Contents

Chapter 8

Techniques of Integration

8.1 Integration by Parts

Some Examples of Integration

Example 8.1.1.

$$\int_0^{\pi/4} \sqrt{1+\cos 4x}\, dx.$$

Use

$$\cos^2\theta = \frac{1+\cos 2\theta}{2}.$$

Example 8.1.2. Find

$$\int \sec x\, dx.$$

The idea is to multiply $\sec x + \tan x$ both the numerator and denominator:

$$\begin{aligned}
\int \sec x\, dx &= \int \sec x \cdot \frac{\sec x + \tan x}{\sec x + \tan x}\, dx \\
&= \int \frac{\sec^2 x + \sec x \tan x}{\sec x + \tan x}\, dx \\
&= \int \frac{du}{u} \\
&= \ln|\sec x + \tan x| + C.
\end{aligned}$$

Similarly, we obtain

$$\int \csc x\, dx = -\ln|\csc x + \cot x| + C.$$

Integral tables

(1) $\displaystyle\int \frac{1}{a^2+u^2}\,du = \frac{1}{a}\tan^{-1}\frac{u}{a} \quad (a>0).$

(2) $\displaystyle\int \frac{1}{\sqrt{a^2-u^2}}\,du = \sin^{-1}\frac{u}{a} \quad (a>0).$

Example 8.1.3. For $\int 1/(4+9x^2)\,dx$, use substitution first. Let $3x/2 = u$ then $3/2dx = du$, and

$$\begin{aligned}\int \frac{1}{4+9x^2}\,dx &= \frac{1}{4}\int \frac{1}{1+(\frac{3x}{2})^2}\,dx \\ &= \frac{1}{6}\int \frac{1}{1+u^2}\,du \\ &= \frac{1}{6}\tan^{-1}\frac{3}{2}x + C.\end{aligned}$$

Integral by parts

$$\frac{d}{dx}(uv) = u\frac{dv}{dx} + v\frac{du}{dx}.$$

Integrating w.r.t x

$$\begin{aligned}uv &= \int u\frac{dv}{dx}\,dx + \int v\frac{du}{dx}\,dx \\ &= \int u\,dv + \int v\,du.\end{aligned}$$

Thus

Proposition 8.1.4 (Integration by Parts I)**.**

$$\int u\,dv = uv - \int v\,du. \tag{8.1}$$

Proposition 8.1.5 (Integration by Parts II)**.**

$$\int f(x)g'(x)\,dx = f(x)g(x) - \int f'(x)g(x)\,dx. \tag{8.2}$$

Proposition 8.1.6 (Definite integral)**.**

$$\int_a^b f(x)g'(x)\,dx = [f(x)g(x)]_a^b - \int_a^b f'(x)g(x)\,dx.$$

Example 8.1.7. Find the following

(1) $\int_0^{\pi} x \sin x \, dx$

(2) $\int \ln x \, dx$.

sol. (1) Let $u = x$, $dv = \sin x \, dx$. Then $du = dx$, $v = -\cos x$. (Fig 8.1)

$$\begin{aligned}\int_0^{\pi} x \sin x \, dx &= [x(-\cos x)]_0^{\pi} - \int_0^{\pi} (-\cos x) \, dx \\ &= \pi + [\sin x]_0^{\pi} \\ &= \pi.\end{aligned}$$

(2) Let $u = \ln x$, $dv = dx$. Then we have $du = (1/x)dx$, $v = x$.

$$\begin{aligned}\int \ln x \, dx &= (\ln x)x - \int x \cdot \frac{1}{x} \, dx \\ &= x \ln x - x + C.\end{aligned}$$

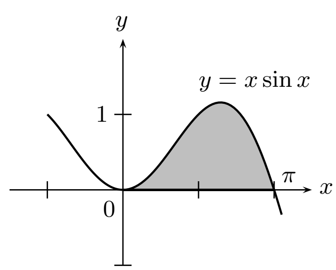


Figure 8.1:

Repeated integration by parts

Example 8.1.8. Find $\int x^2 \sin x \, dx$.

sol. Let $u = x^2$, $dv = \sin x \, dx$. Then $du = 2x dx$, $v = -\cos x$ and hence

$$\begin{aligned}\int x^2 \sin x \, dx &= x^2(-\cos x) - \int (-\cos x) 2x \, dx \\ &= -x^2 \cos x + \int 2x \cos x \, dx.\end{aligned}$$

f and its derivative		g and its integral
x^2	$(+)$	e^x
$2x$	$(-)$	e^x
2	$(+)$	e^x
0		e^x

Again, set $u = 2x$, $dv = \cos x\,dx$. Then $du = 2\,dx$, $v = \sin x$.

$$\begin{aligned}\int x^2 \sin x\,dx &= -x^2\cos x + 2x\sin x - \int 2\sin x\,dx \\ &= -x^2\cos x + 2x\sin x + 2\cos x + C.\end{aligned}$$

Example 8.1.9. Find $\int x^2 e^x\,dx$.

sol. $f(x) = x^2$, $g(x) = e^x$

f and its derivative		g and its integral
x^3	$(+)$	$\sin x$
$3x^2$	$(-)$	$-\cos x$
$6x$	$(+)$	$-\sin x$
6	$(-)$	$\cos x$
0		$\sin x$

Example 8.1.10. Find $\int x^3 \sin x\,dx$.

Use the table above

$$\int x^3 \sin x\,dx = -x^3\cos x + 3x^2\sin x + 6x\cos x - 6\sin x + C.$$

Example 8.1.11. Find $\int e^x \sin x\,dx$.

sol. If $u = e^x$, $dv = \sin x\,dx$, then $du = e^x\,dx$, $v = -\cos x$.

$$\begin{aligned}\int e^x \sin x\,dx &= e^x(-\cos x) - \int e^x(-\cos x)\,dx \\ &= -e^x\cos x + \int e^x\cos x\,dx.\end{aligned}$$

Again let $u = e^x$, $dv = \cos x\,dx$ so that $du = e^x\,dx$, $v = \sin x$.

$$\begin{aligned}\int e^x \sin x\,dx &= -e^x \cos x + \int e^x \cos x\,dx \\ &= -e^x \cos x + e^x \sin x - \int e^x \sin x\,dx.\end{aligned}$$

Solving this for $\int e^x \sin x\,dx$ we obtain

$$\int e^x \sin x\,dx = \frac{1}{2}e^x(\sin x - \cos x) + C.$$

■

Reduction formula

Example 8.1.12. Express $\int \cos^n x\,dx$ in terms of low power of $\cos x$.

sol.

$$\begin{aligned}\int \cos^{n-1} x \cos x\,dx &= \cos^{n-1}\sin x + (n-1)\int \sin^2 x \cos^{n-2} x\,dx \\ &= \cos^{n-1}\sin x + (n-1)\int (1-\cos^2 x)\cos^{n-2} x\,dx \\ &= \cos^{n-1}\sin x + (n-1)\int \cos^{n-2} x\,dx - (n-1)\int \cos^n x\,dx.\end{aligned}$$

So

$$n\int \cos^n x\,dx = \cos^{n-1}\sin x + (n-1)\int \cos^{n-2} x\,dx.$$

■

Example 8.1.13. Prove

$$\int (a^2 \pm x^2)^n\,dx = \frac{x(a^2 \pm x^2)}{2n+1} + \frac{2na^2}{2n+1}\int (a^2 \pm x^2)^{n-1}\,dx, \quad (n \neq -\frac{1}{2}).$$

sol. Integration by parts

$$\begin{aligned}\int (a^2 \pm x^2)^n\,dx &= x(a^2 \pm x^2)^n - \int x \cdot n(a^2 \pm x^2)^{n-1}(\pm 2x)\,dx \\ &= x(a^2 \pm x^2)^n - \int 2n(a^2 \pm x^2)^{n-1}(a^2 \pm x^2 - a^2)\,dx \\ &= x(a^2 \pm x^2)^n - 2n \int (a^2 \pm x^2)^n\,dx \\ &\quad + 2na^2 \int (a^2 \pm x^2)^{n-1}\,dx.\end{aligned}$$

If $n \neq -1/2$,

$$\int (a^2 \pm x^2)^n\,dx = \frac{x(a^2 \pm x^2)^n}{2n+1} + \frac{2na^2}{2n+1} \int (a^2 \pm x^2)^{n-1}\,dx.$$

8.2 Integration of Trigonometric function

Products of powers of Sines and Cosines

Integral of $\sin^m x \cos^n x$

(1) If m is odd, then set $m = 2k+1$ and use $\sin^2 x = 1 - \cos^2 x$ $\sin x\,dx = -d(\cos x)$ to transform it to

$$\int \sin^{2k+1} x \cos^n x\,dx = -\int (1 - \cos^2 x)^k \cos^n x\,d(\cos x).$$

(2) If n is odd $n = 2k+1$, use $\cos^2 x = 1 - \sin^2 x$ $\cos x\,dx = d(\sin x)$ to obtain

$$\int \sin^m x \cos^{2k+1} x\,dx = \int \sin^m x(1 - \sin^2 x)^k\,d(\sin x).$$

(3) If both m, n are even, use $\sin^2 x = (1 - \cos 2x)/2$, $\cos^2 x = (1 + \cos 2x)/2$ to lower the degree and repeat the previous technique.

Example 8.2.1. Find $\displaystyle\int \sin^5 x\,dx$.

sol. $\displaystyle\int \sin^5 x\,dx = -\int (1 - \cos^2 x)^2\,d(\cos x)$

$$= -\int (1 - 2\cos^2 x + \cos^4 x)\, d(\cos x)$$
$$= -\frac{1}{5}\cos^5 x + \frac{2}{3}\cos^3 x - \cos x + C.$$

$\square$

Example 8.2.2. Find $\int \sin^2 x \cos^3 x\, dx$.

sol.
$$\int \sin^2 x \cos^3 x\, dx = \int \sin^2 x(1 - \sin^2 x)\, d(\sin x)$$
$$= -\frac{1}{5}\sin^5 x + \frac{1}{3}\sin^3 x + C.$$

$\square$

Example 8.2.3. Find $\int \sin^4 x \cos^2 x\, dx$.

sol.
$$\int \sin^4 x \cos^2 x\, dx = \int \left(\frac{1-\cos 2x}{2}\right)^2 \left(\frac{1+\cos 2x}{2}\right) dx$$
$$= \frac{1}{8}\int \left(1 - 2\cos 2x + \cos^2 2x\right)\left(1 + \cos 2x\right)\, dx$$
$$= \frac{1}{8}\int \left(1 - \cos 2x - \cos^2 2x + \cos^3 2x\right)\, dx$$
$$= \frac{1}{8}\int \left(1 - \cos 2x - \frac{1+\cos 4x}{2} + (1 - \sin^2 2x)\cos 2x\right)\, dx$$
$$= \frac{1}{16}\int \left(1 - \cos 4x - \sin^2 2x \cdot 2\cos 2x\right)\, dx$$
$$= \frac{1}{16}\left(x - \frac{1}{4}\sin 4x - \frac{1}{3}\sin^3 2x\right) + C.$$

$\square$

Integral of $\sqrt{1 \pm \sin ax}$, $\sqrt{1 \pm \cos ax}$

Use the double angle formula.

$$\sin 2A = 2\sin A\cos A$$
$$\cos 2A = 2\cos^2 A - 1 = 1 - 2\sin^2 A.$$

Change the form $1 \pm \sin ax$, $1 \pm \cos ax$ to a complete square.

Example 8.2.4. Find $\int_0^\pi \sqrt{1 - \sin x}\, dx$.

sol. Use the identity:

$$1 - \sin x = 1 - 2\sin(\frac{x}{2})\cos(\frac{x}{2}) = \left(\sin(\frac{x}{2}) - \cos(\frac{x}{2})\right)^2.$$

$$\begin{aligned}
\int_0^\pi \sqrt{1-\sin x}\,dx &= \int_0^\pi \left|\sin\frac{x}{2} - \cos\frac{x}{2}\right|\,dx \\
&= \int_0^{\pi/2} \left(\cos\frac{x}{2} - \sin\frac{x}{2}\right) + \int_{\pi/2}^{\pi} \left(\sin\frac{x}{2} - \cos\frac{x}{2}\right)\,dx \\
&= \left[2\sin\frac{x}{2} + 2\cos\frac{x}{2}\right]_0^{\pi/2} + \left[-2\cos\frac{x}{2} - 2\sin\frac{x}{2}\right]_{\pi/2}^{\pi} \\
&= (\sqrt{2} + \sqrt{2} - 2) + (-2 + \sqrt{2} + \sqrt{2}) \\
&= 4(\sqrt{2} - 1).
\end{aligned}$$

■

Example 8.2.5. Find $\int_0^{\pi/2} \sqrt{1+\cos 2x}\,dx$.

sol. $1 + \cos 2x = 2\cos^2 x$,

$$\begin{aligned}
\int_0^{\pi/2} \sqrt{1+\cos 2x} &= \sqrt{2}\int_0^{\pi/2} |\cos x|\,dx \\
&= \sqrt{2}\,[\sin x]_0^{\pi/2} \\
&= \sqrt{2}.
\end{aligned}$$

■

Tangent and secant

Recall

$$\begin{aligned}
1 + \tan^2 x &= \sec^2 x, \\
(\tan x)' &= \sec^2 x, \\
(\sec x)' &= \sec x \tan x.
\end{aligned}$$

Example 8.2.6. $\int \sec x\,dx$.

sol. Multiply $\sec x + \tan x$.

$$\begin{aligned}\int \sec x\,dx &= \int \frac{\sec x(\sec x + \tan x)}{\sec x + \tan x}\,dx\\ &= \int \frac{(\sec x + \tan x)'}{\sec x + \tan x}\,dx\\ &= \ln|\sec x + \tan x| + C.\end{aligned}$$

■

Example 8.2.7. $\int \tan^2 x \sec x\,dx$.

sol. Since $\tan^2 x \sec x = (\sec^2 x - 1)\sec x = \sec^3 x - \sec x$, we can find $\int \sec^3 x\,dx$. Let $u = \sec x$, $dv = \sec^2 x\,dx$ then $v = \tan x$, $du = \sec x \tan x\,dx$, we have

$$\begin{aligned}\int \sec^3 x\,dx &= \sec x \tan x - \int (\tan x)\sec x \tan x\,dx\\ &= \sec x \tan x - \int (\sec^2 x - 1)\sec x\,dx\\ &= \sec x \tan x - \int \sec^3 x\,dx + \int \sec x\,dx.\end{aligned}$$

Hence we obtain

$$\int \sec^3 x\,dx = \frac{1}{2}\sec x \tan x + \frac{1}{2}\int \sec x\,dx.$$

Hence

$$\begin{aligned}\int \tan^2 x \sec x\,dx &= \int \sec^3 x\,dx - \int \sec x\,dx\\ &= \frac{1}{2}\sec x \tan x - \frac{1}{2}\int \sec x\,dx\\ &= \frac{1}{2}\sec x \tan x - \frac{1}{2}\ln|\sec x + \tan x| + C.\end{aligned}$$

■

Example 8.2.8. $\int \tan^6 x\,dx$.

sol. Since $\tan^2 x = \sec^2 x - 1$

$$
\begin{aligned}
\int \tan^6 x\,dx &= \int \tan^4 x(\sec^2 x - 1)\,dx \\
&= \int \tan^4 x \sec^2 x\,dx - \int \tan^4 x\,dx \\
&= \int \tan^4 x \sec^2 x\,dx - \int \tan^2 x(\sec^2 x - 1)\,dx \\
&= \int \tan^4 x \sec^2 x\,dx - \int \tan^2 x \sec^2 x\,dx + \int \tan^2 x\,dx \\
&= \int \tan^4 x \sec^2 x\,dx - \int \tan^2 x \sec^2 x\,dx + \int (\sec^2 x - 1)\,dx \\
&= \frac{1}{5}\tan^5 x - \frac{1}{3}\tan^3 x + \tan x - x + C.
\end{aligned}
$$

□

Remark 8.2.9. For $\cot x$ or $\csc x$, use

$$
\begin{aligned}
1 + \cot^2 x &= \csc^2 x, \\
(\cot x)' &= -\csc^2 x, \\
(\csc x)' &= -\csc x \cot x.
\end{aligned}
$$

Products such as $\sin mx \sin nx$, $\sin mx \cos nx$, $\cos mx \cos nx$

Addition formula:

$$
\begin{aligned}
\sin(A + B) &= \sin A \cos B + \cos A \sin B \\
\sin(A - B) &= \sin A \cos B - \cos A \sin B \\
\cos(A + B) &= \cos A \cos B - \sin A \sin B \\
\cos(A - B) &= \cos A \cos B + \sin A \sin B.
\end{aligned}
$$

From these we get(with $A = mx$, $B = nx$)

$$
\begin{aligned}
\sin mx \sin nx &= \frac{1}{2}[\cos(m - n)x - \cos(m + n)x] \\
\sin mx \cos nx &= \frac{1}{2}[\sin(m - n)x + \sin(m + n)x] \\
\cos mx \cos nx &= \frac{1}{2}[\cos(m - n)x + \cos(m + n)x].
\end{aligned}
$$

Example 8.2.10. $\int_0^{\pi/6} \sin 4x \sin 3x \, dx$.

sol.

$$\begin{aligned}\int_0^{\pi/6} \sin 4x \sin 3x \, dx &= \frac{1}{2}\int_0^{\pi/6} (\cos x - \cos 7x) \, dx \\ &= \frac{1}{2}\left[\sin x - \frac{1}{7}\sin 7x\right]_0^{\pi/6} = \frac{2}{7}.\end{aligned}$$

8.3 Trig Substitution

Quadratic term

For the terms of the forms $a^2 - u^2$, $a^2 + u^2$ $u^2 - a^2$, we can try to substitute $u = a\sin\theta$, $u = a\tan\theta$, $u = a\sec\theta$ resp.

$$a^2 - u^2 = a^2 - a^2\sin^2\theta = a^2(1 - \sin^2\theta) = a^2\cos^2\theta \tag{8.3}$$
$$a^2 + u^2 = a^2 + a^2\tan^2\theta = a^2(1 + \tan^2\theta) = a^2\sec^2\theta \tag{8.4}$$
$$u^2 - a^2 = a^2\sec^2\theta - a^2 = a^2(\sec^2\theta - 1) = a^2\tan^2\theta. \tag{8.5}$$

Note the domain of definition

(1) $u = a\sin\theta$ is defined on $-\pi/2 \le \theta \le \pi/2$.

