Chapter 7

Integrals and transcendental function

In chapter 2 we introduced natural logarithmic function as the inverse of the exponential function e^x, where the number e was chosen to satisfy certain slope condition. In this chapter, we introduce an alternative theory for exponential and log. function.

7.1 Logarithm defined as integral

Definition 7.1.1.

$$\ln x = \int_1^x \frac{1}{t}\,dt, \ \ (x > 0)$$

Definition 7.1.2. For $x > 0$, the (natural) logarithmic function is defined by

$$\ln x = \int_1^x \frac{1}{t}\,dt.$$

Thus by fundamental theorem,

$$\frac{d}{dx}\ln x = \frac{1}{x}. \tag{7.1}$$

If $u(x)$ is any positive differentiable function,

$$\frac{d}{dx}\ln u(x) = \frac{1}{u}\frac{du}{dx}. \tag{7.2}$$

Properties:

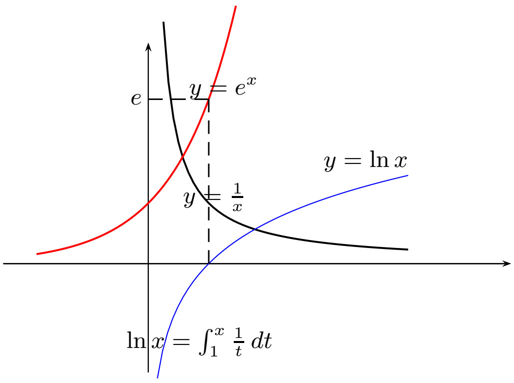

Figure 7.1: Graph of $\ln x$ and e^x

(1) $\ln bx = \ln b + \ln x$

(2) $\ln \frac{b}{x} = \ln b - \ln x$

(3) $\ln \frac{1}{x} = -\ln x$

(4) $\ln x^r = r \ln x$ (For rational number r).

Exponential function

Definition 7.1.3. Define the (natural) exponential function $exp(x) := \ln^{-1} x$ as the inverse function of $\ln x$. Thus

$$y = exp(x) \Leftrightarrow x = \ln y.$$

Thus

$$exp(\ln x) = x, \ (x > 0) \tag{7.3}$$
$$\ln(exp(x)) = x. \tag{7.4}$$

The number e is defined as

$$e = exp(1) = \ln^{-1}(1) = 2.718281828\cdots . \tag{7.5}$$

The function e^x

We can raise the number e to a rational power such as:

$$e^2 = e \cdot e, \ e^{1/2} = \sqrt{e}, \ldots$$

For rational number x, the value e^x is well defined. We see

$$\ln e^x = x \ln e = x. \tag{7.6}$$

The rational power e^x is the same as the $exp(x)$. Thus it is natural to define

Definition 7.1.4. For all real number x, we let

$$\boxed{e^x = exp(x)}. \tag{7.7}$$

The exponential function $exp(x)$ satisfies the usual rule for exponentiation such as $e^{a+b} = e^a e^b$.

Exponential function a^x

Since $a = e^{\ln a}$ for any positive number a, we can define a^x by

$$\begin{aligned} a^x &= e^{\ln a^x} \\ &= e^{x \ln a} \\ &= e^{(\ln a)x}. \end{aligned}$$

Definition 7.1.5. If a is a positive number and x is any number, we define

$$\boxed{a^x = e^{x \ln a}.} \tag{7.8}$$

Since $\ln e^x = x$ for all real x, we have

$$\ln x^n = \ln(e^{n \ln x}) = n \ln x, x > 0.$$

One can also use the definition of $\ln x = \int_1^x dt$ to prove it.

Example 7.1.6. [Power rule] The derivative of x^n for any number n:

$$\begin{aligned} \frac{d}{dx} x^n &= \frac{d}{dx} e^{n \ln x} \ (x > 0) \\ &= e^{n \ln x} \cdot \frac{d}{dx}(n \ln x) \\ &= x^n \cdot \frac{n}{x} \\ &= n x^{n-1}. \end{aligned}$$

Derivative of a^x

By definition, $a^x = e^{x \ln a}$. Thus

$$\frac{d}{dx}a^x = \frac{d}{dx}e^{x \ln a} = \ln a e^{x \ln a} = a^x \ln a.$$

$$\frac{d}{dx}a^u = a^u \ln a \frac{du}{dx}$$

General logarithmic function $\log_a x$

$y = \log_a x$ is defined as the inverse function of $y = a^x (a > 0, a \neq 1)$. Thus

$$\log_a x = y \Leftrightarrow a^y = x$$

$$\log_a (a^x) = x, \text{ for all } x, \text{ and } a^{(\log_a x)} = x, (x > 0)$$

$\log_{10} x$ is written as $\log x$ and called *common logarithmic function*

Properties

(1) Product rule: $\log_a xy = \log_a x + \log_a y$.

