Chapter 3

Differentiation

3.1 Tangents and Derivatives at a point

Finding tangent to the graph of a function

Definition 3.1.1. The slope of the curve $y = f(x)$ at a point $P = (x_0, f(x_0))$ is the number

$$\lim_{h \to 0} \frac{f(x_0 + h) - f(x_0)}{h}$$

provided it exists. The **tangent line** to the curve at the point P is the line through P with this slope.

The rate of change of $y = f(x)$ between $P(x_0, y_0)$ and $Q(x_1, y_1)$ is

$$m_{sec} = \frac{f(x_1) - f(x_0)}{x_1 - x_0}$$

In the limit, it is the slope of tangent line.

Hence the slope of tangent line at P is

$$m_{tan} = \lim_{x_1 \to x_0} \frac{f(x_1) - f(x_0)}{x_1 - x_0} = \lim_{h \to 0} \frac{f(x_0 + h) - f(x_0)}{h}$$

Rate of change:Derivative at a point

Definition 3.1.2. The derivative of a function $f(x)$ at a point is given by

$$f'(x_0) = \lim_{h \to 0} \frac{f(x_0 + h) - f(x_0)}{h}$$

exists, it is called **derivative** at $x = x_0$.

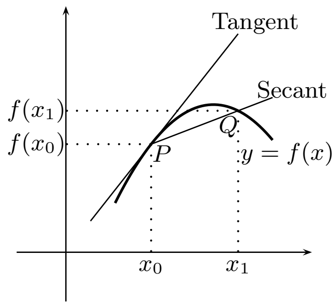


Figure 3.1: Tangent and secant

Example 3.1.3. Interpretations:

(1) The slope of $y = f(x)$ at x_0

(2) The slope of tangent to the curve $y = f(x)$ at x_0

(3) The rate of change of f with respect to x at x_0

(4) The derivative $f'(x_0)$ at the point.

3.2 Derivative as a function

Definition 3.2.1. If the derivative of a function $f(x)$ at a

$$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h}$$

exists, it is called the **derivative** (function) of $f(x)$. (Treated as a function) f is said to be **differentiable** at x. If f is differentiable at all points of domain we say f is **differentiable.** We also use the notation df/dx, $(d/dx)f$ for f'.

Alternative formula for the derivative is

$$f'(x) = \lim_{z\to x} \frac{f(z) - f(x)}{z - x}$$

Try some examples in the text.

One sided derivative

Definition 3.2.2. Suppose f is defined on $[a,b]$. Then at each end point the one sided derivative is defined by

$$f'(a^+) = \lim_{h\to 0^+} \frac{f(a+h)-f(a)}{h},$$
$$f'(b^-) = \lim_{h\to 0^-} \frac{f(b+h)-f(b)}{h} = \lim_{h\to 0^+} \frac{f(b)-f(b-h)}{h}.$$

Example 3.2.3. Using definition, find $(d/dx)x^3$.

sol. Set $f(x) = x^3$

$$f(x+h)-f(x) = h\{(x+h)^2 + x(x+h) + x^2\}$$

$$\lim_{h\to 0} \frac{1}{h}\{f(x+h)-f(x)\} = \lim_{h\to 0}\{(x+h)^2 + x(x+h) + x^2\} = 3x^2$$

So $f'(x) = 3x^2$

Example 3.2.4. The one sided derivatives of $f(x) = |x|$ at $x = 0$ are $f'(0^-) = -1$ and $f'(0^+) = 1$. Hence f is not differentiable at $x = 0$. (Figure 3.2)

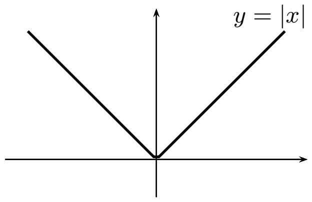


Figure 3.2: $y = |x|$

When does a function do not have derivative at a point?

Example 3.2.5. The function defined by

$$f(x) = \begin{cases} x\sin(1/x), & x \neq 0 \\ 0, & x = 0 \end{cases}$$

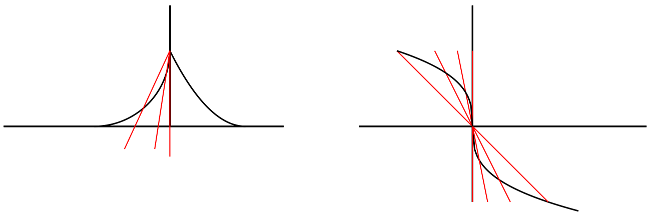


Figure 3.3: Cases with no derivative

is conti at $x = 0$ but not differentiable.

sol. Since $-|x| \le f(x) \le$ and $\lim_{x\to 0}|x| = \lim_{x\to 0} = 0$ by theorem f is conti. at $x = 0$. But the limit

$$\lim_{h\to 0}\frac{1}{h}\left\{h\sin\frac{1}{h} - 0\right\} = \lim_{h\to 0}\sin\frac{1}{h}$$

does not exist, f is not differentiable $x = 0$.

Differentiable functions are continuous

Theorem 3.2.6. *If f is differentiable at $x = a$, then f is conti. at $x = a$.*

Proof. By definition of derivative we have

$$\lim_{h\to 0}\{f(a+h) - f(a)\} = \left(\lim_{h\to 0}\frac{1}{h}\{f(a+h) - f(a)\}\right)\cdot\left(\lim_{h\to 0} h\right) = 0.$$

Hence $f(x)$ is conti. at $x = a$.