(2) $u = a\tan\theta$ $\theta = \tan^{-1}(u/a)$ on $-\pi/2 < \theta < \pi/2$.

(3) $u = a\sec\theta$ $\theta = \sec^{-1}(u/a)$ Since $|u| \ge a$ $0 \le \theta < \pi/2$ (if $u \ge a$), or $\pi/2 < \theta \le \pi$ (if $u \le -a$).

Example 8.3.1. $\int \frac{du}{a^2 + u^2}$.

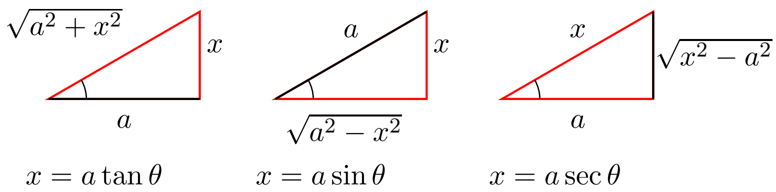


Figure 8.2: trig substitution

sol. Use substitution $u = a\tan\theta$, $du = a\sec^2\theta\, d\theta$ to get

$$\begin{aligned}\int \frac{du}{a^2+u^2} &= \int \frac{a\sec^2\theta\, d\theta}{a^2\sec^2\theta}\\ &= \int \frac{d\theta}{a}\\ &= \frac{\theta}{a} + C\\ &= \frac{1}{a}\tan^{-1}\frac{u}{a} + C.\end{aligned}$$

Example 8.3.2. Find $\int \sqrt{a^2-u^2}\, du$, $(a>0)$.

sol. Use $u = a\sin\theta$, $du = a\cos\theta\, d\theta$ to get

$$\begin{aligned}\int \sqrt{a^2-u^2}\, du &= \int a\cos\theta\cdot a\cos\theta\, d\theta\\ &= \frac{a^2}{2}\int (1+\cos 2\theta)\, d\theta\\ &= \frac{a^2}{2}\left(\theta + \frac{\sin 2\theta}{2}\right) + C\\ &= \frac{a^2}{2}(\theta + \sin\theta\cos\theta) + C\\ &= \frac{a^2}{2}\left(\sin^{-1}\frac{u}{a} + \frac{u}{a}\sqrt{1-\frac{u^2}{a^2}}\right) + C\\ &= \frac{a^2}{2}\sin^{-1}\frac{u}{a} + \frac{1}{2}u\sqrt{a^2-u^2} + C.\end{aligned}$$

Example 8.3.3. Find $\displaystyle\int \frac{du}{\sqrt{u^2-a^2}}$, $(|u| > a > 0)$.

sol. Let $u = a\sec\theta$

$$\begin{aligned} u^2 - a^2 &= a^2(\sec^2\theta - 1) \\ &= a^2\tan^2\theta, \\ du &= a\sec\theta\tan\theta\, d\theta. \end{aligned}$$

Then

$$\begin{aligned} \int \frac{du}{\sqrt{u^2-a^2}} &= \int \frac{a\sec\theta\tan\theta\, d\theta}{a|\tan\theta|} \\ &= \begin{cases} \displaystyle\int \sec\theta\, d\theta & (0<\theta<\pi/2) \\ \displaystyle -\int \sec\theta\, d\theta & (\pi/2<\theta<\pi) \end{cases} \\ &= \begin{cases} \ln|\sec\theta+\tan\theta| + C & (0<\theta<\pi/2) \\ -\ln|\sec\theta+\tan\theta| + C & (\pi/2<\theta<\pi) \end{cases} \\ &= \begin{cases} \ln\left|\dfrac{u}{a} + \dfrac{\sqrt{u^2-a^2}}{a}\right| + C & (u>a) \\ \\ -\ln\left|\dfrac{u}{a} - \dfrac{\sqrt{u^2-a^2}}{a}\right| + C & (u<-a). \end{cases} \end{aligned}$$

On the other hand,

$$\ln\left|\frac{u}{a} + \frac{\sqrt{u^2-a^2}}{a}\right| = \ln\left|u + \sqrt{u^2-a^2}\right| - \ln a.$$

$$
\begin{aligned}
-\ln\left|\frac{u}{a}-\frac{\sqrt{u^2-a^2}}{a}\right| &= \ln\left|\frac{a}{u-\sqrt{u^2-a^2}}\right| \\
&= \ln\left|\frac{a(u+\sqrt{u^2-a^2})}{(u-\sqrt{u^2-a^2})(u+\sqrt{u^2-a^2})}\right| \\
&= \ln\left|\frac{a(u+\sqrt{u^2-a^2})}{a^2}\right| \\
&= \ln\left|\frac{u+\sqrt{u^2-a^2}}{a}\right| \\
&= \ln\left|u+\sqrt{u^2-a^2}\right| - \ln a.
\end{aligned}
$$

Hence

$$\int \frac{du}{\sqrt{u^2-a^2}} = \ln\left|u+\sqrt{u^2-a^2}\right| + C'.$$

■

Example 8.3.4. $\int \frac{dx}{\sqrt{x^2+9}}$.

sol. Let $x = 3\tan\theta$ $(-\pi/2 < \theta < \pi/2)$, $dx = 3\sec^2\theta\, d\theta$,

$$
\begin{aligned}
\int \frac{dx}{\sqrt{x^2+9}} &= \int \frac{3\sec^2\theta}{3\sec\theta}\, d\theta \\
&= \int \sec\theta\, d\theta \\
&= \ln|\sec\theta + \tan\theta| + C \\
&= \ln\left|\sqrt{\left(\frac{x}{3}\right)^2+1}+\frac{x}{3}\right| + C \\
&= \ln\left|x+\sqrt{x^2+9}\right| + C.
\end{aligned}
$$

■

Involving ax^2+bx+c — Completing the square

For factors like ax^2+bx+c, $(a, b \neq 0)$, use $u = x + b/(2a)$ to get $ax^2+bx+c = a(u^2 \pm p^2)$.

Example 8.3.5. Find $\int \sqrt{2x - x^2}\, dx$.

sol. Since $2x - x^2 = 1 - (x-1)^2$ $u = x - 1$ we have as in example 8.3.2 with $a = 1$,

$$\begin{aligned}\int \sqrt{2x - x^2}\, dx &= \int \sqrt{1 - u^2}\, du \\ &= \frac{1}{2}\sin^{-1} u + \frac{1}{2}u\sqrt{1-u^2} + C \\ &= \frac{1}{2}\sin^{-1}(x-1) + \frac{1}{2}(x-1)\sqrt{2x - x^2} + C.\end{aligned}$$

■

Example 8.3.6. $\int \frac{dx}{x^2 + x + 1}$.

sol. $x^2 + x + 1 = (x + 1/2)^2 + 3/4$ $u = x + 1/2$ $a = \sqrt{3}/2$

$$\begin{aligned}\int \frac{dx}{x^2 + x + 1} &= \int \frac{du}{u^2 + 3/4} \\ &= \frac{2}{\sqrt{3}}\tan^{-1}\frac{2u}{\sqrt{3}} + C \\ &= \frac{2}{\sqrt{3}}\tan^{-1}\frac{2x+1}{\sqrt{3}} + C.\end{aligned}$$

■

8.4 Integration of Rational functions by partial fraction

When $p(x)$, $q(x)$ are rational functions, we can always write it as

$$\frac{p(x)}{q(x)} = Q(x) + \frac{r(x)}{q(x)}$$

for some polynomial $Q(x), r(x)$, where the degree of $r(x)$ is less than that of $q(x)$. (i.e., the fraction must be proper)

Distinct linear factors

Suppose $\alpha_1, \ldots, \alpha_r$ are distinct and $p(x)$ is polynomial of degree of is less than r. Then we can set

$$\frac{p(x)}{(x-\alpha_1)\cdots(x-\alpha_r)} = \frac{A_1}{x-\alpha_1} + \cdots + \frac{A_r}{x-\alpha_r}. \tag{8.6}$$

Here A_i's can be obtained by method of undetermined coefficients.(There is another method, called Heaviside cover up method, see below)

$$\int \frac{dx}{(x-\alpha_1)\cdots(x-\alpha_r)} = \sum_{i=1}^{r} A_i \ln|x-\alpha_i| + C.$$

Example 8.4.1. Find $\displaystyle\int \frac{x+1}{x(x+2)}\,dx$.

sol. One can find the following partial fraction

$$\frac{x+1}{x(x+2)} = \frac{1}{2x} + \frac{1}{2(x+2)}.$$

$$\begin{aligned}\int \frac{x+1}{x(x+2)}\,dx &= \frac{1}{2}\int \left(\frac{1}{x} + \frac{1}{x+2}\right) dx \\ &= \frac{1}{2}\ln|x(x+2)| + C.\end{aligned}$$

■

Example 8.4.2. Find $\displaystyle\int \frac{2x+1}{x^3-x}\,dx$.

sol. Since $x^3 - x = x(x-1)(x+1)$ we can set

$$\frac{2x+1}{x^3-x} = \frac{A}{x} + \frac{B}{x-1} + \frac{C}{x+1}.$$

Solving for A, B, C we get $A = -1,\ B = 3/2,\ C = -1/2$. Hence

$$\begin{aligned}\int \frac{2x+1}{x^3-x}\,dx &= \int \left(\frac{-1}{x} + \frac{3/2}{x-1} + \frac{-1/2}{x+1}\right) dx \\ &= -\ln|x| + \frac{3}{2}\ln|x-1| - \frac{1}{2}\ln|x+1| + C.\end{aligned}$$

Repeated linear factor

Assume the degree of $p(x)$ is less than that of $r(x)$. Then

$$\frac{p(x)}{(x-\alpha)^r} = \frac{A_1}{x-\alpha} + \frac{A_2}{(x-\alpha)^2} + \cdots + \frac{A_r}{(x-\alpha)^r}.$$

To find the coefficients $A_1, A_2, \ldots, A_r$, multiply $(x-\alpha)^r$. Then

$$p(x) = A_1(x-\alpha)^{r-1} + A_2(x-\alpha)^{r-1} + \cdots + A_r.$$

Now use method of undetermined coefficients to find A_i's. Another nice way of finding $A_i's$ by derivative will be introduced below. Once $A_i's$ are known, we can find the integral:

$$\begin{aligned}\int \frac{p(x)}{(x-\alpha)^r}\,dx &= \int \left(\frac{A_1}{x-\alpha} + \frac{A_2}{(x-\alpha)^2} + \cdots + \frac{A_r}{(x-\alpha)^r}\right) dx \\ &= A_1 \ln|x-\alpha| - \frac{A_2}{x-\alpha} - \cdots - \frac{(r-1)A_r}{(x-\alpha)^{r-1}} + C.\end{aligned}$$

Example 8.4.3. Find $\displaystyle\int \frac{x^2}{(x-2)^3}\,dx$.

sol. Since $x^2 = (x-2)^2 + 4(x-2) + 4$, we have

$$\frac{x^2}{(x-2)^3} = \frac{1}{x-2} + \frac{4}{(x-2)^2} + \frac{4}{(x-2)^3}.$$

Hence

$$\begin{aligned}\int \frac{x^2}{(x-2)^3}\,dx &= \int \left(\frac{1}{x-2} + \frac{4}{(x-2)^2} + \frac{4}{(x-2)^3}\right) dx \\ &= \ln|x-2| - \frac{4}{x-2} - \frac{8}{(x-2)^2} + C.\end{aligned}$$

Irreducible quadratic factor

Suppose $x^2+\beta_1 x+\gamma_1,\dots,x^2+\beta_r x+\gamma_r$ are distinct quadratic factor without having real roots(we say irreducible quadratic factor). Suppose $p(x)$ is polynomial of degree less than $2r$. So we have

$$\frac{p(x)}{(x^2+\beta_1 x+\gamma_1)\cdots(x^2+\beta_r x+\gamma_r)}=\sum_{i=1}^{r}\frac{B_i x+C_i}{x^2+\beta_i x+\gamma_i}$$

for some $B_1,\dots,B_r$ and $C_1,\dots,C_r$. Hence

$$\int\frac{p(x)}{(x^2+\beta_1 x+\gamma_1)\cdots(x^2+\beta_r x+\gamma_r)}\,dx=\sum_{i=1}^{r}\int\frac{B_i x+C_i}{x^2+\beta_i x+\gamma_i}\,dx.$$

Again we can find the coefficients by method of undetermined coefficients. Now since

$$\begin{aligned}B_i x+C_i&=\frac{B_i}{2}(2x+\beta_i)+D_i,\ (D_i=C_i-B_i\beta_i/2)\\&=\frac{B_i}{2}(x^2+\beta_i x+\gamma_i)'+D_i,\end{aligned}$$

we have

$$\begin{aligned}\int\frac{B_i x+C_i}{x^2+\beta_i x+\gamma_i}\,dx&=\int\left(\frac{B_i}{2}\frac{(x^2+\beta_i x+\gamma_i)'}{x^2+\beta_i x+\gamma_i}+\frac{D_i}{x^2+\beta_i x+\gamma_i}\right)dx\\&=\frac{B_i}{2}\ln(x^2+\beta_i x+\gamma_i)+\int\frac{D_i}{x^2+\beta_i x+\gamma_i}\,dx.\end{aligned}$$

For $D_i/(x^2+\beta_i x+\gamma_i)$ use the formula:

$$\int\frac{du}{u^2+a^2}=\frac{1}{a}\tan^{-1}\frac{u}{a}+C.$$

Example 8.4.4. Find $\displaystyle\int\frac{2x}{x^4+x^2+1}\,dx$.

sol. Since $x^4+x^2+1=(x^2-x+1)(x^2+x+1)$, we set

$$\frac{2x}{x^4+x^2+1}=\frac{B_1 x+C_1}{x^2-x+1}+\frac{B_2 x+C_2}{x^2+x+1}.$$

By comparing, we obtain $B_1=B_2=0$, $C_1=1$, $C_2=-1$. Since

$$x^2\pm x+1=(x\pm 1/2)^2+(\sqrt{3}/2)^2,$$

we see

$$\begin{aligned}&\int \frac{2x}{x^4+x^2+1}\,dx\\ &\quad=\int\left(\frac{1}{(x-1/2)^2+(\sqrt{3}/2)^2}-\frac{1}{(x+1/2)^2+(\sqrt{3}/2)^2}\right)dx\\ &\quad=\frac{2}{\sqrt{3}}\left(\tan^{-1}\frac{2x-1}{\sqrt{3}}-\tan^{-1}\frac{2x+1}{\sqrt{3}}\right)+C.\end{aligned}$$

$\square$

Repeated irreducible quadratic factor

Suppose $p(x)$ is polynomial of degree less than $2r$, and $x^2+\beta x+\gamma$ does not have real roots. Then we can set

$$\frac{p(x)}{(x^2+\beta x+\gamma)^r}=\frac{B_1x+C_1}{x^2+\beta x+\gamma}+\frac{B_2x+C_2}{(x^2+\beta x+\gamma)^2}+\cdots+\frac{B_rx+C_r}{(x^2+\beta x+\gamma)^r}$$

for some $B_1, B_2, \ldots, B_r, C_1, C_2, \ldots, C_r$. Then

$$\begin{aligned}&\int \frac{p(x)}{(x^2+\beta x+\gamma)^r}\,dx\\ &\quad=\int\left(\frac{B_1x+C_1}{x^2+\beta x+\gamma}+\frac{B_2x+C_2}{(x^2+\beta x+\gamma)^2}+\cdots+\frac{B_rx+C_r}{(x^2+\beta x+\gamma)^r}\right)dx.\end{aligned}$$