(2) Quotient rule: $\log_a \frac{x}{y} = \log_a x - \log_a y$.

(3) Product rule: $\log_a \frac{1}{y} = -\log_a y$.

(4) Power rule: $\log_a x^y = y \log_a x$.

Inverse properties

(1) Base a: $a^{\log_a x} = x$, $\log_a (a^x) = x (a > 0, a \neq 1, x > 0)$.

(2) Base e: $e^{\ln x} = x$, $\ln(e^x) = x (x > 0)$.

Derivative of $\log_a x$

We have

$$\log_a x = \frac{\ln x}{\ln a}. \tag{7.9}$$

So

$$\frac{d}{dx}\log_a x = \frac{1}{x \ln a}$$

and

$$\frac{d}{dx}\log_a u = \frac{1}{u \ln a}\frac{du}{dx}.$$

Logarithmic Differentiation

Find dy/dx if $y = \frac{(x^2+1)^{1/3}(x-3)^{1/2}}{x+5}$.

Example 7.1.7. (1) $\displaystyle\int_0^2 \frac{2x}{x^2-5}dx = \ln|u|_{-5}^{-1}$.

(2) $\displaystyle\int_{-\pi/2}^{\pi/2} \frac{4\cos\theta}{3+2\sin\theta}d\theta = \int_1^5 \frac{2}{u}\,du.$

7.2 Exponential change and separable differentiable equations

The decay of radioactive material or money earning interests in bank account, temperature between a cup of hot water and room air it sits, etc follows the law of *exponential change* Suppose $y(t)$ denotes some quantity which changes according to the exponential law: The rate of change of y is proportional to y.

$$\frac{dy}{dt} = ky$$

with I.C. Then $y = Ae^{kt}$.

Example 7.2.1. Assume a disease is spreading "Entero virus", "A.I" Let y be the number of people infected by disease. Assume we cure people as much as possible. Then dy/dt is proportional to y.(The more people, the more infected, the more cured) Suppose for each year the number is reduced by 20% and 10,000 people infected today, how many years will it take to reduce to $1,000$?

sol. $y = Ae^{kt}$, $A = 10,000$ Since it is reduced by 0.2 each year, we see

$$0.8 = e^{k\cdot 1} \to k = \ln 0.8 < 0$$

So we have $y = 10,000e^{(\ln 0.8)t}$ we want $10,000e^{(\ln 0.8)t} = 1,000$. So $e^{(\ln 0.8)t} = \frac{1}{10}$. $\ln(0.8)t = \ln(0.1)$. $t = \frac{\ln(0.1)}{\ln(0.8)} \approx 10.32$ yrs.

□

Example 7.2.2 (Half life of a radioactive material)**.** $y_0e^{-kt} = \frac{1}{2}y_0$. so $t = \ln 2/k$.

Example 7.2.3 (Carbon 14)**.** It is estimated the half life of Carbon 14 is 5700 yrs. AS wooden artifact was found from an ancient site. This contains carbon 14 about 10% less than the living tree. How old is the site? $k = \ln 2$ Half life is $\ln 2/5700$. $y = y_0 e^{-kt} = 0.9 y_0$ So $e^{-kt} = 0.9$ or $t = -5700\frac{\ln 0.9}{\ln 2} = 866$ yrs.

Example 7.2.4 (Law of Cooling)**.** If H is the temperature of an object and H_s the surrounding temperature. Then the rate of change(cooling) is proportional to the temperature difference. Thus

$$\frac{dH}{dt} = -k(H - H_s).$$

Solving

$$H - H_s = (H_0 - H_s)e^{-kt}.$$

A boiled egg at 98^o is put in the sink of 18^o to cool down. In 5 min, the egg was 38^o. how much longer will it take to reach 20^o?

sol.

$$H - 18 = (98 - 18)e^{-kt}, \quad H = 18 + 80e^{-kt}.$$

Set $H = 38, t = 5$. Then $e^{-5k} = 1/4$ and

$$k = -\frac{\ln 1/4}{5} = 0.2 \ln 4 \approx 0.28.$$

$$H = 18 + 80e^{-(0.2 \ln 4)t}.$$

Solving $t \approx 13$ min.

