see

$$(2\mathbf{i}-\mathbf{j}+3\mathbf{k})\times(\mathbf{i}+\mathbf{j}+2\mathbf{k}) = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & -1 & 3 \\ 1 & 1 & 2 \end{vmatrix} = -5\mathbf{i}-\mathbf{j}+3\mathbf{k}.$$

■

A geometric meaning of the cross product

To see the relation with the geometric definition of the cross product, we define the triple product of three vectors: Let

$$\mathbf{a} = a_1\mathbf{i}+a_2\mathbf{j}+a_3\mathbf{k}, \mathbf{b} = b_1\mathbf{i}+b_2\mathbf{j}+b_3\mathbf{k}, \mathbf{c} = c_1\mathbf{i}+c_2\mathbf{j}+c_3\mathbf{k}.$$

The dot product between $(\mathbf{a}\times\mathbf{b})$ and $\mathbf{c}$ is $(\mathbf{a}\times\mathbf{b})\cdot\mathbf{c}$, called the **triple product** $(\mathbf{a}\times\mathbf{b})\cdot\mathbf{c}$ of three vectors, $\mathbf{a},\mathbf{b}$ and $\mathbf{c}$. We see by definition

$$\begin{aligned}
(\mathbf{a}\times\mathbf{b})\cdot\mathbf{c} &= \left(\begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix}\mathbf{i} - \begin{vmatrix} a_1 & a_3 \\ b_1 & b_3 \end{vmatrix}\mathbf{j} + \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix}\mathbf{k}\right)\cdot(c_1\mathbf{i}+c_2\mathbf{j}+c_3\mathbf{k}) \\
&= \begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix} c_1 - \begin{vmatrix} a_1 & a_3 \\ b_1 & b_3 \end{vmatrix} c_2 + \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix} c_3 \\
&= \begin{vmatrix} c_1 & c_2 & c_3 \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}.
\end{aligned}$$

We observe the following properties of $(\mathbf{a}\times\mathbf{b})$:

(1) If $\mathbf{c}$ is a vector in the plane spanned by $\mathbf{a},\mathbf{b}$, then the third row in the determinant is a linear combination of the first and second row, and hence $(\mathbf{a}\times\mathbf{b})\cdot\mathbf{c} = 0$. In other words, *the vector* $\mathbf{a}\times\mathbf{b}$ *is orthogonal to any vector in the plane spanned by* $\mathbf{a}$ *and* $\mathbf{b}$.

(2) We compute length of $\mathbf{a}\times\mathbf{b}$.

$$\begin{aligned}
\|\mathbf{a}\times\mathbf{b}\|^2 &= \begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix}^2 + \begin{vmatrix} a_1 & a_3 \\ b_1 & b_3 \end{vmatrix}^2 + \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix}^2 \\
&= (a_2b_3-a_3b_2)^2 + (a_1b_3-b_1a_3)^2 + (a_1b_2-b_1a_2)^2 \\
&= (a_1^2+a_2^2+a_3^2)(b_1^2+b_2^2+b_3^2) - (a_1b_1+a_2b_2+a_3b_3)^2.
\end{aligned}$$

Hence

$$\|\mathbf{a}\times\mathbf{b}\|^2 = \|\mathbf{a}\|^2\|\mathbf{b}\|^2 - (\mathbf{a}\cdot\mathbf{b})^2 = \|\mathbf{a}\|^2\|\mathbf{b}\|^2(1-\cos^2\theta) = \|\mathbf{a}\|^2\|\mathbf{b}\|^2\sin^2\theta.$$

So we conclude that $\mathbf{a}\times\mathbf{b}$ is a vector perpendicular to the plane $\mathcal{P}$ spanned by $\mathbf{a}$ and $\mathbf{b}$ with length $\|\mathbf{a}\|\|\mathbf{b}\||\sin\theta|$.

(3) Finally, the right handed rule can be checked with $\mathbf{i}\times\mathbf{j}=\mathbf{k}$.

The properties (2) and (3) can be summarized as follows.

Theorem 1.4.12 (Alternative cross product)**.** *For* $\mathbf{a}$, $\mathbf{b}\in\mathbb{R}^3$*, it holds that*

(1) $\|\mathbf{a}\times\mathbf{b}\| = \|\mathbf{a}\|\|\mathbf{b}\|\sin\theta$*, the area of the parallelogram spanned by* $\mathbf{a}$ *and* $\mathbf{b}$.

(2) $\mathbf{a}\times\mathbf{b}$ *is perpendicular to* $\mathbf{a}$ *and* $\mathbf{b}$*, and the triple* $(\mathbf{a},\mathbf{b},\mathbf{a}\times\mathbf{b})$ *form a right-handed rule.*

Hence this alternative definition is the same as the geometric definition of the cross product given earlier.

Component formula using determinant

$$(a_1\mathbf{i}+a_2\mathbf{j}+a_3\mathbf{k})\times(b_1\mathbf{i}+b_2\mathbf{j}+b_3\mathbf{k}) = \begin{vmatrix}\mathbf{i} & \mathbf{j} & \mathbf{k}\\ a_1 & a_2 & a_3\\ b_1 & b_2 & b_3\end{vmatrix}$$
$$= (a_2b_3-a_3b_2)\mathbf{i} - (a_1b_3-a_3b_1)\mathbf{j} + (a_1b_2-a_2b_1)\mathbf{k}.$$

Example 1.4.13. Find $(\mathbf{i}+\mathbf{j})\times(\mathbf{j}-2\mathbf{k})$.

sol. $(\mathbf{i}+\mathbf{j})\times(\mathbf{j}-2\mathbf{k}) = \mathbf{i}\times j - 2\mathbf{i}\times k + \mathbf{j}\times j - 2\mathbf{j}\times k = -2\mathbf{i}+2\mathbf{j}+\mathbf{k}$.

■

Theorem 1.4.14 (Cross product II)**.**

(1) $\|\mathbf{u}\times\mathbf{v}\|^2 = \|\mathbf{u}\|^2\|\mathbf{v}\|^2 - (\mathbf{u}\cdot\mathbf{v})^2$*. In particular,* $\mathbf{u}\times\mathbf{u}=\mathbf{0}$.

(2) If is θ *the angle between* $\mathbf{u}$ *and* $\mathbf{v}$*,* $\|\mathbf{u}\times\mathbf{v}\| = \|\mathbf{u}\|\,\|\mathbf{v}\|\,\sin\theta$*. Hence nec. and suff. condition for* $\mathbf{u}$ *and* $\mathbf{v}$ *are parallel is* $\mathbf{u}\times\mathbf{v}=\mathbf{0}$.

(3) $(\mathbf{u}\times\mathbf{v})\cdot\mathbf{w} = (\mathbf{v}\times\mathbf{w})\cdot\mathbf{u} = (\mathbf{w}\times\mathbf{u})\cdot\mathbf{v}$.

(4) $(\mathbf{u}\times\mathbf{v})\cdot\mathbf{u} = (\mathbf{u}\times\mathbf{v})\cdot\mathbf{v} = 0$, *i.e,* $\mathbf{u}\times\mathbf{v}$ *is orthogonal to* $\mathbf{u}$ *and* $\mathbf{v}$.

(5) $|(\mathbf{a}\times\mathbf{b})\cdot\mathbf{c}|$ *is the volume of parallelepiped formed by three vectors* $\mathbf{a},\mathbf{b}$ *and* $\mathbf{c}$. *(See below)*

Proof. Let $\mathbf{u} = a_1\mathbf{i}+a_2\mathbf{j}+a_3\mathbf{k}$, $\mathbf{v} = b_1\mathbf{i}+b_2\mathbf{j}+b_3\mathbf{k}$, $\mathbf{w} = c_1\mathbf{i}+c_2\mathbf{j}+c_3\mathbf{k}$.

(1) $\|\mathbf{u}\times\mathbf{v}\|^2 = \|\mathbf{u}\|^2\|\mathbf{v}\|^2 - (\mathbf{u}\cdot\mathbf{v})^2$ as shown before.

So $\mathbf{u}\times\mathbf{u} = \mathbf{0}$.