By the same way as before we see, with $D_i=C_i-B_i\beta/2$

$$\begin{aligned}\int\frac{B_ix+C_i}{(x^2+\beta x+\gamma)^i}\,dx&=\int\left(\frac{B_i}{2}\frac{(x^2+\beta x+\gamma)'}{(x^2+\beta x+\gamma)^i}+\frac{D_i}{(x^2+\beta x+\gamma)^i}\right)dx\\ &=-\frac{B_i}{2(i-1)(x^2+\beta x+\gamma)^{i-1}}+\int\frac{D_i}{(x^2+\beta x+\gamma)^i}\,dx.\end{aligned}$$

For the integral of $D_i/(x^2+\beta x+\gamma)^i$ $(i\geq 2)$, use the recurrence relation

$$\int\frac{du}{(u^2+a^2)^i}=\frac{u}{a^2(2i-2)(u^2+a^2)^{i-1}}+\frac{2i-3}{a^2(2i-2)}\int\frac{du}{(u^2+a^2)^{i-1}}.$$

Example 8.4.5. Find $\displaystyle\int\frac{x^4+2x^3+5x^2+6}{(x^2+2)^3}\,dx.$

sol.

$$\frac{x^4+2x^3+5x^2+6}{(x^2+2)^3} = \frac{A_1x+B_1}{x^2+2} + \frac{A_2x+B_2}{(x^2+2)^2} + \frac{A_3x+B_3}{(x^2+2)^3}.$$

Multiply $(x^2+2)^3$ to see

$$\begin{aligned} x^4+2x^3+5x^2+6 &= A_1x^5+B_1x^4+(4A_1+A_2)x^3+(4B_1+B_2)x^2 \\ &\quad + (4A_1+2A_2+A_3)x+4B_1+2B_2+B_3. \end{aligned}$$

Comparing, we get $A_1=0$, $A_2=2$, $A_3=-2$, $B_1=1$, $B_2=1$, $B_3=0$. Hence the integrand is

$$\frac{x^4+2x^3+5x^2+6}{(x^2+2)^3} = \frac{1}{x^2+2} + \frac{2x+1}{(x^2+2)^2} + \frac{-4x}{(x^2+2)^3}.$$

Hence

$$\begin{aligned} &\int \frac{x^4+2x^3+5x^2+6}{(x^2+2)^3}\,dx \\ &= \int \frac{dx}{x^2+2} + \int \frac{2x}{(x^2+2)^2}\,dx + \int \frac{1}{(x^2+2)^2}\,dx + \int \frac{-4x}{(x^2+2)^3}\,dx \\ &= \frac{1}{\sqrt{2}}\tan^{-1}\frac{x}{\sqrt{2}} - \frac{1}{x^2+2} + \frac{x}{4(x^2+2)} + \frac{1}{4}\int \frac{1}{x^2+2}\,dx + \frac{1}{(x^2+2)^2} \\ &= \frac{5}{4\sqrt{2}}\tan^{-1}\frac{x}{\sqrt{2}} + \frac{x-4}{4(x^2+2)} + \frac{1}{(x^2+2)^2} + C. \end{aligned}$$

Heaviside cover up method for linear factors

Example 8.4.6.

$$\frac{x^2+1}{(x-1)(x-2)(x-3)} = \frac{A}{x-1} + \frac{B}{x-2} + \frac{C}{x-3}.$$

Here

$$A = \frac{(1)^2+1}{\underbrace{(x-1)}_{cover}(1-2)(1-3)}$$

$$B = \frac{(2)^2+1}{(2-1)\boxed{(x-2)}_{cover}(2-3)} = \frac{5}{(1)(-1)} = -5$$

$$C = \frac{(3)^2+1}{(3-1)(3-2)\boxed{(x-3)}_{cover}} = \frac{10}{(2)(1)} = 5.$$

Example 8.4.7. Do the same with

$$\int \frac{x+4}{x(x-2)(x+5)}.$$

sol. Note

$$\begin{aligned} \frac{x+4}{x(x-2)(x+5)} &= \frac{A}{x} + \frac{B}{x-2} + \frac{C}{x+5} \\ A &= \frac{0+4}{\boxed{x}(0-2)(0+5)} = -\frac{2}{5} \\ B &= \frac{2+4}{2\boxed{(x-2)}(2+5)} = \frac{3}{7} \\ C &= \frac{-5+4}{(-5)(-5-2)\boxed{(x+5)}} = -\frac{1}{35}. \end{aligned}$$

Using differentiation-repeated factors

Example 8.4.8.

$$\frac{x-1}{(x+1)^3} = \frac{A}{x+1} + \frac{B}{(x+1)^2} + \frac{C}{(x+1)^3}.$$

Write

$$x-1 = A(x+1)^2 + B(x+1) + C.$$

Substitute $x=-1$ to get $C=-2$. Then take derivative

$$1 = 2A(x+1) + B$$

and substitute $x=-1$ to get $B=1$. Finally, taking derivative again, we see $A=0$.

8.5 Numerical Integration

Trapezoidal Rule

$$\Delta x = h = \frac{b-a}{n}.$$

$$x_0 = a, x_1 = a + \Delta x, \cdots, x_{n-1} = a + (n-1)\Delta x, x_n = b.$$

With $y_{x_i} = f(x_i)$

$$\int_a^b f(x)\,dx \approx \frac{h}{2}(y_0 + 2y_1 + \cdots + 2y_{n-1} + y_n).$$

$$|E_T| \leq \frac{M(b-a)^3}{12n^2}.$$

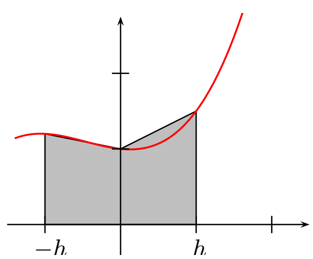


Figure 8.3: Trapezoidal Rule

Simpson's Rule

Replace the definite integral by an integral of quadratic interpolation. Exact for poly. of degree three. Assume $y = Ax^2 + Bx + C$ is an interpolating polynomial of f in the sense that $y(x_i) = f(x_i)$ for $x_0 = -h, x_1 = 0, x_2 = h$

$$\begin{aligned}
\int_{-h}^{h} f(x)\,dx &\approx \int_{-h}^{h} (Ax^2 + Bx + C)\,dx \\
&= \left.\frac{Ax^3}{3} + \frac{Bx^2}{2} + Cx\right]_{-h}^{h} \\
&= \frac{2Ah^3}{3} + 2Ch = \frac{h}{3}(2Ah^2 + 6C).
\end{aligned}$$

Since

$$y_0 = Ah^2 - Bh + C,\ y_1 = C,\ y_2 = Ah^2 + Bh + C$$

we see

$$A = \frac{y_0 - 2y_1 + y_2}{2h^2},\ B = \frac{y_2 - y_0}{2h},\ C = y_1$$

and the the integral is

$$\frac{h}{3}(y_0 - 2y_1 + y_2 + 6y_1) = \frac{h}{3}(y_0 + 4y_1 + y_2).$$

Since this formula is exact for x^3, it is in general third order formula. When

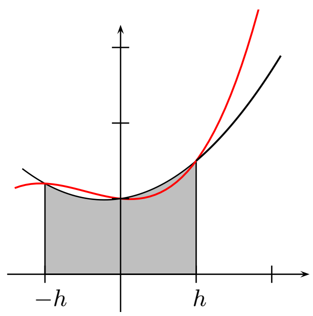


Figure 8.4: Simpson's Rule

the general interval $[a, b]$ is divided by an even number of intervals, we can apply it repeatedly to get

$$\int_a^b f(x)\,dx \approx \frac{h}{3}(y_0 + 4y_1 + 2y_2 + 4y_3 + \cdots + 2y_{n-2} + 4y_{n-1} + y_n).$$

$$|E_S| \le \frac{M(b-a)^5}{180n^4}.$$

Example 8.5.1. Find an upper bound for the error in estimating $\int_0^2 5x^4\,dx$ using Simpson's rule with $n = 5$.

sol. Let $f(x) = 5x^4$. Then $f^{(4)} = 120$. So $M = 120$. $b - a = 2$ and $n = 4$. The error bound is

$$|E_S| \le \frac{M(b-a)^5}{180n^4} = \frac{120(2)^5}{1804^4} = \frac{1}{12}.$$

Example 8.5.2. What is the minimum number of intervals needed to approximate above example using the Simpson's rule with an error less than 10^{-4}.

sol. We set

$$\frac{M(b-a)^5}{180n^4} < 10^{-4}.$$

Then

$$\begin{aligned} \frac{120(2)^5}{180n^4} &< 10^{-4} \\ n^4 &> \frac{64(10)^4}{3} \\ n &> 10\left(\frac{64}{3}\right)^{1/4} \approx 21.5. \end{aligned}$$

■

8.6 Improper Integral

Improper Integral

Example 8.6.1. Find the area surrounded by $y = 1/\sqrt{x}$, x-axis, y-axis, $x = 1$(fig 8.5).

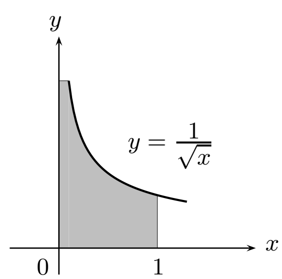


Figure 8.5: Improper Integral

sol. The function $1/\sqrt{x}$ is not defined at $x = 0$. So one cannot define the area like

$$\int_0^1 \frac{dx}{\sqrt{x}}.$$

But we can use limit such as

$$\begin{aligned}(\text{Area}) &= \lim_{\varepsilon\to 0^+}\int_\varepsilon^1 \frac{dx}{\sqrt{x}}\\ &= \lim_{\varepsilon\to 0^+}\left[2x^{1/2}\right]_\varepsilon^1\\ &= \lim_{\varepsilon\to 0^+}\left(2-2\varepsilon^{1/2}\right)\\ &= 2.\end{aligned}$$

□

Computation of Improper integral

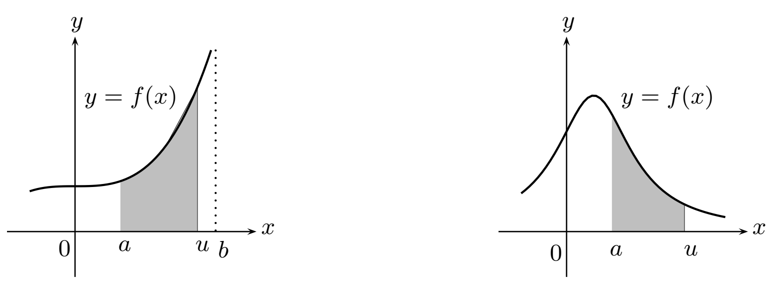


Figure 8.6: Improper integral on $[a, b)$

Definition 8.6.2 (Convergence of Improper integral)**.**

(1) Suppose $f(x)$ is integrable on all closed subinterval of $[a, b)$ and we have either $\lim_{x\to b^-} f(x) = \pm\infty$. If the limit

$$L = \lim_{u\to b^-}\int_a^u f(x)\,dx \tag{8.7}$$

exists then we say the **improper integral converges** and write its limit

$$\int_a^b f(x)\,dx = \lim_{u\to b^-}\int_a^u f(x)\,dx.$$

(2) The same definition holds when $\lim_{x\to a^+} f(x) = \pm\infty$. We write

$$\int_a^b f(x)\,dx = \lim_{\ell\to a^+}\int_\ell^b f(x)\,dx \tag{8.8}$$

if the latter limit exists. Otherwise, we say the integral **diverges**.

(3) The discontinuity can happen at an interior point. In this case, we can still apply the above definitions.

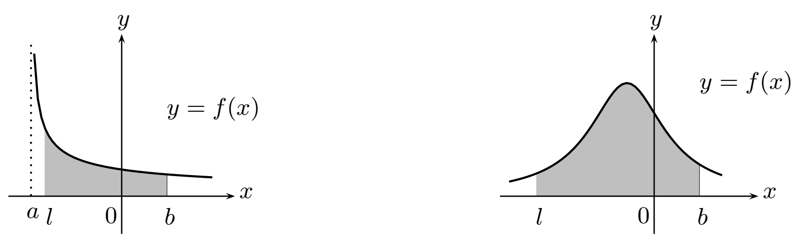


Figure 8.7: Improper integral on $(a, b]$

Example 8.6.3. $\displaystyle\int_{-1}^{1} \frac{1}{\sqrt{1-x^2}}\,dx.$

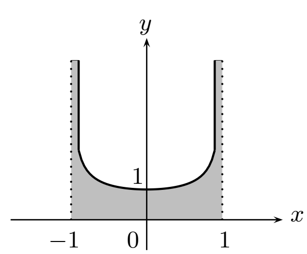


Figure 8.8:

sol. We distinguish two case: $(-1,0]$ and $[0,1)$.

$$\begin{aligned}\int_{-1}^{0}\frac{1}{\sqrt{1-x^2}}\,dx &= \lim_{\ell\to-1^+}\int_{\ell}^{0}\frac{1}{\sqrt{1-x^2}}\,dx\\ &= \lim_{\ell\to-1^+}\left[\sin^{-1}x\right]_{\ell}^{0}\\ &= -\sin^{-1}(-1)\\ &= \frac{\pi}{2}.\\ \int_{0}^{1}\frac{1}{\sqrt{1-x^2}}\,dx &= \lim_{u\to1^-}\int_{0}^{u}\frac{1}{\sqrt{1-x^2}}\,dx\\ &= \lim_{u\to1^-}\left[\sin^{-1}x\right]_{0}^{u}\\ &= \sin^{-1}1\\ &= \frac{\pi}{2}.\end{aligned}$$

Hence

$$\int_{-1}^{1}\frac{1}{\sqrt{1-x^2}}\,dx = \int_{-1}^{0}\frac{1}{\sqrt{1-x^2}}\,dx + \int_{0}^{1}\frac{1}{\sqrt{1-x^2}}\,dx = \pi.$$

Example 8.6.4. $\displaystyle\int_0^2 \frac{dx}{(x-1)^{4/3}}$.

sol. The funciton $1/(x-1)^{4/3}$ is not defined at $x=1$. Hence we separate

$$\int_0^2 \frac{dx}{(x-1)^{4/3}} = \int_0^1 \frac{dx}{(x-1)^{4/3}} + \int_1^2 \frac{dx}{(x-1)^{4/3}}.$$

$$\begin{aligned}\int_0^1 \frac{dx}{(x-1)^{4/3}} &= \lim_{u\to1^-}\int_0^u \frac{dx}{(x-1)^{4/3}}\\ &= \lim_{u\to1^-}\left[-3(x-1)^{-1/3}\right]_0^u\\ &= \lim_{u\to1^-} -\frac{3}{(u-1)^{1/3}} - 3\\ &= \infty.\end{aligned}$$

Since $\displaystyle\int_0^1 \frac{dx}{(x-1)^{4/3}}$ diverges the integral diverges regardless of $\displaystyle\int_1^2 \frac{dx}{(x-1)^{4/3}}$.

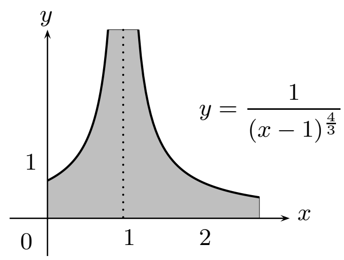


Figure 8.9:

On $(0, 1]$

Example 8.6.5. Find $\displaystyle\int_0^1 \frac{dx}{x^p}$ $(p > 0)$.

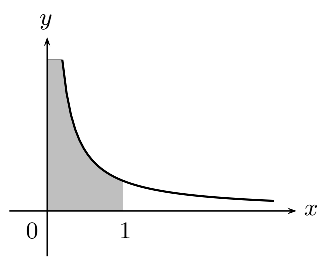


Figure 8.10: On $(0, 1]$

sol.

(1) For $0 < p < 1$

$$\int_0^1 \frac{dx}{x^p} = \lim_{\ell \to 0^+} \int_\ell^1 \frac{dx}{x^p} = \lim_{\ell \to 0^+} \left[\frac{x^{1-p}}{1-p} \right]_\ell^1 = \lim_{\ell \to 0^+} \frac{1 - \ell^{1-p}}{1-p} = \frac{1}{1-p}.$$

(2) For $p = 1$

$$\int_0^1 \frac{dx}{x^p} = \lim_{\ell \to 0^+} \int_1^u \frac{dx}{x} = \lim_{\ell \to 0^+} [\ln x]_\ell^1 = \lim_{\ell \to 0^+} (-\ln \ell) = \infty.$$

(3) For $p > 1$

$$\int_0^1 \frac{dx}{x^p} = \lim_{\ell \to 0^+} \int_\ell^1 \frac{dx}{x^p} = \lim_{\ell \to 0^+} \left[\frac{x^{1-p}}{1-p} \right]_\ell^1 = \lim_{\ell \to 0^+} \frac{1 - \ell^{1-p}}{1-p} = \infty.$$

■

The Case when a or b is ∞

Definition 8.6.6 (Convergence of Improper integral)**.**

(1) Suppose $f(x)$ is continuous on $[a, \infty)$. We set

$$\int_a^\infty f(x)\,dx = \lim_{b \to \infty} \int_a^b f(x)\,dx \tag{8.9}$$

provided the limit exists.