□

Separable Differential Equations

A general differential equation is given in the form

$$\frac{dy}{dx} = f(x, y) \tag{7.10}$$

with certain initial condition such as $y(x_0) = y_0$. Such equation is called **separable** if f is expressed as a product of a function of x and a function of y, i.e,

$$\frac{dy}{dx} = g(x)H(y).$$

We rewrite it in the form

$$\frac{dy}{dx} = \frac{g(x)}{h(y)}$$

and obtain

$$\int h(y)\,dy = \int g(x)\,dx. \tag{7.11}$$

Example 7.2.5. Solve

$$\frac{dy}{dx} = (1+y)e^x,\ \ y > -1. \tag{7.12}$$

Separate variables,

$$\begin{aligned} \frac{dy}{1+y} &= e^x dx \\ \int \frac{dy}{1+y} &= \int e^x dx \\ \ln(1+y) &= e^x + C. \end{aligned}$$

7.3 Hyperbolic function

hyperbolic function

Any $f(x)$ can be written as even part and odd part

$$f(x) = \underbrace{\frac{f(x)+f(-x)}{2}}_{\text{even part}} + \underbrace{\frac{f(x)-f(-x)}{2}}_{\text{odd part}}.$$

Hence e^x can be written as

$$e^x = \frac{e^x + e^{-x}}{2} + \frac{e^x - e^{-x}}{2} \tag{7.13}$$

Definition 7.3.1 (hyperbolic function). [1]

$$\textbf{hyperbolic cosine } \cosh x = \frac{e^x + e^{-x}}{2},$$
$$\textbf{hyperbolic sine } \sinh x = \frac{e^x - e^{-x}}{2},$$
$$\textbf{hyperbolic tangent } \tanh x = \frac{\sinh x}{\cosh x} = \frac{e^x - e^{-x}}{e^x + e^{-x}},$$
$$\textbf{hyperbolic cotangent } \coth x = \frac{1}{\tanh x} = \frac{e^x + e^{-x}}{e^x - e^{-x}},$$
$$\textbf{hyperbolic secant } \operatorname{sech} x = \frac{1}{\cosh x} = \frac{2}{e^x + e^{-x}},$$
$$\textbf{hyperbolic cosecant } \operatorname{csch} x = \frac{1}{\sinh x} = \frac{2}{e^x - e^{-x}}.$$

Some identities of hyperbolic functions:

Proposition 7.3.2.

(1) $\sinh 2x = 2\sinh x \cosh x$

(2) $\cosh 2x = \cosh^2 x + \sinh^2 x$

(3) $\sinh^2 x = \dfrac{\cosh 2x - 1}{2}$

(4) $\cosh^2 x = \dfrac{\cosh 2x + 1}{2}$

(5) $\cosh^2 x - \sinh^2 x = 1$

(6) $\tanh^2 x = 1 - \operatorname{sech}^2 x$

(7) $\coth^2 x = 1 + \operatorname{csch}^2 x$

Proposition 7.3.3.

(1) $\dfrac{d}{dx}(\sinh u) = \cosh u \dfrac{du}{dx}$

(2) $\dfrac{d}{dx}(\cosh u) = \sinh u \dfrac{du}{dx}$

[1] hyperbolic functions have many things in common with trig. functions. We can define trig. functions $\sin x$ and $\cos x$ using complex numbers. We define $e^{i\theta} = \cos\theta + i\sin\theta$. Then $e^{-i\theta} = \cos\theta - i\sin\theta$ and hence

$$\cos\theta = \frac{e^{i\theta} + e^{-i\theta}}{2}, \quad \sin\theta = \frac{e^{i\theta} - e^{-i\theta}}{2i}$$