(2) Since $\mathbf{u}\cdot\mathbf{v} = \|\mathbf{u}\|\,\|\mathbf{v}\|\,\cos\theta$, we have by (1)

$$\begin{aligned}\|\mathbf{u}\times\mathbf{v}\| &= \sqrt{\|\mathbf{u}\|^2\|\mathbf{v}\|^2 - (\mathbf{u}\cdot\mathbf{v})^2}\\ &= \|\mathbf{u}\|\,\|\mathbf{v}\|\,\sqrt{1-\cos^2\theta}\\ &= \|\mathbf{u}\|\,\|\mathbf{v}\|\,\sin\theta.\end{aligned}$$

(3)

$$\begin{aligned}(\mathbf{u}\times\mathbf{v})\cdot\mathbf{w} &= \left(\begin{vmatrix}a_2 & a_3\\ b_2 & b_3\end{vmatrix}\mathbf{i} - \begin{vmatrix}a_1 & a_3\\ b_1 & b_3\end{vmatrix}\mathbf{j} + \begin{vmatrix}a_1 & a_2\\ b_1 & b_2\end{vmatrix}\mathbf{k}\right)\cdot(c_1\mathbf{i}+c_2\mathbf{j}+c_3\mathbf{k})\\ &= \begin{vmatrix}a_2 & a_3\\ b_2 & b_3\end{vmatrix}c_1 - \begin{vmatrix}a_1 & a_3\\ b_1 & b_3\end{vmatrix}c_2 + \begin{vmatrix}a_1 & a_2\\ b_1 & b_2\end{vmatrix}c_3\\ &= \begin{vmatrix}a_1 & a_2 & a_3\\ b_1 & b_2 & b_3\\ c_1 & c_2 & c_3\end{vmatrix}\end{aligned}$$

Expanding w.r.t first row, this is

$$\begin{aligned}&= a_1\begin{vmatrix}b_2 & b_3\\ c_2 & c_3\end{vmatrix} - a_2\begin{vmatrix}b_1 & b_3\\ c_1 & c_3\end{vmatrix} + a_3\begin{vmatrix}b_1 & b_2\\ c_1 & c_2\end{vmatrix}\\ &= (a_1\mathbf{i}+a_2\mathbf{j}+a_3\mathbf{k})\cdot\left(\begin{vmatrix}b_2 & b_3\\ c_2 & c_3\end{vmatrix}\mathbf{i} - \begin{vmatrix}b_1 & b_3\\ c_1 & c_3\end{vmatrix}\mathbf{j} + \begin{vmatrix}b_1 & b_2\\ c_1 & c_2\end{vmatrix}\mathbf{k}\right)\\ &= \mathbf{u}\cdot(\mathbf{v}\times\mathbf{w}) = (\mathbf{v}\times\mathbf{w})\cdot\mathbf{u}.\end{aligned}$$

By the same way this equals with $(\mathbf{w}\times\mathbf{u})\cdot\mathbf{v}$.

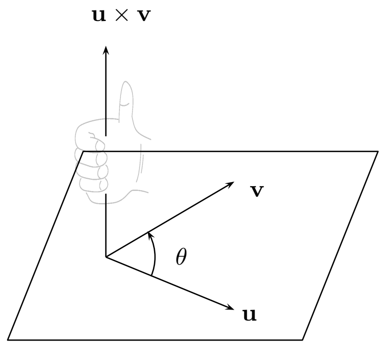


Figure 1.15: right handed rule

(4) Using (3) and corollary 1.4.7, we see

$$(\mathbf{u}\times\mathbf{v})\cdot\mathbf{u} = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ a_1 & a_2 & a_3 \end{vmatrix} = 0.$$

$$(\mathbf{u}\times\mathbf{v})\cdot\mathbf{v} = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ b_1 & b_2 & b_3 \end{vmatrix} = 0.$$

□

Geometric meaning of determinant

2×2 matrix: If $\mathbf{a} = (a_1, a_2)$ and $\mathbf{b} = (b_1, b_2)$ then we can view them as vectors in $\mathbb{R}^3$ and define

$$\mathbf{a}\times\mathbf{b} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a_1 & a_2 & 0 \\ b_1 & b_2 & 0 \end{vmatrix} = \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix}\mathbf{k} = (a_1b_2 - a_2b_1)\mathbf{k}.$$

Hence $\|\mathbf{a}\times\mathbf{b}\|$ is the area of the parallelogram formed by the two vectors.

Example 1.4.15. Find the area of triangle with vertices at $(1,1), (0,2)$ and $(3,2)$.

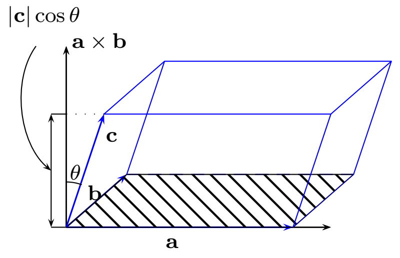


Figure 1.16: Meaning of triple product: $(\mathbf{a} \times \mathbf{b}) \cdot \mathbf{c}$

sol. Two sides are $(0, 2) - (1, 1) = (-1, 1)$ and $(3, 2) - (1, 1) = (2, 1)$. Thus the area is the absolute value of $\frac{1}{2}\begin{vmatrix} -1 & 1 \\ 2 & 1 \end{vmatrix} = -\frac{3}{2}$.

■

Proposition 1.4.16. *The volume of parallelepiped with sides* $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$ *is the absolute value of the triple product* $(\mathbf{a} \times \mathbf{b}) \cdot \mathbf{c}$

$$|(\mathbf{a} \times \mathbf{b}) \cdot \mathbf{c}| = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}.$$

Proof. Consider the parallelogram with sides $\mathbf{a}$, $\mathbf{b}$ as the bottom of the parallelepiped. On the other hand, the height of the parallelepiped is the length of the orthogonal projection of $\mathbf{c}$ onto $\mathbf{a} \times \mathbf{b}$ which is $\left\| \frac{(\mathbf{a}\times\mathbf{b})\cdot\mathbf{c}}{\|\mathbf{a}\times\mathbf{b}\|^2} \mathbf{a} \times \mathbf{b} \right\|$. Hence the volume is

$$Area(bottom) \times height = \|\mathbf{a} \times \mathbf{b}\| \left\| \frac{(\mathbf{a} \times \mathbf{b}) \cdot \mathbf{c}}{\|\mathbf{a} \times \mathbf{b}\|^2} \mathbf{a} \times \mathbf{b} \right\| = |(\mathbf{a} \times \mathbf{b}) \cdot \mathbf{c}|.$$

□

Example 1.4.17. Three points $A(1, 2, 3)$, $B(0, 1, 2)$, $C(0, 3, 2)$ are given. Find the volume of hexahedron having three vectors OA, OB, OC as sides.

sol. By proposition 1.4.16, we have

$$\begin{vmatrix} 1 & 2 & 3 \\ 0 & 1 & 2 \\ 0 & 3 & 2 \end{vmatrix} = 1\begin{vmatrix} 1 & 2 \\ 3 & 2 \end{vmatrix} - 0\begin{vmatrix} 2 & 3 \\ 3 & 2 \end{vmatrix} + 0\begin{vmatrix} 2 & 3 \\ 1 & 2 \end{vmatrix} = -4.$$

■

Torque

Imagine we are trying to fasten a bolt with a wrench. If one apply the force $\mathbf{F}$ at the end of wrench as in figure 1.17, we see the force of turning the bolt is $\|\mathbf{r}\|\|\mathbf{F}\|\sin\theta$.

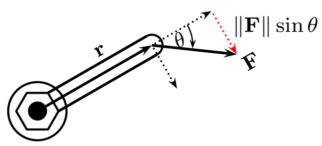


Figure 1.17: Turning a hexagonal bolt with a wrench with force $\mathbf{F}$. Torque vector is $\mathbf{r}\times\mathbf{F}$.

Then

$$\begin{aligned}\text{Size of torque} &= (\text{length of wrench})(\text{component of } \mathbf{F} \perp \text{wrench})\\ &= \|\mathbf{r}\|\|\mathbf{F}\|\sin\theta = \|\mathbf{r}\times\mathbf{F}\|.\end{aligned}$$

Also, note that the direction of $\mathbf{r}\times\mathbf{F}$ is the same as the direction the bolt moves. Hence it is natural to define $\mathbf{r}\times\mathbf{F}$ to be the torque vector.

Rotation of a rigid body

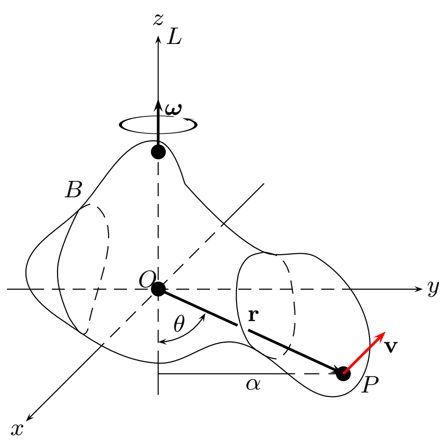


Figure 1.18: velocity $\mathbf{v}$ and angular velocity $\boldsymbol{\omega}$ has relation $\mathbf{v}=\boldsymbol{\omega}\times\mathbf{r}$.

Consider a rigid body B rotating about the axis L. (See fig 1.18) What is the relation between the velocity of a point on the object and the rotational velocity?

First we need to define a vector $\boldsymbol{\omega}$, the angular velocity of the rotation. **The rotational motion of B can be described by a vector along axis of rotation $\boldsymbol{\omega}$.** The vector points along the axis of rotation with its direction determined by the right handed rule. Its magnitude is the angular speed (measured in radians per unit time) at which the object spins. The vector $\boldsymbol{\omega}$ is called the **angular velocity vector** and ω is angular speed, $\omega = ||\boldsymbol{\omega}||$. Next fix a point O(the origin) on the axis of rotation, and let $\mathbf{r}(t) = \vec{OP}$ be the position vector of the point P. Let $\boldsymbol{\omega}$ the vector along z-axis s.t. $\omega = ||\boldsymbol{\omega}||$.