Intermediate Value property of derivatives(Darboux's theorem)

Theorem 3.2.7. *If a, b are any two points in an interval where f is differentiable, then f' takes any value between $f'(a)$ and $f'(b)$.*

3.3 Differentiation Rules

Proposition 3.3.1. *Suppose f, g are differentiable functions. Then*

(1) For any constant C, $\frac{dC}{dx} = 0$ for any constant C.

(2) When n is positive integer , $\frac{d}{dx}x^n = nx^{n-1}$.

(3) For any constant C, $\frac{d(Cu)}{dx} = C\frac{du}{dx}$.

(4) $\frac{d}{dx}(u \pm v) = \frac{du}{dx} \pm \frac{dv}{dx}$.

(5) $\frac{d(uv)}{dx} = v\frac{du}{dx} + u\frac{dv}{dx}$.

(6) $\frac{d}{dx}\left(\frac{u}{v}\right) = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$.

(7) For any real number n, $\frac{d}{dx}x^n = nx^{n-1}$.

Proof. (2) Use

$$z^n - x^n = (z-x)(z^{n-1} + z^{n-2}x + \cdots + x^{n-2}z + x^{n-1})$$

(4)

$$\begin{aligned}
\frac{d}{dx}(uv) &= \lim_{h\to 0}\frac{u(x+h)v(x+h) - u(x)v(x)}{h} \\
&= \lim_{h\to 0}\left\{\frac{u(x+h)-u(x)}{h}v(x+h) + u(x)\frac{v(x+h)-v(x)}{h}\right\} \\
&= u'(x)v(x) + u(x)v'(x).
\end{aligned}$$

Try to prove the product rule and draw Figure for product rule.

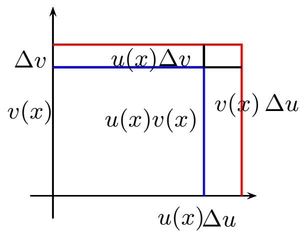


Figure 3.4: product rule

Higher order derivative

$$f''(x) = \frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right)$$

$$y^{(n)} = \frac{dy^{(n-1)}}{dx} = \frac{d^n y}{dx^n}$$

3.4 The Derivative as a Rate of change

Definition 3.4.1. The **instantaneous rate of change** of f at x_0 is

$$f'(x_0) = \lim_{h \to 0} \frac{f(x_0 + h) - f(x_0)}{h}$$

provided the limit exists.

How to describe a moving object?

How fast ? etc. First we assume an object is moving along a line(coordinate line) and its position is given as a function of time:

$$s = f(t)$$

Then the **displacement** of the object over the time interval t to $t + \Delta t$ is

$$\Delta s = f(t + \Delta t) - f(t)$$

and define the **average velocity** of the object over that time interval is

$$v_{av} = \frac{\text{displacement}}{\text{elapsed time}} = \frac{\Delta s}{\Delta t} = \frac{f(t + \Delta t) - f(t)}{\Delta t}$$

Definition 3.4.2. Velocity(instantaneous velocity) is the derivative of a position function w.r.t time. If a moving object position is given by $s = f(t)$, then the **velocity** at t is

$$v(t) = \lim_{h \to 0} \frac{f(t + h) - f(t)}{h}$$

provided the limit exists.

Definition 3.4.3. (Speed) is the absolute value of the velocity, i.e, **Speed**

$$\text{Speed } = |v(t)| = \left| \frac{ds}{dt} \right|$$

Acceleration is

$$a(t) = \frac{dv}{dt} = \frac{d^2 s}{dt^2}$$

Third derivative is called a Jerk

3.5 Derivative of Trig functions

Use definition to find the derivative of $f(x) = \sin x$.

$$\begin{aligned}
&= \lim_{h\to 0} \frac{\sin(x+h) - \sin x}{h} \\
&= \lim_{h\to 0} \frac{\sin x \cos h + \cos x \sin h - \sin x}{h} \\
&= \sin x \lim_{h\to 0} \frac{\cos h - 1}{h} + \cos x \lim_{h\to 0} \frac{\sin h}{h} \\
&= \sin x \cdot 0 + \cos x \cdot 1 = \cos x.
\end{aligned}$$

Hence $\frac{d}{dx} \sin x = \cos x$. Similarly, we have

$$\frac{d}{dx} \cos x = -\sin x$$

Other trigonometric functions are defined by the following relation and their derivatives can be found using differentiation rules:

$$\tan x = \frac{\sin x}{\cos x}, \quad \sec x = \frac{1}{\cos x}, \quad \csc x = \frac{1}{\sin x}, \quad \cot x = \frac{\cos x}{\sin x}$$

The derivative of $\tan x$ is

$$\frac{d}{dx} \tan x = \frac{(\sin x)' \cos x - (\cos x)' \sin x}{\cos^2 x} = \frac{1}{\cos^2 x} = \sec^2 x$$

Summarizing, we have

Proposition 3.5.1. *(1)* $\frac{d}{dx} \sin x = \cos x$

(2) $\frac{d}{dx} \cos x = -\sin x$

(3) $\frac{d}{dx} \tan x = \sec^2 x$

(4) $\frac{d}{dx} \sec x = \sec x \tan x$

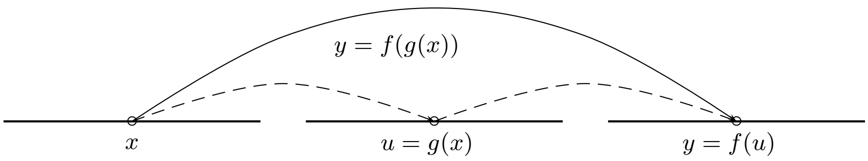


Figure 3.5: Chain rule

(5) $\dfrac{d}{dx}\csc x = -\csc x \cot x$

(6) $\dfrac{d}{dx}\cot x = -\csc^2 x$

3.6 Chain rule

Chain Rule

Theorem 3.6.1. *(1) If $f(u)$ is differentiable at $u = g(x)$ and g is differentiable at x, then the composite function $f \circ g$ is differentiable at x and*

$$(f \circ g)'(x) = f'(g(x))g'(x)$$

If $y = f(u)$, $u = g(x)$ then

$$\left.\frac{dy}{dx}\right|_x = \left.\frac{dy}{du}\right|_{u=g(x)} \cdot \left.\frac{du}{dx}\right|_x$$