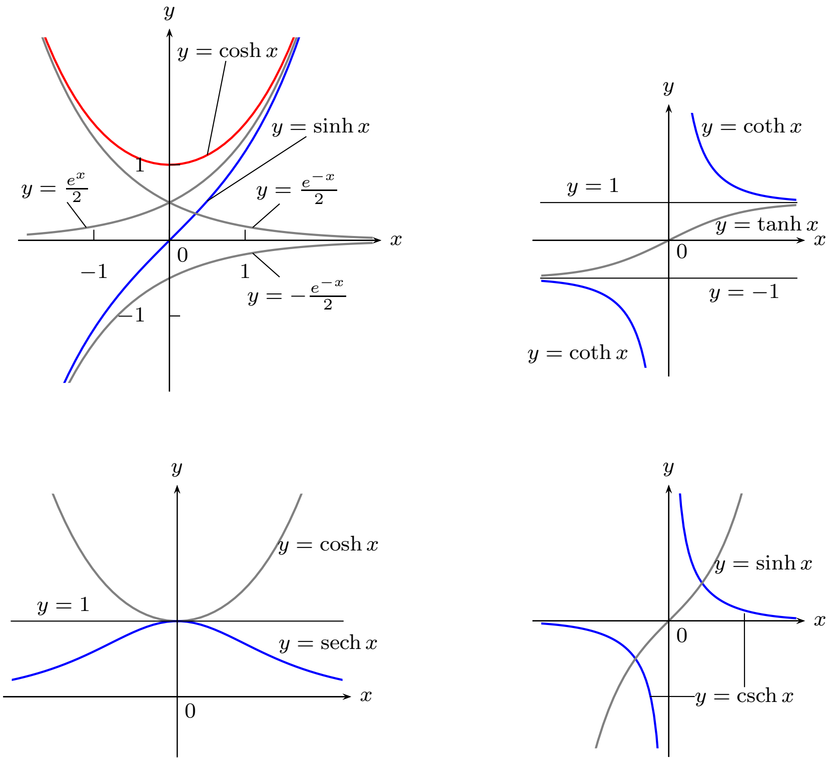

Figure 7.2: hyperbolic functions

(3) $\dfrac{d}{dx}(\tanh u) = \operatorname{sech}^2 u \dfrac{du}{dx}$

(4) $\dfrac{d}{dx}(\coth u) = -\operatorname{csch}^2 u \dfrac{du}{dx}$

(5) $\dfrac{d}{dx}(\operatorname{sech} u) = -\operatorname{sech} u \tanh u \dfrac{du}{dx}$

(6) $\dfrac{d}{dx}(\operatorname{csch} u) = -\operatorname{csch} u \coth u \dfrac{du}{dx}$

Proposition 7.3.4.

(1) $\displaystyle\int \sinh u\, du = \cosh u + C$

(2) $\displaystyle\int \cosh u\, du = \sinh u + C$

(3) $\displaystyle\int \operatorname{sech}^2 u\, du = \tanh u + C$

(4) $\displaystyle\int \operatorname{csch}^2 u\, du = -\coth u + C$

(5) $\displaystyle\int \operatorname{sech} u \tanh u\, du = -\operatorname{sech} u + C$

(6) $\displaystyle\int \operatorname{csch}\ u \coth u du = -\operatorname{csch} u + C$

Example 7.3.5. (1) The indefinite integral of $\sinh^2 x$ can be computed just as that of $\sin^2 x$.

$$\begin{aligned}\int_0^1 \sinh^2 x\, dx &= \int_0^1 \frac{\cosh 2x - 1}{2}\, dx\\ &= \frac{1}{2}\left[\frac{\sinh 2x}{2} - x\right]_0^1\\ &= \frac{\sinh 2}{4} - \frac{1}{2}.\end{aligned}$$

(2) Using the definition of $\sinh x$

$$\begin{aligned}\int_0^{\ln 2} 4e^x \sinh x\, dx &= \int_0^{\ln 2} 4e^x \frac{e^x - e^{-x}}{2}\, dx = \int_0^{\ln 2} (2e^{2x} - 2)\, dx\\ &= \left[e^{2x} - 2x\right]_0^{\ln 2}\\ &= 3 - 2\ln 2.\end{aligned}$$

Inverse hyperbolic function

The function $y = \sinh x$ is defined on $(-\infty, \infty)$ having values in $(-\infty, \infty)$. So (inverse hyperbolic sine) $y = \sinh^{-1} x$ is defined on $(-\infty,\ \infty)$.

The function $y = \cosh x$ restricted to $x \geq 0$ is 1-1 to $[1, \infty)$. So inverse $y = \cosh^{-1}$ inverse hyperbolic cosine is defined on $[1, \infty)$.