Assume L is z-axis and α is distance from P to L. Then $\alpha = ||\mathbf{r}|| \sin\theta$ ($\mathbf{r}$ points to P). Consider the tangent vector $\mathbf{v}$ at P. Since P moves around a circle of radius α perpendicular to $\boldsymbol{\omega}$ (parallel to xy-plane, counterclockwise), we see,

$$\begin{aligned}\Delta\mathbf{r} &\approx \text{(radius of circle)(angle swept by } Q)\\ &= ||\mathbf{r}|| \sin\theta(\Delta\phi).\end{aligned}$$

Thus

$$\left|\left|\frac{\Delta\mathbf{r}}{\Delta t}\right|\right| \approx ||\mathbf{r}|| \sin\theta \frac{\Delta\phi}{\Delta t}.$$

As $\Delta t \to 0$, we obtain the (line) velocity and angular velocity by

$$\mathbf{v} = \lim_{\Delta t\to 0} \frac{\Delta\mathbf{r}}{\Delta t}, \quad \omega = \lim_{\Delta t\to 0} \frac{\Delta\phi}{\Delta t}.$$

Hence

$$||\mathbf{v}|| = \omega\alpha = \omega||\mathbf{r}|| \sin\theta = ||\boldsymbol{\omega}||||\mathbf{r}|| \sin\theta. \tag{1.5}$$

Then by definition of cross product,

$$\mathbf{v} = \boldsymbol{\omega} \times \mathbf{r}. \tag{1.6}$$

1.5 Equations of Planes

Let $\mathcal{P}$ be a plane and $P_0 = (x_0, y_0, z_0)$ a point on that plane, and suppose that $\mathbf{n} = A\mathbf{i} + B\mathbf{j} + C\mathbf{k}$ is a normal vector. Let $P = (x, y, z)$ be any point in $\mathbb{R}^3$. Then P lies in the plane iff the vector $\overrightarrow{P_0P} = (x - x_0, y - y_0, z - z_0)$ is perpendicular to $\mathbf{n}$, that is, $\overrightarrow{P_0P} \cdot \mathbf{n} = 0$. In other words,

$$(A\mathbf{i} + B\mathbf{j} + C\mathbf{k}) \cdot [(x - x_0)\mathbf{i} + (y - y_0)\mathbf{j} + (z - z_0)\mathbf{k}] = 0.$$

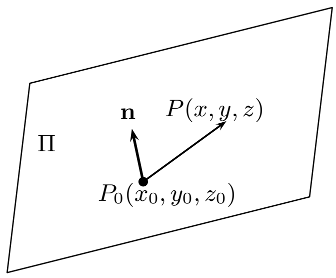


Figure 1.19: A plane is det'd by a point and normal vector

Proposition 1.5.1. *Equation of plane through* (x_0, y_0, z_0) *that has normal vector* $\mathbf{n}$ *is*

$$A(x - x_0) + B(y - y_0) + C(z - z_0) = 0,$$

or

$$Ax + By + Cz - D = 0,$$

where $D = -(Ax_0 + By_0 + Cz_0)$.

Example 1.5.2. Find the equation of plane through the points $A(-3, 0, -1)$, $B(-2, 3, 2)$, $C(1, 1, 3)$.

sol. Draw some graph describing the normal vector.

Find a vector $\mathbf{n}$ orthogonal to plane.

$$\begin{aligned}
\mathbf{n} &= \overrightarrow{AB} \times \overrightarrow{AC} \\
&= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -2-(-3) & 3-0 & 2-(-1) \\ 1-(-3) & 1-0 & 3-(-1) \end{vmatrix} \\
&= \begin{vmatrix} 3 & 3 \\ 1 & 4 \end{vmatrix} \mathbf{i} - \begin{vmatrix} 1 & 3 \\ 4 & 4 \end{vmatrix} \mathbf{j} + \begin{vmatrix} 1 & 3 \\ 4 & 1 \end{vmatrix} \mathbf{k} \\
&= 9\mathbf{i} + 8\mathbf{j} - 11\mathbf{k}.
\end{aligned}$$

By proposition 1.5.1, the equation is

$$9(x+3)+8(y-0)-11(z+1)=0$$

or $9x+8y-11z+16=0$.

■

Distance from a point to plane

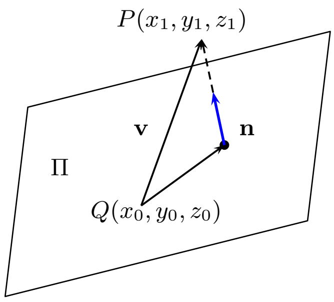


Figure 1.20: Distance from a point to plane

Proposition 1.5.3. *The distance from $P(x_1, y_1, z_1)$ to the plane $Ax+By+Cz+D=0$ is*

$$\frac{|Ax_1+By_1+Cz_1+D|}{\sqrt{A^2+B^2+C^2}}.$$

Proof. Let $\mathbf{n}$ be a normal vector to the plane. If $Q(x_0, y_0, z_0)$ lies in the plane, the distance from P to the plane is the orthogonal projection of $\overrightarrow{PQ}$ along $\mathbf{n}$. Note that from $A(x-x_0)+B(y-y_0)+C(z-z_0)=0$, we see $\mathbf{n}//A\mathbf{i}+B\mathbf{j}+C\mathbf{k}$. Hence length of the orthogonal projection of $\overrightarrow{PQ}$ along $\mathbf{n}$ is

$$\begin{aligned}
\left\| \frac{\mathbf{n}\cdot\overrightarrow{PQ}}{\|\mathbf{n}\|^2}\mathbf{n} \right\| &= \frac{|\mathbf{n}\cdot\overrightarrow{PQ}|}{\|\mathbf{n}\|} \\
&= \frac{|A(x_0-x_1)+B(y_0-y_1)+C(z_0-z_1)|}{\sqrt{A^2+B^2+C^2}} \\
&= \frac{|Ax_0+By_0+Cz_0-Ax_1-By_1-Cz_1|}{\sqrt{A^2+B^2+C^2}} \\
&= \frac{|-D-Ax_1-By_1-Cz_1|}{\sqrt{A^2+B^2+C^2}} = \frac{|Ax_1+By_1+Cz_1+D|}{\sqrt{A^2+B^2+C^2}}.
\end{aligned}$$

□

Example 1.5.4. Find the distance from $(3,4,-2)$ to the plane $2x-y+z-4=0$.

sol. Using above proposition, distance is

$$\frac{|2\cdot 3-1\cdot 4+1\cdot(-2)-4|}{\sqrt{4+1+1}}=\frac{|-4|}{\sqrt{6}}=\frac{2\sqrt{6}}{3}.$$

□

Example 1.5.5. Find a unit vector perpendicular to the plane $4x-3y+z-4=0$ and express it as a cross product of two unit orthogonal vectors lying in the plane.

sol. Let $\mathcal{S}$ be the given plane. By proposition 1.5.1 we see $4\mathbf{i}-3\mathbf{j}+\mathbf{k}$ is orthogonal to $\mathcal{S}$. Hence a unit normal vector is

$$\mathbf{n}=\pm\frac{4\mathbf{i}-3\mathbf{j}+\mathbf{k}}{\sqrt{4^2+(-3)^2+1^2}}=\pm\frac{1}{\sqrt{26}}(4\mathbf{i}-3\mathbf{j}+\mathbf{k}).$$

Now in order to express this as a cross product of two vectors lying in the plane, we choose three arbitrary points in $\mathcal{S}$. For example, we choose $(1,0,0),(0,0,4),(2,1,-1)$. Then we obtain two vectors

$$\begin{aligned}\mathbf{u}&=(1,0,0)-(2,1,-1)=-\mathbf{i}-\mathbf{j}+\mathbf{k}\\ \mathbf{v}&=(0,0,4)-(2,1,-1)=-2\mathbf{i}-\mathbf{j}+5\mathbf{k}\end{aligned}$$

which lie in the plane $\mathcal{S}$. Now we orthogonalize them.

