Chapter 10

Infinite Sequence and Series

10.1 Sequences

Example 10.1.1. (1)

$$1, 3, 5, 7, \ldots$$

(2) n-th term is given by $(-1)^{n+1}1/n$:

$$1, -\frac{1}{2}, \frac{1}{3}, -\frac{1}{4}, \ldots, (-1)^{n+1}\frac{1}{n}, \ldots$$

(3) Certain rules

$$1, \frac{1}{2}, \frac{1}{2}, -\frac{1}{3}, -\frac{1}{3}, -\frac{1}{3}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4}, \ldots$$

Sequence as graph

Example 10.1.2. (1) $a_n = (n-1)/n$.

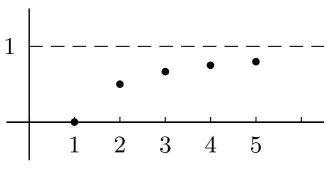


Figure 10.1: $a_n = (n-1)/n$

(2) $a_n = (-1)^n 1/n$.

(3) $a_n = \sin(n\pi/6)$.

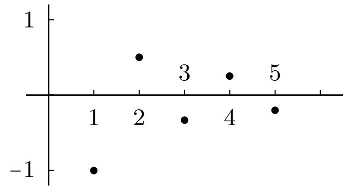


Figure 10.2: $a_n = (-1)^n 1/n$

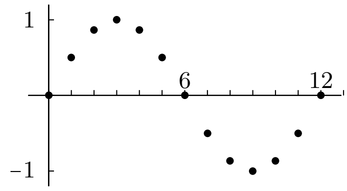


Figure 10.3: $a_n = \sin(n\pi/6)$

Recursive relation

Some sequence are defined through recursive relation such as

$$\begin{aligned} a_1 &= 1, \\ a_{n+1} &= 2a_n + 1, \quad n = 1, 2, 3, \ldots \end{aligned}$$

or

$$\begin{aligned} a_1 &= 1,\ a_2 = 2, \\ a_{n+2} &= a_{n+1} + a_n, \quad n = 1, 2, 3, \ldots \end{aligned}$$

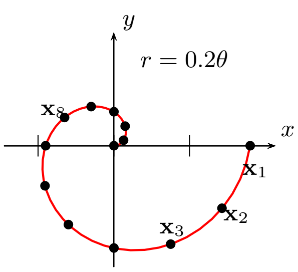


Figure 10.4: 2-D sequence

10.1.1 Convergence of a sequence

Definition 10.1.3. We say $\{a_n\}$ **converges** to L, if for any $\varepsilon > 0$, there exists some N s.t. for all $n > N$ it holds that

$$|a_n - L| < \varepsilon.$$

In this case we write

$$\lim_{n\to\infty} a_n = L \quad \text{or} \quad \{a_n\} \to L.$$

L is the called the **limit** of a_n. If $\{a_n\}$ does not converge, we say $\{a_n\}$ **diverges**.

Properties of limit

Example 10.1.4. (1) $\lim_{n\to\infty} \ln n/n = 0$.

(2) $\lim_{n\to\infty} n(e^{1/n} - 1) = 1$.

(3) Find $\lim_{n\to\infty} \left(\frac{n+1}{n-1}\right)^n$.

sol. (1) Let $f(x) = \ln x/x$. Then

$$\lim_{n\to\infty} f(n) = \lim_{x\to\infty} f(x) = \lim_{x\to\infty} \frac{(\ln x)'}{x'} = \lim_{x\to\infty} \frac{1}{x} = 0.$$

$$\lim_{n\to\infty} \ln n/n = 0.$$

(2) Set $x = 1/n$. Then it corresponds to the limit of $f(x) = (e^x - 1)/x$ as $x \to 0$. By L'Hopital's rule

$$\lim_{x\to 0} f(x) = \lim_{x\to 0} e^x = 1.$$

$$\lim_{n\to\infty} n(e^{1/n} - 1) = 1.$$

□

Theorem 10.1.5 (Sanwich theorem)**.** *Suppose a_n, b_n, c_n satisfy $a_n \le b_n \le c_n$ and $\lim_{n\to\infty} a_n = \lim_{n\to\infty} c_n = L$. Then $\lim_{n\to\infty} b_n = L$.*

Limits arising often

Proposition 10.1.6.

(1) $\lim_{n\to\infty} \frac{\ln n}{n} = 0.$

(2) $\lim_{n\to\infty} \sqrt[n]{n} = 1.$

(3) $\lim_{n\to\infty} x^{1/n} = 1, \; x > 0.$

(4) $\lim_{n\to\infty} x^n = 0, \; |x| < 1.$

(5) $\lim_{n\to\infty} \left(1 + \frac{x}{n}\right)^n = e^x, \; x \in \mathbb{R}.$

(6) $\lim_{n\to\infty} \frac{x^n}{n!} = 0, \; x \in \mathbb{R}.$

Proof. (1) See Example 10.1.4.

(2) Let $a_n = n^{1/n}$. Then $\ln a_n = \ln n^{1/n} = \frac{\ln n}{n}$. Since this approaches 0 and e^x is continuous at 0, we have $a_n = e^{\ln a_n} \to e^0 = 1$ by theorem.

(3) Set $a_n = x^{1/n}$. Since the limit of $\ln a_n = \ln x^{1/n} = \frac{\ln x}{n}$ is 0, we see $x^{1/n} = a_n = e^{\ln a_n}$ converges to $e^0 = 1$.

(4) Use the definition. Given $\varepsilon > 0$, we must find n, s.t. for $|x| < \varepsilon^{1/n}$ $|x^n - 0| < \varepsilon$ holds. Since $\lim_{n\to\infty} \varepsilon^{1/n} = 1$ there is an N s.t $|x| < \varepsilon^{1/N}$ holds. Now if $n > N$ we have $|x|^n < |x^N| < \varepsilon$.

(5) Let $a_n = (1 + x/n)^n$. Then $\lim_{n\to\infty} \ln a_n = \lim_{n\to\infty} \ln(1 + x/n)^n = n \ln(1 + x/n)$ and by L'Hopital's rule we see

$$\lim_{n\to\infty} \frac{\ln(1 + x/n)}{1/n} = \lim_{n\to\infty} \frac{x}{1 + x/n} = x.$$

Hence $a_n = (1 + x/n)^n = e^{\ln a_n}$ converges to e^x.

(6) First we will show that

$$-\frac{|x|^n}{n!} \le \frac{x^n}{n!} \le \frac{|x|^n}{n!}$$

and $|x|^n/n! \to 0$. Then use Sandwich theorem. If M is any number greater than $|x|$, then $|x|/M < 1$ and hence $(|x|/M)^n \to 0$. If $n > M$

$$\frac{|x|^n}{n!} = \frac{|x|^n}{1 \cdot 2 \cdots M(M+1) \cdots n} \le \frac{|x|^n}{M! M^{n-M}} = \frac{M^M}{M!} \left(\frac{|x|}{M}\right)^n$$

holds. But $M^M/M!$ is fixed number. As $n \to \infty$ $(|x|/M)^n$ approaches 0. So $|x|^n/n!$ approaches 0. Finally by Sandwich theorem we get the result.

□

Example 10.1.7. (1) $\lim_{n\to\infty}\left(\frac{1}{1000}\right)^{1/n} = 1.$

(2) $\lim_{n\to\infty}\left(10^{1000}n^2\right)^{1/n} = \lim_{n\to\infty}(10^{1/n})^{1000}\lim_{n\to\infty}n^{2/n} = 1\cdot\lim_{n\to\infty}\left(n^{1/n}\right)^2 = 1.$

(3) $\lim_{n\to\infty}\left(1-\frac{2}{n}\right)^n = e^{-2}.$

(4) $\lim_{h\to 0^+}(1+h)^{1/h} = \lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n = e.$

(5) $\lim_{n\to\infty}\frac{10^n}{n!} = 0.$

(6) The set of all x satisfying $\lim_{n\to\infty}\frac{|x|^n}{5^n} = 0$ is $|x| < 5$.

Example 10.1.8. $\lim_{n\to\infty}\sqrt[n]{5n+1} = 1.$

sol. Since $\ln(5n+1)^{1/n} = \ln(5n+1)/n \to 0$ above limit is $e^0 = 1$.

□

Example 10.1.9. Show that $\lim_{n\to\infty}\ln n/n^{\varepsilon} = 0$ for any $\varepsilon > 0$.

sol. By L'Hopital rule 3.6.5

$$\lim_{n\to\infty}\frac{\ln n}{n^{\varepsilon}} = \lim_{n\to\infty}\frac{1/n}{\varepsilon n^{\varepsilon-1}} = \lim_{n\to\infty}\frac{1}{\varepsilon n^{\varepsilon}} = 0.$$

□

Monotone Sequence

Definition 10.1.10. If a_n satisfies

$$a_1 \le a_2 \le \cdots \le a_n \le \cdots$$

then a_n is called an **increasing sequence (nondecreasing sequence)**.

Definition 10.1.11. If there is a number M such that $a_n \leq M$ for all n, then this sequence is said to be **bounded from above**. Any such M is called an **upper bound**. If the smallest number exists among all upper bound, then it is called the **least upper bound**. Similarly, we say a sequence **bounded from below** if there is a number N such that $a_n \geq N$ for all n, Any such an N is called a **lower bound**. If the largest number exists among all lower bound, then it is called the **greatest lower bound**. If a sequence has both lower bound and upper bound, then we say it is **bounded**.

Example 10.1.12. For the sequence $a_n = 1 - 1/2^n$, $M = 1$ is an upper bound and any number bigger than 1 is an upper bound. The smallest such number(if exists) is **least upper bound**.

Theorem 10.1.13. *If an increasing sequence has an upper bound, it converges(to the least upper bound).*

Suppose L is a least upper bound, we observe two things:

(1) $a_n \leq L$ for all n, and

(2) for any $\varepsilon > 0$ there is a term a_N greater than $L - \varepsilon$.