$y = \operatorname{sech} x$ restricted to $x \geq 0$ is one-to-one. Hence its inverse $y = \operatorname{sech}^{-1} x$ is defined on $(0, 1]$. Meanwhile $y = \tanh x$, $y = \coth x$, $y = \operatorname{csch} x$ are one-to-one on $(-\infty, \infty)$. Hence their inverses $y = \tanh^{-1} x$, $y = \coth^{-1} x$, $y = \operatorname{csch}^{-1} x$ are defined accordingly. The graphs are as in figure 7.3

Proposition 7.3.6. *Inverse hyperbolic functions can be represented by log functions.*

(1) $\sinh^{-1} x = \ln\bigl(x + \sqrt{x^2 + 1}\bigr), \quad -\infty < x < \infty$

(2) $\cosh^{-1} x = \ln\bigl(x + \sqrt{x^2 - 1}\bigr), \quad x \geq 1$ *(formula at p. 442 of the book is wrong)*

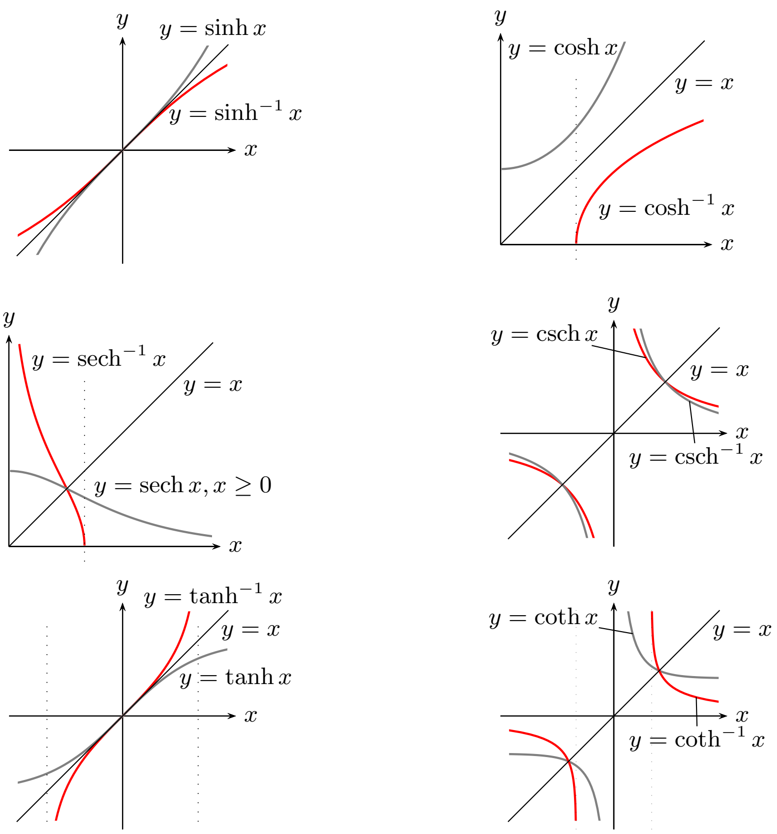

Figure 7.3: Inverse hyperbolic functions

(3) $\tanh^{-1} x = \dfrac{1}{2} \ln \dfrac{1+x}{1-x}, \quad |x| < 1$

(4) $\operatorname{sech}^{-1} x = \ln\left(\dfrac{1+\sqrt{1-x^2}}{x}\right), \quad 0 < x \leq 1$

(5) $\operatorname{csch}^{-1} x = \ln\left(\dfrac{1}{x} + \dfrac{\sqrt{1+x^2}}{|x|}\right), \quad x \neq 0$

(6) $\coth^{-1} x = \dfrac{1}{2} \ln \dfrac{x+1}{x-1}, \quad |x| > 1.$

Proof. (1) We prove the formula for $\sinh^{-1} x$.

$$y = \sinh x = \frac{e^x - e^{-x}}{2},$$
$$e^x - e^{-x} = 2y,$$
$$e^{2x} - 2ye^x - 1 = 0,$$
$$e^x = y + \sqrt{y^2+1}. \quad \text{(Since } y - \sqrt{y^2+1} \text{ is negative, we drop it.)}$$

Hence $x = \ln(y + \sqrt{y^2+1})$, changing variable, we have $y = \ln(x + \sqrt{x^2+1})$.

(4) The formula for $\operatorname{sech}^{-1} x$.

$$\begin{aligned} y = \operatorname{sech}^{-1} x \Rightarrow \operatorname{sech} y = x \Rightarrow x &= \frac{2}{e^y + e^{-y}}, \\ e^y + e^{-y} &= \frac{2}{x}, \\ e^{2y} - \frac{2}{x} e^y + 1 &= 0, \\ e^y = \frac{1}{x} \pm \sqrt{\frac{1}{x^2} - 1} &= \frac{1 \pm \sqrt{1-x^2}}{x}. \end{aligned}$$

Hence $y = \ln\left(\frac{1+\sqrt{1-x^2}}{x}\right), \quad 0 < x \le 1.$

Proposition 7.3.7.