Let $\mathbf{a}$ be the orthogonal projection of $\mathbf{v}$ onto $\mathbf{u}$. Then let $\mathbf{b}=\mathbf{v}-\mathbf{a}$.

$$\begin{aligned}\mathbf{a}&=\frac{\mathbf{u}\cdot\mathbf{v}}{\|\mathbf{u}\|^2}\mathbf{u}=\frac{8}{3}(-\mathbf{i}-\mathbf{j}+\mathbf{k})\\ \mathbf{b}&=\mathbf{v}-\frac{\mathbf{u}\cdot\mathbf{v}}{\|\mathbf{u}\|^2}\mathbf{u}=(-2\mathbf{i}-\mathbf{j}+5\mathbf{k})-\frac{8}{3}(-\mathbf{i}-\mathbf{j}+\mathbf{k})\\ &=\frac{1}{3}(2\mathbf{i}+5\mathbf{j}+7\mathbf{k}).\end{aligned}$$

Now normalize them.

$$\mathbf{a}_1=\frac{\mathbf{a}}{\|\mathbf{a}\|}=\frac{1}{\sqrt{3}}(-\mathbf{i}-\mathbf{j}+\mathbf{k}),\quad \mathbf{b}_1=\frac{\mathbf{b}}{\|\mathbf{b}\|}=\frac{1}{\sqrt{78}}(2\mathbf{i}+5\mathbf{j}+7\mathbf{k}).$$

We can check that

$$\begin{aligned}
\mathbf{a}_1 \cdot \mathbf{b}_1 &= \frac{(-1)\cdot 2 + (-1)\cdot 5 + 1\cdot 7}{\sqrt{3}\cdot\sqrt{78}} = 0 \text{ (orthogonal)} \\
\mathbf{a}_1 \times \mathbf{b}_1 &= \frac{1}{3\sqrt{26}} \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -1 & -1 & 1 \\ 2 & 5 & 7 \end{vmatrix} \\
&= \frac{1}{3\sqrt{26}} \left(\begin{vmatrix} -1 & 1 \\ 5 & 7 \end{vmatrix} \mathbf{i} - \begin{vmatrix} -1 & 1 \\ 2 & 7 \end{vmatrix} \mathbf{j} + \begin{vmatrix} -1 & -1 \\ 2 & 5 \end{vmatrix} \mathbf{k} \right) \\
&= -\frac{1}{\sqrt{26}}(4\mathbf{i} - 3\mathbf{j} + \mathbf{k}).
\end{aligned}$$

■

Parametric equation of a plane

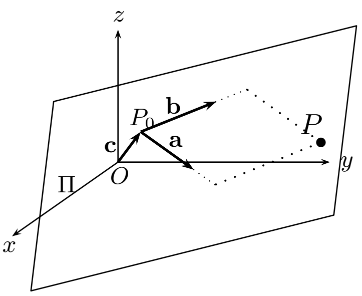


Figure 1.21: A plane is det'd by a point and two vectors

Proposition 1.5.6. *A parametric equation for the plane passing the point* $P_0 = (c_1, c_2, c_3)$ *and parallel to two vectors* $\mathbf{a}$ *and* $\mathbf{b}$ *is given by*

$$\mathbf{x}(s,t) = s\mathbf{a} + t\mathbf{b} + \mathbf{c}.$$

Distance between a point and a line

Example 1.5.7. Find the distance from the point $P_0(2,1,3)$ to the line $\ell(t) = t(-1,1,-2) + (2,3,-2)$.

sol. Choose any point B on the line and find an orthogonal decomposition of $\overrightarrow{BP_0}$ onto the direction vector $\mathbf{a} = (-1,1,-2)$ of the line. Then the length

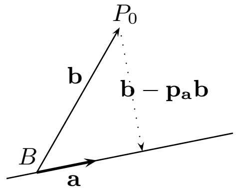


Figure 1.22: Distance from a point to a line

of the orthogonal complement is the distance. Choose $B = (2, 3, -2)$. Then

$$\begin{aligned}\overrightarrow{BP_0} := \mathbf{b} &= (2,1,3)-(2,3,-2)\\ &= (0,-2,5).\end{aligned}$$

Hence the orthogonal projection onto $\mathbf{a}$ is

$$\begin{aligned}\mathbf{p_a b} &= \frac{\mathbf{a}\cdot\mathbf{b}}{\|\mathbf{a}\|^2}\mathbf{a}\\ &= (2,-2,4).\end{aligned}$$

Thus the distance is

$$\|\mathbf{b}-\mathbf{p_a b}\| = \|(0,-2,5)-(2,-2,4)\| = \sqrt{5}.$$

■

Distance between two parallel planes

To find the distance between two parallel planes, we first compute a normal vector common to both planes. Now choose one point from each plane, say P_i from the plane $\Pi_i (i = 1, 2)$. Then find the projection of $\overrightarrow{P_1P_2}$ onto the common normal vector.

Distance between two skewed lines

Two lines are said to be **skewed** if they are neither intersecting nor parallel. It follows that they must *lie in two parallel planes* and the distance between the lines is equal to the distance between the planes. Let us describe how to find the distance between them.

Assume we have two parallel planes Π_1 and Π_2 containing each lines. They

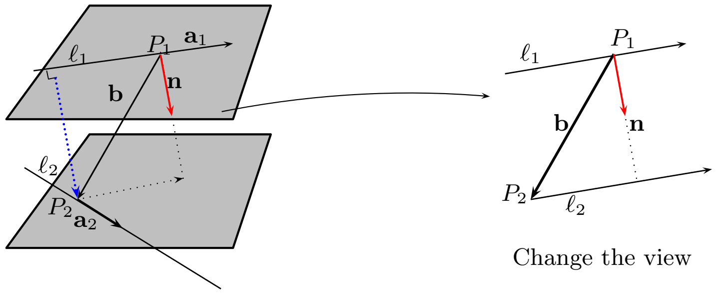


Figure 1.23: Distance between two lines is the length of $proj_{\mathbf{n}}\mathbf{b}$(blue)

share a common normal vector $\mathbf{n}$. Assume $\mathbf{a}_1$ and $\mathbf{a}_2$ are two direction vectors of the planes.(verify) Then the normal vector is obtained by taking cross product of $\mathbf{a}_1$ and $\mathbf{a}_2$. Let $P_1 \in \ell_1$, $P_2 \in \ell_2$ be any two points on each line. Then we compute the projection of $\overrightarrow{P_1P_2}$ onto $\mathbf{n}$. Moving the projection along the line ℓ_1 so that the head ends at P_2, we see its length is the desired distance.

Example 1.5.8. Find the distance between the two lines

$$\ell_1(t) = (0,5,-1) + t(2,1,3), \text{ and } \ell_2(t) = (-1,2,0) + t(1,-1,0).$$

sol. We have $\mathbf{a}_1 = (2,1,3)$ and $\mathbf{a}_2 = (1,-1,0)$. Choose $P_1 = (2,6,2)$ and $P_2 = (0,1,0)$. Then $\mathbf{b} = (2,6,2) - (0,1,0) = (2,5,2)$. While

$$\mathbf{n} = \mathbf{a}_1 \times \mathbf{a}_2 = (2,1,3) \times (1,-1,0) = (3,3,-3).$$

Normalizing, we let $\mathbf{n} = (1,1,-1)/\sqrt{3}$. Now the projection of $\mathbf{b}$ onto $\mathbf{n}$ is

$$proj_{\mathbf{n}}\mathbf{b} = (\mathbf{b}\cdot\mathbf{n})\mathbf{n} = \frac{(2+5-2)}{\sqrt{3}}\frac{(1,1,-1)}{\sqrt{3}} = \frac{5}{3}(1,1,-1).$$

Hence the distance is

$$\left\|\frac{5}{3}(1,1,-1)\right\| = \frac{5}{\sqrt{3}}.$$

□

1.6 n-dim Euclidean space

Vectors in n -dim space

The set of all points with n-coordinates

$$\mathbb{R}^n = \{(a_1, a_2, \dots, a_n) \mid a_1, a_2, \dots, a_n \in \mathbb{R}\}$$

is called **n-dimensional Euclidean space**. Addition and scalar multiplication can be defined as

$$(a_1, a_2, \dots, a_n) + (b_1, b_2, \dots, b_n) = (a_1 + b_1, a_2 + b_2, \dots, a_n + b_n)$$
$$s(a_1, a_2, \dots, a_n) = (sa_1, sa_2, \dots, sa_n).$$

The identity $(0, 0, \dots, 0)$ in $\mathbb{R}^n$ is the **zero element**. The inverse of $(a_1, a_2, \dots, a_n)$ is $(-a_1, -a_2, \dots, -a_n)$, or $-(a_1, a_2, \dots, a_n)$. For two points $P(a_1, a_2, \dots, a_n)$ and $Q(b_1, b_2, \dots, b_n)$, the set

$$\overline{PQ} = \{(1-t)(a_1, a_2, \dots, a_n) + t(b_1, b_2, \dots, b_n) \mid 0 \le t \le 1\}$$

is called the **line segment** PQ and

$$\sqrt{(a_1 - b_1)^2 + (a_2 - b_2)^2 + \cdots + (a_n - b_n)^2}$$

is the **length of** PQ. Also the set

$$\overleftrightarrow{PQ} = \{(1-t)(a_1, a_2, \dots, a_n) + t(b_1, b_2, \dots, b_n) \mid -\infty < t < \infty\}$$

is the line PQ.