Proof. (Intuitive) Let $\Delta u = g(x+\Delta x) - g(x)$ be the change of u corresponding to the change of Δx. (We assume $g(x + \Delta x) \neq g(x)$) We might consider

$$\frac{\Delta y}{\Delta x} = \frac{\Delta y}{\Delta u}\frac{\Delta u}{\Delta x}, (\Delta u \neq 0)$$

$$\begin{aligned}
\lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} &= \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta u}\frac{\Delta u}{\Delta x} \\
&= \lim_{\Delta u \to 0} \frac{\Delta y}{\Delta u} \lim_{\Delta x \to 0} \frac{\Delta u}{\Delta x} \\
&= \frac{dy}{du}\frac{du}{dx}
\end{aligned}$$

Repeated Use

Example 3.6.2. $y = \cos(1 + x^4)^5$

sol. Given function is the composite of $y = \cos u$ and $u = (1 + x^4)^5$. Hence

$$\frac{dy}{du} = -\sin u \frac{du}{dx}.$$

On the other hand, since $(1 + x^4)^5$ is a composite function of $u = v^5$ and $v = 1 + x^4$, use Chain rule again

$$\frac{du}{dx} = 5v^4 \cdot 4x^3.$$

Hence

$$\frac{dy}{dx} = -\sin(1 + x^4)^5 \cdot 20(1 + x^4)^4 x^3.$$ □

Outside-Inside Rule

Example 3.6.3. The derivative of $|x|$ can be computed as follows: Notice that $|x| = \sqrt{x^2}$. Hence

$$\frac{d}{dx}\sqrt{x^2} = \frac{1}{2\sqrt{x^2}} \cdot 2x = \frac{x}{|x|}, \quad x \neq 0.$$

Example 3.6.4. Find slope of tangent line to $y = (1 - 2x)^3$ at $x = 1$.

Derivatives of Power function

Find derivative of $f(x) = u^r(x)$ for any real r and $x > 0$.

Theorem 3.6.5. *Since* $u^r = e^{r \ln u}$

$$\frac{d}{dx}u^r = ru^r \frac{d \ln u}{dx} = ur^r \frac{1}{u}\frac{du}{dx} = ru^{r-1}\frac{du}{dx}.$$

3.7 Implicit differentiation

There are situation where some relation between x and y defines some graph, but not graph of any function. Suppose x, y satisfy $y^5 + \sin xy = x^3 y$. This relation defines a function implicitly.

Implicitly defined function

Assuming y is a differentiable function of x satisfying some relation like $F(x, y) = 0$, we take derivative of $F(x, y) = 0$ w.r.t x using the chain rule. Then solving for dy/dx we find the derivative. This procedure is called an **Implicit differentiation**.

$$\frac{d}{dx}\left(y^5\right) + \frac{d}{dx}(\sin xy) = \frac{d}{dx}\left(x^3 y\right)$$
$$5y^4\frac{dy}{dx} + (\cos xy)\left(y + x\frac{dy}{dx}\right) = 3x^2 y + x^3\frac{dy}{dx}.$$

Hence

$$\frac{dy}{dx} = \frac{3x^2 y - y\cos xy}{5y^4 + x\cos xy - x^3}.$$

Example 3.7.1. (1) $x^3 + y^3 = 3xy$. Find dy/dx and d^2y/dx^2.

(2) Find equation of tangent line to $x^3 + y^3 = 3xy$ at $(3/2, 3/2)$.

sol.

(1) Taking derivative we have

$$(3y^2 - 3x)\frac{dy}{dx} = 3y - 3x^2.$$

By chain rule

$$\left(6y\frac{dy}{dx} - 3\right)\frac{dy}{dx} + (3y^2 - 3x)\frac{d^2y}{dx^2} = 3\frac{dy}{dx} - 6x.$$

Hence

$$\frac{dy}{dx} = \frac{y - x^2}{y^2 - x},$$
$$\frac{d^2y}{dx^2} = \frac{-2\left((y - x^2)/(y^2 - x)\right)^2 y + 2(y - x^2)/(y^2 - x) - 2x}{y^2 - x}.$$

(2) At $(3/2, 3/2)$, $dy/dx = -1$. The tangent line is

$$y = -\left(x - \frac{3}{2}\right) + \frac{3}{2} = -x + 3.$$ □

3.8 Inverse functions and Their Derivatives

Definition 3.8.1. A function f is one-to-one on a domain D if $f(x_1) \neq f(x_2)$ whenever $x_1 \neq x_2$.