(2) Similarly, if $f(x)$ is continuous on $(-\infty, b]$, we set

$$\int_{-\infty}^b f(x)\,dx = \lim_{a \to -\infty} \int_a^b f(x)\,dx \tag{8.10}$$

provided the limit exists.

(3) If $f(x)$ is continuous on $(-\infty, \infty)$ then we set

$$\int_{-\infty}^b f(x)\,dx = \lim_{a \to -\infty} \int_a^b f(x)\,dx \tag{8.11}$$

provided the limit exists. In these cases, we say the **improper integral converges**. Otherwise, we say the integral **diverges**.

The function $1/x^p$

The integral of $1/x^p$ on $(0, 1]$ or $[1, \infty)$ depends on the value of p. In particular, the integral on $[1, \infty)$ is used to decide the convergence of the series $\sum 1/n^p$.

On $[1, \infty)$

Example 8.6.7. Find $\displaystyle\int_1^\infty \frac{dx}{x^p}$ $(p > 0)$.

sol.

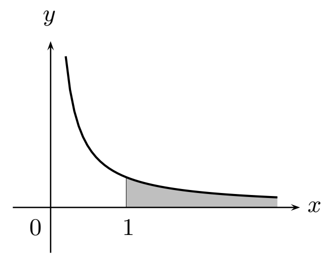


Figure 8.11: Improper integral on $[1, \infty)$

(1) For $0 < p < 1$,

$$\int_1^\infty \frac{dx}{x^p} = \lim_{u\to\infty} \int_1^u \frac{dx}{x^p} = \lim_{u\to\infty} \left[\frac{x^{1-p}}{1-p}\right]_1^u = \lim_{u\to\infty} \frac{u^{1-p}-1}{1-p} = \infty.$$

(2) For $p = 1$

$$\int_1^\infty \frac{dx}{x^p} = \lim_{u\to\infty} \int_1^u \frac{dx}{x} = \lim_{u\to\infty} [\ln x]_1^u = \lim_{u\to\infty} \ln u = \infty.$$

(3) For $p > 1$

$$\int_1^\infty \frac{dx}{x^p} = \lim_{u\to\infty} \int_1^u \frac{dx}{x^p} = \lim_{u\to\infty} \left[\frac{x^{1-p}}{1-p}\right]_1^u = \lim_{u\to\infty} \frac{u^{1-p}-1}{1-p} = \frac{1}{p-1}.$$

Example 8.6.8.

$$\int_1^\infty \frac{\ln}{x^2}\, dx$$

Example 8.6.9.

$$\int_0^\infty \frac{1}{1+x^2}\, dx$$

Test for Convergence

Theorem 8.6.10 (Comparison test)**.** *Let $0 \le f(x) \le g(x)$ for all $x > a$. Then*

(1) If $\int_a^\infty g(x)\, dx$ converges, then $\int_a^\infty f(x)\, dx$ also converges.

(2) If $\int_a^\infty f(x)\, dx$ diverges, then $\int_a^\infty g(x)\, dx$ also diverges.

Example 8.6.11. Test whether $\displaystyle\int_0^\infty \frac{dx}{1+x^3}$ converges or not?

sol. We see, for all $x \geq 1$, $1/(1+x^3) \leq 1/x^3$ holds. By example 8.6.7 we see $\int_1^\infty 1/x^3\,dx = 1/2$. Hence by Comparison test $\int_1^\infty 1/(1+x^3)\,dx$ converges. On the other hand, the integra $\int_0^1 1/(1+x^3)\,dx$ is well defined on $[0,1]$. Hence $\int_0^\infty 1/(1+x^3)\,dx$ converges and the value is $\int_0^1 1/(1+x^3)\,dx + \int_1^\infty 1/(1+x^3)\,dx$. (See Fig 8.12)

Theorem 8.6.12 (Limit Comparison Test)**.** *Assume $f(x)$, $g(x)$ are positive on $[a,\infty)$ and suppose*

$$\lim_{x\to\infty} \frac{f(x)}{g(x)} = L > 0.$$

Then the two integra $\displaystyle\int_a^\infty f(x)\,dx$ and $\displaystyle\int_a^\infty g(x)\,dx$ both converge or both diverge.

Proof. (1) Suppose $\int_a^\infty g(x)\,dx$ converges: Then there is $N > a$ such that $f(x)/g(x) \leq L+1$ holds for all $x \geq N$. So we have $0 \leq f(x) \leq (L+1)g(x)$ and by Limit Comparison Test, $\int_N^\infty f(x)\,dx$ converge. Hence $\int_a^\infty f(x)\,dx$ converges to $\int_a^N f(x)\,dx + \int_N^\infty f(x)\,dx$.

(2) Suppose $\int_a^\infty g(x)\,dx$ diverges:There exists $N > a$ s.t. for all $x \geq N$, $f(x)/g(x) \geq L - L/2 = L/2$ holds. Hence $f(x) \geq (L/2)g(x) \geq 0$ and by Limit Comparison Test $\int_N^\infty f(x)\,dx$ diverges. So does $\int_a^\infty f(x)\,dx$.

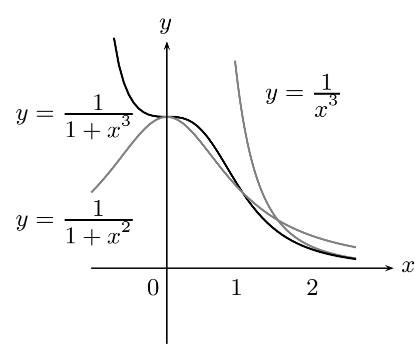


Figure 8.12:

Example 8.6.13. Test whether $\displaystyle\int_0^\infty \frac{dx}{1+e^x}$ converges or not?

sol. Let $f(x) = 1/(1+e^x)$, $g(x) = 1/e^x$. Then

$$\lim_{x\to\infty}\frac{f(x)}{g(x)} = \lim_{x\to\infty}\frac{e^x}{1+e^x} = 1$$

and

$$\int_0^\infty \frac{dx}{e^x} = \lim_{u\to\infty}\int_0^u \frac{dx}{e^x} = \lim_{u\to\infty}\left[-e^{-x}\right]_0^u = \lim_{u\to\infty}\left(-e^{-u}+1\right) = 1.$$

Hence by Limit Comparison Test, $\int_0^\infty 1/(1+e^x)\,dx$ converges.

Example 8.6.14. Test for convergence $\displaystyle\int_2^\infty \sqrt{\frac{x}{x^2-1}}\,dx$.

sol. Set $f(x) = \sqrt{\dfrac{x}{x^2-1}}$ and $g(x) = \dfrac{1}{\sqrt{x}}$.
Then

$$\lim_{x\to\infty}\frac{f(x)}{g(x)} = \lim_{x\to\infty}\sqrt{\frac{x^2}{x^2-1}} = 1.$$

$$\int_2^\infty \frac{dx}{\sqrt{x}} = \lim_{u\to\infty}\left[2\sqrt{x}\right]_2^u = \lim_{u\to\infty}\left(2\sqrt{u}-2\sqrt{2}\right) = \infty.$$

By Limit Comparison Test $\displaystyle\int_2^\infty \sqrt{\frac{x}{x^2-1}}\,dx$ converges.

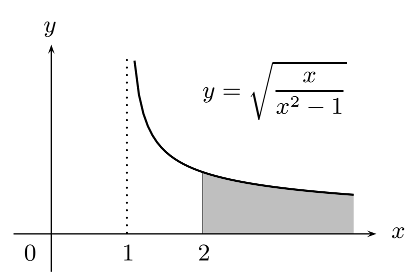


Figure 8.13:

Chapter 10

Infinite Sequence and Series

10.1 Sequences

Example 10.1.1. (1)

$$1, 3, 5, 7, \ldots$$

(2) n-th term is given by $(-1)^{n+1}1/n$:

$$1, -\frac{1}{2}, \frac{1}{3}, -\frac{1}{4}, \ldots, (-1)^{n+1}\frac{1}{n}, \ldots$$

(3) Certain rules

$$1, \frac{1}{2}, \frac{1}{2}, -\frac{1}{3}, -\frac{1}{3}, -\frac{1}{3}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4}, \ldots$$

(4) Constant sequence :

$$3, 3, 3, \ldots$$

(5) Digits after decimal point of $\sqrt{2}$

$$4, 1, 4, 1, 5, 9, \ldots$$

n-th term a_n

Definition 10.1.2. A **sequence** is a function with the set of natural numbers as domain.

Sequence as graph

Example 10.1.3. (1) $a_n = (n-1)/n$.

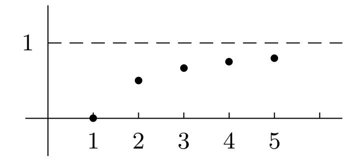


Figure 10.1: $a_n = (n-1)/n$

(2) $a_n = (-1)^n 1/n$.

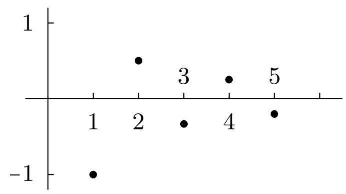


Figure 10.2: $a_n = (-1)^n 1/n$

(3) $a_n = \sqrt{n}$.

(4) $a_n = \sin(n\pi/6)$.

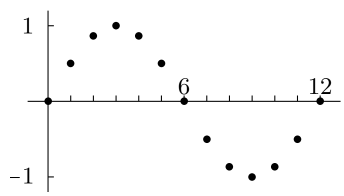


Figure 10.3: $a_n = \sin(n\pi/6)$

(5) a_n is the n-th digit of π after decimal point.

Among these (1), (3), (4) are functions $(x-1)/x$, $\sqrt{x}$, $\ln x$ are restricted to N.

Subsequence

If all the terms of $\{a_n\}$ appears as some term in $\{b_n\}$ without changing orders we say $\{a_n\}$ is a **subsequence** of $\{b_n\}$.

Example 10.1.4. (1) $1, 1, 1, 1, \ldots$ is a subsequence of $1, -1, 1, -1, \ldots$.

(2) $\{9n\}$ $(n = 1, 2, 3, \dots)$ is a subsequence of $\{3n\}$ $(n = 1, 2, 3, \dots)$.

(3) $\{1+1/4^n\}$ $(n = 1, 2, 3, \dots)$ is a subsequence of $\{1+1/2^n\}$ $(n = 1, 2, 3, \dots)$.

Recursive relation

Some sequence are defined through recursive relation such as

$$\begin{aligned} a_1 &= 1, \\ a_{n+1} &= 2a_n + 1, \quad n = 1, 2, 3, \dots \end{aligned}$$

or

$$\begin{aligned} a_1 &= 1,\ a_2 = 2, \\ a_{n+2} &= a_{n+1} + a_n, \quad n = 1, 2, 3, \dots \end{aligned}$$

10.1.1 Convergence of a sequence

Definition 10.1.5. We say $\{a_n\}$ **converges** to L, if for any $\varepsilon > 0$, there exists some N s.t. for all $n > N$ it holds that

$$|a_n - L| < \varepsilon.$$

Otherwise, we say $\{a_n\}$ is said to **diverge**. If $\{a_n\}$ converges to L, we write

$$\lim_{n\to\infty} a_n = L \quad \text{or} \quad \{a_n\} \to L.$$

L is the called the **limit** of a_n.

Example 10.1.6. Show that $\{(n-1)/n\}$converges to 1.

sol. We can expect $L = 1$. For any ε, $|(n-1)/n - 1| < \varepsilon$ holds for n satisfying $|1/n| > \varepsilon$.

■

Example 10.1.7. Show that $\{\sqrt{n+2} - \sqrt{n}\}$ converges to 0.

sol. Let ε be given. We want to choose so that

$$|\sqrt{n+2} - \sqrt{n} - 0| = \frac{2}{\sqrt{n+2} + \sqrt{n}}$$

is less than ε for all n greater than certain N. Since

$$\frac{2}{\sqrt{n+2}+\sqrt{n}} < \frac{1}{\sqrt{n}},$$

we choose n such that

$$\frac{1}{\sqrt{n}} < \varepsilon.$$

So if N is any natural number greater than $1/\varepsilon^2$, it satisfies the goal.

∎

Theorem 10.1.8. *Suppose and subsequence b_n of a_n converges to L, then a_n also converges to L.*

Theorem 10.1.9 (Uniqueness)**.** *If $\{a_n\}$ converges, it has unique limit.*

Proof. Suppose $\{a_n\}$ has two limits L_1, L_2. Choose $\varepsilon = |L_1 - L_2|/2$ There exist N_1 s.t. for $n > N_1$ the following holds

$$|a_n - L_1| < \varepsilon.$$

Similarly, there exist N_2 s.t. for all $n > N_2$ it holds that

$$|a_n - L_2| < \varepsilon.$$

Let N be the greater one of N_1, N_2. Then for all $n > N$

$$\begin{aligned} |L_1 - L_2| &= |L_1 - a_n + a_n - L_2| \le |L_1 - a_n| + |a_n - L_2| \\ &< \varepsilon + \varepsilon = |L_1 - L_2| \end{aligned}$$

holds. A contradiction. So $L_1 = L_2$.

Corollary 10.1.10. *If $\{a_n\}$ converges, we have $\lim\limits_{n\to\infty}(a_n - a_{n+1}) = 0$.*

Remark 10.1.11. The above condition is not a sufficient for convergence. For example, the sequence $a_n = \ln(n+1)/n$ satisfies $a_{n+1} - a_n = \ln(n+1)/n \to 0$ but $\lim_{n\to\infty} a_n = \infty$.

Properties of limit

Theorem 10.1.12. *Suppose $\lim\limits_{n\to\infty} a_n = A$, $\lim\limits_{n\to\infty} b_n = B$. Then we have*

(1) $\lim_{n\to\infty}\{a_n + b_n\} = A + B$

(2) $\lim_{n\to\infty}\{a_n - b_n\} = A - B$

(3) $\lim_{n\to\infty}\{ka_n\} = kA$

(4) $\lim_{n\to\infty}\{a_n \cdot b_n\} = A \cdot B$

(5) $\lim_{n\to\infty}\left\{\frac{a_n}{b_n}\right\} = A/B,\ B \neq 0.$

$$\lim_{n\to\infty}\frac{n^2-n}{n^2} = \lim_{n\to\infty} 1 - \frac{1}{n} = 1 - 0 = 1.$$
$$\lim_{n\to\infty}\frac{2-3n^5}{n^5+1} = \lim_{n\to\infty}\frac{2/n^5-3}{1+1/n^5} = -3.$$

Theorem 10.1.13 (Continuous function)**.** *Suppose the limit of a_n is L and a function f is defined on an interval containing all values of a_n and L, and continuous at L, then*

$$\lim_{n\to\infty} f(a_n) = f(L).$$

Proof. Since f is continuous at L, we have for any ε there is a δ such that for all a_n with $|a_n - L| < \delta$ it holds that $|f(a_n) - f(L)| < \varepsilon$. Since a_n converges to L, there is a natural number N s.t. for $n > N$ it holds that $|a_n - L| < \delta$. Hence $|f(a_n) - f(L)| < \varepsilon$ holds.

Example 10.1.14. (1) $\lim_{n\to\infty}\sin(n\pi/(2n+1)) = 1$ (2) $\lim_{n\to\infty} 2^{1/\sqrt{n}} = 1$

sol. (1) Since the limit of $n\pi/(2n+1)$ is $\pi/2$ and the function $\sin x$ is continuous at $\pi/2$, we have $\lim_{n\to\infty}\sin(n\pi/(2n+1)) = 1$.

(2) Since $f(x) = 2^{\sqrt{x}}$ is continuous at $x = 0^+$ we have

$$\lim_{n\to\infty} 2^{1/\sqrt{n}} = 1.$$

Theorem 10.1.15. *Suppose $f(x)$ is defined for $x \geq 0$ and if $\{a_n\}$ is given by $a_n = f(n)$, $n = 1, 2, 3, \ldots$ and if $\lim_{x\to\infty} f(x) = L$ then $\lim_{n\to\infty} a_n = L$.*

This theorem holds when $f(x) \to +\infty$ or $f(x) \to -\infty$.

Example 10.1.16. (1) $\lim_{n\to\infty}\ln n/n = 0$

(2) $\lim_{n\to\infty} n(e^{1/n} - 1) = 1$

(3) Find $\lim_{n\to\infty} \left(\frac{n+1}{n-1}\right)^n$.

sol. (1) Let $f(x) = \ln x/x$. Then

$$\lim_{n\to\infty} f(n) \lim_{x\to\infty} f(x) = \lim_{x\to\infty} \frac{(\ln x)'}{x'} = \lim_{x\to\infty} \frac{1}{x} = 0.$$

$$\lim_{n\to\infty} \ln n/n = 0.$$

(2) Set $x = 1/n$. Then it corresponds to the limit of $f(x) = (e^x - 1)/x$ as $x \to 0$. By L'Hopital's rule

$$\lim_{x\to 0} f(x) = \lim_{x\to 0} e^x = 1.$$

$$\lim_{n\to\infty} n(e^{1/n} - 1) = 1.$$

Theorem 10.1.17 (Sanwich theorem)**.** *Suppose a_n, b_n, c_n satisfy $a_n \leq b_n \leq c_n$ and $\lim_{n\to\infty} a_n = \lim_{n\to\infty} c_n = L$. Then $\lim_{n\to\infty} b_n = L$.*

Limit used Often

Proposition 10.1.18.