The first assertion is trivial. For the second, suppose there does not exist such a_N, it holds that $a_n \leq L - \varepsilon$ for all n, which is a contradiction. Thus for $n \geq N$

$$L - \varepsilon < a_n \leq L.$$

Hence $|L - a_n| < \varepsilon$ for $n \geq N$ and we have $a_n \to L$.

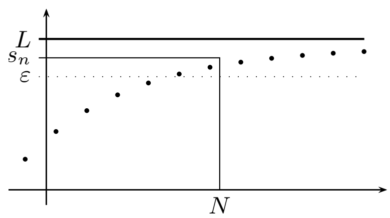


Figure 10.5: Nondecreasing(increasing) sequence and least upper bound L

For decreasing sequence, we can define similar concept.

Definition 10.1.14. If a_n satisfies

$$a_1 \geq a_2 \geq \cdots \geq a_n \cdots$$

a_n is called a **decreasing sequence**. If $s_n \geq N$, then N is called a **lower bound**(lower bound). The largest such number is called the **greatest lower bound**.

10.2 Infinite Series

A sequence given as the sum of an infinite sequence of numbers is called **infinite series**.

Example 10.2.1. If we denote the sum of first n- term of $a_n = 1/2^n$ by s_n then

$$\begin{aligned}
s_1 &= a_1 = \frac{1}{2} \\
s_2 &= a_1 + a_2 = \frac{1}{2} + \frac{1}{4} = \frac{3}{4} \\
s_3 &= a_1 + a_2 + a_3 = \frac{1}{2} + \frac{1}{4} + \frac{1}{8} = \frac{7}{8} \\
&\vdots
\end{aligned}$$

In general, the series $\{s_n\}$ is written as

$$s_n = a_1 + a_2 + a_3 + \cdots + a_n = \sum_{k=1}^{n} a_k.$$

We write the infinite series as $\sum_{n=1}^{\infty} a_n$ or $\sum a_n$ (whether it converges or not!)

Definition 10.2.2. a_n is called n**-th term** and $s_n = \sum_{k=1}^{n} a_k$ is called n-th **partial sum**. If the limit of $\{s_n\}$ is L, then we say $\sum a_n$ converges to L and write $\sum_{n=1}^{\infty} a_n = L$ or $a_1 + a_2 + a_3 + \cdots = L$. If a series does not converges, we say it **diverges**.

Example 10.2.3 (Telescoping Series)**.** Find the sum $\displaystyle\sum_{n=1}^{\infty} 1/n(n+1)$.

sol. We use the identity $1/n(n+1) = 1/n - 1/(n+1)$ to see

$$s_n = \left(\frac{1}{1} - \frac{1}{2}\right) + \left(\frac{1}{2} - \frac{1}{3}\right) + \cdots + \left(\frac{1}{n} - \frac{1}{n+1}\right).$$

Since $s_n = 1 - 1/(n+1)$, we see $s_n \to 1$.

□

Divergent Series

Example 10.2.4. $\sum_{n=1}^{\infty}(n+1)/n$ diverges since n-th term is greater than 1.

Example 10.2.5. $\sum_{n=1}^{\infty}\sin(\pi n/2)$ diverges.

sol.

$$1, 0, -1, 0, 1, \ldots$$

Since $s_4 = s_8 = \cdots = s_{4n} = 0$ and $s_2 = s_6 = \cdots = s_{4n+2} = 1$ the sequence s_n oscillates between 0 and 1.

□

Theorem 10.2.6 (n-th term test)**.** *If $\sum a_n$ converges then $a_n \to 0$.*

Proof. Suppose $\sum_{n=1}^{\infty} a_n$ converges then s_n and s_{n-1} must have the same limit. Since $a_n = s_n - s_{n-1}$ we see $\lim a_n = \lim s_n - \lim s_{n-1} = 0$. □

Theorem 10.2.7 (nth term test)**.** *If $\lim a_n \not\to 0$ or $\lim a_n$ does not exists, then $\sum a_n$ diverges.*

Example 10.2.8. $\sum(n-1)/n$ diverges since $a_n = (n-1)/n \to 1$.

Example 10.2.9. $\sum(-1)^n \ln(\ln n)$ diverges since $\ln(\ln n) \to \infty$.

Theorem 10.2.10. *Suppose $\sum a_n, \sum b_n$ converges. Then*

(1) $\sum(a_n + b_n) = \sum a_n + \sum b_n$,

(2) $\sum(a_n - b_n) = \sum a_n - \sum b_n$,

(3) $\sum k a_n = k \sum a_n$.

Example 10.2.11.

(1) $$\sum_{n=1}^{\infty} \frac{2^n - 1}{3^n} = \sum_{n=1}^{\infty} \frac{2^n}{3^n} - \sum_{n=1}^{\infty} \frac{1}{3^n} = \frac{2}{3}\frac{1}{1-2/3} - \frac{1}{3}\frac{1}{1-1/3} = \frac{3}{2}.$$

(2) $$\sum_{n=1}^{\infty} \frac{3^n - 2^n}{6^n} = \sum_{n=1}^{\infty} \frac{3^n}{6^n} - \sum_{n=1}^{\infty} \frac{2^n}{6^n} = \sum_{n=1}^{\infty} \frac{1}{2^n} - \sum_{n=1}^{\infty} \frac{1}{3^n} = \frac{1}{2}.$$

What's wrong with the following argument?

$$1 = \sum_{n=1}^{\infty}\left(\frac{1}{n} - \frac{1}{n+1}\right) = \sum \frac{1}{n} - \sum \frac{1}{n+1} = \infty - \infty.$$

Example 10.2.12.

(1) $$\sum_{n=1}^{\infty}\frac{2^n-1}{3^n}=\sum_{n=1}^{\infty}\frac{2^n}{3^n}-\sum_{n=1}^{\infty}\frac{1}{3^n}=\frac{2}{3}\frac{1}{1-2/3}-\frac{1}{3}\frac{1}{1-1/3}=\frac{3}{2}.$$

(2) $$\sum_{n=1}^{\infty}\frac{3^n-2^n}{6^n}=\sum_{n=1}^{\infty}\frac{3^n}{6^n}-\sum_{n=1}^{\infty}\frac{2^n}{6^n}=\sum_{n=1}^{\infty}\frac{1}{2^n}-\sum_{n=1}^{\infty}\frac{1}{3^n}=\frac{1}{2}.$$

Example 10.2.13 (Snowflake - See exer. 94)**.** (1) Find the length ℓ_n of curve C_n.

(1) It suffices to consider one side. After the first step the length is

$$\ell_2=\ell_1-\frac{1}{3}\ell_1+\frac{2}{3}\ell_1=\frac{4}{3}\ell_1.$$

After the second step the length is $(\frac{4}{3})^2\ell_1$ and so on, so $\ell_n=(\frac{4}{3})^n\ell_1\to\infty$.

(2) Find the area of the region A_n enclosed by the curve C_n, and show that $\lim_n A_n=(8/5)A_1$. After a few steps the area is

$$\begin{aligned}
A_2 &= A_1+3\frac{1}{3^2}A_1=\frac{4}{3}A_1,\\
A_3 &= A_1+3\frac{1}{3^2}A_1+12\frac{1}{3^4}A_1,\\
A_4 &= A_1+3\frac{1}{3^2}A_1+12\frac{1}{3^4}A_1+24\frac{1}{3^6}A_1,\\
A_n &= A_1+3\frac{1}{3^2}A_1+\cdots+3\times 2^{n-1}\frac{1}{3^{2n-1}}A_1.
\end{aligned}$$

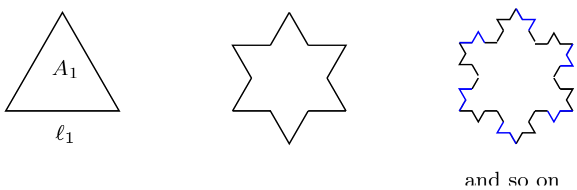


Figure 10.6: Growing Snowflake

10.3 Integral Test

Example 10.3.1. Determine whether the following series converges or not.

$$\sum\frac{1}{n^2}=1+\frac{1}{4}+\frac{1}{9}+\cdots+\frac{1}{n^2}+\cdots$$

sol. Set $f(x) = 1/x^2$. Then

$$s_n = 1 + \frac{1}{4} + \frac{1}{9} + \cdots + \frac{1}{n^2} = f(1) + f(2) + f(3) + \cdots + f(n)$$

and

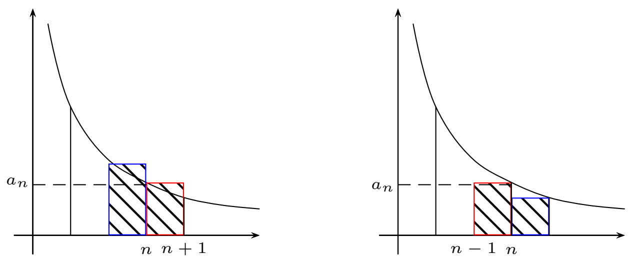
$$\begin{aligned} f(2) &= \frac{1}{2^2} < \int_1^2 \frac{1}{x^2}\,dx \\ f(3) &= \frac{1}{3^2} < \int_2^3 \frac{1}{x^2}\,dx \\ &\vdots \\ f(n) &= \frac{1}{n^2} < \int_{n-1}^n \frac{1}{x^2}\,dx. \end{aligned}$$

$$s_n = f(1) + f(2) + f(3) + \cdots + f(n) < 1 + \int_1^n \frac{1}{x^2}\,dx = 2 - \frac{1}{n}.$$

Thus s_n is bounded, increasing, and hence converges.