Definition 3.8.2. Suppose a function f is one-to-one on a domain D with range R. The **inverse function** f^{-1} exists and is defined by

$$f^{-1}(b) = a \text{ if } f(a) = b.$$

The domain of f^{-1} is R and range is D.

$$(f^{-1} \circ f)(x) = x, \quad x \in D$$
$$(f \circ f^{-1})(y) = y, \quad y \in R$$

Derivatives of inverse function

Theorem 3.8.3. *Suppose f is differentiable in I. If $f'(x)$ is never zero, then f^{-1} exists, differentiable. Furthermore for $a \in I$, $f(a) = b$,*

$$(f^{-1})'(b) = \frac{1}{f'(a)}.$$

Set $y = f(x)$. Then the inverse function is $x = f^{-1}(y)$, and its derivative is

$$\left.\frac{dx}{dy}\right|_{y=f(a)} = \frac{1}{dy/dx|_{x=a}}, \quad a \in I$$

Proof. Differentiate $x = (f^{-1} \circ f)(x) = f^{-1}(f(x)) = f^{-1}(y)$ w.r.t x using the Chain rule, we have

$$1 = (f^{-1})'(f(x))f'(x).$$

Setting $x = a$, we see $1 = (f^{-1})'(f(a))f'(a)$. Thus

$$(f^{-1})'(b) = 1/f'(a).$$

Usually, we use the notation $y = f^{-1}(x)$. The graph of $y = f(x)$ and that of $y = f^{-1}(x)$ are symmetric w.r.t the line $y = x$.

Example 3.8.4. (1) $f(x) = x^7 + 8x^3 + 4x - 2$. Find $(f^{-1})'(-2)$.

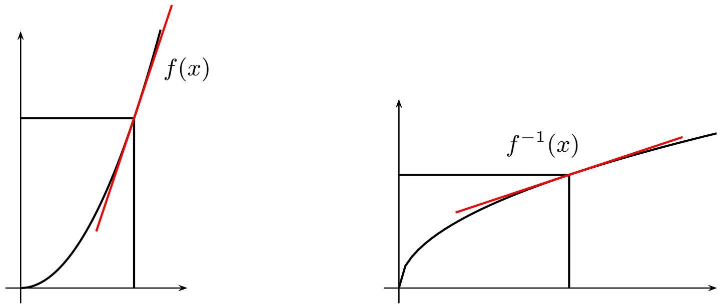


Figure 3.6: Slope of inverse function

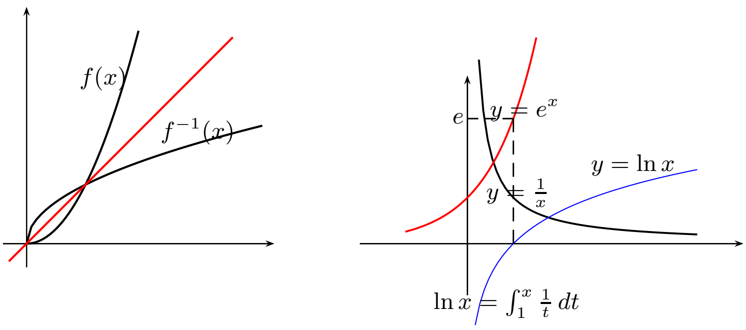


Figure 3.7: Graph of inverse functions, Graph of $\ln x$ and e^x

(2) $f(x) = \sin^{-1} x$. Find f'.

sol. (1) Since $f' = 7x^6 + 24x^2 + 4 \geq 4$ inverse f^{-1} exists. Since $f(0) = -2$ we have

$$(f^{-1})'(-2) = (f^{-1})'(f(0)) = \frac{1}{f'(0)} = \frac{1}{4}.$$

(2) $y = \sin^{-1} x$, $x = \sin y$. Hence

$$\begin{aligned} \frac{d}{dx}\sin^{-1} x &= \frac{dy}{dx} = \frac{1}{dx/dy} = \frac{1}{(d/dy)\sin y} \\ &= \frac{1}{\cos y} = \frac{1}{\sqrt{1-\sin^2 y}} = \frac{1}{\sqrt{1-x^2}}. \end{aligned}$$

■

Derivative of the natural Logarithmic function

$f(x) = e^x$. The derivative of its inverse function $f^{-1}(x) = \ln x, x > 0$ is

$$\begin{aligned}(f^{-1})'(x) &= \frac{1}{f'(f^{-1}(x))} \\ &= \frac{1}{e^{f^{-1}(x)}} \\ &= \frac{1}{e^{\ln x}} = \frac{1}{x}.\end{aligned}$$

Alterative way: Let $y = \ln x$. Implicit differentiation w.r.t. x gives

$$\begin{aligned}e^y &= x \\ e^y \frac{dy}{dx} &= 1 \\ \frac{dy}{dx} &= \frac{1}{e^y} = \frac{1}{x}.\end{aligned}$$

If $u(x)$ is any positive differentiable function,

$$\frac{d}{dx} \ln u(x) = \frac{1}{u} \frac{du}{dx}, u > 0. \tag{3.1}$$

When $x < 0$ one can use $u = bx, (b = -1)$ and the chain rule to derive

$$\begin{aligned}\frac{d}{dx} \ln(bx) &= \frac{1}{bx} \cdot \frac{d}{dx}(bx) \\ &= \frac{1}{bx} \cdot \frac{d}{dx}(bx) \\ &= \frac{1}{x}.\end{aligned}$$

Thus we have

$$\boxed{\frac{d}{dx} \ln |x| = \frac{1}{x}, \ x \neq 0.} \tag{3.2}$$

Example 3.8.5. The point where the line through of origin $y = mx$ is tangent to the graph of $y = \ln x$.

sol. We must have $m = \frac{1}{x}$ and $mx = \ln x$. Hence we get $m = \frac{1}{e}$ and $x = e$.