(1) $\lim_{n\to\infty} \frac{\ln n}{n} = 0$

(2) $\lim_{n\to\infty} \sqrt[n]{n} = 1$

(3) $\lim_{n\to\infty} x^{1/n} = 1,\ x > 0$

(4) $\lim_{n\to\infty} x^n = 0,\ |x| < 1$

(5) $\lim_{n\to\infty} \left(1 + \frac{x}{n}\right)^n = e^x,\ x \in \mathbb{R}$

(6) $\lim_{n\to\infty} \frac{x^n}{n!} = 0,\ x \in \mathbb{R}.$

Proof. (1) See Example 10.1.16.

(2) Let $a_n = n^{1/n}$ and take ln $\ln a_n = \ln n^{1/n} = \frac{\ln n}{n}$. Since this approaches 0 and e^x is continuous at 0 $a_n = e^{\ln a_n} \to e^0 = 1$ by theorem 10.1.15.

(3) Set $a_n = x^{1/n}$. Since the limit of $\ln a_n = \ln x^{1/n} = \frac{\ln x}{n}$ is 0, we see $x^{1/n} = a_n = e^{\ln a_n}$ converges to $e^0 = 1$.

(4) Use the definition. given $\varepsilon > 0$, we must find n, s.t. for $|x| < \varepsilon^{1/n}$ $|x^n - 0| < \varepsilon$ holds. Since $\lim_{n\to\infty} \varepsilon^{1/n} = 1$ there is an N s.t $|x| < \varepsilon^{1/N}$ holds. Now if $n > N$ we have $|x|^n < |x^N| < \varepsilon$.

(5) Let $a_n = (1 + x/n)^n$. Then $\lim_{n\to\infty} \ln a_n = \lim_{n\to\infty} \ln (1 + x/n)^n = n \ln (1 + x/n)$ and by L'Hopital's rule we see

$$\lim_{n\to\infty} \frac{\ln(1 + x/n)}{1/n} = \lim_{n\to\infty} \frac{x}{1 + x/n} = x.$$

Hence $a_n = (1 + x/n)^n = e^{\ln a_n}$ converges to e^x.

(6) First we will show that

$$-\frac{|x|^n}{n!} \le \frac{x^n}{n!} \le \frac{|x|^n}{n!}$$

and $|x|^n/n! \to 0$. Then use Sandwich theorem. If $|x|$ is greater than M, then $|x|/M < 1$ and hence $(|x|/M)^n \to 0$. If $n > M$

$$\frac{|x|^n}{n!} = \frac{|x|^n}{1 \cdot 2 \cdots M(M+1)\cdots n} \le \frac{|x|^n}{M!M^{n-M}} = \frac{M^M}{M!}\left(\frac{|x|}{M}\right)^n$$

holds. But $M^M/M!$ is fixed number. As $n\infty$ $(|x|/M)^n$ approaches 0. So $|x|^n/n!$ approaches 0. Finally by Sandwich theorem 10.1.17 we get the result. $x^n/n! \to 0$.

Example 10.1.19. (1) $\lim_{n\to\infty} \left(\frac{1}{1000}\right)^{1/n} = 1.$

(2) $\lim_{n\to\infty} \left(10^{1000} n^2\right)^{1/n} = \lim_{n\to\infty} (10^{1/n})^{1000} \lim_{n\to\infty} n^{2/n} = 1 \cdot \lim_{n\to\infty} \left(n^{1/n}\right)^2 = 1.$

(3) $\lim_{n\to\infty} \left(1 - \frac{2}{n}\right)^n = e^{-2}.$

(4) $\lim_{h\to 0^+} (1 + h)^{1/h} = \lim_{n\to\infty} \left(1 + \frac{1}{n}\right)^n = e.$

(5) $\lim_{n\to\infty} \frac{10^n}{n!} = 0.$

(6) The set of all x satisfying $\lim_{n\to\infty} \frac{|x|^n}{5^n} = 0$ is, $|x/5| < 1$, $|x| < 5$.

Example 10.1.20. $\lim_{n\to\infty} \sqrt[n]{5n+1} = 1.$

sol. Since $\ln(5n+1)^{1/n} = \ln(5n+1)/n \to 0$ above limit is $e^0 = 1$.

■

Example 10.1.21. Show that $\lim_{n\to\infty} \ln n/n^{\varepsilon} = 0$ for any $\varepsilon > 0$.

sol. By L'Hopital rule 3.6.5

$$\lim_{n\to\infty} \frac{\ln n}{n^{\varepsilon}} = \lim_{n\to\infty} \frac{1/n}{\varepsilon n^{\varepsilon-1}} = \lim_{n\to\infty} \frac{1}{\varepsilon n^{\varepsilon}} = 0. \qquad \square$$

■

Monotone Sequence

Definition 10.1.22. If a_n satisfies

$$a_1 \le a_2 \le \cdots \le a_n \le \cdots$$

then a_n is called an **increasing sequence, nondecreasing sequence**.

Definition 10.1.23. If there is a number M such that $a_n \le M$ for all n, then this sequence is said to be **bounded from above**. Any such M is called an **upper bound**. If the smallest number exists among all upper bound, then it is called the **least upper bound**. Similarly, we say a sequence **bounded from below** if there is a number N such that $a_n \ge N$ for all n, Any such an N is called a **lower bound**. If the largest number exists among all lower bound, then it is called the **greatest lower bound**. If a sequence has both lower bound and upper bound, then we say it is **bounded**.

Example 10.1.24. $a_n = 1 - 1/2^n$ $M = 1$ is an upper bound and any number bigger than 1 is an upper bound. The smallest such number(if exists) is **least upper bound**.

Theorem 10.1.25. *If a nondecreasing sequence has an upper bound, it converges(to the least upper bound).*

Suppose L is a least upper bound, we observe two things:

(1) $a_n \leq L$ for all n, and

(2) for any $\varepsilon > 0$ there is a term a_N greater than $L - \varepsilon$.

Suppose there does not exist such a_N, it holds that $a_n \leq L - \varepsilon$ for all n, which is a contradiction. Thus for $n \geq N$

$$L - \varepsilon < a_n \leq L$$

$|L - a_n| < \varepsilon$ and we see $a_n \to L$.

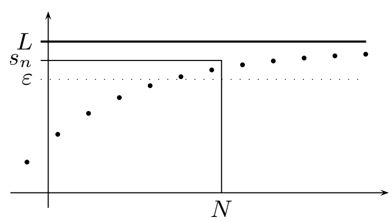


Figure 10.4: Nondecreasing(increasing) sequence and least upper bound L

For decreasing sequence, we can define similar concept.

Definition 10.1.26. If a_n satisfies

$$a_1 \geq a_2 \geq \cdots \geq a_n \cdots$$

a_n is called a **decreasing sequence**. If $s_n \geq N$, then N is called a **lower bound**(lower bound) The largest such number is called the **greatest lower bound**.

10.2 Infinite Series

A sequence given as the sum of an infinite sequence of numbers is called **infinite series**.

Example 10.2.1. If we denote the sum of first n- term of $a_n = 1/2^n$ by s_n then

$$s_1 = a_1 = \frac{1}{2}$$
$$s_2 = a_1 + a_2 = \frac{1}{2} + \frac{1}{4} = \frac{3}{4}$$
$$s_3 = a_1 + a_2 + a_3 = \frac{1}{2} + \frac{1}{4} + \frac{1}{8} = \frac{7}{8}$$
$$\vdots$$

The general term $\{s_n\}$ is

$$s_n = a_1 + a_2 + a_3 + \cdots + a_n = \sum_{k=1}^{n} a_k.$$

We write the infinite series as $\sum_{n=1}^{\infty} a_n$ or $\sum a_n$.(whether it converges or not!)

Definition 10.2.2. a_n is called n**-th term** and $s_n = \sum_{k=1}^{n} a_k$ is called n-th **partial sum**. If the limit of $\{s_n\}$ is L, then we say $\sum a_n$ converges to L and write $\sum_{n=1}^{\infty} a_n = L$ or $a_1 + a_2 + a_3 + \cdots = L$. If s series does not converges, we say it **diverges**.

Example 10.2.3 (Repeating decimals)**.** Write $0.1111\cdots$ as series.

sol. Writing $0.111\cdots = 0.1 + 0.01 + 0.001 + \cdots$ we see

$$a_1 = 0.1,$$
$$a_2 = 0.01,$$
$$\vdots$$
$$a_n = (0.1)^n.$$

Hence $0.111 = \sum_{k=1}^{\infty} 10^{-k}$.

Definition 10.2.4.

$$a + ar + ar^2 + \cdots$$

is called a **geometric series** and r is called a **ratio**.

$$s_n = a + ar + \cdots + ar^{n-1}$$
$$rs_n = ar + ar^2 + \cdots + ar^n.$$

$$s_n - rs_n = a - ar^n.$$

Thus $s_n = a(1 - r^n)/(1 - r)$.

Example 10.2.5 (Telescoping Series)**.** Find the sum $\sum_{n=1}^{\infty} 1/n(n+1)$.

sol. We use the identity $1/n(n+1) = 1/n - 1/(n+1)$ to see

$$s_n = \left(\frac{1}{1} - \frac{1}{2}\right) + \left(\frac{1}{2} - \frac{1}{3}\right) + \cdots + \left(\frac{1}{n} - \frac{1}{n+1}\right).$$

Since $s_n = 1 - 1/(n+1)$, we see $s_n \to 1$.

■

Divergent Series

Example 10.2.6. $\sum_{n=1}^{\infty}(n+1)/n$ diverges since n-th term is greater than 1.

Example 10.2.7. $\sum_{n=1}^{\infty} \sin(\pi n/2)$ diverges.

sol.

$$1, 0, -1, 0, 1, \ldots$$

$$s_4 = s_8 = \cdots = s_{4n} = 0$$

but

$$s_2 = s_6 = \cdots = s_{4n+2} = 1.$$

So s_n oscillates between 0 and 1.

■

Theorem 10.2.8 (n-th term test)**.** *If* $\sum a_n$ *converges then* $a_n \to 0$.

Proof. Suppose $\sum_{n=1}^{\infty} a_n$ converges then s_n and s_{n-1} must have the same limit. Since $a_n = s_n - s_{n-1}$ we see $\lim a_n = \lim s_n - \lim s_{n-1} = 0$.

Theorem 10.2.9 (nth term test)**.** *If* $\lim a_n \not\to 0$ *or* $\lim a_n$ *does not exists, then* $\sum a_n$ *diverges.*

Example 10.2.10. $\sum(n-1)/n$ diverges since $a_n = (n-1)/n \to 1$.

Example 10.2.11. $\sum(-1)^n \ln(\ln n)$ diverges since $\ln(\ln n) \to \infty$.

Theorem 10.2.12. *Suppose* $\sum a_n, \sum b_n$ *converges. Then*

(1) $\sum(a_n + b_n) = \sum a_n + \sum b_n$,

(2) $\sum(a_n - b_n) = \sum a_n - \sum b_n$,

(3) $\sum ka_n = k \sum a_n$.

Example 10.2.13.

(1) $$\sum_{n=1}^{\infty} \frac{2^n - 1}{3^n} = \sum_{n=1}^{\infty} \frac{2^n}{3^n} - \sum_{n=1}^{\infty} \frac{1}{3^n} = \frac{2}{3}\frac{1}{1-2/3} - \frac{1}{3}\frac{1}{1-1/3} = \frac{3}{2}.$$

(2) $$\sum_{n=1}^{\infty} \frac{3^n - 2^n}{6^n} = \sum_{n=1}^{\infty} \frac{3^n}{6^n} - \sum_{n=1}^{\infty} \frac{2^n}{6^n} = \sum_{n=1}^{\infty} \frac{1}{2^n} - \sum_{n=1}^{\infty} \frac{1}{3^n} = \frac{1}{2}.$$

What's wrong with the following argument?

$$1 = \sum_{n=1}^{\infty} \left(\frac{1}{n} - \frac{1}{n+1} \right) = \sum \frac{1}{n} - \sum \frac{1}{n+1} = \infty - \infty.$$

10.3 Integral Test

Example 10.3.1. Determine whether the following series converges or not.

$$\sum \frac{1}{n^2} = 1 + \frac{1}{4} + \frac{1}{9} + \cdots + \frac{1}{n^2} + \cdots$$

sol. Set $f(x) = 1/x^2$. Then

$$s_n = 1 + \frac{1}{4} + \frac{1}{9} + \cdots + \frac{1}{n^2} = f(1) + f(2) + f(3) + \cdots + f(n)$$

and

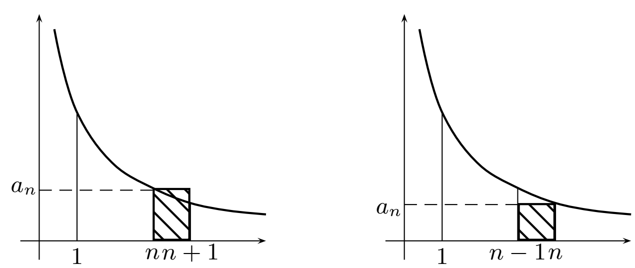
$$f(2) = \frac{1}{2^2} < \int_1^2 \frac{1}{x^2}\,dx$$
$$f(3) = \frac{1}{3^2} < \int_2^3 \frac{1}{x^2}\,dx$$
$$\vdots$$
$$f(n) = \frac{1}{n^2} < \int_{n-1}^n \frac{1}{x^2}\,dx.$$

$$s_n = f(1) + f(2) + f(3) + \cdots + f(n) < 1 + \int_1^n \frac{1}{x^2}\,dx = 2 - \frac{1}{n}.$$

Thus s_n is bounded, increasing, and hence converges.

□

Theorem 10.3.2 (Integral Test)**.** *Suppose $f(x)$ is nonnegative, non-increasing for $x \geq 1$ and $a_n = f(n)$. Then the series $\sum_{n=1}^{\infty} a_n$ converges iff $\int_1^{\infty} f(x)\,dx$ converges.*



(a) $\int_n^{n+1} f(x)\,dx \leq a_n$ (b) $a_n \leq \int_{n-1}^{n} f(x)\,dx$

Figure 10.5: Integral Test

Proof. Since f is decreasing and $f(n) = a_n$, we see from figure 10.5(a) $\int_n^{n+1} f(x)\,dx \leq a_n$. So

$$\int_1^{n+1} f(x)\,dx \leq a_1 + a_2 + \cdots + a_n.$$

Also from figure 10.5 (b), we see $a_n \le \int_{n-1}^{n} f(x)\,dx$, $(n = 2, 3, 4, \dots)$. Hence we have

$$a_2 + a_3 + \cdots + a_n \le \int_1^n f(x)\,dx$$

and finally

$$\int_1^{n+1} f(x)\,dx \le a_1 + a_2 + \cdots + a_n \le a_1 + \int_1^n f(x)\,dx.$$

Example 10.3.3 (p-series)**.** Let p be a fixed number. Then

$$\sum_1^\infty \frac{1}{n^p} = \frac{1}{1^p} + \frac{1}{2^p} + \cdots + \frac{1}{n^p} + \cdots$$

converges when $p > 1$ and diverges when $p \le 1$. For $p = 1$ we see

$$\int_1^\infty \frac{1}{x}\,dx = \lim_{b\to\infty} [\ln b]_1^b = \infty.$$

So the harmonic series

$$1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n} + \cdots$$

diverges.

Example 10.3.4. Test the convergence of

$$\sum_1^\infty \frac{1}{1+n^2}.$$

We see

$$\int_1^\infty \frac{1}{1+x^2}\,dx = \lim_{b\to\infty} [\tan^{-1} x]_1^b = \lim_{b\to\infty} [\tan^{-1} b - \tan^{-1} 1] = \frac{\pi}{4}.$$

10.3.1 Series with nonnegative terms

$$\sum \frac{1}{n^3}, \quad \sum \frac{1}{3^n+1}, \quad \sum \frac{(-1)^n}{\sqrt{n}}.$$

Example 10.3.5. Investigate $\displaystyle\sum_{n=1}^\infty a_n = \sum_{n=1}^\infty \frac{1}{n^2}$.

sol. Use the inequality $1/n^2 < 1/n(n-1)$ and partial fraction,

$$\begin{aligned} s_n &= \frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \cdots + \frac{1}{n^2} \\ &< \frac{1}{1\cdot 1} + \frac{1}{1\cdot 2} + \frac{1}{2\cdot 3} + \cdots + \frac{1}{n(n-1)} \\ &= 1 + \left(1 - \frac{1}{2}\right) + \left(\frac{1}{2} - \frac{1}{3}\right) + \cdots + \left(\frac{1}{n-1} - \frac{1}{n}\right) \\ &= 2 - \frac{1}{n} < 2. \end{aligned}$$

Hence s_n is bounded above and monotonic increasing hence converges.

