Contents

Chapter 3

Differentiation

3.8 Inverse functions and Their Derivatives

Definition 3.8.1. A function f is one-to-one on a domain D if $f(x_1) \neq f(x_2)$ whenever $x_1 \neq x_2$.

Definition 3.8.2. Suppose a function f is one-to-one on a domain D with range R. The **inverse function** f^{-1} exists and is defined by

$$f^{-1}(b) = a \text{ if } f(a) = b.$$

The domain of f^{-1} is R and range is D.

$$(f^{-1} \circ f)(x) = x, \quad x \in D$$
$$(f \circ f^{-1})(y) = y, \quad y \in R$$

Derivatives of inverse function

Theorem 3.8.3. *Suppose f is differentiable in I. If $f'(x)$ is never zero, then f^{-1} exists, differentiable. Furthermore for $a \in I$, $f(a) = b$,*

$$(f^{-1})'(b) = \frac{1}{f'(a)}.$$

Set $y = f(x)$. Then the inverse function is $x = f^{-1}(y)$, and its derivative is

$$\left.\frac{dx}{dy}\right|_{y=f(a)} = \frac{1}{dy/dx|_{x=a}}, \quad a \in I$$

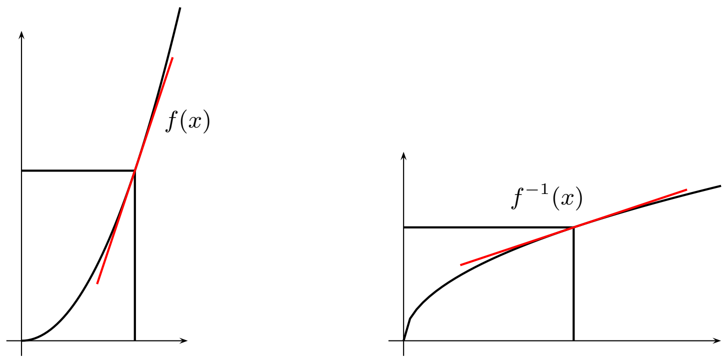

Figure 3.1: Slope of inverse function

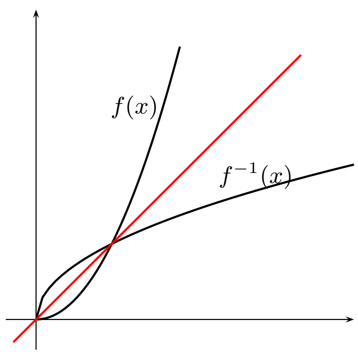

Figure 3.2: Graph of inverse function is symmetric about $y = x$

Proof. Differentiate $x = (f^{-1} \circ f)(x) = f^{-1}(f(x)) = f^{-1}(y)$ w.r.t x using the Chain rule, we have

$$1 = (f^{-1})'(f(x))f'(x).$$

Setting $x = a$, we see $1 = (f^{-1})'(f(a))f'(a)$. Thus

$$(f^{-1})'(b) = 1/f'(a).$$

Usually, we use the notation $y = f^{-1}(x)$. The graph of $y = f(x)$ and that of $y = f^{-1}(x)$ are symmetric w.r.t the line $y = x$.

Example 3.8.4. (1) $f(x) = x^7 + 8x^3 + 4x - 2$. Find $(f^{-1})'(-2)$.

(2) $f(x) = \sin^{-1} x$. Find $(f^x)'$.

sol. (1) Since $f' = 7x^6 + 24x^2 + 4 \geq 4$ inverse f^{-1} exists. Since $f(0) = -2$

we have

$$(f^{-1})'(-2) = (f^{-1})'(f(0)) = \frac{1}{f'(0)} = \frac{1}{4}.$$

(2) $y = \sin^{-1} x$, $x = \sin y$. Hence

$$\begin{aligned}\frac{d}{dx}\sin^{-1} x &= \frac{dy}{dx} = \frac{1}{dx/dy} = \frac{1}{(d/dy)\sin y} \\ &= \frac{1}{\cos y} = \frac{1}{\sqrt{1-\sin^2 y}} = \frac{1}{\sqrt{1-x^2}}.\end{aligned}$$

□

3.9 Logarithmic functions

Definition 3.9.1. For $x > 0$, the (natural) logarithmic function is defined by

$$\ln x = \int_1^x \frac{1}{t}\,dt.$$

Thus by fundamental theorem,

$$\frac{d}{dx}\ln x = \frac{1}{x}. \tag{3.1}$$

If $u(x)$ is any positive differentiable function,

$$\frac{d}{dx}\ln u(x) = \frac{1}{u}\frac{du}{dx}. \tag{3.2}$$

Properties:

(1) $\ln bx = \ln b + \ln x$

(2) $\ln \frac{b}{x} = \ln b - \ln x$

(3) $\ln \frac{1}{x} = -\ln x$

(4) $\ln x^r = r \ln x$ (For rational number r).