The number e is sometimes defined as

$$e = exp(1) = \ln^{-1}(1) = 2.718281828 \cdots . \tag{3.3}$$

Derivative of a^x and $\log_a x$

By definition, $a^x = e^{x \ln a}$. Thus

$$\frac{d}{dx} a^x = \frac{d}{dx} e^{x \ln a} = \ln a e^{x \ln a} = a^x \ln a.$$

$$\frac{d}{dx} a^u = a^u \ln a \frac{du}{dx}$$

$y = \log_a x$ is defined as the inverse function of $y = a^x (a > 0, a \neq 1)$. Thus

$$\log_a x = y \Leftrightarrow a^y = x$$

$$\log_a (a^x) = x, \text{ for all } x, \text{ and } a^{(\log_a x)} = x, (x > 0).$$

$\log_{10} x$ is written as $\log x$ and called *common logarithmic function.*

Derivative of $\log_a x$

We have

$$\log_a x = \frac{\ln x}{\ln a}. \tag{3.4}$$

Proof.

$$\begin{aligned} a^{(\log_a x)} &= x \\ \ln a^{(\log_a x)} &= \ln x \\ \log_a x \cdot \ln a &= \ln x \\ \log_a x &= \frac{\ln x}{\ln a}. \end{aligned}$$

So

$$\frac{d}{dx} \log_a x = \frac{1}{x \ln a}$$

and

$$\frac{d}{dx} \log_a u = \frac{1}{u \ln a} \frac{du}{dx}.$$

Logarithmic Differentiation

Find dy/dx if $y = \frac{(x^2+1)^{1/3}(x-3)^{1/2}}{x+5}$.

Derivatives of Power function

We prove the following theorem which was stated earlier.

Theorem 3.8.6. *For any real* r, $\frac{d}{dx}u^r = u^{r-1}\frac{du}{dx}$.

Proof. Since $u^r = e^{r\ln u}$ we have

$$\frac{d}{dx}u^r = ru^r\frac{d\ln u}{dx} = ur^r\frac{1}{u}\frac{du}{dx} = ru^{r-1}\frac{du}{dx}.$$

Example 3.8.7. Differentiate $f(x) = x^x, x > 0$

sol. Write $f(x) = x^x = e^{x\ln x}$. So

$$\begin{aligned} f'(x) &= \frac{d}{dx}(e^{x\ln x}) \\ &= (e^{x\ln x})\frac{d}{dx}(x\ln x) \\ &= e^{x\ln x}(\ln x + x\cdot\frac{1}{x}) \\ &= x^x(\ln x + 1). \end{aligned}$$

Example 3.8.8. Sketch the graph of $x^{1/x}$. To do this, we first investigate the behavior of the function as $x \to 0^+$ and $x \to \infty$. To study the limit, we take the logarithm:

$$\lim_{x\to\infty} \ln x^{1/x} = \lim_{x\to\infty} \frac{\ln x}{x} = 0$$

Hence

$$\lim_{x\to\infty} x^{1/x} = \exp(\lim_{x\to\infty} \frac{\ln x}{x}) = e^0 = 1$$

Meanwhile

$$\lim_{x\to 0^+} \ln x^{1/x} = \lim_{x\to 0^+} \frac{\ln x}{x} = -\infty.$$

Hence

$$\lim_{x\to 0^+} x^{1/x} = \exp(\lim_{x\to 0^+} \frac{\ln x}{x}) = e^{-\infty} = 0.$$

To see the local extrema, take the derivative and find the critical point. $f'(x) = (1 - \ln x)/x^2 = 0$ for $x = e$. By checking the sign of $f'(x)$ near $x = e$, we conclude $x = e$ is a point of local maximum.

The number e as a limit

Theorem 3.8.9. *The number* e *satisfies*

$$e = \lim_{x\to 0}(1 + x)^{1/x}.$$

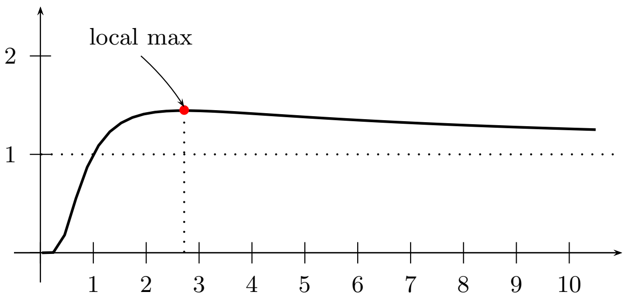


Figure 3.8: Graph of $y = x^{1/x}$

Proof. If $f(x) = \ln x$. Then $f'(1) = \frac{1}{x}|_{x=1} = 1$. By definition of derivative

$$\begin{aligned} 1 &= f'(1) = \lim_{x\to 0^+} \frac{\ln(1+x) - \ln 1}{x} = \lim_{x\to 0^+} \ln[(1+x)^{\frac{1}{x}}] \\ &= \ln[\lim_{x\to 0^+} (1+x)^{\frac{1}{x}}]. \end{aligned}$$

Now exponentiate.

3.9 Inverse trig functions

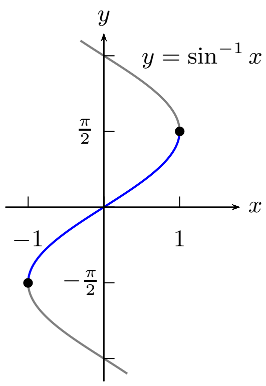


Figure 3.9: $y = \sin^{-1} x$

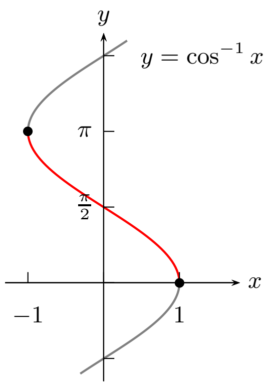


Figure 3.10: $y = \cos^{-1} x$

Inverse sine

Restrict the function $\sin x$ on $[-\pi/2, \pi/2]$. Then $\sin x\colon [-\pi/2, \pi/2] \to [-1, 1]$ is one-to - one function. So the inverse exists. Define

$$\sin^{-1} x\colon [-1, 1] \longrightarrow [-\pi/2, \pi/2].$$

whenever $x = \sin y$ for $x \in [-\pi/2, \pi/2]$. Graph is as in figure 3.9. $\sin^{-1} x$ is sometimes written as $\arcsin x$.