If the three points $P(a_1, \dots, a_n)$, $Q(b_1, \dots, b_n)$, $R(c_1, \dots, c_n)$ are not lying in the same line, then the set

$$\{r(a_1, \dots, a_n) + s(b_1, \dots, b_n) + t(c_1, \dots, c_n) \mid -\infty < r, s, t < \infty, r + s + t = 1\}$$

is called the plane **determined by** $P,\ Q,\ R$.

Standard basis vector

We let

$$\begin{aligned}
\mathbf{e}_1 &= (1, 0, 0, \dots, 0) \\
\mathbf{e}_2 &= (0, 1, 0, \dots, 0) \\
\mathbf{e}_3 &= (0, 0, 1, \dots, 0) \\
&\vdots \\
\mathbf{e}_n &= (0, 0, 0, \dots, 1).
\end{aligned}$$

Then any vector in $\mathbb{R}^n$ can be written as a scalar combination of $\mathbf{e}_1,\ \mathbf{e}_2, \dots,\ \mathbf{e}_n$:

$$(a_1, a_2, \dots, a_n) = a_1\mathbf{e}_1 + a_2\mathbf{e}_2 + \cdots + a_n\mathbf{e}_n.$$

The vectors $\mathbf{e}_1,\ \mathbf{e}_2, \dots,\ \mathbf{e}_n$ are the **standard basis vectors**[1] of $\mathbb{R}^n$. Clearly, we have

$$\mathbf{e}_i \cdot \mathbf{e}_j = \begin{cases} 0 & (i \neq j) \\ 1 & (i = j). \end{cases}$$

Theorem 1.6.1. *We have the following:*

(i) $(\alpha\mathbf{x} + \beta\mathbf{y}) \cdot \mathbf{z} = \alpha\mathbf{x} \cdot \mathbf{z} + \beta\mathbf{y} \cdot \mathbf{z}$ *(associate law)*

(ii) $\mathbf{x} \cdot \mathbf{y} = \mathbf{y} \cdot \mathbf{x}$ *(commutative law)*

(iii) $\mathbf{x} \cdot \mathbf{x} \geq 0$

(iv) $\mathbf{x} \cdot \mathbf{x} = 0$ *iff* $\mathbf{x} = 0$.

Example 1.6.2. Let $\mathbf{u} = 3\mathbf{e}_1 - 4\mathbf{e}_2 + 2\mathbf{e}_4$, $\mathbf{v} = \mathbf{e}_1 + 2\mathbf{e}_2 + 2\mathbf{e}_3 - 3\mathbf{e}_4$ be in $\mathbb{R}^4$. Express $2\mathbf{u} - 7\mathbf{v}$ using standard basis vector.

sol. Using standard basis vector, $2\mathbf{u} - 7\mathbf{v}$ is

$$\begin{aligned}
2\mathbf{u} - 7\mathbf{v} &= 2(3\mathbf{e}_1 - 4\mathbf{e}_2 + 2\mathbf{e}_4) - 7(\mathbf{e}_1 + 2\mathbf{e}_2 + 2\mathbf{e}_3 - 3\mathbf{e}_4) \\
&= (6\mathbf{e}_1 - 8\mathbf{e}_2 + 4\mathbf{e}_4) + (-7\mathbf{e}_1 - 14\mathbf{e}_2 - 14\mathbf{e}_3 + 21\mathbf{e}_4) \\
&= (6 - 7)\mathbf{e}_1 + (-8 - 14)\mathbf{e}_2 + (0 - 14)\mathbf{e}_3 + (4 + 21)\mathbf{e}_4 \\
&= -\mathbf{e}_1 - 22\mathbf{e}_2 - 14\mathbf{e}_3 + 25\mathbf{e}_4.
\end{aligned}$$

[1]By definition 1.1.7, the vectors $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$ are denoted by $\mathbf{i}, \mathbf{j}, \mathbf{k}$ in $\mathbb{R}^3$

■

For two vector $\mathbf{u} = a_1\mathbf{e}_1 + a_2\mathbf{e}_2 + \cdots + a_n\mathbf{e}_n$, $\mathbf{v} = b_1\mathbf{e}_1 + b_2\mathbf{e}_2 + \cdots + b_n\mathbf{e}_n$, their **inner product** is defined as

$$\mathbf{u} \cdot \mathbf{v} = a_1b_1 + a_2b_2 + \cdots + a_nb_n = \sum_{i=1}^{n} a_ib_i$$

This satisfies proposition 1.3.3. The length of a vector $\mathbf{u}$ is defined as

$$\|\mathbf{u}\| = (a_1^2 + \cdots + a_n^2)^{1/2} = \sqrt{\mathbf{u} \cdot \mathbf{u}}$$

and the distance between two vectors $\mathbf{u}$ and $\mathbf{v}$ is defined as $\|\mathbf{u} - \mathbf{v}\|$.

One can even define the angle between $\mathbf{u}$ and $\mathbf{v}$ by

$$\theta = \cos^{-1} \frac{\mathbf{u} \cdot \mathbf{v}}{\|\mathbf{u}\| \, \|\mathbf{v}\|} = \cos^{-1} \frac{a_1b_1 + \cdots + a_nb_n}{(a_1^2 + \cdots + a_n^2)^{1/2}(b_1^2 + \cdots + b_n^2)^{1/2}}.$$

Example 1.6.3. Find the inner product of $\mathbf{u} = \mathbf{e}_1 - 2\mathbf{e}_2 + 3\mathbf{e}_3 + 2\mathbf{e}_4$, $\mathbf{v} = 2\mathbf{e}_1 + \mathbf{e}_2 - 3\mathbf{e}_3 - \mathbf{e}_4$.

sol.

$$\mathbf{u} \cdot \mathbf{v} = 2 - 2 - 9 - 2 = -11.$$

■

Example 1.6.4. Find the angle between $\mathbf{u} = \mathbf{e}_1 - \mathbf{e}_2 + \mathbf{e}_4$, $\mathbf{v} = -\mathbf{e}_2 - \mathbf{e}_3 + 2\mathbf{e}_4$.

sol. The angle between $\mathbf{u}$ and $\mathbf{v}$ is

$$\cos^{-1} \frac{0 + 1 + 0 + 2}{\sqrt{(1 + 1 + 0 + 1)(0 + 1 + 1 + 4)}} = \cos^{-1}(\frac{1}{\sqrt{2}}) = \frac{\pi}{4}.$$

■

Theorem 1.6.5 (Cauchy-Schwarz inequality)**.** *For any two vectors* $\mathbf{a}$, $\mathbf{b}$ *in* n*-dim space the following holds. Equality holds iff* $\mathbf{a}$ *and* $\mathbf{b}$ *are parallel.*

$$|\mathbf{a} \cdot \mathbf{b}| \leq \|\mathbf{a}\| \, \|\mathbf{b}\|.$$

Proof. For $n > 3$ our early proof is unclear. Thus we prove this again. We may assume none of the vectors are zero. Recall the orthogonal decomposition of $\mathbf{b}$ onto $\mathbf{a}$, i.e, we write

$$\mathbf{b} = k\mathbf{a} + \mathbf{c},$$

where $k\mathbf{a}$ is the projection of $\mathbf{b}$ and $\mathbf{c} = \mathbf{b} - k\mathbf{a}$ is the orthogonal complement. By orthogonality($\mathbf{a} \cdot \mathbf{c} = 0$),

$$\|\mathbf{a}\|^2\|\mathbf{b}\|^2 = \|\mathbf{a}\|^2(k^2\|\mathbf{a}\|^2 + \|\mathbf{c}\|^2) \geq k^2\|\mathbf{a}\|^2\|\mathbf{a}\|^2.$$

Thus

$$k^2\|\mathbf{a}\|^2 \leq \|\mathbf{b}\|^2.$$

Since $k = \frac{\mathbf{b}\cdot\mathbf{a}}{\|\mathbf{a}\|^2}$, we see

$$\|\mathbf{a} \cdot \mathbf{b}\|^2 \leq \|\mathbf{a}\|^2\|\mathbf{b}\|^2.$$

□

Theorem 1.6.6 (Triangle inequality)**.** *For any two vectors* $\mathbf{u}$, $\mathbf{v}$ *in n-dim space the following holds. Equality holds iff* $\mathbf{u}$ *and* $\mathbf{v}$ *are parallel and same direction.*

$$\|\mathbf{u} + \mathbf{v}\| \leq \|\mathbf{u}\| + \|\mathbf{v}\|.$$

Proof.

$$\begin{aligned}
\|\mathbf{u} + \mathbf{v}\|^2 &= (\mathbf{u} + \mathbf{v}) \cdot (\mathbf{u} + \mathbf{v}) \\
&= \mathbf{u} \cdot \mathbf{u} + 2\mathbf{u} \cdot \mathbf{v} + \mathbf{v} \cdot \mathbf{v} \\
&\leq \mathbf{u} \cdot \mathbf{u} + 2\|\mathbf{u}\|\|\mathbf{v}\| + \mathbf{v} \cdot \mathbf{v} \\
&= (\|\mathbf{u}\| + \|\mathbf{v}\|)^2.
\end{aligned}$$

□

General matrix

Let m, n be any natural numbers. The arrays a_{ij} $(1 \leq i \leq m,\ 1 \leq j \leq n)$

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}$$

is said to be an $m \times n$ **matrix** and denote by

$$\begin{bmatrix} a_{ij} \end{bmatrix}_{\substack{1 \le i \le m \\ 1 \le j \le n}}, \quad \begin{bmatrix} a_{ij} \end{bmatrix}_{m \times n} \quad \text{or} \quad [a_{ij}]$$

If $m = 1$, then $1 \times n$ matrix consists of one row and is called **row vector**, and if $n = 1$ then $m \times 1$ matrix is **column vector**. If $m = n$, it is called **square matrix**. a_{ij} is called ij**-entry**. The $1 \times n$ matrix

$$\begin{bmatrix} a_{i1} & a_{i2} & \cdots & a_{in} \end{bmatrix}$$

is i**-th row vector**, $m \times 1$ matrix

$$\begin{bmatrix} a_{1j} \\ a_{2j} \\ \vdots \\ a_{mj} \end{bmatrix}$$

is j**-th column vector**.