The proof of (4) is in Example 3.9.5.

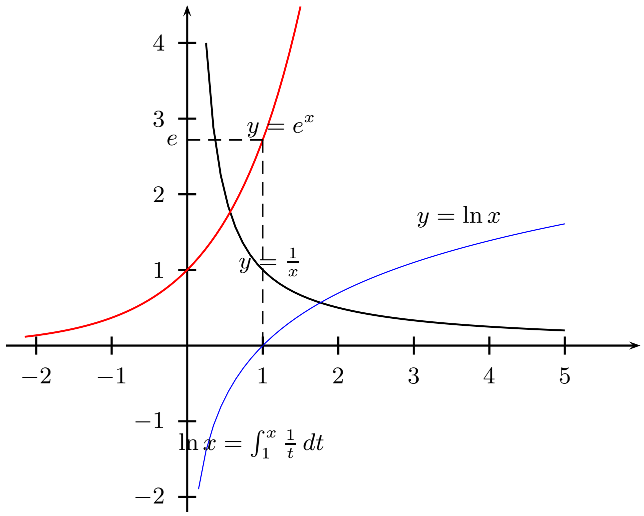

Figure 3.3: Graph of $\ln x$ and e^x

Exponential function

Definition 3.9.2. Define the (natural) exponential function $exp(x) := \ln^{-1} x$ as the inverse function of $\ln x$. Thus

$$y = exp(x) \Leftrightarrow x = \ln y.$$

Thus

$$exp(\ln x) = x, \ (x > 0) \tag{3.3}$$

$$\ln(exp(x)) = x. \tag{3.4}$$

The number e is defined as

$$e = exp(1) = \ln^{-1}(1) = 2.718281828\cdots . \tag{3.5}$$

The function e^x

We can raise the number e to a rational power such as:

$$e^2 = e \cdot e, \ e^{1/2} = \sqrt{e}, \ldots$$

For rational number x, the value e^x is well defined. We see

$$\ln e^x = x \ln e = x. \tag{3.6}$$

Thus the rational power e^x is the same as the $exp(x)$. Thus it is natural to define

Definition 3.9.3. For all real number x, we let

$$\boxed{e^x = exp(x)}. \tag{3.7}$$

The exponential function $exp(x)$ satisfies the usual rule for exponentiation such as $e^{a+b} = e^a e^b$.

Exponential function a^x

Since $a = e^{\ln a}$ for any positive number a, we can define a^x by

$$\begin{aligned} a^x &= e^{\ln a^x} \\ &= e^{x \ln a} \\ &= e^{(\ln a)x}. \end{aligned}$$

Definition 3.9.4. If a is a positive number and x is any number, we define

$$\boxed{a^x = e^{x \ln a}.} \tag{3.8}$$

Since $\ln e^x = x$ for all real x, we have

$$\ln x^n = \ln(e^{n \ln x}) = n \ln x, x > 0.$$

One can also use the definition of $\ln x = \int_1^x dt$ to prove it.

Example 3.9.5. [Power rule] The derivative of x^n for any number n:

$$\begin{aligned} \frac{d}{dx} x^n &= \frac{d}{dx} e^{n \ln x} \ (x > 0) \\ &= e^{n \ln x} \cdot \frac{d}{dx}(n \ln x) \\ &= x^n \cdot \frac{n}{x} \\ &= n x^{n-1}. \end{aligned}$$

Derivative of a^x

By definition, $a^x = e^{x \ln a}$. Thus

$$\frac{d}{dx}a^x = \frac{d}{dx}e^{x \ln a} = \ln a e^{x \ln a} = a^x \ln a.$$

$$\frac{d}{dx}a^u = a^u \ln a \frac{du}{dx}$$

General logarithmic function $\log_a x$

$y = \log_a x$ is defined as the inverse function of $y = a^x (a > 0, a \neq 1)$. Thus

$$\log_a x = y \Leftrightarrow a^y = x$$

$$\log_a(a^x) = x, \text{ for all } x, \text{ and } a^{(\log_a x)} = x, (x > 0)$$

$\log_{10} x$ is written as $\log x$ and called *common logarithmic function*

Properties

(1) Product rule: $\log_a xy = \log_a x + \log_a y$.