□

Theorem 10.3.2 (Integral Test)**.** *Suppose $f(x)$ is nonnegative, non-increasing for $x \geq 1$ and $a_n = f(n)$. Then the series $\sum_{n=1}^{\infty} a_n$ converges iff $\int_1^{\infty} f(x)\,dx$ converges.*



(a) $\int_n^{n+1} f(x)\,dx \leq a_n$ (b) $a_n \leq \int_{n-1}^{n} f(x)\,dx$

Figure 10.7: Integral Test

Proof. Since f is decreasing and $f(n) = a_n$, we see from figure 10.7(a) that

$\int_n^{n+1} f(x)\,dx \le a_n$. So

$$\int_1^{n+1} f(x)\,dx \le a_1 + a_2 + \cdots + a_n. \tag{10.1}$$

On the other hand, we see from figure 10.7 (b), that $a_n \le \int_{n-1}^n f(x)\,dx$, $(n = 2, 3, 4, \dots)$. Hence we have

$$a_2 + a_3 + \cdots + a_n \le \int_1^n f(x)\,dx.$$

Finally together with (10.1) we see

$$\int_1^{n+1} f(x)\,dx \le a_1 + a_2 + \cdots + a_n \le a_1 + \int_1^n f(x)\,dx.$$

Letting $n \to \infty$, we obtain the result. □

Example 10.3.3 (p-series)**.** Let p be a fixed number. Then

$$\sum_1^\infty \frac{1}{n^p} = \frac{1}{1^p} + \frac{1}{2^p} + \cdots + \frac{1}{n^p} + \cdots$$

converges when $p > 1$ and diverges when $p \le 1$. For $p = 1$, we see

$$\int_1^\infty \frac{1}{x}\,dx = \lim_{b\to\infty} [\ln b]_1^b = \infty.$$

So the harmonic series

$$1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n} + \cdots$$

diverges.

Example 10.3.4. Test the convergence of

$$\sum_1^\infty \frac{1}{1+n^2}.$$

We see

$$\int_1^\infty \frac{1}{1+x^2}\,dx = \lim_{b\to\infty} [\tan^{-1} x]_1^b = \lim_{b\to\infty} [\tan^{-1} b - \tan^{-1} 1] = \frac{\pi}{4}.$$

10.3.1 Series with nonnegative terms

$$\sum \frac{1}{n^3}, \quad \sum \frac{1}{3^n+1}.$$

Example 10.3.5. Investigate the convergence of $\sum_{n=1}^{\infty} \frac{1}{n^2}$.

sol. Use the inequality $1/n^2 < 1/n(n-1)$ and partial fraction,

$$\begin{aligned} s_n &= \frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \cdots + \frac{1}{n^2} \\ &< \frac{1}{1\cdot 1} + \frac{1}{1\cdot 2} + \frac{1}{2\cdot 3} + \cdots + \frac{1}{n(n-1)} \\ &= 1 + \left(1 - \frac{1}{2}\right) + \left(\frac{1}{2} - \frac{1}{3}\right) + \cdots + \left(\frac{1}{n-1} - \frac{1}{n}\right) \\ &= 2 - \frac{1}{n} < 2. \end{aligned}$$

Hence s_n is bounded above and monotonic increasing hence converges.

□

Example 10.3.6 (Harmonic series).

$$\sum \frac{1}{n} = 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n} + \cdots$$

diverges since

$$1 + \frac{1}{2} + \underbrace{\frac{1}{3} + \frac{1}{4}}_{>2/4} + \underbrace{\frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8}}_{>4/8} + \underbrace{\frac{1}{9} + \frac{1}{10} + \cdots + \frac{1}{16}}_{>8/16} + \cdots$$

10.4 Comparison Test

Theorem 10.4.1 (The Comparison Test). *Let $a_n \ge 0$.*

(a) The series $\sum a_n$ converges if $a_n \le c_n$ for all $n > N$ and $\sum c_n$ converges

(b) The series $\sum a_n$ diverges if $a_n \ge d_n$ for all $n > N$ and $\sum d_n$ diverge.

Proof. In (a), the partial sum is bounded by

$$M = a_1 + a_2 + \cdots a_n + \sum_{n=N+1}^{\infty} c_n.$$

Hence if $\sum c_n$ converges, then $\sum a_n$ converges by Theorem 10.1.13. In (b), the partial sum is greater than

$$M^* = a_1 + a_2 + \cdots a_n + \sum_{n=N+1}^{\infty} d_n.$$

But the series $\sum_{n=N+1}^{\infty} d_n$ diverges. Hence so does $\sum a_n$.

□

Example 10.4.2. Look at the tail part of

$$3 + 600 + 5000 + \frac{1}{3!} + \frac{1}{4!} + \frac{1}{5!} + \cdots + \frac{1}{n!} + \cdots .$$

Then $1/n! < 1/2^n$ for $n = 4, 5, 6, \ldots$ and $\sum 1/2^n$ converges. Hence the series converges.

Limit Comparison Test

Example 10.4.3. Investigate the convergence of

$$\sum_{1}^{\infty} \frac{n}{2n^3 - n + 3}.$$

$\boxed{\textbf{sol.}}$ Let

$$a_n = \frac{n}{2n^3 - n + 3} = \frac{1}{2n^2 - 1 + 3/n}$$

and use the fact that a_n behaves similar to $1/2n^2$. If $c_n = 1/2n^2$ then $\lim_{n\to\infty} a_n/c_n = 1$. Hence for any ε there is N such that if $n > N$ for some N then the following holds:

$$1 - \varepsilon \leq \frac{a_n}{c_n} \leq 1 + \varepsilon.$$

In other words,

$$(1 - \varepsilon)c_n \leq a_n \leq (1 + \varepsilon)c_n.$$

Since $\sum_{n\geq N} c_n$ converges $\sum_{n\geq N} a_n$ converges by comparison.

□

Theorem 10.4.4 (Limit Comparison Test)**.** *(1) Suppose* $a_n > 0$ *and there* *is a series* $\sum c_n$ $(c_n > 0)$ *which converges and if*

$$\lim_{n\to\infty} \frac{a_n}{c_n} = L > 0$$

then $\sum a_n$ *converges.*

(2) *Suppose* $a_n > 0$ *and there is a series* $\sum d_n$ $(d_n > 0)$ *which diverges and if*

$$\lim_{n\to\infty} \frac{a_n}{d_n} = L > 0$$

then $\sum a_n$ *diverges.*

Proof. We prove (1) only. Since $L/2 > 0$ there is an N such that for all $n > N$ we have

$$\left|\frac{a_n}{c_n} - L\right| < \frac{L}{2}.$$

Hence

$$-\frac{L}{2} < \frac{a_n}{c_n} - L < \frac{L}{2} \text{ or } \frac{L}{2} < \frac{a_n}{c_n} < \frac{3L}{2}.$$

Hence

$$\frac{L}{2}c_n < a_n < \frac{3L}{2}c_n.$$

$\square$

Example 10.4.5. (1) $\sum_1^\infty \frac{n+1}{100n^3+n+1}$ converges since $\sum_1^\infty \frac{1}{n^2}$ converges

(2) $\sum_{20}^\infty \frac{1}{3^n-1000n}$ converges since $\sum_1^\infty \frac{1}{3^n}$ converge

(3) $\sum_1^\infty \frac{2n+1}{n^2+4n+1}$

(4) Does $\sum_2^\infty \frac{\ln n}{n^{3/2}}$ converge ?

(5) Compare $\sum_1^\infty \frac{(\ln n)^{1/2}}{(n \ln n+1)}$ with $\sum_2^\infty \frac{1}{n(\ln n)^{1/2}}$. Use integral test.

$$\int_2^\infty \frac{dx}{x(\ln x)^{1/2}} = \int_{\ln 2}^\infty \frac{du}{u^{1/2}} = \infty.$$

10.5 Ratio test and Root Tests

There are many series whose behvior is similar to geometric series. In such cases, we may compare with certain geometric series. The idea is to assume $a_n \sim C\rho^n$ for some $\rho > 0$ and try to estimate r. There are two ways:

(1) compute the ratio: $\lim_{n\to\infty} \frac{a_{n+1}}{a_n}$, or

(2) compute the n-th root: $\lim_{n\to\infty} \sqrt[n]{a_n}$.

Ratio Test

Example 10.5.1. It is not easy to find general term of $a_1 = 1,\ a_{n+1} = \frac{na_n}{3n+2}$. But its ratio is easy to compute.

Theorem 10.5.2 (Ratio Test)**.** *Suppose $a_n > 0$ and if the limit exists.*

$$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \rho.$$

Then exactly one of the following holds.

(1) The sum $\sum a_n$ converges if $\rho < 1$

(2) The sum $\sum a_n$ diverges if $\rho > 1$

(3) The test is inconclusive if $\rho = 1$.

Proof. (1) Let $\rho < 1$. Then choose any r between ρ and 1 and set $\varepsilon = r - \rho$. Then since

$$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \rho,$$

there exists a natural number N such that for all $n > N$,

$$\left| \frac{a_{n+1}}{a_n} - \rho \right| < \varepsilon$$

holds. Since $a_{n+1}/a_n < \rho + \varepsilon = r$, we see

$$\begin{aligned} a_{N+1} &< ra_N \\ a_{N+2} &< ra_{N+1} < r^2 a_N \\ &\vdots \\ a_{N+m} &< ra_{N+m-1} < r^m a_N. \end{aligned}$$

We compare a_n with a series general term is $r^m a_N$. Since $\sum_{m=1}^{\infty} r^m a_N$ converges, $\sum_{n=N+1}^{\infty} a_n$ converges. (2) Suppose $\rho > 1$. Then exist an M such that for $n > M$, it holds that

$$\frac{a_{n+1}}{a_n} > 1.$$

Hence the series diverges:

$$a_M < a_{M+1} < a_{M+2} < \cdots .$$

(3) The case $\rho = 1$. Both the series $\sum 1/n^2$ and $\sum 1/n$. But the former converges and the latter diverges. □

Example 10.5.3.

(1) $\sum \frac{n!n!}{(2n)!}$

(2) $\sum \frac{(2^n+5)}{3^n}$

(3) $\sum \frac{2^n}{n!}$

sol.