Example 1.6.7. What is 4-th row and second column of the 4×3 matrix?

$$\begin{bmatrix} 0 & -2 & 12 \\ 3 & 1 & 4 \\ -1 & 0 & 5 \\ 1 & -3 & 7 \end{bmatrix}$$

sol. 4-th row and second column is

$$\begin{bmatrix} 1 & -3 & 7 \end{bmatrix}, \quad \begin{bmatrix} -2 \\ 1 \\ 0 \\ -3 \end{bmatrix}$$

□

Matrix addition, multiplication

Let A and B be two $m \times n$ matrices. Then the **matrix sum** $A + B$ is the $m \times n$ matrix whose (i, j) entries are the sum of $a_{i,j}$ and $b_{i,j}$. If k is any scalar, define the scalar multiplication kA by

$$(kA)_{ij} = ka_{ij}.$$

i.e, each entry is multiplied by k.

Definition 1.6.8 (Matrix multiplication)**.** If $A = [a_{ij}]$ is $m \times n$ matrix and $B = [b_{kl}]$ is $n \times p$ matrix, then the $m \times p$ matrix

$$\left[\sum_{k=1}^{n} a_{ik} b_{kj}\right]_{\substack{1 \le i \le m \\ 1 \le j \le p}}$$

is the **product** of A and B denoted by AB. In other words, the product of A and B is AB and its ij-component is the inner product of $i-th$ row of A and $j-th$ column of B.

Example 1.6.9. Product of 2×3 and 3×4 matrices

$$\begin{bmatrix} 2 & 3 & 1 \\ -1 & 1 & 4 \end{bmatrix} \begin{bmatrix} 0 & 2 & 3 & -2 \\ -2 & 1 & 5 & -3 \\ 2 & 0 & -2 & 1 \end{bmatrix} = \begin{bmatrix} -4 & 9 & 19 & -12 \\ 6 & -1 & -6 & 3 \end{bmatrix}$$

Example 1.6.10. Product of 1×3 and 3×2 matrices

$$\begin{bmatrix} 2 & 1 & -3 \end{bmatrix} \begin{bmatrix} -3 & 5 \\ 1 & -3 \\ 0 & -2 \end{bmatrix} = \begin{bmatrix} -5 & 13 \end{bmatrix}$$

Example 1.6.11. Product of 3×4 and 4×1 matrices

$$\begin{bmatrix} 0 & 2 & 3 & 1 \\ -1 & 2 & 0 & -3 \\ 2 & 0 & 1 & 4 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix} = \begin{bmatrix} 2y + 3z + w \\ -x + 2y - 3w \\ 2x + z + 4w \end{bmatrix}$$

Definition 1.6.12. The following $n \times n$ matrix is

$$\begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \end{bmatrix}$$

$n \times n$ **identity matrix** and denote it by I_n.

Proposition 1.6.13 (Properties of matrix multiplication)**.** *Let A, B and C are matrices where the multiplication AB and BC etc, makes sense. Then*

(1) $A(BC) = (AB)C$.

(2) $k(AB) = (kA)B = A(kB)$

(3) $A(B + C) = AB + AC$

(4) $(A + B)C = AC + BC$

whenever the multiplication makes sense.

Transpose.

$$\begin{aligned} (A^T)^T &= A \\ (AB)^T &= B^T A^T. \end{aligned}$$

$$\mathbf{a} \cdot \mathbf{b} = \mathbf{a}^T \mathbf{b}.$$

Lemma 1.6.14. *For nay $m \times n$ matrix A and $n \times p$ matrix B, we have*

$$A\, I_n = A, \qquad I_n\, B = B.$$

Also, for any $n \times n$ matrix A, it holds that

$$A\, I_n = I_n\, A = A.$$

I_n is identity element in multiplication.

Example 1.6.15.

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Definition 1.6.16. If for any vector $\mathbf{u},\ \mathbf{v} \in \mathbb{R}^n$ and scalar $\alpha \in \mathbb{R}$, a function $T\colon \mathbb{R}^n \to \mathbb{R}^m$ is defined by

(1) $T(\mathbf{u}+\mathbf{v}) = T(\mathbf{u}) + T(\mathbf{v})$

(2) $T(\alpha\,\mathbf{u}) = \alpha\,T(\mathbf{u})$

we say T is a **linear transformation(mapping, function).**

Example 1.6.17. Express a given linear transformation $T\colon \mathbb{R}^n \to \mathbb{R}^m$ using the standard basis vector.

Since any vector in $\mathbb{R}^n$ can be written as $a_1\mathbf{e}_1 + a_2\mathbf{e}_2 + \cdots + a_n\mathbf{e}_n$ and T is determined by the values at these vectors.

$$\begin{aligned} T(a_1\mathbf{e}_1 + a_2\mathbf{e}_2 + \cdots + a_n\mathbf{e}_n) &= T(a_1\mathbf{e}_1) + T(a_2\mathbf{e}_2) + \cdots + T(a_n\mathbf{e}_n) \\ &= a_1T(\mathbf{e}_1) + a_2T(\mathbf{e}_2) + \cdots + a_nT(\mathbf{e}_n). \end{aligned}$$

Since $T(\mathbf{e}_1), T(\mathbf{e}_2), \ldots, T(\mathbf{e}_n)$ are in $\mathbb{R}^m$, we can write it as linear combinations of $\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_m$. Hence there exist numbers t_{ij} $(1 \le i \le m, 1 \le j \le n)$ s.t.

$$T(\mathbf{e}_j) = \sum_{i=1}^{m} t_{ij}\mathbf{e}_i \quad (1 \le j \le n). \tag{1.7}$$

Hence

$$T(a_1\mathbf{e}_1 + a_2\mathbf{e}_2 + \cdots + a_n\mathbf{e}_n) = \sum_{j=1}^{n} a_jT(\mathbf{e}_j) = \sum_{i=1}^{m}\left(\sum_{j=1}^{n} t_{ij}a_j\right)\mathbf{e}_i. \tag{1.8}$$

This procedure can be written in matrix form Eq. (1.7). The matrix having t_{ij} as ij-th component

$$\mathrm{mat}(T) = \begin{bmatrix} t_{11} & t_{12} & \cdots & t_{1n} \\ t_{21} & t_{22} & \cdots & t_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ t_{m1} & t_{m2} & \cdots & t_{mn} \end{bmatrix}$$

is called **matrix of** T. Let us multiply the column vector $a_1\mathbf{e}_1 + a_2\mathbf{e}_2 + \cdots +$

$a_n\mathbf{e}_n$ to the right of this matrix.

$$\begin{bmatrix} t_{11} & t_{12} & \cdots & t_{1n} \\ t_{21} & t_{22} & \cdots & t_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ t_{m1} & t_{m2} & \cdots & t_{mn} \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{bmatrix} = \begin{bmatrix} t_{11}a_1 + t_{12}a_2 + \cdots + t_{1n}a_n \\ t_{21}a_1 + t_{22}a_2 + \cdots + t_{2n}a_n \\ \vdots \\ t_{m1}a_1 + t_{m2}a_2 + \cdots + t_{mn}a_n \end{bmatrix}$$

Compare this with equation (1.8). Then rhs vector has $T(a_1\mathbf{e}_1 + a_2\mathbf{e}_2 + \cdots + a_n\mathbf{e}_n)$ as its component. Conversely, any $m \times n$ matrix $[t_{ij}]$ is given, then it determines linear transformation $T\colon \mathbb{R}^n \to \mathbb{R}^m$ as in equation (1.8). Hence linear transformation $T\colon \mathbb{R}^n \to \mathbb{R}^m$ has one-one correspondence with $m \times n$ matrix as follows:

$$\text{mat}\colon T \mapsto \Big[\ \mathbf{e}_i \cdot T(\mathbf{e}_j)\ \Big]_{\substack{1 \le i \le m \\ 1 \le j \le n}}$$