(2) Quotient rule: $\log_a \frac{x}{y} = \log_a x - \log_a y$.

(3) Product rule: $\log_a \frac{1}{y} = -\log_a y$.

(4) Power rule: $\log_a x^y = y \log_a x$.

Inverse properties

(1) Base a: $a^{\log_a x} = x$, $\log_a(a^x) = x (a > 0, a \neq 1, x > 0)$.

(2) Base e: $e^{\ln x} = x$, $\ln(e^x) = x (x > 0)$.

Derivative of $\log_a x$

We have

$$\log_a x = \frac{\ln x}{\ln a}. \tag{3.9}$$

Proof.

$$\begin{aligned} a^{(\log_a x)} &= x \\ \ln a^{(\log_a x)} &= \ln x \\ \log_a x \cdot \ln a &= \ln x \\ \log_a x &= \frac{\ln x}{\ln a}. \end{aligned}$$

So

$$\frac{d}{dx}\log_a x = \frac{1}{x \ln a}$$

and

$$\frac{d}{dx}\log_a u = \frac{1}{u \ln a}\frac{du}{dx}.$$

Logarithmic Differentiation

Find dy/dx if $y = \frac{(x^2+1)^{1/3}(x-3)^{1/2}}{x+5}$.

Derivatives of Power function

We prove the following theorem which was stated earlier.

Theorem 3.9.6. *For any real r, $\frac{d}{dx}u^r = u^{r-1}\frac{du}{dx}$.*

Proof. Since $u^r = e^{r \ln u}$ we have

$$\frac{d}{dx}u^r = ru^r\frac{d\ln u}{dx} = ur^r\frac{1}{u}\frac{du}{dx} = ru^{r-1}\frac{du}{dx}.$$

Example 3.9.7. Differentiate $f(x) = x^x, x > 0$

sol. Write $f(x) = x^x = e^{x \ln x}$. So

$$\begin{aligned} f'(x) &= \frac{d}{dx}(e^{x\ln x}) \\ &= (e^{x \ln x})\frac{d}{dx}(x \ln x) \\ &= e^{x \ln x}(\ln x + x \cdot \frac{1}{x}) \\ &= x^x(\ln x + 1). \end{aligned}$$

■

Example 3.9.8. Sketch the graph of $x^{1/x}$. To do this, we first investigate the behavior of the function as $x \to 0^+$ and $x \to \infty$. To study the limit, we take the logarithm:

$$\lim_{x\to\infty} \ln x^{1/x} = \lim_{x\to\infty} \frac{\ln x}{x} = 0$$

Hence

$$\lim_{x\to\infty} x^{1/x} = \exp(\lim_{x\to\infty} \frac{\ln x}{x}) = e^0 = 1$$

Meanwhile

$$\lim_{x\to 0^+} \ln x^{1/x} = \lim_{x\to 0^+} \frac{\ln x}{x} = -\infty.$$

Hence

$$\lim_{x\to 0^+} x^{1/x} = \exp(\lim_{x\to 0^+} \frac{\ln x}{x}) = e^{-\infty} = 0.$$

To see the local extrema, take the derivative and find the critical point. $f'(x) = (1 - \ln x)/x^2 = 0$ for $x = e$. By checking the sign of $f'(x)$ near $x = e$, we conclude $x = e$ is a point of local maximum.