(1) Ratio Test

$$\begin{aligned}\frac{a_{n+1}}{a_n} &= \frac{(n+1)!(n+1)!(2n)!}{n!n!(2n+2)(2n+1)(2n)!}\\ &= \frac{(n+1)(n+1)}{(2n+2)(2n+1)} = \frac{n+1}{4n+2} \to \frac{1}{4}.\end{aligned}$$

(2) $\dfrac{a_{n+1}}{a_n} = \dfrac{(2^{n+1}+5)3^n}{3^{n+1}(2^n+5)} = \dfrac{2^{n+1}+5}{3(2^n+5)} \to \dfrac{2}{3}$

(3) $\dfrac{a_{n+1}}{a_n} = \dfrac{2^{n+1}n!}{(n+1)!2^n} = \dfrac{2}{n+1} \to 0$

□

Example 10.5.4. Find the range of x which makes the following converge.

$$1 + \frac{x^2}{2} + \frac{x^4}{4} + \frac{x^6}{6} + \cdots .$$

sol. For $n > 1$, $a_n = x^{2n-2}/(2n-2)$

$$\frac{a_{n+1}}{a_n} = \frac{x^{2n}(2n-2)}{2nx^{2n-2}} = \frac{(2n-2)x^2}{2n} \to x^2.$$

So converges if $|x| < 1$ and diverges if $|x| > 1$. When $|x| = 1$ the series diverges since it behaves like

$$1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{6} \cdots = 1 + \frac{1}{2}\left(1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots\right).$$

$\square$

Estimation of the error $\sum_{n=1}^{\infty} a_n - S_N = \sum_{n=1}^{N+1} a_n$

For $\rho < 1$ If the series is approximated by its N-th partial sum, then the error is

$$a_{N+1} + a_{N+2} + \cdots .$$

So if N is large, for some r with $\rho < r < 1$ we have

$$\frac{a_{n+1}}{a_n} < r, \quad n \geq N.$$

Here the estimate of errors is

$$a_{N+1} + a_{N+2} + \cdots \leq r a_N + r^2 a_N + \cdots = a_N \cdot \frac{r}{1-r}.$$

Example 10.5.5. Investigate

$$\frac{1}{3} + \frac{2}{9} + \frac{1}{27} + \frac{4}{81} + \cdots + \frac{f(n)}{3^n} + \cdots .$$

$$f(n) = \begin{cases} n, & n \text{ even} \\ 1, & n \text{ odd.} \end{cases}$$

sol. Since $a_n = \frac{f(n)}{3^n}$ we have

$$\frac{a_{n+1}}{a_n} = \frac{f(n+1)}{3f(n)} = \begin{cases} \frac{1}{3n}, & n \text{ even} \\ \frac{n+1}{3}, & n \text{ odd.} \end{cases}$$

So we cannot use ratio test. However if we take n-th root,

$$\sqrt[n]{a_n} = \frac{\sqrt[n]{f(n)}}{3} = \begin{cases} \frac{\sqrt[n]{n}}{3}, & n \text{ even} \\ \frac{1}{3}, & n \text{ odd.} \end{cases}$$

and $\sqrt[n]{n}$ converges to 1. Since

$$\lim_{n\to\infty} \sqrt[n]{a_n} = \frac{1}{3}$$

we can compare this series with $(\frac{1}{3})^n$ and conclude that it converges (as above example).

$\square$

n-th Root Test

Theorem 10.5.6 (n-th Root Test)**.** *Suppose $\sqrt[n]{a_n} \to \rho$. Then*

(1) $\sum a_n$ converges if $\rho < 1$.

(2) $\sum a_n$ diverges if $\rho > 1$.

(3) We cannot tell anything if $\rho = 1$.

Proof. (1) Suppose $\rho < 1$. Choose r between ρ and 1 and set $\varepsilon = \rho - r > 0$. Since $\sqrt[n]{a_n}$ converges to ρ there is some N s.t. when n is greater than N, it holds that

$$|\sqrt[n]{a_n} - \rho| < \varepsilon.$$

In other words, $\sqrt[n]{a_n} < \rho + \varepsilon = r < 1$. Hence

$$a_n < (\rho + \varepsilon)^n$$

holds. So $\sum(\rho+\varepsilon)^n$ converges and by comparison test $\sum_{n=N}^{\infty} a_n$ converges.

(2) Suppose $\rho > 1$ then $\sqrt[n]{a_n} > 1$ for suff. large n $a_n > 1$. So diverges.

(3) The case $\rho = 1$: No conclusion can be drawn since both the series $\sum 1/n^2$ and $\sum 1/n$ have $\rho = 1$ while one converges and the other not. $\square$

Example 10.5.7. $\sum_{n=1}^{\infty} \frac{n}{2^n}$ converges since $\sqrt[n]{\frac{n}{2^n}} = \sqrt[n]{\frac{n}{2}} \to \frac{1}{2}$.

Example 10.5.8. $\sum_{n=1}^{\infty} \frac{3^n}{n^n}$ converges since $\sqrt[n]{\frac{3^n}{n^n}} = \frac{3}{n} \to 0$.

10.6 Absolute and conditional convergence

Alternating Series

Definition 10.6.1. Suppose $a_n > 0$ for all n. A series of the form

$$a_1 - a_2 + a_3 - a_4 + \cdots$$

is called an **alternating series**.

The following are examples of alternating series.

$$1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \cdots$$
$$1 - 2 + 3 - 4 + 5 - 6 + \cdots$$

But

$$1 - \frac{1}{2} - \frac{1}{3} + \frac{1}{4} + \frac{1}{5} - \frac{1}{6} - \frac{1}{7} + \cdots$$

is not an alternating series.

Theorem 10.6.2 (Alternating Series Test, Leibniz theorem). *Suppose the following three conditions hold.*

(1) $a_n > 0$.

(2) $a_n \geq a_{n+1}$.

(3) $a_n \to 0$.

Then $\sum_{n=1}^{\infty}(-1)^{n+1}a_n$ *converges.*

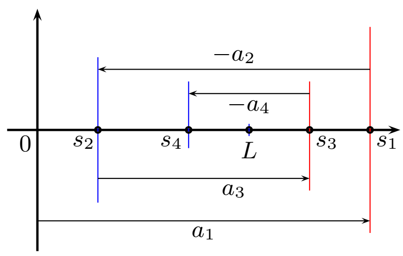


Figure 10.8: Partial sum of alternating series

Proof. The idea is to show that the sum of even number of terms form a bounded, increasing sequence so that it converges by Theorem 10.1.13. Suppose n is even ($n = 2m$) then the partial sum

$$s_{2m} = (a_1 - a_2) + (a_3 - a_4) + \cdots + (a_{2m-1} - a_{2m})$$

is **increasing**. Hence $s_{2m+2} \geq s_{2m}$. But we also see

$$s_{2m} = a_1 - (a_2 - a_3) - (a_4 - a_5) - \cdots - (a_{2m-2} - a_{2m-1}) - a_{2m}.$$

Hence s_{2m} is less than a_1. In other words, s_{2m} is **bounded above**, hence converges. Let $L = \lim s_{2m}$ be its limit. Now suppose n is odd ($n = 2m+1$). Then

$$s_{2m+1} = s_{2m} + a_{2m+1}.$$

Then since $a_{2m+1} \to 0$, $\lim s_{2m+1} = \lim(s_{2m} + a_{2m+1}) = L$.

□

Example 10.6.3. The series

$$\sum (-1)^{n+1} \frac{1}{n} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots$$

converges.

Example 10.6.4.

$$\sum (-1)^{n+1} \frac{1}{\sqrt{n}} = 1 - \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} - \frac{1}{\sqrt{4}} + \cdots$$

converges.

Example 10.6.5.

$$\sum (-1)^{n+1} \frac{\sqrt{n}}{\sqrt{n+1}} = \frac{1}{\sqrt{2}} - \frac{\sqrt{2}}{\sqrt{3}} + \frac{\sqrt{3}}{\sqrt{4}} - \frac{\sqrt{4}}{\sqrt{5}} + \cdots$$

diverges by n-th term test.

Example 10.6.6.

$$\frac{2}{1} - \frac{1}{1} + \frac{2}{3} - \frac{1}{3} + \frac{2}{5} - \frac{1}{5} + \cdots + \frac{2}{2n-1} - \frac{1}{2n-1} + \cdots$$

is alternating. But it is not monotonically decreasing. But

$$\left(\frac{2}{1}-\frac{1}{1}\right)+\left(\frac{2}{3}-\frac{1}{3}\right)+\left(\frac{2}{5}-\frac{1}{5}\right)+\cdots$$
$$+\left(\frac{2}{2n-1}-\frac{1}{2n-1}\right)+\cdots=1+\frac{1}{3}+\frac{1}{5}+\cdots+\frac{1}{2n-1}+\cdots$$

So diverges.

Example 10.6.7. Investigate $\displaystyle\sum_{n=2}^{\infty}(-1)^n\frac{\ln n}{n+1}$.

$\boxed{\textbf{sol.}}$ We let

$$f(x)=\frac{\ln x}{x+1}.$$

Then $f(n)=\ln n/(n+1)$ and $f'(x)=((x+1)/x-\ln x)/(x+1)^2$. For sufficiently large x, $(x+1)/x-\ln x<0$. Hence $f(x)$ is decreasing function. For example, for $x\geq 8$, $f(x)$ is decreasing. So $a_n=f(n)$ is decreasing for $n\geq 8$. By Leibniz theorem the series converges.