Inverse cosine

Restrict $\cos x$ to $[0, \pi]$, we obtain $\cos^{-1} x$ as

$$\cos^{-1} x\colon [-1, 1] \longrightarrow [0, \pi].$$

If $\cos x = y$ for any $x \in [0, \pi]$ then $\cos^{-1} y = x$ is defined and figure is in 3.10 written as $\cos^{-1} x$ or $\arccos x$.

Example 3.9.1. (1) $\sin^{-1}(1/2) = \pi/6$

(2) $\sin^{-1} 1 = \pi/2$

Example 3.9.2. (1) $\cos^{-1}(1/2) = \pi/3$

(2) $\cos^{-1} 0 = \pi/2$

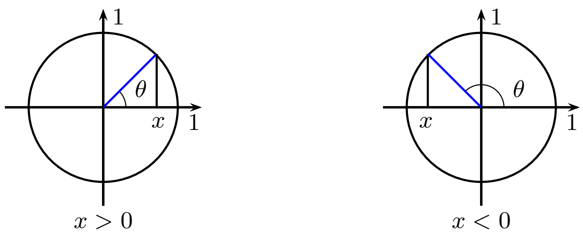


Figure 3.11: $\theta = \cos^{-1} x$

Example 3.9.3.

$$\sin^{-1} x + \cos^{-1} x = \frac{\pi}{2}, \quad \cos^{-1} x + \cos^{-1}(-x) = \pi$$

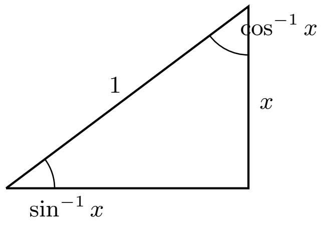


Figure 3.12: $\sin^{-1} x$

Inverse of $\tan x$

The function $\tan x$ is one to one on $(-\pi/2, \pi/2)$, thus it has an inverse $\tan^{-1} x$

$$\tan^{-1} x \colon \mathbb{R} \longrightarrow (-\pi/2, \pi/2)$$

for any $x \in \mathbb{R}$. Thus $\tan x = \alpha$ iff $\tan^{-1} \alpha = x$. See figure 3.13. It is written as $\tan^{-1} x$ or $\arctan x$.

$\tan^{-1} 1 = \pi/4$ $\tan^{-1} 0 = 0$.

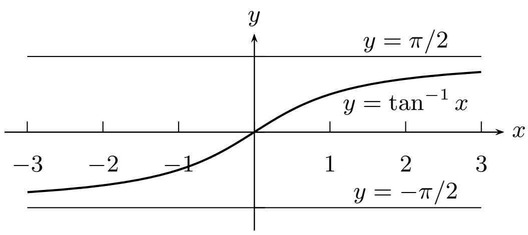


Figure 3.13: $y = \tan^{-1} x$

Example 3.9.4. Find the derivative of $\tan^{-1} x$.

From $y = f(x) = \tan x$, we see by Theorem 3.8.3

$$\begin{aligned} (f^{-1})'(y) &= \frac{1}{f'(x)} \\ &= \frac{1}{1 + \tan^2 x} \\ &= \frac{1}{1 + y^2}. \end{aligned}$$

Thus $(f^{-1})'(x) = \frac{1}{1+x^2}$.

Example 3.9.5. Find derivatives

(1) $y = \sin^{-1} x, \quad (|x| \le 1)$.

(2) $y = \sec^{-1} x, \quad (|x| \ge 1)$.

sol. (3) Let $y = \sec^{-1} x$. Then $x = \sec y$. Taking derivative w.r.t x, we get $1 = \sec y \tan y (dy/dx)$. Thus

$$\frac{dy}{dx} = \frac{1}{\sec y \tan y}.$$

We need to change it to expression in x.

For $x > 1$, $\tan y = \sqrt{x^2 - 1}$. Hence, we have

$$\frac{dy}{dx} = \frac{1}{x\sqrt{x^2 - 1}}, \quad x > 1.$$

For $x < -1$, use $(x \to -x)$ to get

$$\frac{dy}{dx} = \frac{1}{-x\sqrt{x^2 - 1}}, \quad x < -1.$$

Hence

$$\frac{d}{dx} \sec^{-1} x = \frac{1}{|x|\sqrt{x^2 - 1}}, \quad |x| > 1.$$

Other inverse trig functions

Inverses of $\csc x$, $\sec x$, $\cot x$

$\csc^{-1} x : \mathbb{R} - (-1, 1) \to [-\pi/2, \pi/2] - \{0\}$

$\sec^{-1} x : \mathbb{R} - (-1, 1) \to [0, \pi] - \{\pi/2\}$

$\cot^{-1} x : \mathbb{R} \to (0, \pi)$. (Note that the range is different from that of $\tan^{-1} x$)