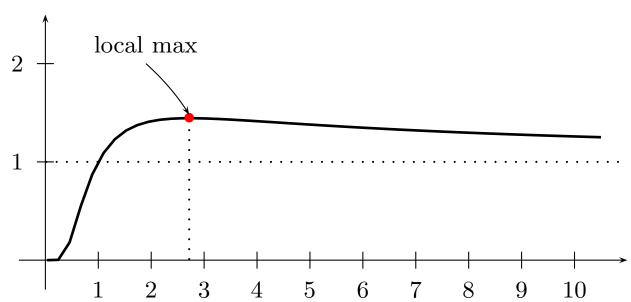

Figure 3.4: Graph of $y = x^{1/x}$

The number e as a limit

Theorem 3.9.9. *The number e satisfies*

$$e = \lim_{x\to 0}(1 + x)^{1/x}.$$

Proof. If $f(x) = \ln x$. Then $f'(1) = \frac{1}{x}|_{x=1} = 1$. By definition,

$$1 = f'(1) = \lim_{x\to 0^+} \frac{\ln(1 + x) - \ln 1}{x} = \lim_{x\to 0^+} \ln[(1 + x)^{\frac{1}{x}}] = \ln[\lim_{x\to 0^+} (1 + x)^{\frac{1}{x}}].$$

Now exponentiate.

3.10 Inverse trig functions

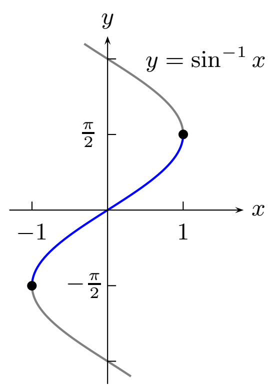

Figure 3.5: $y = \sin^{-1} x$

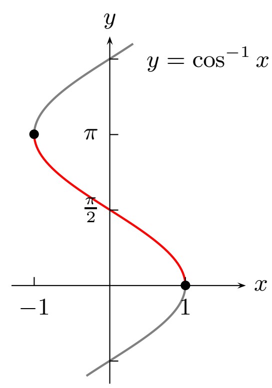

Figure 3.6: $y = \cos^{-1} x$

Inverse sine

Restrict the function $\sin x$ on $[-\pi/2, \pi/2]$. Then $\sin x\colon [-\pi/2, \pi/2] \to [-1, 1]$ is one-to - one function. So the inverse exists. Define

$$\sin^{-1} x\colon [-1, 1] \longrightarrow [-\pi/2, \pi/2].$$

whenever $x = \sin y$ for $x \in [-\pi/2, \pi/2]$. Graph is as in figure 3.5. $\sin^{-1} x$ is sometimes written as $\arcsin x$.

Inverse cosine

Restrict $\cos x$ to $[0, \pi]$, we obtain $\cos^{-1} x$ as

$$\cos^{-1} x\colon [-1, 1] \longrightarrow [0, \pi].$$

If $\cos x = y$ for any $x \in [0, \pi]$ then $\cos^{-1} y = x$ is defined and figure is in 3.6 written as $\cos^{-1} x$ or $\arccos x$.

Example 3.10.1. (1) $\sin^{-1}(1/2) = \pi/6$

(2) $\sin^{-1} 1 = \pi/2$

Example 3.10.2. (1) $\cos^{-1}(1/2) = \pi/3$

(2) $\cos^{-1} 0 = \pi/2$

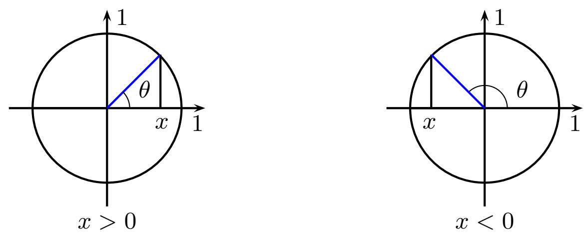

Figure 3.7: $\theta = \cos^{-1} x$

Example 3.10.3.

$$\sin^{-1} x + \cos^{-1} x = \frac{\pi}{2}, \quad \cos^{-1} x + \cos^{-1}(-x) = \pi$$

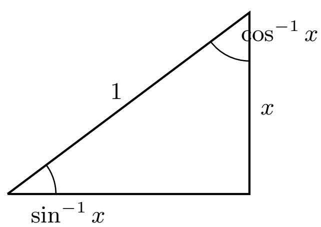

Figure 3.8: $\sin^{-1} x$

Inverse of $\tan x$

The function $\tan x$ is one to one on $(-\pi/2, \pi/2)$, thus it has an inverse $\tan^{-1} x$

$$\tan^{-1} x \colon \mathbb{R} \longrightarrow (-\pi/2, \pi/2)$$

for any $x \in \mathbb{R}$. Thus $\tan x = \alpha$ iff $\tan^{-1} \alpha = x$. See figure 3.9. It is written as $\tan^{-1} x$ or $\arctan x$.