■

Example 10.3.6 (Harmonic series).

$$\sum \frac{1}{n} = 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n} + \cdots$$

diverges since

$$1 + \frac{1}{2} + \underbrace{\frac{1}{3} + \frac{1}{4}}_{>2/4} + \underbrace{\frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8}}_{>4/8} + \underbrace{\frac{1}{9} + \frac{1}{10} + \cdots + \frac{1}{16}}_{>8/16} + \cdots$$

10.4 Comparison Test

Theorem 10.4.1 (The Comparison Test). *Let $a_n \geq 0$.*

(a) The series $\sum a_n$ converges if $a_n \leq c_n$ for all $n > N$ and $\sum c_n$ converges

(b) The series $\sum a_n$ diverges $a_n \geq d_n$ for all $n > N$ and $\sum d_n$ diverge.

Proof. In (a), the partial sum is bounded by

$$M = a_1 + a_2 + \cdots a_n + \sum_{n=N+1}^{\infty} c_n.$$

Hence if $\sum c_n$ converges, then $\sum a_n$ converges by Theorem 10.1.25. In (b), the partial sum is greater than

$$M^* = a_1 + a_2 + \cdots a_n + \sum_{n=N+1}^{\infty} d_n.$$

But the series $\sum_{n=N+1}^{\infty} d_n$ diverges. Hence so does $\sum a_n$.

Example 10.4.2. Look at the tail part of

$$3 + 600 + 5000 + \frac{1}{3!} + \frac{1}{4!} + \frac{1}{5!} + \cdots + \frac{1}{n!} + \cdots .$$

Then $1/n! < 1/2^n$ for $n = 4, 5, 6, \ldots$ and $\sum 1/2^n$ converges. Hence the series converges.

Limit Comparison Test

Example 10.4.3. Investigate the convergence of

$$\sum_{1}^{\infty} \frac{n}{2n^3 - n + 3}.$$

sol. Let

$$a_n = \frac{n}{2n^3 - n + 3} = \frac{1}{2n^2 - 1 + 3/n}$$

and use the fact that a_n behaves similar to $1/2n^2$. If $c_n = 1/2n^2$ then $\lim_{n\to\infty} a_n/c_n = 1$. Hence for any ε there is N such that if $n > N$ for some N then the following holds:

$$1 - \varepsilon \le \frac{a_n}{c_n} \le 1 + \varepsilon.$$

In other words,

$$(1 - \varepsilon)c_n \le a_n \le (1 + \varepsilon)c_n.$$

Since $\sum_{n \ge N} c_n$ converges $\sum_{n \ge N} a_n$ converges by comparison.

■

Theorem 10.4.4 (Limit Comparison Test). *(1) Suppose $a_n > 0$ and there is a series $\sum c_n$ $(c_n > 0)$ which converges and if*

$$\lim_{n\to\infty} \frac{a_n}{c_n} = c > 0$$

then $\sum a_n$ converges.

(2) Suppose $a_n > 0$ and there is a series $\sum d_n$ $(d_n > 0)$ which diverges and

if

$$\lim_{n\to\infty} \frac{a_n}{d_n} = c > 0$$

then $\sum a_n$ *diverges.*

Proof. We prove part 1. Since $c/2 > 0$ there is an N such that for all $n > N$ we have

$$\left|\frac{a_n}{b_n} - c\right| < \frac{c}{2}.$$

Hence

$$-\frac{c}{2} < \frac{a_n}{b_n} - c < \frac{c}{2} \text{ or } \frac{c}{2} < \frac{a_n}{b_n} < \frac{3c}{2}.$$

Hence

$$(\frac{c}{2})b_n < a_n < \frac{3c}{2}b_n.$$

Example 10.4.5. (1) $\sum_1^\infty \frac{n+1}{100n^3+n+1}$ converges since $\sum_1^\infty \frac{1}{n^2}$ converges

(2) $\sum_{20}^\infty \frac{1}{3^n-1000n}$ converges since $\sum_1^\infty \frac{1}{3^n}$ converge

(3) $\sum_1^\infty \frac{2n+1}{n^2+4n+1}$

(4) Does $\sum_2^\infty \frac{\ln n}{n^{3/2}}$ converge ?

(5) Compare $\sum_1^\infty \frac{(\ln n)^{1/2}}{(n \ln n+1)}$ with $\sum_2^\infty \frac{1}{n(\ln n)^{1/2}}$. Use integral test.

$$\int_2^\infty \frac{dx}{x(\ln x)^{1/2}} = \int_{\ln 2}^\infty \frac{du}{u^{1/2}} = \infty.$$

10.5 Ratio test and Root Tests

Example 10.5.1. It is not easy to find general term of $a_1 = 1,\ a_{n+1} = \frac{na_n}{3n+2}$. But its ratio is clearly seen.

Ratio Test

Theorem 10.5.2 (Ratio Test)**.** *Suppose* $a_n > 0$ *and if the limit exists.*

$$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \rho.$$

Then exactly one of the following holds.

(1) The sum $\sum a_n$ *converges if* $\rho < 1$

(2) The sum $\sum a_n$ *diverges if* $\rho > 1$

(3) The test is inconclusive if $\rho = 1$.

Proof. (1) Let $\rho < 1$. Then choose any r between ρ and 1 and set $\varepsilon = r - \rho$. Then since

$$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \rho,$$

there exists a natural number N such that for all $n > N$,

$$\left|\frac{a_{n+1}}{a_n} - \rho\right| < \varepsilon$$

holds. Since $a_{n+1}/a_n < \rho + \varepsilon = r$, we see

$$\begin{aligned} a_{N+1} &< ra_N \\ a_{N+2} &< ra_{N+1} < r^2 a_N \\ &\vdots \\ a_{N+m} &< ra_{N+m-1} < r^m a_N. \end{aligned}$$

We compare a_n with a series general term is $r^m a_N$. Since $\sum_{m=1}^{\infty} r^m a_N$ converges, $\sum_{n=N+1}^{\infty} a_n$ converges. (2) Suppose $\rho > 1$. Then exist an M such that for $n > M$, it holds that

$$\frac{a_{n+1}}{a_n} > 1.$$

Hence the series diverges:

$$a_M < a_{M+1} < a_{M+2} < \cdots .$$

(3) The case $\rho = 1$. Both the series $\sum 1/n^2$ and $\sum 1/n$. But the former converges and the latter diverges.

Example 10.5.3.

(1) $\sum \dfrac{n!n!}{(2n)!}$

(2) $\sum \dfrac{(2^n + 5)}{3^n}$

(3) $\sum \dfrac{2^n}{n!}$

sol. (1) Ratio Test

$$\frac{a_{n+1}}{a_n} = \frac{(n+1)!(n+1)!(2n)!}{n!n!(2n+2)(2n+1)(2n)!}$$
$$= \frac{(n+1)(n+1)}{(2n+2)(2n+1)} = \frac{n+1}{4n+2} \to \frac{1}{4}.$$

(2) $\frac{a_{n+1}}{a_n} = \frac{(2^{n+1}+5)3^n}{3^{n+1}(2^n+5)} = \frac{2^{n+1}+5}{3(2^n+5)} \to \frac{2}{3}$ (3) $\frac{a_{n+1}}{a_n} = \frac{2^{n+1}n!}{(n+1)!2^n} = \frac{2}{n+1} \to 0$

Example 10.5.4. Find the range of x which makes the following converge.

$$1 + \frac{x^2}{2} + \frac{x^4}{4} + \frac{x^6}{6} + \cdots.$$

sol. For $n > 1$, $a_n = x^{2n-2}/(2n-2)$

$$\frac{a_{n+1}}{a_n} = \frac{x^{2n}(2n-2)}{2nx^{2n-2}} = \frac{(2n-2)x^2}{2n} \to x^2.$$

So converges if $|x| < 1$ and diverges if $|x| > 1$. When $|x| = 1$ the series diverges since it behaves like

$$1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{6} \cdots = 1 + \frac{1}{2}\left(1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots\right).$$

Estimate error

For $\rho < 1$ If the series is approximated by its N- partial sum, then the error is

$$a_{N+1} + a_{N+2} + \cdots$$

So if N is large, for some r with $\rho < r < 1$ we have

$$\frac{a_{n+1}}{a_n} < r, \quad n \geq N.$$

Here the estimate of errors is

$$a_{N+1} + a_{N+2} + \cdots \leq r a_N + r^2 a_N + \cdots = a_N \cdot \frac{r}{1-r}.$$

Example 10.5.5. Investigate

$$\frac{1}{3} + \frac{2}{9} + \frac{1}{27} + \frac{4}{81} + \cdots + \frac{f(n)}{3^n} + \cdots .$$

$$f(n) = \begin{cases} n, & n \text{ even} \\ 1, & n \text{ odd.} \end{cases}$$

sol. Since $a_n = \frac{f(n)}{3^n}$ we have

$$\frac{a_{n+1}}{a_n} = \frac{f(n+1)}{3f(n)} = \begin{cases} \frac{1}{3n}, & n \text{ even} \\ \frac{n+1}{3}, & n \text{ odd.} \end{cases}$$

So we cannot use ratio test. However if we take n-th root,

$$\sqrt[n]{a_n} = \frac{\sqrt[n]{f(n)}}{3} = \begin{cases} \frac{\sqrt[n]{n}}{3}, & n \text{ even} \\ \frac{1}{3}, & n \text{ odd.} \end{cases}$$

and $\sqrt[n]{n}$ converges to 1

$$\lim_{n\to\infty} \sqrt[n]{a_n} = \frac{1}{3}$$

we can compare this series with $(\frac{1}{3})^n$.

■

n-th Root Test

Theorem 10.5.6 (n-th Root Test)**.** *Suppose* $\sqrt[n]{a_n} \to \rho$. *Then*

(1) $\sum a_n$ *converges if* $\rho < 1$.

(2) $\sum a_n$ *diverges if* $\rho > 1$.

(3) We cannot tell anything if $\rho = 1$.

Proof. (1) Suppose $\rho < 1$. Choose r between ρ and 1 and set $\varepsilon = \rho - r > 0$. Since $\sqrt[n]{a_n}$ converges to ρ there is some N s.t. when n is greater than N, it holds that

$$|\sqrt[n]{a_n} - \rho| < \varepsilon.$$

In other words, $\sqrt[n]{a_n} < \rho + \varepsilon = r < 1$. Hence

$$a_n < (\rho + \varepsilon)^n$$

holds. So $\sum(\rho+\varepsilon)^n$ converges and by comparison test $\sum_{n=N}^{\infty} a_n$ converges.

(2) Suppose $\rho > 1$ then $\sqrt[n]{a_n} > 1$ for suff. large n $a_n > 1$. So diverges.

(3) The case $\rho = 1$: No conclusion can be drawn since both the series $\sum 1/n^2$ and $\sum 1/n$ have $\rho = 1$ while one converges and the other not.

Example 10.5.7. $\sum_{n=1}^{\infty} \frac{n}{2^n}$ converges since $\sqrt[n]{\frac{n}{2^n}} = \sqrt[n]{\frac{n}{2}} \to \frac{1}{2}$.

Example 10.5.8. $\sum_{n=1}^{\infty} \frac{3^n}{n^n}$ converges since $\sqrt[n]{\frac{3^n}{n^n}} = \frac{3}{n} \to 0$.

10.6 Alternating Series, absolute and conditional convergence

Alternating Series

Definition 10.6.1. Suppose $a_n > 0$ for all n. A series of the form

$$a_1 - a_2 + a_3 - a_4 + \cdots$$

is called an **alternating series**.

The following are examples of alternating series.

$$1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \cdots$$
$$1 - 2 + 3 - 4 + 5 - 6 + \cdots$$

But

$$1 - \frac{1}{2} - \frac{1}{3} + \frac{1}{4} + \frac{1}{5} - \frac{1}{6} - \frac{1}{7} + \cdots$$

is not an alternating series.

Theorem 10.6.2 (Alternating Series Test, Leibniz theorem)**.** *Suppose the following three conditions hold.*

(1) $a_n > 0$.

(2) $a_n \geq a_{n+1}$.

(3) $a_n \to 0$.

Then $\sum_{n=1}^{\infty}(-1)^{n+1}a_n$ *converges.*

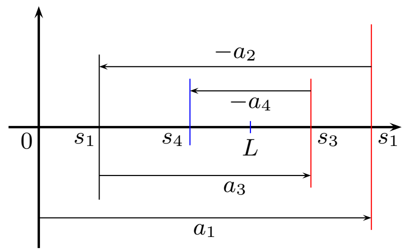


Figure 10.6: Partial sum of alternating series

Proof. The idea is to show that the sum of even number of terms form a bounded, increasing sequence so that it converges by Theorem 10.1.25. Suppose n is even ($n = 2m$) then the partial sum

$$s_{2m} = (a_1 - a_2) + (a_3 - a_4) + \cdots + (a_{2m-1} - a_{2m})$$

is **increasing**. Hence $s_{2m+2} \geq s_{2m}$. But we also see

$$s_{2m} = a_1 - (a_2 - a_3) - (a_4 - a_5) - \cdots - (a_{2m-2} - a_{2m-1}) - a_{2m}.$$

Hence s_{2m} is less than a_1. In other words, s_{2m} is **bounded above**, hence converges. Let $L = \lim s_{2m}$ be its limit. Now suppose n is odd ($n = 2m + 1$). Then

$$s_{2m+1} = s_{2m} + a_{2m+1}.$$

Then since $a_{2m+1} \to 0$, $\lim s_{2m+1} = \lim(s_{2m} + a_{2m+1}) = L$.

Example 10.6.3. The series

$$\sum(-1)^{n+1}\frac{1}{n} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots$$

converges.

Example 10.6.4.

$$\sum(-1)^{n+1}\frac{1}{\sqrt{n}} = 1 - \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} - \frac{1}{\sqrt{4}} + \cdots$$

converges.

Example 10.6.5.

$$\sum(-1)^{n+1}\frac{\sqrt{n}}{\sqrt{n+1}} = \frac{1}{\sqrt{2}} - \frac{\sqrt{2}}{\sqrt{3}} + \frac{\sqrt{3}}{\sqrt{4}} - \frac{\sqrt{4}}{\sqrt{5}} + \cdots$$

diverges by n-th term test.

Example 10.6.6.

$$\frac{2}{1} - \frac{1}{1} + \frac{2}{3} - \frac{1}{3} + \frac{2}{5} - \frac{1}{5} + \cdots + \frac{2}{2n-1} - \frac{1}{2n-1} + \cdots$$

is alternating. But it is not monotonically decreasing. But

$$\left(\frac{2}{1} - \frac{1}{1}\right) + \left(\frac{2}{3} - \frac{1}{3}\right) + \left(\frac{2}{5} - \frac{1}{5}\right) + \cdots$$
$$+ \left(\frac{2}{2n-1} - \frac{1}{2n-1}\right) + \cdots = 1 + \frac{1}{3} + \frac{1}{5} + \cdots + \frac{1}{2n-1} + \cdots$$

So diverges.

Example 10.6.7. Investigate $\displaystyle\sum_{n=2}^{\infty}(-1)^n\frac{\ln n}{n+1}$.

sol. We let

$$f(x) = \frac{\ln x}{x+1}.$$

Then $f(n) = \ln n/(n+1)$ and $f'(x) = ((x+1)/x - \ln x)/(x+1)^2$. For sufficiently large x, $(x+1)/x - \ln x < 0$. Hence $f(x)$ is decreasing function. For example, for $x \geq 8$, $f(x)$ is decreasing. So $a_n = f(n)$ is decreasing for $n \geq 8$. By Leibniz theorem the series converges.

∎

Partial Sum of Alternating Series

We look at the partial sums of an alternating series:

$$
\begin{aligned}
s_1 &= a_1, \\
s_2 &= a_1 - a_2, \text{ So } s_2 < s_1. \\
s_3 &= a_1 - a_2 + a_3 = a_1 - (a_2 - a_3), \text{ So } s_2 < s_3 < s_1. \\
s_4 &= a_1 - a_2 + a_3 - a_4 = a_1 - a_2 + (a_3 - a_4), \text{ So } s_2 < s_4 < s_3 < s_1.
\end{aligned}
$$

Thus s_{2m+1} is decreasing and s_{2m} is increasing. Let L be its sum. Then

$$
\underbrace{\underbrace{s_{2m} < s_{2m+2} < \cdots < L}_{|s_{2m}-L|} < \cdots < s_{2m+1}}_{|s_{2m}-s_{2m+1}|} < s_{2m-1}
$$

But since

$$
\begin{aligned}
|s_{2m} - L| &< |s_{2m} - s_{2m+1}| = a_{2m+1}, \\
|s_{2m+1} - L| &< |s_{2m+2} - s_{2m+1}| = a_{2m+2}
\end{aligned}
$$

we see

$$
|s_n - L| < a_{n+1}.
$$

In other words, the partial sum is a good approximation to the true sum with error bound a_{n+1}. Since a_n is decreasing s_{n+1} is better approximation than s_n.