□

Partial Sum of Alternating Series

We look at the partial sums of an alternating series:

$$\begin{aligned}
&s_1=a_1,\\
&s_2=a_1-a_2,\ \text{So } s_2<s_1.\\
&s_3=a_1-a_2+a_3=a_1-(a_2-a_3),\ \text{So } s_2<s_3<s_1.\\
&s_4=a_1-a_2+a_3-a_4=a_1-a_2+(a_3-a_4),\ \text{So } s_2<s_4<s_3<s_1.
\end{aligned}$$

Thus s_{2m+1} is decreasing and s_{2m} is increasing. Let L be its sum. Then

$$\underbrace{\underbrace{s_{2m}<s_{2m+2}<\cdots<L}_{|s_{2m}-L|}<\cdots<s_{2m+1}}_{|s_{2m}-s_{2m+1}|}<s_{2m-1}$$

But since

$$\begin{aligned}
|s_{2m}-L|&<|s_{2m}-s_{2m+1}|=a_{2m+1},\\
|s_{2m+1}-L|&<|s_{2m+2}-s_{2m+1}|=a_{2m+2}
\end{aligned}$$

we see

$$|s_n - L| < a_{n+1}.$$

In other words, the partial sum is a good approximation to the true sum with error bound a_{n+1}. Since a_n is decreasing s_{n+1} is better approximation than s_n.

Theorem 10.6.8 (Alternating Series Estimation Theorem). *Suppose* $\sum(-1)^{n+1}a_n$ *is an alternating series satisfying the conditions of Leibniz theorem. Then the partial sum*

$$s_n = a_1 - a_2 + a_3 + \cdots + (-1)^{n+1}a_n$$

is a good approximation with an error bound less than a_{n+1}.

Example 10.6.9. estimate

$$\sum_{n=0}^{\infty} \frac{(-1)^n}{2^n} = 1 - \frac{1}{2} + \frac{1}{4} + \cdots = \frac{2}{3}$$

with first six term.

$\boxed{\textbf{sol.}}$ The error bound is $a_7 = 1/64$. The true value up to six terms is

$$s_6 = 1 - \frac{1}{2} + \frac{1}{4} - \frac{1}{8} + \frac{1}{16} - \frac{1}{32} = \frac{21}{32}.$$

So the true error is $|2/3 - 21/32| = 1/96$ which is less than $a_7 = 1/64$.

□

Example 10.6.10. Use s_{10} or s_{100} to estimate

$$\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} = 1 - \frac{1}{2} + \frac{1}{3} - \cdots = \ln 2 = 0.69314\cdots$$

$\boxed{\textbf{sol.}}$ True error of

$$s_{10} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots - \frac{1}{10} = 0.64563\cdots$$

is $0.0475\cdots < a_{11} = 1/11$. The true error of

$$s_{100} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots - \frac{1}{100} = 0.68881\cdots$$

is $0.00433\cdots < a_{111} = 1/111$.

□

Absolute convergence and Conditional Convergence

Definition 10.6.11. If $\sum |a_n|$ converges then $\sum a_n$ is said to **converge absolutely**.

Theorem 10.6.12. *If $\sum |a_n|$ converges then so does $\sum a_n$.*

Proof.

$$-|a_n| \leq a_n \leq |a_n|$$

holds for all n. Hence

$$0 \leq a_n + |a_n| \leq 2|a_n|.$$

Since $\sum |a_n|$ converges and $a_n + |a_n| \geq 0$

$$\sum (a_n + |a_n|)$$

converges by comparison. Subtracting converging series, we have

$$\sum a_n = \sum (a_n + |a_n|) - \sum |a_n|$$

and so $\sum a_n$ converges. □

Corollary 10.6.13. *If $\sum a_n$ diverges, so does $\sum |a_n|$.*

Example 10.6.14. (1) $\sum_{n=1}^{\infty} (-1)^{n+1} \frac{1}{n^2} = 1 - \frac{1}{4} + \frac{1}{9} + \cdots +$. Its n-th term $a_n = \frac{(-1)^{n+1}}{n^2}$ satisfies $|a_n| = \frac{1}{n^2}$. Since $\sum \frac{1}{n^2}$ converges we see the series $\sum_{n=1}^{\infty} (-1)^{n+1} \frac{1}{n^2}$ converges absolutely. The series $\sum_{n=1}^{\infty} (-1)^{n+1} \frac{1}{n^2}$, of course converges.

(2) The n-th term of $\sum \frac{\cos n}{n^2}$ satisfies $|a_n| = \frac{|\cos n|}{n^2} \leq \frac{1}{n^2}$. Since $\sum \frac{1}{n^2}$ converges, $\sum \frac{\cos n}{n^2}$ converges.

(3)
$$\sum (-1)^{n+1} \frac{1}{n} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots$$

But $\sum |a_n| = \sum \frac{1}{n}$ diverges by integral test. Thus the series does not converge absolutely. Still, this series converges (by Leibniz theorem).

(4) $\sum \frac{(-1)^n}{n^p}$ converges absolutely for $p > 1$ but does not converges absolutely for $p \leq 1$. However, the series converges for all $p > 0$.

Definition 10.6.15. A series which converges but does not converge not absolutely **converges conditionally**.

Rearrangement of Series for Absolutely Convergent Series

Theorem 10.6.16 (Rearrangement of Series)**.** *Suppose $\sum a_n$ converges absolutely and b_n is a rearrangement of a_n. Then $\sum b_n$ converges absolutely and and*

$$\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} b_n.$$

Here we have $b_k = a_{n(k)}$ for some 1-1 function $n(k)$.

Proof. See exercise 68 of the text book. □

Example 10.6.17. We know the following converges absolutely:

$$1 - \frac{1}{2} + \frac{1}{4} - \frac{1}{8} + \frac{1}{16} - \frac{1}{32} + \cdots = \frac{2}{3}.$$

Hence rearranging it in any order we get

$$1 + \frac{1}{4} - \frac{1}{2} + \frac{1}{16} + \frac{1}{64} - \frac{1}{8} + \cdots$$

We can guarantee this series converges to $\frac{2}{3}$. We know the series

$$1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \cdots$$

converges but not absolutely. Hence its rearrangement may not converge. In fact, even if it converges it may converge to a different value.

Consider one rearrangement:

$$\left(1 - \frac{1}{2}\right) + \left(\frac{1}{3} + \frac{1}{5} - \frac{1}{4}\right) + \left(\frac{1}{7} + \frac{1}{9} - \frac{1}{6}\right) + \left(\frac{1}{11} + \frac{1}{13} - \frac{1}{8}\right) + \cdots$$

Then sum may be bigger than $\ln 2 = 0.69314\cdots$.

Product of two series

Suppose $\sum_{n=0}^{\infty} a_n, \sum_{n=0}^{\infty} b_n$ converge absolutely. Then

$$\left(\sum_{n=0}^{\infty} a_n\right) \times \left(\sum_{n=0}^{\infty} b_n\right) = (a_0 + a_1 + \cdots + a_n + \cdots) \times (b_0 + b_1 + \cdots + b_n + \cdots).$$

Finite partial sum is

$$(a_0 + a_1 + \cdots + a_n) \times (b_0 + b_1 + \cdots + b_n).$$

We can write it as

$$\begin{aligned}&a_0b_0 + (a_0b_1 + a_1b_0) + (a_0b_2 + a_1b_1 + a_2b_0) + \cdots \\ &+ \cdots + (a_0b_n + a_1b_{n-1} + \cdots + a_{n-1}b_1 + a_nb_0).\end{aligned}$$

In the limit,

$$\begin{aligned}&\left(\sum_{n=0}^{\infty} a_n\right) \times \left(\sum_{n=0}^{\infty} b_n\right) \\ &= a_0b_0 + (a_0b_1 + a_1b_0) + (a_0b_2 + a_1b_1 + a_2b_0) + \cdots \\ &+ \cdots (a_0b_n + a_1b_{n-1} + \cdots + a_{n-1}b_1 + a_nb_0) + \cdots\end{aligned}$$

Since it converges absolutely, it value does not change.

Theorem 10.6.18. *Suppose both $\sum_{n=0}^{\infty} a_n$ and $\sum_{n=0}^{\infty} b_n$ converge absolutely. If we set $c_n = \sum_{n=0}^{k} a_k b_{n-k}$ then $\sum c_n$ converge absolutely and*

$$\sum_{n=0}^{\infty} c_n = \left(\sum_{n=0}^{\infty} a_n\right) \times \left(\sum_{n=0}^{\infty} b_n\right).$$

10.7 Power Series

Definition 10.7.1. A **power series** about $x = 0$ is a series of the form

$$\sum_{n=0}^{\infty} a_n x^n = a_0 + a_1x + a_2x^2 + \cdots + a_nx^n + \cdots$$

A power series about $x = a$ is a series of the form

$$\sum_{n=0}^{\infty} a_n (x-a)^n.$$

Here a_n are the **coefficients** and a is called the **center**.

Example 10.7.2. (1) (Geometric series) $\sum_{n=1}^{\infty} \frac{(x-1)^n}{2^n} = \frac{1}{2^1} + \frac{(x-1)^2}{2^2} + \frac{(x-1)^3}{2^3} + \cdots$

(2) $\sum_{n=1}^{\infty}(-1)^{n-1}\frac{x^n}{n} = x - \frac{x^2}{2} + \frac{x^3}{3} - \cdots$

(3) $\sum_{n=1}^{\infty}(-1)^{n-1}\frac{x^{2n-1}}{2n-1} = x - \frac{x^3}{3} + \frac{x^5}{5} - \cdots$

(4) $\sum_{n=0}^{\infty}\frac{x^n}{n!} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$

(5) $\sum_{n=0}^{\infty} n!x^n = 1 + x + 2!x^2 + 3!x^3 + \cdots$

Theorem 10.7.3 (Convergenec of Power Series). *Given a power series* $\sum_{n=0}^{\infty} a_n(x-x_0)^n$

(1) Suppose it converges at a point $x_1\,(\neq x_0)$*. Then it converges absolutely for all points* x *satisfying* $|x-x_0| < |x_1-x_0|$*.*

(2) Suppose it diverges at x_2 *it. Then it diverges for all* x *with* $|x-x_0| > |x_2-x_0|$*.*

Proof. Suppose $\sum_{n=0}^{\infty} a_n(x_1-x_0)^n$ converges. Then $\lim_{n\to\infty} a_n(x_1-x_0)^n = 0$. Hence for suff. large n, it holds that $|a_n(x_1-x_0)^n| \leq 1$ and we see

$$|a_n(x-x_0)^n| = |a_n(x_1-x_0)^n|\left|\frac{x-x_0}{x_1-x_0}\right|^n \leq \left|\frac{x-x_0}{x_1-x_0}\right|^n.$$

Hence for all x with $|x-x_0| < |x_1-x_0|$, the series $\sum_{n=0}^{\infty} a_n(x_1-x_0)^n$ converges absolutely. Now suppose the series $\sum_{n=0}^{\infty} a_n(x_2-x_0)^n$ diverges and $\sum_{n=0}^{\infty} a_n(x-x_0)^n$ converge for some x satisfying $|x-x_0| > |x_2-x_0|$. Then by (1) the series $\sum_{n=0}^{\infty} a_n(x_2-x_0)^n$ must converge, which is a contradiction. Hence the series $\sum_{n=0}^{\infty} a_n(x-x_0)^n$ must diverges for any x with $|x-x_0| > |x_2-x_0|$. □

By Theorem 10.7.3, there are three possibilities for the series $\sum_{n=0}^{\infty} a_n(x-x_0)^n$:

(1) It converges for x_0 only;

(2) It converges absolutely for all x;

(3) There exists an R such that for all x with $|x-x_0| < R$ it converges absolutely and diverges for all x with $|x-x_0| > R$.