$\tan^{-1} 1 = \pi/4$ $\tan^{-1} 0 = 0$.

Example 3.10.4. Find the derivative of $\tan^{-1} x$.

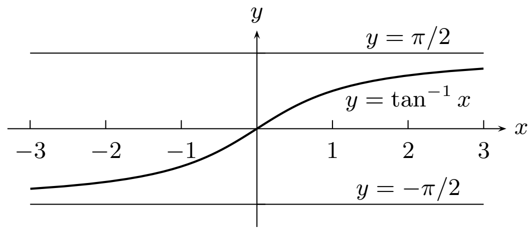

Figure 3.9: $y = \tan^{-1} x$

From $y = f(x) = \tan x$, we see by Theorem 3.8.3

$$\begin{aligned}(f^{-1})'(y) &= \frac{1}{f'(x)} \\ &= \frac{1}{1+\tan^2 x} \\ &= \frac{1}{1+y^2}.\end{aligned}$$

Thus $(f^{-1})'(x) = \frac{1}{1+x^2}$.

Example 3.10.5. Find derivatives

(1) $y = \sin^{-1} x, \quad (|x| \le 1)$.

(2) $y = \sec^{-1} x, \quad (|x| \ge 1)$.

sol. (3) Let $y = \sec^{-1} x$. Then $x = \sec y$. Taking derivative w.r.t x, we get $1 = \sec y \tan y (dy/dx)$. Thus

$$\frac{dy}{dx} = \frac{1}{\sec y \tan y}.$$

We need to change it to expression in x.

For $x > 1$, $\tan y = \sqrt{x^2-1}$. Hence, we have

$$\frac{dy}{dx} = \frac{1}{x\sqrt{x^2-1}}, \quad x > 1.$$

For $x < -1$, use $(x \to -x)$ to get

$$\frac{dy}{dx} = \frac{1}{-x\sqrt{x^2-1}}, \quad x < -1.$$

Hence

$$\frac{d}{dx}\sec^{-1}x = \frac{1}{|x|\sqrt{x^2-1}}, \quad |x| > 1.$$

Other inverse trig functions

Inverses of $\csc x$, $\sec x$, $\cot x$

$\csc^{-1}x : \mathbb{R} - (-1,1) \to [-\pi/2, \pi/2] - \{0\}$

$\sec^{-1}x : \mathbb{R} - (-1,1) \to [0,\pi] - \{\pi/2\}$

$\cot^{-1}x : \mathbb{R} \to (0,\pi)$. (Note that the range is different from that of $\tan^{-1}x$)