Theorem 10.6.8 (Alternating Series Estimation Theorem). *Suppose* $\sum(-1)^{n+1}a_n$ *is an alternating series satisfying the conditions of Leibniz theorem. Then the partial sum*

$$
s_n = a_1 - a_2 + a_3 + \cdots + (-1)^{n+1}a_n
$$

is a good approximation with an error bound less than a_{n+1}.

Example 10.6.9. estimate

$$
\sum_{n=0}^{\infty} \frac{(-1)^n}{2^n} = 1 - \frac{1}{2} + \frac{1}{4} + \cdots = \frac{2}{3}
$$

with first five term.

sol. The error bound is $a_6 = 1/64$. The true value up to five terms is

$$s_5 = 1 - \frac{1}{2} + \frac{1}{4} - \frac{1}{8} + \frac{1}{16} - \frac{1}{32} = \frac{21}{32}.$$

So the true error is $|2/3 - 21/32| = 1/96$ which is less than $a_6 = 1/64$.

Example 10.6.10. Use s_{10} or s_{100} to estimate

$$\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} = 1 - \frac{1}{2} + \frac{1}{3} - \cdots = \ln 2 = 0.69314\cdots$$

sol. True error of

$$s_{10} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots - \frac{1}{10} = 0.64563\cdots$$

is $0.0475\cdots < a_{11} = 1/11$. The true error of

$$s_{100} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots - \frac{1}{100} = 0.68881\cdots$$

is $0.00433\cdots < a_{111} = 1/111$.

Absolute convergence and Conditional Convergence

Definition 10.6.11. If $\sum |a_n|$ converges then $\sum a_n$ is said to **converge absolutely**.

Theorem 10.6.12. *If $\sum |a_n|$ converges then so does $\sum a_n$.*

Proof.

$$-|a_n| \le a_n \le |a_n|$$

holds for all n. Hence

$$0 \le a_n + |a_n| \le 2|a_n|.$$

Since $\sum |a_n|$ converges and $a_n + |a_n| \ge 0$

$$\sum (a_n + |a_n|)$$

converges by comparison. Subtracting converging series, we have

$$\sum a_n = \sum(a_n + |a_n|) - \sum |a_n|$$

and so $\sum a_n$ converges.

Corollary 10.6.13. *If $\sum a_n$ diverges, so does $\sum |a_n|$.*

Example 10.6.14. (1) $\sum_{n=1}^{\infty}(-1)^{n+1}\frac{1}{n^2} = 1 - \frac{1}{4} + \frac{1}{9} + \cdots +$. Its n-th term $a_n = \frac{(-1)^{n+1}}{n^2}$ satisfies $|a_n| = \frac{1}{n^2}$. Since $\sum \frac{1}{n^2}$ converges we see the series $\sum_{n=1}^{\infty}(-1)^{n+1}\frac{1}{n^2}$ converges absolutely. The series $\sum_{n=1}^{\infty}(-1)^{n+1}\frac{1}{n^2}$, of course converges.

(2) The n-th term of $\sum \frac{\cos n}{n^2}$ satisfies $|a_n| = \frac{|\cos n|}{n^2} \le \frac{1}{n^2}$. Since $\sum \frac{1}{n^2}$ converges, $\sum \frac{\cos n}{n^2}$ converges.

(3)

$$\sum(-1)^{n+1}\frac{1}{n} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots$$

But $\sum |a_n| = \sum \frac{1}{n}$ diverges by integral test. Thus the series does not converge absolutely. Still, this series converges (by Leibniz theorem).

(4) $\sum \frac{(-1)^n}{n^p}$ converges absolutely for $p > 1$ but does not converges absolutely for $p \le 1$. However, the series converges for all $p > 0$.

Definition 10.6.15. A series which converges but does not converge not absolutely **converges conditionally**.

Rearrangement of Series for Absolutely Convergent Series

Theorem 10.6.16 (Rearrangement of Series)**.** *Suppose $\sum a_n$ converges absolutely and b_n is a rearrangement of a_n. Then $\sum b_n$ converges absolutely and and*

$$\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} b_n.$$

Here we have $b_k = a_{n(k)}$ for some 1-1 function $n(k)$.

Example 10.6.17. We know the following converges absolutely:

$$1 - \frac{1}{2} + \frac{1}{4} - \frac{1}{8} + \frac{1}{16} - \frac{1}{32} + \cdots = \frac{2}{3}.$$

Hence rearranging it in any order we get

$$1 + \frac{1}{4} - \frac{1}{2} + \frac{1}{16} + \frac{1}{64} - \frac{1}{8} + \cdots$$

We can guarantee this series converges to $\frac{2}{3}$. We know the series

$$1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \cdots$$

converges but not absolutely. Hence its rearrangement may not converge. In fact, even if it converges it may converge to a different value.

Consider one rearrangement:

$$\left(1 - \frac{1}{2}\right) + \left(\frac{1}{3} + \frac{1}{5} - \frac{1}{4}\right) + \left(\frac{1}{7} + \frac{1}{9} - \frac{1}{6}\right) + \left(\frac{1}{11} + \frac{1}{13} - \frac{1}{8}\right) + \cdots$$

Then sum may be bigger than $\ln 2 = 0.69314\cdots$.

Product of two series

Suppose $\sum_{n=0}^{\infty} a_n, \sum_{n=0}^{\infty} b_n$ converge absolutely. Then

$$\left(\sum_{n=0}^{\infty} a_n\right) \times \left(\sum_{n=0}^{\infty} b_n\right) = (a_0 + a_1 + \cdots + a_n + \cdots) \times (b_0 + b_1 + \cdots + b_n + \cdots).$$

Finite partial sum is

$$(a_0 + a_1 + \cdots + a_n) \times (b_0 + b_1 + \cdots + b_n).$$

We can write it as

$$a_0b_0 + (a_0b_1 + a_1b_0) + (a_0b_2 + a_1b_1 + a_2b_0) + \cdots$$
$$+ \cdots + (a_0b_n + a_1b_{n-1} + \cdots + a_{n-1}b_1 + a_nb_0).$$

In the limit,

$$\left(\sum_{n=0}^{\infty} a_n\right) \times \left(\sum_{n=0}^{\infty} b_n\right)$$
$$= a_0b_0 + (a_0b_1 + a_1b_0) + (a_0b_2 + a_1b_1 + a_2b_0) + \cdots$$
$$+ \cdots (a_0b_n + a_1b_{n-1} + \cdots + a_{n-1}b_1 + a_nb_0) + \cdots$$

Since it converges absolutely, it value does not change.

Theorem 10.6.18. *Suppose both* $\sum_{n=0}^{\infty} a_n$ *and* $\sum_{n=0}^{\infty} b_n$ *converge absolutely. If we set* $c_n = \sum_{n=0}^{k} a_k b_{n-k}$ *then* $\sum c_n$ *converge absolutely and*

$$\sum_{n=0}^{\infty} c_n = \left(\sum_{n=0}^{\infty} a_n\right) \times \left(\sum_{n=0}^{\infty} b_n\right).$$

10.7 Power Series

Definition 10.7.1. A **power series** about $x = 0$ is a series of the form

$$\sum_{n=0}^{\infty} a_n x^n = a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n + \cdots$$

A power series about $x = a$ is a series of the form

$$\sum_{n=0}^{\infty} a_n (x-a)^n.$$

Here a_n are the **coefficients** and x_0 is the **center**.

Example 10.7.2. (1) (Geometric series) $\sum_{n=1}^{\infty} \frac{(x-1)^n}{2^n} = \frac{1}{2^1} + \frac{(x-1)^2}{2^2} + \frac{(x-1)^3}{2^3} + \cdots$

(2) $\sum_{n=1}^{\infty} (-1)^{n-1} \frac{x^n}{n} = x - \frac{x^2}{2} + \frac{x^3}{3} - \cdots$

(3) $\sum_{n=1}^{\infty} (-1)^{n-1} \frac{x^{2n-1}}{2n-1} = x - \frac{x^3}{3} + \frac{x^5}{5} - \cdots$

(4) $\sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$

(5) $\sum_{n=0}^{\infty} n! x^n = 1 + x + 2! x^2 + 3! x^3 + \cdots$

Theorem 10.7.3 (Convergenec of Power Series)**.** *Given a power series* $\sum_{n=0}^{\infty} a_n (x - x_0)^n$

(1) Suppose it converges at a point $x_1 (\neq x_0)$*. Then it converges absolutely for all points* x *satisfying* $|x - x_0| < |x_1 - x_0|$*.*

(2) Suppose it diverges at x_2 *it. Then it diverges for all* x *with* $|x - x_0| > |x_2 - x_0|$*.*

Proof. Suppose $\sum_{n=0}^{\infty} a_n(x_1-x_0)^n$ converges. Then $\lim_{n\to\infty} a_n(x_1-x_0)^n = 0$. Hence for suff. large n, it holds that $|a_n(x_1-x_0)^n| \le 1$ and

$$|a_n(x-x_0)^n| \le |a_n(x_1-x_0)^n| \left|\frac{x-x_0}{x_1-x_0}\right|^n \le \left|\frac{x-x_0}{x_1-x_0}\right|^n.$$

Hence for all x with $|x-x_0| < |x_1-x_0|$, the series $\sum_{n=0}^{\infty} a_n(x_1-x_0)^n$ converges absolutely. Now suppose the series $\sum_{n=0}^{\infty} a_n(x_2-x_0)^n$ diverges and $\sum_{n=0}^{\infty} a_n(x-x_0)^n$ converge for some x satisfying $|x-x_0| > |x_2-x_0|$. Then by (1) the series $\sum_{n=0}^{\infty} a_n(x_2-x_0)^n$ must converge, which is a contradiction. Hence the series $\sum_{n=0}^{\infty} a_n(x-x_0)^n$ must diverges for any x with $|x-x_0| > |x_2-x_0|$.

From Theorem 10.7.3, there are three possibilities for the series $\sum_{n=0}^{\infty} a_n(x-x_0)^n$:

(1) It converges for x_0 only;

(2) It converges absolutely for all x;

(3) There exists an R such that for all x with $|x-x_0| < R$ it converges absolutely and diverges for all x with $|x-x_0| > R$.

We see that in case (1) $R = 0$, and in case (2) $R = \infty$. In general, the number $R\,(0 \le R \le \infty)$ is called **the radius of convergence** of $\sum_{n=0}^{\infty} a_n(x-x_0)^n$.

Theorem 10.7.4. *For $\sum_{n=0}^{\infty} a_n(x-x_0)^n$, the radius of convergence is given as follows:*

$$R = \lim_{n\to\infty} \left|\frac{a_n}{a_{n+1}}\right| \tag{10.1}$$

$$R = \lim_{n\to\infty} \frac{1}{\sqrt[n]{|a_n|}} \tag{10.2}$$

provided that either of the limit exists.

Proof. Suppose the limit in (10.1) exists. Then

$$\lim_{n\to\infty} \left|\frac{a_{n+1}(x-x_0)^{n+1}}{a_n(x-x_0)^n}\right| = \lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| |x-x_0| = \frac{|x-x_0|}{R}.$$

Now by ratio test (Theorem 10.5.2), the power series converges absolutely for $|x-x_0|/R < 1$ and diverges if $|x-x_0|/R > 1$. Hence R given by (10.1) is the radius of convergence. One can show (10.2) holds if we use n-th root test (Theorem 10.5.6).

The set of all point for which the series converges form an interval I (called the **interval of convergence**) and I satisfies

$$(x_0 - R, x_0 + R) \subset I \subset [x_0 - R, x_0 + R].$$

Example 10.7.5. Find the interval of convergence.

(1) $\displaystyle\sum_{n=0}^{\infty} n^n x^n$

(2) $\displaystyle\sum_{n=1}^{\infty} \frac{x^n}{n^2}$

(3) $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n-1} x^n}{n}$

(4) $\displaystyle\sum_{n=0}^{\infty} \frac{x^n}{n!}$.

sol. (2)

$$R = \lim_{n\to\infty} \frac{(n+1)^2}{n^2} = 1.$$

When $x = \pm 1$, $\sum_{n=1}^{\infty}((\pm 1)^n/n^2)$ converges absolutely. Hence $I = [-1, 1]$.

(3)

$$R = \lim_{n\to\infty} \frac{n+1}{n} = 1.$$

For $x = 1$, $\sum_{n=1}^{\infty}((-1)^{n-1}/n)$ is alternating, so conditionally converges. While $x = -1$ $\sum_{n=1}^{\infty}(-1/n)$ diverges. $I = (-1, 1]$.

(4)

$$R = \lim_{n\to\infty} \frac{(n+1)!}{n!} = \infty.$$

Theorem 10.7.6 (Term by term differentiation)**.** *Suppose* $\sum_{n=0}^{\infty} a_n (x - x_0)^n$ *converges for all* $|x - x_0| < R$ *for some* $R > 0$*, i.e.,*

$$f(x) = \sum_{n=0}^{\infty} a_n (x - x_0)^n, \quad |x - x_0| < R. \tag{10.3}$$

Then

(i) $f(x)$ *is differentiable on* $(x_0 - R, x_0 + R)$ *and its derivative is*

$$f'(x) = \sum_{n=1}^{\infty} n a_n (x - x_0)^{n-1}, \quad |x - x_0| < R. \tag{10.4}$$

(ii) $f(x)$ *is integrable on* $(x_0 - R, x_0 + R)$ *and*

$$\int f(x)\, dx = \sum_{n=0}^{\infty} a_n \frac{(x - x_0)^{n+1}}{n+1} + C, \quad |x - x_0| < R. \tag{10.5}$$

The radius convergence of (10.4) *and* (10.5) *are also* R*.*

Proof. Suppose

$$R = \lim_{n\to\infty} \left| \frac{a_n}{a_{n+1}} \right|.$$

The radius of convergence of (10.4) is given by Theorem 10.7.4

$$\lim_{n\to\infty} \left| \frac{(n+1)a_{n+1}}{(n+2)a_{n+2}} \right| = \lim_{n\to\infty} \left| \frac{a_{n+1}}{a_{n+2}} \right| = R.$$

Similarly, the radius of convergence of (10.5) is obtained.

Corollary 10.7.7. *The series in Theorem* 10.7.6 *is differentiable infinitely many times on* $(x_0 - R, x_0 + R)$ *and its* k*-th derivative is given by*

$$f^{(k)}(x) = \sum_{n=k}^{\infty} n(n-1)\cdots(n-k+1) a_n (x - x_0)^{n-k}, \quad |x - x_0| < R, \tag{10.6}$$

$k = 0, 1, \ldots.$

Product of two Power series

Theorem 10.7.8. *Suppose both* $A(x) = \sum_{n=0}^{\infty} a_n x^n$, $B(x) = \sum_{n=0}^{\infty} b_n x^n$ *converge absolutely for* $|x| < R$ *and*

$$c_n = a_0 b_n + a_1 b_{n-1} + \cdots + a_n b_0 = \sum_{n=0}^{k} a_k b_{n-k}.$$

Then $\sum_{n=0}^{\infty} c_n x^n$ *converge absolutely for* $|x| < R$ *also, and*

$$\left(\sum_{n=0}^{\infty} a_n x^n\right) \times \left(\sum_{n=0}^{\infty} b_n x^n\right) = \sum_{n=0}^{\infty} c_n x^n.$$

Example 10.7.9. Use

$$\sum_{n=0}^{\infty} x^n = 1 + x + x^2 + \cdots = \frac{1}{1-x}, \text{ for } |x| < 1$$

to get the power series for $1/(1-x)^2$.

sol. We let $A(x) = B(x) = \sum_{n=0}^{\infty} x^n$. Then we see

$$c_n = a_0 b_n + a_1 b_{n-1} + \cdots + a_n b_0 = \sum_{n=0}^{k} a_k b_{n-k} = n + 1.$$

Hence

$$A(x)B(x) = \sum_{n=0}^{\infty} c_n x^n = \sum_{n=0}^{\infty} (n+1) x^n.$$

This series could be obtained by differentiation.

10.8 Taylor and Maclaurin Series

In the previous discussions we have seen that a power series defines a continuous function on I. How about its converse? Suppose f is differentiable n-times. Is it possible to express it in power series ? A power series $\sum_{n=0}^{\infty} a_n (x-a)^n$ represents a function on its interval of convergence I

$$f(x) = \sum_{n=0}^{\infty} a_n (x-a)^n, \quad x \in I.$$

We shall later show

$$\begin{aligned}&\sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n \\ =& f(a) + f'(a)(x-a) + \cdots + \frac{f^{(n)}(a)}{n!}(x-a)^n + \cdots .\end{aligned}$$

This is called **Taylor series** of $f(x)$ at a (If $a = 0$, it is also called **Maclaurin series**).