We see that in case (1) $R = 0$, and in case (2) $R = \infty$. In general, the number R is called **the radius of convergence** of $\sum_{n=0}^{\infty} a_n(x-x_0)^n$.

Theorem 10.7.4. *For $\sum_{n=0}^{\infty} a_n(x-x_0)^n$, the radius of convergence is given as follows:*

$$R = \lim_{n\to\infty}\left|\frac{a_n}{a_{n+1}}\right| \tag{10.2}$$

$$R = \lim_{n\to\infty}\frac{1}{\sqrt[n]{|a_n|}} \tag{10.3}$$

provided that either of the limit exists.

Proof. Suppose the limit in (10.2) exists. Then

$$\lim_{n\to\infty}\left|\frac{a_{n+1}(x-x_0)^{n+1}}{a_n(x-x_0)^n}\right| = \lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right||x-x_0| = \frac{|x-x_0|}{R}.$$

Now by ratio test (Theorem 10.5.2), the power series converges absolutely for $|x-x_0|/R < 1$ and diverges if $|x-x_0|/R > 1$. Hence R given by (10.2) is the radius of convergence. One can show (10.3) holds if we use n-th root test (Theorem 10.5.6). □

The set of all point for which the series converges form an interval I (called the **interval of convergence**) and I satisfies

$$(x_0 - R, x_0 + R) \subset I \subset [x_0 - R, x_0 + R].$$

Example 10.7.5. Find the interval of convergence.

(1) $\displaystyle\sum_{n=0}^{\infty} n^n x^n$

(2) $\displaystyle\sum_{n=1}^{\infty} \frac{x^n}{n^2}$

(3) $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n-1}x^n}{n}$

(4) $\displaystyle\sum_{n=0}^{\infty} \frac{x^n}{n!}$.

sol. (1)

$$R = \lim_{n\to\infty}\frac{n^n}{(n+1)^{n+1}} = \lim_{n\to\infty}\left(\frac{n}{n+1}\right)^n\frac{1}{n+1} = 0.$$

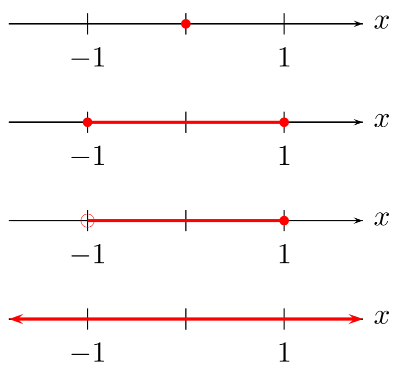


Figure 10.9: interval of convergence

(2)

$$R = \lim_{n\to\infty} \frac{(n+1)^2}{n^2} = 1.$$

When $x = \pm 1$, $\sum_{n=1}^{\infty}((\pm 1)^n/n^2)$ both converges. Hence $I = [-1, 1]$.

(3)

$$R = \lim_{n\to\infty} \frac{n+1}{n} = 1.$$

For $x = 1$, $\sum_{n=1}^{\infty}((-1)^{n-1}/n)$ is alternating, so converges. While for $x = -1$, $\sum_{n=1}^{\infty}(1/n)$ diverges. So $I = (-1, 1]$.

(4) Since

$$R = \lim_{n\to\infty} \frac{(n+1)!}{n!} = \infty$$

the interval of convergence is $(-\infty, \infty)$.

□

Theorem 10.7.6 (Term by term differentiation)**.** *Suppose $\sum_{n=0}^{\infty} a_n(x - x_0)^n$ converges for all $|x - x_0| < R$ for some $R > 0$, i.e., the function $f(x)$ is defined by*

$$f(x) = \sum_{n=0}^{\infty} a_n(x - x_0)^n, \quad |x - x_0| < R. \tag{10.4}$$

Then

(i) $f(x)$ is differentiable on $(x_0 - R, x_0 + R)$ and its derivative is

$$f'(x) = \sum_{n=1}^{\infty} n a_n (x - x_0)^{n-1}, \quad |x - x_0| < R. \tag{10.5}$$

(ii) $f(x)$ is integrable on $(x_0 - R, x_0 + R)$ and

$$\int f(x)\,dx = \sum_{n=0}^{\infty} a_n \frac{(x - x_0)^{n+1}}{n+1} + C, \quad |x - x_0| < R. \tag{10.6}$$

The radius convergence of the power series (10.5) *and* (10.6) *are also R.*

Proof. Suppose

$$R = \lim_{n\to\infty} \left| \frac{a_n}{a_{n+1}} \right|.$$

The radius of convergence of the power series (10.5) is given by Theorem 10.7.4

$$\lim_{n\to\infty} \left| \frac{(n+1)a_{n+1}}{(n+2)a_{n+2}} \right| = \lim_{n\to\infty} \left| \frac{a_{n+1}}{a_{n+2}} \right| = R.$$

Similarly, the radius of convergence of (10.6) is also R. $\square$

Corollary 10.7.7. *The series in Theorem* 10.7.6 *is differentiable infinitely many times on $(x_0 - R, x_0 + R)$ and its k-th derivative is given by*

$$\begin{gathered} f^{(k)}(x) = \sum_{n=k}^{\infty} n(n-1)\cdots(n-k+1)a_n(x - x_0)^{n-k}, \\ |x - x_0| < R, \end{gathered} \tag{10.7}$$

$k = 0, 1, \ldots$.

Example 10.7.8 (Caution!)**.** The (not a power!) series

$$\sum_{n=1}^{\infty} \frac{\sin(n!x)}{n^2}$$

converges for all x, but if we differentiate it we get

$$\sum_{n=1}^{\infty} \frac{n!\cos(n!x)}{n^2}$$

which does not converges for any x.

Product of two Power series

Theorem 10.7.9. *Suppose both* $A(x) = \sum_{n=0}^{\infty} a_n x^n,\ B(x) = \sum_{n=0}^{\infty} b_n x^n$ *converge absolutely for* $|x| < R$ *and*

$$c_n = a_0 b_n + a_1 b_{n-1} + \cdots + a_n b_0 = \sum_{n=0}^{k} a_k b_{n-k}.$$

Then $\sum_{n=0}^{\infty} c_n x^n$ *converge absolutely to* $A(x)B(x)$ *for* $|x| < R$ *also, and*

$$\left(\sum_{n=0}^{\infty} a_n x^n\right) \times \left(\sum_{n=0}^{\infty} b_n x^n\right) = \sum_{n=0}^{\infty} c_n x^n.$$

Example 10.7.10. Use

$$\sum_{n=0}^{\infty} x^n = 1 + x + x^2 + \cdots = \frac{1}{1-x}, \text{ for } |x| < 1$$

to get the power series for $1/(1-x)^2$.

sol. We let $A(x) = B(x) = \sum_{n=0}^{\infty} x^n$. Then we see

$$c_n = a_0 b_n + a_1 b_{n-1} + \cdots + a_n b_0 = \sum_{n=0}^{k} a_k b_{n-k} = n + 1.$$

Hence

$$A(x)B(x) = \sum_{n=0}^{\infty} c_n x^n = \sum_{n=0}^{\infty} (n+1) x^n.$$

This series could be obtained by differentiation.

□

Example 10.7.11. Compute the first few terms of

$$\left(\sum_{n=0}^{\infty} (n+1) x^n\right) \left(\sum_{n=0}^{\infty} (-1)^n x^n\right)$$

sol.

$$\begin{aligned} & \left(1 + x + 2x^2 + 3x^3 + \cdots\right)\left(1 - x + x^2 - x^3 + x^4 \cdots\right) \\ = \; & 1 + 2x^2 + x^3 + 3x^4 + 2x^5 \cdots \end{aligned}$$

□

10.8 Taylor and Maclaurin Series

In the previous discussions we have seen that a power series defines a continuous function on I. How about its converse? Suppose f is differentiable n-times. Is it possible to express it in power series ? A power series $\sum_{n=0}^{\infty} a_n(x-a)^n$ represents a function on its interval of convergence I

$$f(x) = \sum_{n=0}^{\infty} a_n(x-a)^n, \quad x \in I.$$

We shall later show

$$\begin{aligned}&\sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n \\ =& f(a) + f'(a)(x-a) + \cdots + \frac{f^{(n)}(a)}{n!}(x-a)^n + \cdots .\end{aligned}$$

This is called **Taylor series** of $f(x)$ at a (If $a = 0$, it is also called **Maclaurin series**).