(1) $\operatorname{sech}^{-1} x = \cosh^{-1} \dfrac{1}{x}$

(2) $\operatorname{csch}^{-1} x = \sinh^{-1} \dfrac{1}{x}$

(3) $\coth^{-1} x = \tanh^{-1} \dfrac{1}{x}$

Derivatives of inverse hyperbolic functions

Proposition 7.3.8.

(1) $\dfrac{d(\sinh^{-1} u)}{dx} = \dfrac{1}{\sqrt{1+u^2}} \dfrac{du}{dx}$

(2) $\dfrac{d(\cosh^{-1} u)}{dx} = \dfrac{1}{\sqrt{u^2-1}} \dfrac{du}{dx}, \quad u > 1$

(3) $\dfrac{d(\tanh^{-1} u)}{dx} = \dfrac{1}{1-u^2} \dfrac{du}{dx}, \quad |u| < 1$

(4) $\dfrac{d(\coth^{-1} u)}{dx} = \dfrac{1}{1-u^2} \dfrac{du}{dx}, \quad |u| > 1$

(5) $\dfrac{d(\operatorname{sech}^{-1} u)}{dx} = \dfrac{-du/dx}{u\sqrt{1-u^2}}, \quad 0 < u < 1$

(6) $\dfrac{d(\operatorname{csch}^{-1} u)}{dx} = \dfrac{-du/dx}{|u|\sqrt{1+u^2}}, \quad u \ne 0$

Proof. (5) The formula for $\operatorname{sech}^{-1} x$.

$$\begin{aligned} y = \operatorname{sech}^{-1} x &\Rightarrow \operatorname{sech} y = x, \\ -\operatorname{sech} y \tanh y \frac{dy}{dx} &= 1, \\ \frac{dy}{dx} &= -\frac{1}{\operatorname{sech} y \tanh y}, \\ &= -\frac{1}{x\sqrt{1-x^2}}, \quad 0 < x \le 1. \end{aligned}$$

Proposition 7.3.9.

(1) $\displaystyle\int \frac{du}{\sqrt{1+u^2}} = \sinh^{-1} u + C$

(2) $\displaystyle\int \frac{du}{\sqrt{u^2-1}} = \cosh^{-1} u + C, \quad u > 1$

(3) $\displaystyle\int \frac{du}{1-u^2} = \begin{cases} \tanh^{-1} u + C, & \text{if } |u| < 1, \\ \coth^{-1} u + C, & \text{if } |u| > 1 \end{cases}$

(4) $\displaystyle\int \frac{du}{u\sqrt{1-u^2}} = -\operatorname{sech}^{-1} |u| + C = -\cosh^{-1}\left(\frac{1}{|u|}\right) + C$

(5) $\displaystyle\int \frac{du}{u\sqrt{1+u^2}} = -\operatorname{csch}^{-1} |u| + C = -\sinh^{-1}\left(\frac{1}{|u|}\right) + C$

Example 7.3.10.

$$\begin{aligned} \int \frac{2\,dx}{\sqrt{3+4x^2}} &= \int \frac{du}{\sqrt{a^2+u^2}}, \ (u = 2x, a = \sqrt{3}) \\ &= \sinh^{-1}(\frac{u}{a}) + C = \sinh^{-1}(\frac{2x}{\sqrt{3}}) + C \end{aligned}$$

7.4 Relative Rate of Growth

Definition 7.4.1. Suppose $f(x), g(x)$ are positive for sufficiently large x.

(1) f **grows faster than** g as $x \to \infty$ if

$$\lim_{x\to\infty} \frac{f(x)}{g(x)} = \infty$$

(2) $f(x)$ **grows at a smaller order than** $g(x)$ as $x \to \infty$ if

$$\lim_{x\to\infty} \frac{f(x)}{g(x)} = 0.$$

In this case we write $f = o(g)$.

(3) f **grows at the same rate** as g as $x \to \infty$ if

$$\lim_{x\to\infty} \frac{f(x)}{g(x)} = L, \text{ for some postive finite number } L.$$

In this case we write $f = O(g)$.

Example 7.4.2. (1) e^x grows faster than x^3 as $x \to \infty$

(2) 3^x grows faster than 2^x as $x \to \infty$

(3) x grows faster than $\ln x$ as $x \to \infty$

Example 7.4.3. (1) $\ln x = o(x)$ as $x \to \infty$

(2) $x^2 = o(x^3)$ as $x \to \infty$

(3) $x + \sin x = O(x)$