Example 10.8.1. Find Taylor series of $f(x) = 1/x$ at $a = 2$.

sol.

$$f(x) = \frac{1}{x}, \quad f'(x) = -x^{-2}, \quad f''(x) = 2!x^{-3}, \cdots, \quad f^{(n)}(x) = (-1)^n n! x^{-(n+1)},$$

$$f(2) = \frac{1}{2}, \quad f'(2) = -\frac{1}{2^2}, \quad \frac{f''(x)}{2!} = \frac{1}{2^{-3}}, \cdots, \quad \frac{f^{(n)}(2)}{n!} = \frac{(-1)^n}{2^{n+1}}.$$

■

Taylor Polynomial

Consider

$$y = P_1(x) := f(a) + f'(x_0)(x - a).$$

This is linear approximation to $f(x)$. Similarly we can consider

$$y = P_2(x) := f(a) + f'(a)(x - a) + \frac{f''(a)}{2}(x - a)^2.$$

which has same derivative up to second order. By the same way one can find a polynomial $P_n(x)$ of degree n. It is called **a Taylor polynomial of degree** n Then we see

$$P_n^{(k)}(a) = f^{(k)}(a), \quad k = 0, 1, \cdots, n.$$

$$P_n(x) = f(a) + f'(x_0)(x - a) + \cdots + \frac{f^{(n)}(a)}{n!}(x - a)^n. \tag{10.7}$$

The difference(error) is defined as

$$R_n(x) = f(x) - P_n(x)$$

and called the **remainder**

$$f(x) = P_n(x) + R_n(x)$$

is called n-th **Taylor formula** of $f(x)$ at a.

Example 10.8.2. Find Taylor polynomial for $\cos x$.

Example 10.8.3.

$$f(x) = \begin{cases} \exp(-1/x^2), & x \neq 0 \\ 0, & x = 0. \end{cases}$$

is infinitely differentiable at 0, but the Taylor series converges only at $x = 0$. In fact we can show that $f^{(n)}(0) = 0$, $n = 0, 1, \ldots$. So the Taylor polynomial $P_n(x) = 0$ and $R_n(x) = f(x)$. Hence $P_n(x) \not\to f(x)$.

10.9 Convergence of Taylor Series, Error estimates

Theorem 10.9.1 (Taylor's Theorem with Remainder)**.** *Suppose $f(x)$ is differentiable $n+1$ times on an open interval I containing a and $P_n(x)$ is the Taylor polynomial given by* (10.7)*. Then*

$$R_n(x) = \frac{f^{(n+1)}(c)}{(n+1)!}(x-a)^{n+1}. \tag{10.8}$$

Definition 10.9.2. Suppose $f(x)$ is infinitely differentiable on I and

$$\lim_{n\to\infty} R_n(x) = 0, \quad x \in I$$

then we say the Taylor series at a **converges** to $f(x)$ and we we write

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n, \quad x \in I.$$

Here $R_n(x) = f(x) - P_n(x)$ is the remainder.

Corollary 10.9.3. *Suppose there is some M such that $f(x)$ satisfies $|f^{(n+1)}(x)| \leq M$ for all $x \in I$. Then*

$$|R_n(x)| \leq M\frac{|x-x_0|^{n+1}}{(n+1)!}, \quad x \in I. \tag{10.9}$$

Example 10.9.4. At $a = 0$, we have

$$e^x = 1 + x + \cdots + \frac{x^n}{n!} + R_n(x).$$

Here

$$|R_n(x)| \leq e^c \frac{x^{n+1}}{(n+1)!}.$$

Example 10.9.5. (1) Maclaurin series of $\sin x$, $\cos x$, e^x:

$$\sin x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!}, \quad -\infty < x < \infty$$
$$\cos x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!}, \quad -\infty < x < \infty$$
$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}, \quad -\infty < x < \infty$$

(2) Maclaurin series of $\ln(1+x)$ on $(0,\infty)$

$$\ln(1+x) = \sum_{n=1}^{\infty} \frac{(-1)^{n-1} x^n}{n}, \quad -1 < x \leq 1$$

(3) Maclaurin series of $1/(1-x)$

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n, \quad -1 < x < 1$$

(4) Taylor series of $\sqrt{x}$ is at 1.

Example 10.9.6 (Substitution)**.** Find series for $\cos x^2$ near $x = 0$.

Example 10.9.7 (Multiplication)**.** Find series for $x \sin x^2$ near $x = 0$.

Example 10.9.8 (Truncation Error)**.** For what values of x can we replace $\sin x$ by $\sin x \approx x - \frac{x^3}{3!}$ with error less than 3×10^{-4}?

$$\sin x \approx x - \frac{x^3}{3!}.$$

Since the error term is $R_3(x) = \frac{|x|^5}{5!}$, we let

$$\frac{|x|^5}{5!} \leq 3 \times 10^{-4}.$$

Proof of Taylor's Formula with Remainder

With

$$P_n(x) = f(a) + f'(a)(x-a) + \cdots + \frac{f^{(n)}(a)}{n!}(x-a)^n$$

we set

$$\phi_n(x) = P_n(x) + K(x-a)^{n+1}.$$

This function has same first n-derivative as f at a. We can choose K so that $\phi_n(x)$ agrees with $f(x)$. The idea is to fix $x = b$ and choose K so that $\phi_n(b)$ agrees with $f(b)$. So

$$f(b) = P_n(b) + K(b-a)^{n+1}, \text{ or } K = \frac{f(b) - P_n(b)}{(b-a)^{n+1}} \tag{10.10}$$

and

$$F(x) = f(x) - \phi_n(x)$$

is the error. We use Rolle's theorem. First, since $F(b) = F(a) = 0$, we have

$$F'(c_1) = 0, \qquad \text{for some } c_1 \in (a, b).$$

Next, because $F'(a) = F'(c_1) = 0$, we have

$$F''(c_2) = 0, \qquad \text{for some } c_2 \in (a, c_1).$$

Now repeated application of Rolle's theorem to F'', etc show there exist

$$\begin{array}{lll} c_3 & \text{in } (a, c_2) & \text{such that } F'''(c_3) = 0, \\ c_4 & \text{in } (a, c_3) & \text{such that } F^{(4)}(c_4) = 0, \\ & \vdots & \\ c_n & \text{in } (a, c_{n-1}) & \text{such that } F^{(n)}(c_n) = 0, \\ c_{n+1} & \text{in } (a, c_n) & \text{such that } F^{(n+1)}(c_{n+1}) = 0. \end{array}$$

But since $F(x) = f(x) - \phi_n(x) = f(x) - P_n(x) - K(x-a)^{n+1}$, we see

$$F^{(n+1)}(c) = f^{(n+1)}(c) - 0 - (n+1)!K.$$

Hence

$$K = \frac{f^{(n+1)}(c)}{(n+1)!}, \quad c = c_{n+1}.$$

So

$$f(b) = P_n(b) + \frac{f^{(n+1)}(c)}{(n+1)!}(b-a)^{n+1}. \tag{10.11}$$

Now since b is arbitrary, we can set $b = x$. Furthermore, if $R_n \to$ as $n \to \infty$, we obtain Taylor's theorem.

10.10 Application

Binomial Series

First assume m is a positive integer and consider the binomial expansion

$$(1+x)^m = 1 + mx + \frac{m(m+1)}{2!}x^2 + \cdots + \binom{m}{k}x^k + \cdots + x^m.$$

Here

$$\binom{m}{k} = \frac{m(m-1)\cdots(m-k+1)}{k!}, \quad n = 0, 1, 2, \ldots .$$

We now consider the Taylor series of $(1+x)^m$. Since

$$\begin{aligned} f(x) &= (1+x)^m \\ f'(x) &= m(1+x)^{m-1} \\ f''(x) &= m(m-1)(1+x)^{m-2} \\ &\cdots \\ f^{(k)}(x) &= m(m-1)(m-2)\cdots(m-k+1)(1+x)^{m-k} \end{aligned} \tag{10.12}$$

we obtain the Taylor series

$$(1+x)^m = 1 + mx + \frac{m(m+1)}{2!}x^2 + \cdots + \binom{m}{k}x^k + \cdots . \tag{10.13}$$

We can show the radius of convergence is $R = 1$. When m is an integer, the derivatives $f^{(k)}(x) = 0$ for $k \geq m$, and we obtain the usual binomial expansion as a special case.

Example 10.10.1.

$$\frac{1}{(1+x)} = 1 - x + x^2 - x^3 + \cdots + (-1)^k x^k + \cdots$$

$$(1+x)^{1/2} = 1+\frac{x}{2}+\frac{(\frac{1}{2})(-\frac{1}{2})}{2!}x^2+\frac{(\frac{1}{2})(-\frac{1}{2})(-\frac{3}{2})}{3!}x^3+\frac{(\frac{1}{2})(-\frac{1}{2})(-\frac{3}{2})(-\frac{5}{2})}{4!}x^4+\cdots$$
$$= 1 + \frac{x}{2} - \frac{x^2}{8} + \frac{x^3}{16} - \frac{5x^4}{128} + \cdots . \quad (10.14)$$

Substitution gives

$$\sqrt{1-x^2} = 1 - \frac{x^2}{2} - \frac{x^4}{8} + \cdots, \ |x^2| < 1$$

or

$$\sqrt{1-x^3} = 1 - \frac{x^3}{2} - \frac{x^6}{8} + \cdots \ |x^3| < 1$$

or even

$$\sqrt{1-\frac{1}{x}} = 1 - \frac{1}{2x} - \frac{1}{8x^2} + \cdots \ |\frac{1}{x}| < 1$$

are possible.

Example 10.10.2. Find $\sqrt{1.2}$ up to two decimal point.

sol. Let $f(x) = \sqrt{1+x}$. Then $\sqrt{1.2} = f(0.2)$. Hence from equation (10.13) We see Taylor series at $x_0 = 0$ is

$$f(x) = 1 + \frac{1}{2}x + \cdots + \binom{1/2}{n}x^n + R_{n+1}(x),$$
$$R_{n+1}(x) = \frac{1}{(n+1)!}f^{(n+1)}(\bar{x})x^{n+1} \quad (0 \le \bar{x} \le 0.2.)$$

For $n = 1$ $R_2(0.2) = (1/2)f''(\bar{x})(0.2)^2 = -0.005(1+\bar{x})^{-3/2}$ $(0 \le \bar{x} \le 0.2)$. Hence $\sqrt{1.2} \approx 1 + (1/2)(0.2) = 1.1$ and the error satisfies $|R_2(0.2)| < 0.005$.

■

Example 10.10.3. Find $\int \sin^2 x\,dx$ as power series.

Estimate $\int_0^1 \sin^2 x\,dx$ within error less than 0.001.

Example 10.10.4. Find Maclaurin series of $\arctan x$.

sol. Note that for $|x| < 1$ the $\arctan x$ has convergent power series:

$$(\arctan x)' = \frac{1}{1+x^2} = \sum_{n=0}^{\infty}(-1)^n x^{2n}.$$

Integrate it from 0 to x

$$\begin{aligned}\arctan x &= \int_0^x \sum_{n=0}^{\infty}(-1)^n t^{2n}\,dt \\ &= \sum_{n=0}^{\infty}\frac{(-1)^n x^{2n+1}}{2n+1}, \quad |x| < 1.\end{aligned}$$

Thus

$$\arctan x = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \cdot$$

This formula can be used to compute π. For example,

$$\frac{\pi}{4} = \arctan 1 = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdot$$

The error with n-term is $1/(2n+1)$. So to get the error less than 10^{-3}, we need $2n + 1 \approx 1000$, $n = 500$ terms. Because of its slowness, we suggest another methods. For example, if

$$\alpha = \tan^{-1}\frac{1}{2}, \quad \beta = \tan^{-1}\frac{1}{3},$$

then

$$\tan(\alpha+\beta) = \frac{\tan\alpha + \tan\beta}{1 - \tan\alpha\tan\beta} = \frac{\frac{1}{2}+\frac{1}{3}}{1-\frac{1}{6}} = 1 = \tan\frac{\pi}{4}$$

and

$$\frac{\pi}{4} = \alpha + \beta = \tan^{-1}\frac{1}{2} + \tan^{-1}\frac{1}{3}.$$

Now use the Taylor series for $\tan^{-1} x$ with $x = \frac{1}{2}$ and $x = \frac{1}{3}$. This is faster.

For example

$$\tan^{-1}\frac{1}{2} = (\frac{1}{2}) - \frac{1}{3}(\frac{1}{2})^3 + \frac{1}{5}(\frac{1}{2})^5 - \frac{1}{7}(\frac{1}{2})^7 + R_8^1 = 0.463467...$$

$$\tan^{-1}\frac{1}{3} = (\frac{1}{3}) - \frac{1}{3}(\frac{1}{3})^3 + \frac{1}{5}(\frac{1}{3})^5 + R_6^2 = 0.321810..$$

Here $|R_8^1| \le \frac{1}{9}(\frac{1}{2})^9 = \frac{1}{4{,}500}$ and $|R_6^2| \le \frac{1}{7}(\frac{1}{3})^7 = \frac{1}{15{,}309}$ and

$$\tan^{-1}\frac{1}{2} + \tan^{-1}\frac{1}{3} = 0.7852777$$

Multiply by 4 we get

$$\pi \approx 3.14111...$$

which is accurate at least three decimals.

Similar idea can be used to the following problem:

Example 10.10.5. Estimate

$$\ln 2 = \ln(1+1) = 1 - \frac{1}{2} + \cdots + \frac{(-1)^{n-1}}{n} + R_{n+1}(1).$$

Since

$$|R_{n+1}(1)| \le \frac{1}{n+1}$$

we need to take large n. However, we can do the following:

$$\ln 2 = \ln\frac{4}{3} \cdot \ln\frac{3}{2}2 = \ln(1+\frac{1}{3}) + \ln(1+\frac{1}{2})$$

and use Taylor series.

Example 10.10.6. Estimate $\int_0^1 \sin x^2\, dx$ with error less than 0.001.

sol. First note that

$$\sin x^2 = x^2 - \frac{x^6}{3!} + \frac{x^{10}}{5!} - \frac{x^{14}}{7!} + \cdots$$

Integrating

$$\int_0^1 \sin x^2\, dx = \frac{1}{3} - \frac{1}{7\cdot 3!} + \frac{1}{11\cdot 5!} - \frac{1}{15\cdot 7!} + \cdots$$

Since

$$\frac{1}{11\cdot 5!} < 0.00076$$

it suffices to take two terms.

Example 10.10.7. Estimate sin(0.1) up to third digit 3.

sol. Taylor polynomial of $\sin x$ at $x_0 = 0$

$$\sin x = \sum_{k=0}^{n} \frac{1}{k!} \left(\frac{d}{dx}\right)^k \sin x \bigg|_{x=0} x^k + R_{n+1}(x).$$

Since $|\sin x| \le 1$, for $|\cos x| \le 1$

$$|R_{n+1}(x)| \le \frac{|x|^{n+1}}{(n+1)!}.$$

If $n = 2$

$$|R_3(0.1)| \le \frac{(0.1)^3}{3!} < 10^{-3}$$

we have $\sin(0.1) \approx 0.1$ and the error is less than $\pm(1/6) \times 10^{-3}$.

Indeterminate forms

Example 10.10.8. Find

$$\lim_{x\to 1} \frac{\ln x}{x-1}.$$

Use the Taylor series of $\ln x$ ar $x = 1$.

Example 10.10.9. Find

$$\lim_{x\to 0} \frac{\sin x - x + (x^3/6)}{x^4}.$$

sol. $x_0 = 0$. Taylor polynomial of $\sin x$ at$x_0 = 0$ is

$$\sin x = x - \frac{x^3}{6} + R_5(x) \quad \text{and} \quad |R_5(x)| \le \frac{|x|^5}{5!}.$$

Hence

$$\left|\frac{\sin x - x + (x^3/6)}{x^4}\right| = \left|\frac{R_5(x)}{x^4}\right| \le \frac{|x|}{5!}$$

and limit is 0.

Example 10.10.10. Find

$$\lim_{x\to 0} \left(\frac{1}{\sin x} - \frac{1}{x}\right).$$

sol.

$$\begin{aligned}
\frac{1}{\sin x} - \frac{1}{x} &= \frac{x - \sin x}{x \sin x} \\
&= \frac{x - \left(x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots\right)}{x \cdot \left(x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots\right)} \\
&= \frac{x^3 \left(\frac{1}{3!} - \frac{x^2}{5!} + \cdots\right)}{x^2 \left(1 - \frac{x^2}{3!} + \cdots\right)}
\end{aligned}$$

Euler's identity

$$\begin{aligned}
e^{i\theta} &= 1 + \frac{i\theta}{1!} + \frac{i^2\theta^2}{2!} + \frac{i^3\theta^3}{3!} + \frac{i^4\theta^4}{4!} + \cdots \\
&= \left(1 - \frac{\theta}{2!} + \frac{\theta^4}{4!} - \frac{\theta^6}{6!} + \cdots\right) + i\left(\theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - \cdots\right) \\
&= \cos\theta + i\sin\theta.
\end{aligned}$$