Proposition 1.6.18. *For two linear transformations $T\colon \mathbb{R}^n \to \mathbb{R}^m$, $U\colon \mathbb{R}^p \to \mathbb{R}^n$ it holds that*

$$\text{mat}(T \circ U) = \text{mat}(T)\ \text{mat}(U).$$

Example 1.6.19. For the given two linear transformations $T\colon \mathbb{R}^n \to \mathbb{R}^m$, $U\colon \mathbb{R}^p \to \mathbb{R}^n$ check Proposition 1.6.18 holds.

$$\begin{aligned} T(x, y, z) &= (3y - z, x + y) \\ U(s, t) &= (2s - t, s + 2t, -3s). \end{aligned}$$

sol. The matrices for T and U are

$$\text{mat}(T) = \begin{bmatrix} 0 & 3 & -1 \\ 1 & 1 & 0 \end{bmatrix}, \quad \text{mat}(U) = \begin{bmatrix} 2 & -1 \\ 1 & 2 \\ -3 & 0 \end{bmatrix}$$

Hence

$$\text{mat}(T)\ \text{mat}(U) = \begin{bmatrix} 0 & 3 & -1 \\ 1 & 1 & 0 \end{bmatrix} \begin{bmatrix} 2 & -1 \\ 1 & 2 \\ -3 & 0 \end{bmatrix} = \begin{bmatrix} 6 & 6 \\ 3 & 1 \end{bmatrix}.$$

On the other hand

$$\begin{aligned}(T\circ U)(s,t) &= T(2s-t, s+2t, -3s)\\ &= (3(s+2t)-(-3s), (2s-t)+(s+2t))\\ &= (6s+6t, 3s+t).\end{aligned}$$

So

$$\mathrm{mat}(T\circ U) = \begin{bmatrix} 6 & 6\\ 3 & 1\end{bmatrix}.$$

Hence the following holds.

$$\mathrm{mat}(T\circ U) = \mathrm{mat}(T)\ \mathrm{mat}(U).$$

■

Determinant

We have seen 3×3 2×2. Using these, we define determinant of $n\times n$ matrix by induction. We expand w.r.t 1st row.

$$\begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1n}\\ a_{21} & a_{22} & \cdots & a_{2n}\\ \cdots & \cdots & \ddots & \vdots\\ a_{n1} & a_{n2} & \cdots & a_{nn}\end{vmatrix} = a_{11}\begin{vmatrix} a_{22} & a_{23} & \cdots & a_{2n}\\ a_{32} & a_{33} & \cdots & a_{3n}\\ \cdots & \cdots & \ddots & \vdots\\ a_{n2} & a_{n3} & \cdots & a_{nn}\end{vmatrix} - a_{12}\begin{vmatrix} a_{21} & a_{23} & \cdots & a_{2n}\\ a_{31} & a_{33} & \cdots & a_{3n}\\ \cdots & \cdots & \ddots & \vdots\\ a_{n1} & a_{n3} & \cdots & a_{nn}\end{vmatrix}$$

$$\cdots(-1)^{1+i}a_{1i}\begin{vmatrix} a_{21} & \cdots & \check{a}_{2,i} & \cdots & a_{2n}\\ a_{31} & \cdots & \check{a}_{3,i} & \cdots & a_{3n}\\ \vdots & \ddots & \vdots & \ddots & \vdots\\ a_{n1} & \cdots & \check{a}_{n,i} & \cdots & a_{nn}\end{vmatrix} + \cdots(-1)^{1+n}a_{1n}\begin{vmatrix} a_{21} & \cdots & a_{2,n-1}\\ a_{31} & \cdots & a_{3,n-1}\\ \cdots & \ddots & \vdots\\ a_{n1} & \cdots & a_{n,n-1}\end{vmatrix}$$

Here $\check{a}_{2,i}$ etc. means skip that entry. The i-th term on the right is $(-1)^{1+i}a_{1i}$ times the determinant of $(n-1)\times(n-1)$ obtained by deleting first row and i-column.

Theorem 1.4.6 and corollary 1.4.7 hold for any square matrices.

Expansion with respect to any row

Let A_{ij} be the $(n-1)\times(n-1)$ matrix obtained by deleting i-row and j-th column. We can expand the determinant w.r.t i-th row, i.e.,

$$|A| = \sum_{j=1}^{n}(-1)^{i+j}a_{ij}|A_{ij}|.$$

If we expand w.r.t j-th column, we see

$$|A| = \sum_{i=1}^{n}(-1)^{i+j}a_{ij}|A_{ij}|.$$

Example 1.6.20. Expand w.r.t 2nd row

$$\begin{aligned}
\begin{vmatrix} 2 & -1 & 3 & 0 \\ 0 & -1 & 0 & 4 \\ 3 & 1 & 0 & 2 \\ 2 & 0 & -3 & 0 \end{vmatrix} &= -0\begin{vmatrix} -1 & 3 & 0 \\ 1 & 0 & 2 \\ 0 & -3 & 0 \end{vmatrix} + (-1)\begin{vmatrix} 2 & 3 & 0 \\ 3 & 0 & 2 \\ 2 & -3 & 0 \end{vmatrix} \\
&\quad - 0\begin{vmatrix} 2 & -1 & 0 \\ 3 & 1 & 2 \\ 2 & 0 & 0 \end{vmatrix} + 4\begin{vmatrix} 2 & -1 & 3 \\ 3 & 1 & 0 \\ 2 & 0 & -3 \end{vmatrix} \\
&= -2\begin{vmatrix} 0 & 2 \\ -3 & 0 \end{vmatrix} + 3\begin{vmatrix} 3 & 2 \\ 2 & 0 \end{vmatrix} + 4\cdot 2\begin{vmatrix} 1 & 0 \\ 0 & -3 \end{vmatrix} \\
&\quad + 4\cdot 1\begin{vmatrix} 3 & 0 \\ 2 & -3 \end{vmatrix} + 4\cdot 3\begin{vmatrix} 3 & 1 \\ 2 & 0 \end{vmatrix} \\
&= -148.
\end{aligned}$$

Cramer's rule

The solution to $A\mathbf{x} = \mathbf{b}$ is $\mathbf{x} = (x_1, x_2, \cdots, x_n)$ where

$$x_i = \frac{1}{\det(A)}|A_i|.$$

Here A_i is the matrix obtained by replacing i-th column by $\mathbf{b}$.

Example 1.6.21. Solve

$$\begin{array}{ccccccc} 3x & + & 2y & + & z & = & 1 \\ & & y & + & z & = & 0 \\ x & + & y & & & = & 3. \end{array}$$

sol. Use Cramer's rule. First the determinant is

$$\begin{vmatrix} 3 & 2 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 0 \end{vmatrix} = 3\begin{vmatrix} 1 & 1 \\ 1 & 0 \end{vmatrix} - 0\begin{vmatrix} 2 & 1 \\ 1 & 0 \end{vmatrix} + 1\begin{vmatrix} 2 & 1 \\ 1 & 1 \end{vmatrix} = -2.$$

Then x_1, x_2 and x_3 are

$$x_1 = \frac{1}{-2}\begin{vmatrix} 1 & 2 & 1 \\ 0 & 1 & 1 \\ 3 & 1 & 0 \end{vmatrix} = -1,\ x_2 = \frac{1}{-2}\begin{vmatrix} 3 & 1 & 1 \\ 0 & 0 & 1 \\ 1 & 3 & 0 \end{vmatrix} = 4,\ x_3 = \frac{1}{-2}\begin{vmatrix} 3 & 2 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 3 \end{vmatrix} = -4.$$

■

Example 1.6.22. (1) Show for all vectors that $\|\mathbf{a}-\mathbf{b}\| \le \|\mathbf{a}-\mathbf{c}\| + \|\mathbf{c}-\mathbf{b}\|$

(2) Find a condition that $\|\mathbf{a}\|^2 + \|\mathbf{a}\|^2 = \|\mathbf{a}+\mathbf{b}\|^2$

(3) Show the determinant of a triangular matrix is given by

$$\begin{vmatrix} a_{11} & a_{12} & \cdots & \\ 0 & a_{22} & \cdots & \\ \vdots & 0 & a_{ii} & \vdots \\ 0 & 0 & \cdots & a_{nn} \end{vmatrix} = a_{11}a_{22}\cdots a_{nn}$$

Solve

$$\begin{array}{ccccccc} 3x & + & 2y & + & z & = & 1 \\ & & y & + & z & = & 0 \\ x & + & y & & & = & 3. \end{array}$$

1.7 Cylindrical and spherical coordinate

Cylindrical coordinate system

Given a point $P = (x, y, z)$, we can use polar coordinate for (x, y)-plane. Then it holds that

$$\text{Cylindrical to Cartesain} \begin{cases} x &= r\cos\theta, \\ y &= r\sin\theta, \\ z &= z. \end{cases}$$

We say (r, θ, z) is **cylindrical coordinate** of P. Conversely, the cylindrical coordinate (r, θ, z) is given by

$$\text{Cartesain to Cylindrical} \begin{cases} r^2 &= x^2 + y^2, \\ \tan\theta &= \frac{y}{x}, \\ z &= z. \end{cases}$$

The expression (r, θ, z) is not unique.