Proposition 3.9.6. *The derivatives of inverse trig. functions :*

(1) $\dfrac{d}{dx} \sin^{-1} x = \dfrac{1}{\sqrt{1 - x^2}}$

(2) $\dfrac{d}{dx} \cos^{-1} x = -\dfrac{1}{\sqrt{1 - x^2}}$

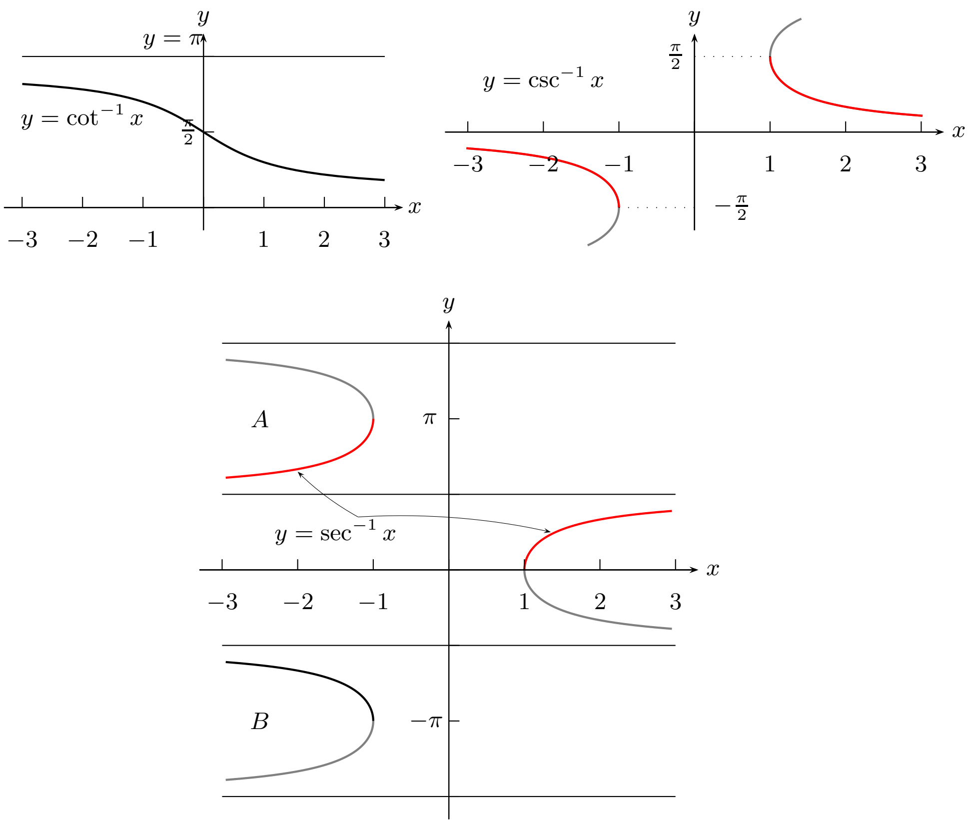


Figure 3.14:

(3) $$\frac{d}{dx}\tan^{-1} x = \frac{1}{1+x^2}$$

(4) $$\frac{d}{dx}\csc^{-1} x = -\frac{1}{|x|\sqrt{x^2-1}}, \ |x| > 1$$

(5) $$\frac{d}{dx}\sec^{-1} x = \frac{1}{|x|\sqrt{x^2-1}}, \ |x| > 1$$

(6) $$\frac{d}{dx}\cot^{-1} x = -\frac{1}{1+x^2}.$$

Proposition 3.9.7. *The following relations hold.*

$$\begin{array}{|lcl|}\hline \cos^{-1} x &=& (\pi/2) - \sin^{-1} x \\ \cot^{-1} x &=& (\pi/2) - \tan^{-1} x \\ \csc^{-1} x &=& (\pi/2) - \sec^{-1} x \\ \cot^{-1} x &=& \tan^{-1}(1/x) \\ \sec^{-1} x &=& \cos^{-1}(1/x) \\ \csc^{-1} x &=& \sin^{-1}(1/x) \\ \hline \end{array}$$

Example 3.9.8. (1) Find $\sin(\cos^{-1}(3/5))$

(2) Simplify $\tan(\sin^{-1} a)$

sol. (1) Let $\theta = \cos^{-1}(3/5)$. Then $\cos\theta = 3/5$ and $0 \le \theta \le \pi$. Hence

$$\sin\theta = \sqrt{1 - \frac{9}{25}} = \frac{4}{5}.$$

(2) Let $\theta = \sin^{-1} a$. Then $\sin\theta = a$ and $-\pi/2 \le \theta \le \pi/2$.

$$\cos\theta = \sqrt{1 - a^2}.$$

Hence

$$\tan\theta = \sin\theta/\cos\theta = a/\sqrt{1 - a^2}. \tag{3.5}$$

Integral of $\tan x, \cot x, \sec x$ and $\csc x$

$$\begin{aligned} \int \tan x\, dx &= \int \frac{\sin x}{\cos x}\, dx \\ &= -\int \frac{du}{u} \\ &= -\ln|u| + C \\ &= -\ln|\cos x| + C \\ &= \ln\frac{1}{|\cos x|} + C \\ &= \ln|\sec x| + C. \end{aligned}$$

For $\sec x$ we need special trick:

$$\begin{aligned}
\int \sec x\,dx &= \int \sec x \frac{(\sec x + \tan x)}{(\sec x + \tan x)}\,dx \\
&= \int \frac{(\sec^2 x + \sec x \tan x)}{\sec x + \tan x}\,dx \\
&= \int \frac{du}{u} \\
&= \ln|u| + C \\
&= \ln|\sec x + \tan x| + C.
\end{aligned}$$

For $\csc x$ we do similarly. Thus we have

$$\boxed{\begin{aligned}
\int \sec x\,dx &= \ln|\sec x + \tan x| + C \\
\int \csc x\,dx &= -\ln|\csc x + \cot x| + C.
\end{aligned}}$$

3.10 Related Rates

Skip this section.