Example 10.8.1. Find Taylor series of $f(x) = 1/x$ at $a = 2$.

sol.

$$f(x) = \frac{1}{x}, \quad f'(x) = -x^{-2}, \quad f''(x) = 2!x^{-3}, \cdots, \quad f^{(n)}(x) = (-1)^n n! x^{-(n+1)},$$

$$f(2) = \frac{1}{2}, \quad f'(2) = -\frac{1}{2^2}, \quad \frac{f''(x)}{2!} = \frac{1}{2^{-3}}, \cdots, \quad \frac{f^{(n)}(2)}{n!} = \frac{(-1)^n}{2^{n+1}}.$$

□

Taylor Polynomial

Consider

$$y = P_1(x) := f(a) + f'(x_0)(x-a).$$

This is linear approximation to $f(x)$. Similarly we can consider

$$y = P_2(x) := f(a) + f'(a)(x-a) + \frac{f''(a)}{2}(x-a)^2.$$

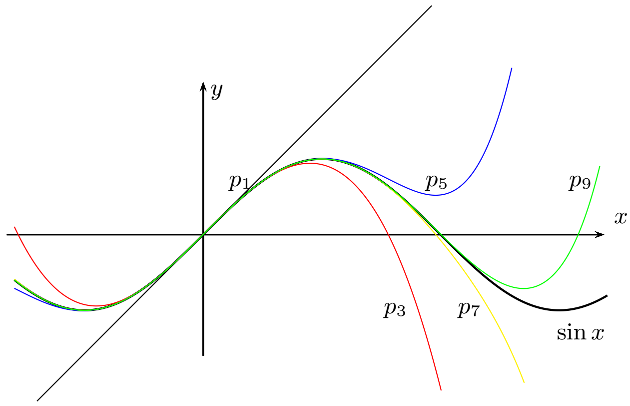


Figure 10.10: Taylor polynomials for $\sin x$

which has same derivative up to second order. By the same way one can find a polynomial $P_n(x)$ of degree n. It is called **a Taylor polynomial of degree** n Then we see

$$P_n^{(k)}(a) = f^{(k)}(a), \quad k = 0, 1, \cdots, n.$$

$$P_n(x) = f(a) + f'(a)(x-a) + \cdots + \frac{f^{(n)}(a)}{n!}(x-a)^n. \tag{10.8}$$

The difference(error) is defined as

$$R_n(x) = f(x) - P_n(x)$$

and called the **remainder**

$$f(x) = P_n(x) + R_n(x)$$

is called n-th **Taylor formula** of $f(x)$ at a.

Example 10.8.2. Find Taylor polynomial for $\cos x$.

Example 10.8.3.

$$f(x) = \begin{cases} \exp(-1/x^2), & x \neq 0 \\ 0, & x = 0. \end{cases}$$

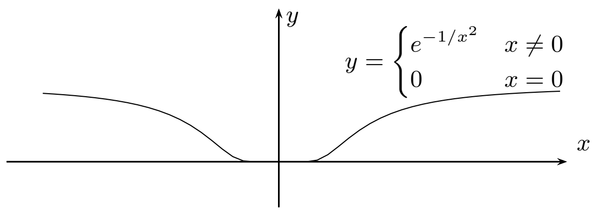


Figure 10.11: Non convergent Taylor polynomials for e^{-1/x^2}

is infinitely differentiable at 0, but the Taylor series converges only at $x = 0$. In fact we can show that $f^{(n)}(0) = 0$, $n = 0, 1, \ldots$. So the Taylor polynomial $P_n(x) = 0$ and $R_n(x) = f(x)$. Hence $P_n(x) \not\to f(x)$.

10.9 Convergence of Taylor Series, Error estimates

Theorem 10.9.1 (Taylor's Theorem with Remainder)**.** *Suppose $f(x)$ is differentiable $n+1$ times on an open interval I containing a and $P_n(x)$ is the Taylor polynomial given by* (10.8)*. Then*

$$R_n(x) = \frac{f^{(n+1)}(c)}{(n+1)!}(x-a)^{n+1}. \tag{10.9}$$

Definition 10.9.2. Suppose $f(x)$ is infinitely differentiable on I and

$$\lim_{n\to\infty} R_n(x) = 0, \quad x \in I$$

then we say the Taylor series at a **converges** to $f(x)$ and we we write

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n, \quad x \in I.$$

Here $R_n(x) = f(x) - P_n(x)$ is the remainder.

Corollary 10.9.3. *Suppose there is some M such that $f(x)$ satisfies $|f^{(n+1)}(x)| \leq$*

M for all $x \in I$. Then

$$|R_n(x)| \le M \frac{|x - x_0|^{n+1}}{(n+1)!}, \quad x \in I. \tag{10.10}$$

Example 10.9.4. At $a = 0$, we have

$$e^x = 1 + x + \cdots + \frac{x^n}{n!} + R_n(x).$$

Here

$$|R_n(x)| \le e^c \frac{x^{n+1}}{(n+1)!}.$$

Example 10.9.5. (1) Maclaurin series of $\sin x$, $\cos x$, e^x:

$$\sin x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!}, \quad -\infty < x < \infty$$

$$\cos x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!}, \quad -\infty < x < \infty$$

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}, \quad -\infty < x < \infty$$

(2) Maclaurin series of $\ln(1+x)$ on $(0, \infty)$

$$\ln(1+x) = \sum_{n=1}^{\infty} \frac{(-1)^{n-1} x^n}{n}, \quad -1 < x \le 1$$

(3) Maclaurin series of $1/(1-x)$

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n, \quad -1 < x < 1$$

(4) Taylor series of $\sqrt{x}$ is at 1.

Example 10.9.6 (Substitution)**.** Find series for $\cos x^2$ near $x = 0$.

Example 10.9.7 (Multiplication)**.** Find series for $x \sin x^2$ near $x = 0$.

Example 10.9.8. Find Taylor expansion of

$$e^x \cos x.$$

$\boxed{\textbf{sol.}}$

$$\begin{aligned} e^x \cos x &= \left(1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots\right) \cdot \left(1 - \frac{x^2}{2!} + \frac{x^4}{4!} + \cdots\right) \\ &= \left(1 + x - \frac{x^3}{3} - \frac{x^4}{6} \cdots\right). \end{aligned}$$

$\square$

Example 10.9.9 (Truncation Error)**.** For what values of x can we replace $\sin x$ by $\sin x \approx x - \frac{x^3}{3!}$ with error less than 3×10^{-4}?

$$\sin x \approx x - \frac{x^3}{3!} = p_3(x) = p_4(x).$$

Hence we see the error term is either $R_3(x)$ or $R_4(x)$. We use the latter since it is more accurate value. $R_4(x) = \frac{f^{(5)}(c)}{5!}|x|^5$, we let

$$\frac{|x|^5}{5!} \le 3 \times 10^{-4}.$$

Proof of Taylor's Formula with Remainder

With

$$P_n(x) = f(a) + f'(a)(x-a) + \cdots + \frac{f^{(n)}(a)}{n!}(x-a)^n$$

we set

$$\phi_n(x) = P_n(x) + K(x-a)^{n+1}.$$

This function has same first n-derivative as f at a. We can choose K so that $\phi_n(x)$ agrees with $f(x)$. The idea is to fix $x = b$ and choose K so that $\phi_n(b)$ agrees with $f(b)$. So

$$f(b) = P_n(b) + K(b-a)^{n+1}, \text{ or } K = \frac{f(b) - P_n(b)}{(b-a)^{n+1}} \tag{10.11}$$

and

$$F(x) = f(x) - \phi_n(x)$$

is the error. We use Rolle's theorem. First, since $F(b) = F(a) = 0$, we have

$$F'(c_1) = 0, \qquad \text{for some } c_1 \in (a, b).$$

Next, because $F'(a) = F'(c_1) = 0$, we have

$$F''(c_2) = 0, \quad \text{for some } c_2 \in (a, c_1).$$

Now repeated application of Rolle's theorem to F'', etc show there exist

$$\begin{array}{lll} c_3 & \text{in } (a, c_2) & \text{such that } F'''(c_3) = 0, \\ c_4 & \text{in } (a, c_3) & \text{such that } F^{(4)}(c_4) = 0, \\ & \vdots & \\ c_n & \text{in } (a, c_{n-1}) & \text{such that } F^{(n)}(c_n) = 0, \\ c_{n+1} & \text{in } (a, c_n) & \text{such that } F^{(n+1)}(c_{n+1}) = 0. \end{array}$$

But since $F(x) = f(x) - \phi_n(x) = f(x) - P_n(x) - K(x-a)^{n+1}$, we see

$$F^{(n+1)}(c) = f^{(n+1)}(c) - 0 - (n+1)!K.$$

Hence

$$K = \frac{f^{(n+1)}(c)}{(n+1)!}, \quad c = c_{n+1}.$$

So

$$f(b) = P_n(b) + \frac{f^{(n+1)}(c)}{(n+1)!}(b-a)^{n+1}. \tag{10.12}$$

Now since b is arbitrary, we can set $b = x$. Furthermore, if $R_n \to$ as $n \to \infty$, we obtain Taylor's theorem.

10.10 Binomial Series and Applications

Binomial Series

First assume m is a positive integer and consider the binomial expansion

$$(1+x)^m = 1 + mx + \frac{m(m+1)}{2!}x^2 + \cdots + \binom{m}{k}x^k + \cdots + x^m.$$

Here

$$\binom{m}{k} = \frac{m(m-1)\cdots(m-k+1)}{k!}, \quad n = 0, 1, 2, \dots .$$

We now consider the Taylor series of $(1+x)^m$ for any real m. Since

$$\begin{aligned} f(x) &= (1+x)^m \\ f'(x) &= m(1+x)^{m-1} \\ f''(x) &= m(m-1)(1+x)^{m-2} \\ &\cdots \\ f^{(k)}(x) &= m(m-1)(m-2)\cdots(m-k+1)(1+x)^{m-k}, \end{aligned} \tag{10.13}$$

we obtain the Taylor series

$$(1+x)^m = 1 + mx + \frac{m(m+1)}{2!}x^2 + \cdots + \binom{m}{k}x^k + \cdots . \tag{10.14}$$

We can show the radius of convergence is $R = 1$. When m is an integer, the derivatives $f^{(k)}(x) = 0$ for $k \geq m$, and we obtain the usual binomial expansion as a special case.