Example 1.7.1. The set of all points $r = a$ in cylindrical coordinate is

$$\{(x, y, z) \mid x^2 + y^2 = a^2\}.$$

This is a cylinder (Figure 1.24).

Example 1.7.2. $r = 3\cos\theta$ gives

$$r^2 = 3r\cos\theta \Rightarrow x^2 + y^2 = 3x.$$

This is again a cylinder.

Example 1.7.3. Change cylindrical coordinate $(6, \pi/3, 4)$ to Cartesian coordinate.

sol.

$$x = 6\cos(\pi/3) = 3, \quad y = 6\sin(\pi/3) = 3\sqrt{3}, \quad z = 4.$$

So $(x, y, z) = (3, 3\sqrt{3}, 4)$.

■

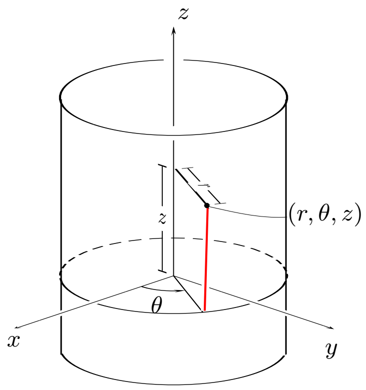


Figure 1.24: cylindrical coordinate

$$\theta = \begin{cases} \tan^{-1}(y/x) & (x > 0,\ y \geq 0) \\ 2\pi + \tan^{-1}(y/x) & (x > 0,\ y < 0) \\ \pi + \tan^{-1}(y/x) & (x < 0) \\ \pi/2 & (x = 0,\ y > 0) \\ 3\pi/2 & (x = 0,\ y < 0) \end{cases}$$

Example 1.7.4. Identify the surface given by the equation $z = 2r$ in cylindrical coordinate.

sol. $z^2 = 4r^2 = 4(x^2 + y^2)$. This is a cone.

■

Example 1.7.5. Change the equation $x^2 + y^2 - z^2 = 1$ to cylindrical coordinate.

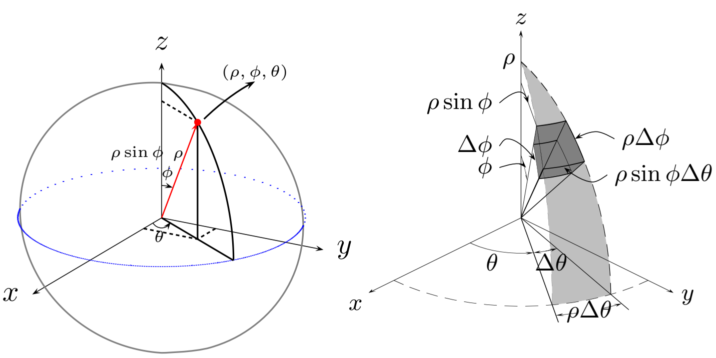
sol. $r^2 - z^2 = 1$.

■

Spherical coordinate system

We call (ρ, ϕ, θ) the **spherical coordinate** of P.

$$\text{Spherical to cylindrical} \begin{cases} r = \rho \sin\phi \\ \theta = \theta \\ z = \rho\cos\phi \end{cases} \quad \begin{cases} \rho^2 = r^2 + z^2 \\ \tan\phi = \dfrac{r}{z} \\ \theta = \theta. \end{cases}$$



$$x = \rho \sin\phi \cos\theta,\ \ y = \rho \sin\phi \sin\theta.$$

Figure 1.25: Spherical coordinate

For $P = (x, y, z)$ we have

$$\text{Spherical to Cartesian} \begin{cases} x = \rho \sin\phi \cos\theta \\ y = \rho \sin\phi \sin\theta \\ z = \rho \cos\phi \end{cases} \qquad \begin{pmatrix} \rho \ge 0 \\ 0 \le \theta < 2\pi \\ 0 \le \phi \le \pi \end{pmatrix}$$

Conversely, we can write ρ, ϕ, θ in terms of x, y, z.

$$\text{Cartesian to spherical} \begin{cases} \rho = \sqrt{x^2 + y^2 + z^2} \\ \cos\phi = \dfrac{z}{\rho} \\ \tan\theta = \dfrac{y}{x}. \end{cases}$$

Now the second condition has an alternate expression:We see from the figure

$$r = \rho \sin\phi, \quad z = \rho \cos\phi.$$

Hence $\cos\phi = \frac{z}{\rho}$ can be replaced by

$$\tan\phi = \frac{r}{z}.$$

Example 1.7.6. (1) Find the spherical coord. of the point $(x, y, z) = (1, -1, 1)$ and plot.

(2) Find the cartesian coord. of $(3, \pi/6, \pi/4)$.

(3) Find the spherical coord. of $(2, -3, 6)$.

(4) Find the spherical coord. of $(-3, -3, \sqrt{6})$.

sol. (1) $\rho = \sqrt{3}$.

$$\phi = \cos^{-1}(\frac{z}{\rho}) = \cos^{-1}(\frac{1}{\sqrt{3}}) \approx 0.955 \approx 54.74^o.$$

Since the point $(1, -1)$ in xy-plane lies in the 4-th quadrant, we see

$$\theta = \arctan(\frac{y}{x}) = \frac{7\pi}{4}.$$

(3) $\rho = \sqrt{2^2 + (-3)^2 + 6^2} = \sqrt{2^2 + (-3)^2 + 6^2} = 7$.

$$\phi = \cos^{-1}(\frac{z}{\rho}) = \cos^{-1}\frac{6}{7}.$$

Also, the point lies in the fourth quadrant, we have

$$\theta = 2\pi + \tan^{-1}(-3/2).$$

(4)

$$\begin{aligned}
\rho &= \sqrt{9+9+6} = 2\sqrt{6} \\
\phi &= \cos^{-1}\frac{\sqrt{6}}{2\sqrt{6}} = \cos^{-1}\frac{1}{2} = \frac{\pi}{3} \\
\theta &= \pi + \tan^{-1}(\frac{-1}{-1}) = \pi + \frac{\pi}{4} = \frac{5\pi}{4}.
\end{aligned}$$

Hence spherical coordinate is $(2\sqrt{6}, \pi/3, 5\pi/4)$.

■

Example 1.7.7. Express the surface (1) $xz = 1$ and (2) $x^2 + y^2 - z^2 = 1$ in spherical coordinate.

sol. (1) Since $xz = \rho^2 \sin\phi \cos\theta \cos\phi = 1$, we have the equation

$$\rho^2 \sin 2\phi \cos\phi = 2.$$

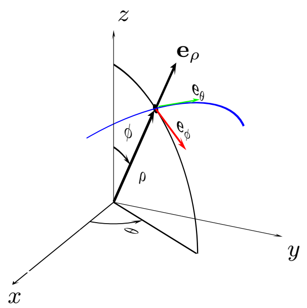


Figure 1.26: Standard basis for spherical coordinate

(2) Since $x^2+y^2-z^2 = x^2+y^2+z^2-2z^2 = \rho^2-2(\rho\cos\phi)^2 = \rho^2(1-2\cos^2\phi)$, the equation is $\rho^2(1-2\cos^2\phi) = 1$.

■

Standard basis for cylindrical and spherical coordinates

For cylindrical coordinates, the following sets are standard basis vectors:

$$\mathbf{e_r} = \frac{x\mathbf{i}+y\mathbf{j}}{\sqrt{x^2+y^2}}, \quad \mathbf{e}_\theta, \quad \mathbf{e}_z = \mathbf{k}.$$

These vary depending on the points and are defined so that only the coordinate indicated by the subscript increases. Now $\mathbf{e}_\theta$ is given by

$$\mathbf{e}_\theta = \mathbf{e_z} \times \mathbf{e_r} = \frac{-y\mathbf{i}+x\mathbf{j}}{\sqrt{x^2+y^2}}.$$

In this way $(\mathbf{e_r}, \mathbf{e}_\theta, \mathbf{e}_z)$ form a right handed coordinate system.

For spherical coordinates the followings are standard basis vectors.

$$\begin{aligned}
\mathbf{e}_\rho &= \frac{x\mathbf{i}+y\mathbf{j}+z\mathbf{j}}{\sqrt{x^2+y^2+z^2}} = \sin\phi\cos\theta\mathbf{i} + \sin\phi\sin\theta\mathbf{j} + \cos\phi\mathbf{k} \\
\mathbf{e}_\phi &= \cos\phi\cos\theta\mathbf{i} + \cos\phi\sin\theta\mathbf{j} - \sin\phi\mathbf{k} \\
\mathbf{e}_\theta &= \frac{-y\mathbf{i}+x\mathbf{j}}{\sqrt{x^2+y^2}} = -\sin\theta\mathbf{i} + \cos\theta\mathbf{j}.
\end{aligned}$$