3.11 Linearization and differential

Definition 3.11.1. Given a differentiable function f, the linear function

$$L(x) = f(a) + f'(a)(x - a)$$

is called the **linearization** of f at a.

Example 3.11.2. (1) Find the linearization of $\cos x$ at $\pi/2$.

(2) Find an approx value of $\sqrt{1.003}$ using the linearization of $\sqrt{1+x}$ at $x = 0$.

(3) Find the linearization of $\frac{1}{\sqrt[3]{x^4+1}}$ at $x = 0$

(4) Find an approx value of $\sqrt{4.8}$

(5) Find the linearization of $\cos x$ at $\pi/2$. Ans $-x + \pi/2$.

(6) Find the linearization of $(1 + x)^k$. Ans $1 + kx$.

Differential

Definition 3.11.3. Let $y = f(x)$ be differentiable. We can treat dx(differential) like an independent variable. In this point of view, the quantity dy defined by

$$dy := f'(x)dx$$

is called the **differential** of f.

The geometric meaning of differential is given in Figure 3.15. We observe

$$\Delta y = f(a + dx) - f(a), \quad f(a + dx) = f(a) + \Delta y \approx f(a) + dy.$$

We see that dy is precisely the change of the tangent line as x changes by an amount of $dx = \Delta x$. In other words, dy is an approximation of exact change Δy.

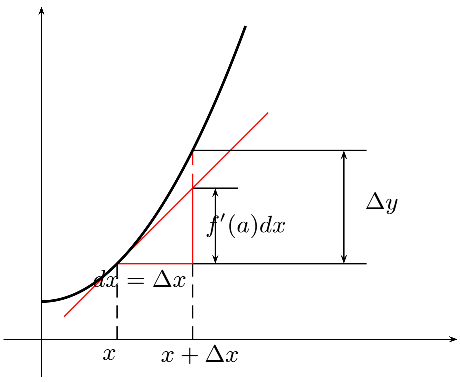


Figure 3.15: Differential $dy = f'(a)dx$ and Δy

Example 3.11.4. Find differential of

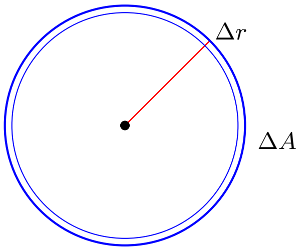
(1) $y = x^3 - \sin x$

(2) $y = \sin u(x)$

(3) $\tan(3x)$

(4) $d(\frac{x}{1+x})$.

Estimating with differentials



Example 3.11.5.

Radius of a circle is enlarged from 10 to 10.1. Use dA to estimate the increase in area. Compare with exact increase. $A = \pi r^2$,

$$dA = 2\pi r dr = 2\pi(10)(0.1) = 2\pi m^2.$$

Actual increase is $A(10.1) - A(10) = 2\pi((10.1)^2 - 100) = 2.01\pi$.

Error in differential approximation

We estimate the change in y in more detail.

Theorem 3.11.6. *We have*

$$\Delta f = f'(a)\Delta x + \epsilon \Delta x,$$

where $\epsilon \to 0$ *as* $\Delta x \to 0$.

Proof.

$$\begin{aligned} \text{approximation error} &= \Delta f - df \\ &= \Delta f - f'(a)\Delta x \\ &= f(a+\Delta x) - f(a) - f'(a)\Delta x \\ &= \left(\frac{f(a+\Delta x) - f(a)}{\Delta x} - f'(a) \right) \Delta x \\ &= \epsilon \Delta x. \end{aligned}$$

Since f is differentiable, we know $\epsilon := \left(\frac{f(a+\Delta x)-f(a)}{\Delta x} - f'(a)\right)$ approaches 0 as Δx approaches 0. Thus

$$\overset{\substack{\text{true}\\\text{change}}}{\Delta f} = \overset{\substack{\text{estimated}\\\text{change}}}{f'(a)\Delta x} + \overset{\text{error}}{\epsilon\Delta x}$$

Proof of Chain rule

Assume $y = f(u)$ is a diff'ble function of u and $u = g(x)$ is a diff'ble function of x. Then the composite function $y = f(g(x))$ is diff'ble and by theorem there exist ϵ_1, ϵ_2 which approaches 0 as $\Delta u, \Delta x$ approaches 0 in such a way that

$$\begin{aligned}\Delta y &= f'(u_0)\Delta u + \epsilon_2 \Delta u\\ \Delta u &= g'(x_0)\Delta x + \epsilon_1 \Delta x.\end{aligned}$$

Hence

$$\Delta y = (f'(u_0) + \epsilon_2)(g'(x_0) + \epsilon_1)\Delta x$$

$$\frac{\Delta y}{\Delta x} = (f'(u_0) + \epsilon_2)(g'(x_0) + \epsilon_1).$$

Let $\Delta x \to 0$. Then we obtain the Chain rule.

Example 3.11.7. Converting mass to energy: The Newton's law

$$F = m\frac{dv}{dt} = ma$$

is not exactly true when an object is moving at very high speed, because the mass increases with velocity. In Einstein's correction, the mass is

$$m = \frac{m_0}{\sqrt{1 - v^2/c^2}} \approx m_0\left(1 + \frac{v^2}{2c^2}\right).$$

So the new mass is

$$m \approx m_0 + \frac{mv^2}{2c^2}.$$

By multiplying c^2

$$(m - m_0)c^2 \approx \frac{1}{2}mv^2 - \frac{1}{2}m0^2 = \Delta(KE).$$

Thus the change in the mass corresponds to the change in the Kinetic Energy.