Example 10.10.1.

$$\frac{1}{(1+x)} = 1 - x + x^2 - x^3 + \cdots + (-1)^k x^k + \cdots$$

$$\begin{aligned} (1+x)^{1/2} &= 1 + \frac{x}{2} + \frac{(\frac{1}{2})(-\frac{1}{2})}{2!}x^2 + \frac{(\frac{1}{2})(-\frac{1}{2})(-\frac{3}{2})}{3!}x^3 + \frac{(\frac{1}{2})(-\frac{1}{2})(-\frac{3}{2})(-\frac{5}{2})}{4!}x^4 + \cdots \\ &= 1 + \frac{x}{2} - \frac{x^2}{8} + \frac{x^3}{16} - \frac{5x^4}{128} + \cdots . \end{aligned} \tag{10.15}$$

Substitution gives

$$\sqrt{1-x^2} = 1 - \frac{x^2}{2} - \frac{x^4}{8} + \cdots , \ |x^2| < 1$$

or

$$\sqrt{1-x^3} = 1 - \frac{x^3}{2} - \frac{x^6}{8} + \cdots \ |x^3| < 1$$

or even

$$\sqrt{1-\frac{1}{x}} = 1 - \frac{1}{2x} - \frac{1}{8x^2} + \cdots \ |\frac{1}{x}| < 1$$

are possible.

Example 10.10.2. Express $\int \sin\sqrt{x}\, dx$ as a power series and use it to evaluate $\int_0^1 \sin\sqrt{x}\,dx$.

$$\begin{aligned}
\sin\sqrt{x} &= x^{1/2} - \frac{x^{3/2}}{3!} + \frac{x^{5/2}}{5!} - \frac{x^{7/2}}{7!} + \cdots \\
\int_0^1 \sin\sqrt{x}\,dx &= \frac{2}{3}x^{3/2} - \frac{2}{5\cdot 3!}x^{5/2} + \frac{2}{7\cdot 5!}x^{7/2} - \frac{2}{9\cdot 7!}x^{9/2} + \cdots \Big|_0^1 \\
&= \frac{2}{3} - \frac{2}{5\cdot 3!} + \frac{2}{7\cdot 5!} - \frac{2}{9\cdot 7!} + \cdots
\end{aligned}$$

Example 10.10.3. Find $\sqrt{1.2}$ up to two decimal point.

sol. Let $f(x) = \sqrt{1+x}$. Then $\sqrt{1.2} = f(0.2)$. Hence from equation (10.14) We see Taylor series at $x_0 = 0$ is

$$\begin{aligned}
f(x) &= 1 + \frac{1}{2}x + \cdots + \binom{1/2}{n}x^n + R_{n+1}(x), \\
R_n(x) &= \frac{1}{(n+1)!}f^{(n+1)}(\bar{x})x^{n+1} \quad (0 \le \bar{x} \le 0.2).
\end{aligned}$$

For $n = 1$,

$$R_1(0.2) = -\frac{1}{2}f''(\bar{x})(0.2)^2 = -\frac{1}{2}\frac{1}{2}(-\frac{1}{2})(1+\bar{x})^{-3/2} = -0.005(1+\bar{x})^{-3/2}, (0 \le \bar{x} \le 0.2)$$

Hence $\sqrt{1.2} \approx 1 + (1/2)(0.2) = 1.1$ and the error satisfies $|R_1(0.2)| < 0.005$.

□

Example 10.10.4. Find Maclaurin series of $\arctan x$.

sol. Note that for $|x| < 1$ the $\arctan x$ has convergent power series:

$$(\arctan x)' = \frac{1}{1+x^2} = \sum_{n=0}^{\infty}(-1)^n x^{2n}.$$

Integrate it from 0 to x

$$\begin{aligned}
\arctan x &= \int_0^x \sum_{n=0}^{\infty}(-1)^n t^{2n}\,dt \\
&= \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{2n+1}, \quad |x| < 1.
\end{aligned}$$

Thus

$$\arctan x = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \cdots$$

□

This formula can be used to compute π. For example,

$$\frac{\pi}{4} = \arctan 1 = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots$$

The error with n-term is $1/(2n+1)$. So to get the error less than 10^{-3}, we need $2n + 1 \approx 1000$, $n = 500$ terms. Because of its slowness, we suggest another methods. For example, if

$$\alpha = \tan^{-1}\frac{1}{2}, \quad \beta = \tan^{-1}\frac{1}{3},$$

then

$$\tan(\alpha+\beta) = \frac{\tan\alpha + \tan\beta}{1 - \tan\alpha\tan\beta} = \frac{\frac{1}{2} + \frac{1}{3}}{1 - \frac{1}{6}} = 1 = \tan\frac{\pi}{4}$$

and

$$\frac{\pi}{4} = \alpha + \beta = \tan^{-1}\frac{1}{2} + \tan^{-1}\frac{1}{3}.$$

Now use the Taylor series for $\tan^{-1} x$ with $x = \frac{1}{2}$ and $x = \frac{1}{3}$. This is faster.

For example

$$\tan^{-1}\frac{1}{2} = (\frac{1}{2}) - \frac{1}{3}(\frac{1}{2})^3 + \frac{1}{5}(\frac{1}{2})^5 - \frac{1}{7}(\frac{1}{2})^7 + R_8^1 = 0.463467...$$

$$\tan^{-1}\frac{1}{3} = (\frac{1}{3}) - \frac{1}{3}(\frac{1}{3})^3 + \frac{1}{5}(\frac{1}{3})^5 + R_6^2 = 0.321810..$$

Here $|R_8^1| \le \frac{1}{9}(\frac{1}{2})^9 = \frac{1}{4{,}500}$ and $|R_6^2| \le \frac{1}{7}(\frac{1}{3})^7 = \frac{1}{15{,}309}$ and

$$\tan^{-1}\frac{1}{2} + \tan^{-1}\frac{1}{3} = 0.7852777$$

Multiply by 4 we get

$$\pi \approx 3.14111...$$

which is accurate at least three decimals.

Similar idea can be used to the following problem:

Example 10.10.5. Estimate

$$\ln 2 = \ln(1+1) = 1 - \frac{1}{2} + \cdots + \frac{(-1)^{n-1}}{n} + R_n(1).$$

Since

$$|R_n(1)| \leq \frac{1}{n+1}$$

we need to take large n. However, we can do the following:

$$\ln 2 = \ln\frac{4}{3} + \ln\frac{3}{2}2 = \ln(1+\frac{1}{3}) + \ln(1+\frac{1}{2})$$

and use Taylor series.

Example 10.10.6. Estimate $\int_0^1 \sin x^2\, dx$ with error less than 0.001.

sol. First note that

$$\sin x^2 = x^2 - \frac{x^6}{3!} + \frac{x^{10}}{5!} - \frac{x^{14}}{7!} + \cdots$$

Integrating

$$\int_0^1 \sin x^2\, dx = \frac{1}{3} - \frac{1}{7\cdot 3!} + \frac{1}{11\cdot 5!} - \frac{1}{15\cdot 7!} + \cdots$$

Since

$$\frac{1}{11\cdot 5!} < 0.00076$$

it suffices to take two terms.

□

Example 10.10.7. Estimate $\sin(0.1)$ up to third digit 3.

sol. Taylor polynomial of $\sin x$ at $x_0 = 0$

$$\sin x = x - \frac{1}{3!}x^3 + \frac{1}{5!}x^5 - \cdots + (-1)^{n-1}\frac{1}{(2n-1)!}x^{2n-1} + R_{2n-1}(x).$$

Since $|\sin x| \leq 1$, for $|\cos x| \leq 1$

$$|R_{2n-1}(x)| = |R_{2n}(x)| \leq \frac{|x|^{2n+1}}{(2n+1)!}.$$

If $n = 1$, $\sin x \sim x$ and the error for $\sin(0.1) \sim 0.1$ is

$$|R_2(0.1)| \leq \frac{(0.1)^3}{3!} < 10^{-3}$$

we have $\sin(0.1) \approx 0.1$ and the error is less than $\pm(1/6)\times 10^{-3}$.

□

Indeterminate forms

Example 10.10.8. Find
$$\lim_{x\to 1} \frac{\ln x}{x-1}.$$
Use the Taylor series of $\ln x$ ar $x = 1$.

Example 10.10.9. Find
$$\lim_{x\to 0} \frac{\sin x - x + (x^3/6)}{x^4}.$$

sol. $x_0 = 0$. Taylor polynomial of $\sin x$ at$x_0 = 0$ is
$$\sin x = x - \frac{x^3}{6} + R_4(x) \quad \text{and} \quad |R_4(x)| \le \frac{|x|^5}{5!}.$$
Hence
$$\left|\frac{\sin x - x + (x^3/6)}{x^4}\right| = \left|\frac{R_4(x)}{x^4}\right| \le \frac{|x|}{5!}$$
and limit is 0.

□

Example 10.10.10. Find
$$\lim_{x\to 0} \left(\frac{1}{\sin x} - \frac{1}{x}\right).$$

sol.
$$\begin{aligned}
\frac{1}{\sin x} - \frac{1}{x} &= \frac{x - \sin x}{x \sin x} \\
&= \frac{x - \left(x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots\right)}{x \cdot \left(x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots\right)} \\
&= \frac{x^3\left(\frac{1}{3!} - \frac{x^2}{5!} + \cdots\right)}{x^2\left(1 - \frac{x^2}{3!} + \cdots\right)}
\end{aligned}$$

□

Euler's identity

$$
\begin{aligned}
e^{i\theta} &= 1 + \frac{i\theta}{1!} + \frac{i^2\theta^2}{2!} + \frac{i^3\theta^3}{3!} + \frac{i^4\theta^4}{4!} + \cdots \\
&= \left(1 - \frac{\theta}{2!} + \frac{\theta^4}{4!} - \frac{\theta^6}{6!} + \cdots\right) + i\left(\theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - \cdots\right) \\
&= \cos\theta + i\sin\theta.
\end{aligned}
$$