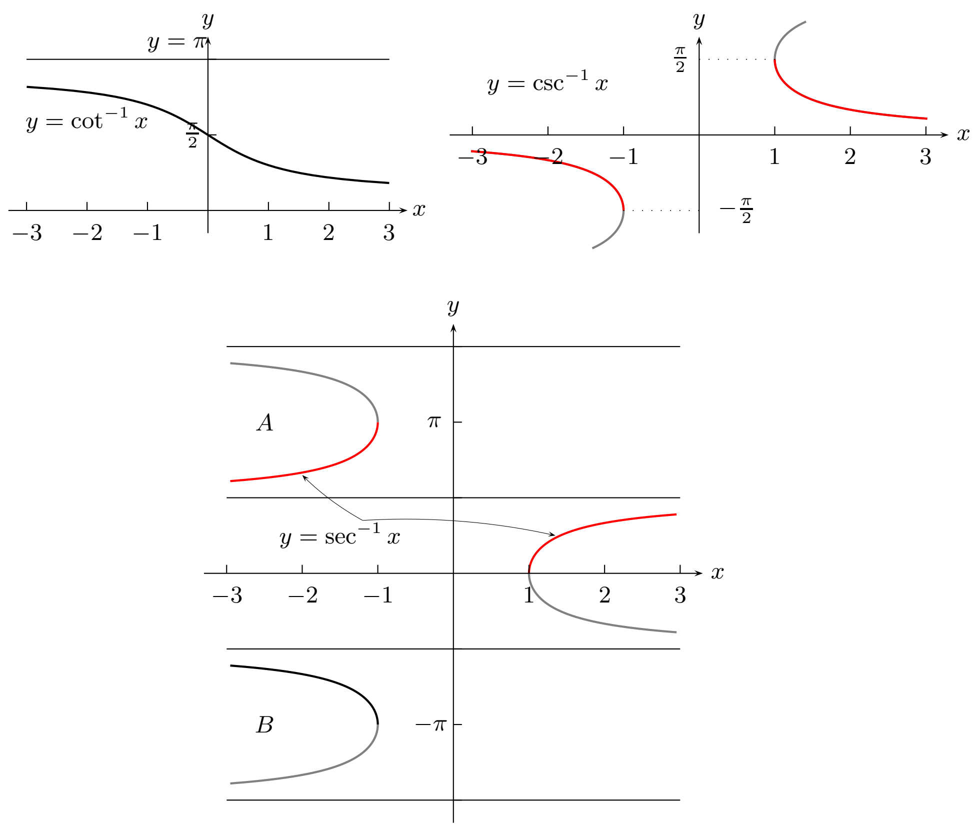

Figure 3.10:

Proposition 3.10.6. *The derivatives of inverse trig. functions :*

(1) $\frac{d}{dx}\sin^{-1}x = \frac{1}{\sqrt{1-x^2}}$

(2) $\frac{d}{dx}\cos^{-1}x = -\frac{1}{\sqrt{1-x^2}}$

(3) $\frac{d}{dx}\tan^{-1}x = \frac{1}{1+x^2}$

(4) $\frac{d}{dx}\csc^{-1}x = -\frac{1}{|x|\sqrt{x^2-1}}, \ |x|>1$

(5) $\frac{d}{dx}\sec^{-1}x = \frac{1}{|x|\sqrt{x^2-1}}, \ |x|>1$

(6) $\frac{d}{dx}\cot^{-1}x = -\frac{1}{1+x^2}.$

Proposition 3.10.7. *The following relations hold.*

$$\begin{array}{|lcl|}\hline \cos^{-1}x & = & (\pi/2)-\sin^{-1}x \\ \cot^{-1}x & = & (\pi/2)-\tan^{-1}x \\ \csc^{-1}x & = & (\pi/2)-\sec^{-1}x \\ \cot^{-1}x & = & \tan^{-1}(1/x) \\ \sec^{-1}x & = & \cos^{-1}(1/x) \\ \csc^{-1}x & = & \sin^{-1}(1/x) \\ \hline\end{array}$$

Example 3.10.8. (1) Find $\sin(\cos^{-1}(3/5))$

(2) Simplify $\tan(\sin^{-1}a)$

sol. (1) Let $\theta = \cos^{-1}(3/5)$. Then $\cos\theta = 3/5$ and $0 \le \theta \le \pi$. Hence

$$\sin\theta = \sqrt{1-\frac{9}{25}} = \frac{4}{5}.$$

(2) Let $\theta = \sin^{-1}a$. Then $\sin\theta = a$ and $-\pi/2 \le \theta \le \pi/2$.

$$\cos\theta = \sqrt{1-a^2}.$$

Hence

$$\tan\theta = \sin\theta/\cos\theta = a/\sqrt{1-a^2}. \tag{3.10}$$

■

Integral of $\tan x, \cot x, \sec x$ **and** $\csc x$

$$\begin{aligned}
\int \tan x\,dx &= \int \frac{\sin x}{\cos x}\,dx \\
&= -\int \frac{du}{u} \\
&= -\ln|u| + C \\
&= -\ln|\cos x| + C \\
&= \ln\frac{1}{|\cos x|} + C \\
&= \ln|\sec x| + C.
\end{aligned}$$

For $\sec x$ we need special trick:

$$\begin{aligned}
\int \sec x\,dx &= \int \sec x \frac{(\sec x + \tan x)}{(\sec x + \tan x)}\,dx \\
&= \int \frac{(\sec^2 x + \sec x \tan x)}{\sec x + \tan x}\,dx \\
&= \int \frac{du}{u} \\
&= \ln|u| + C \\
&= \ln|\sec x + \tan x| + C.
\end{aligned}$$

For $\csc x$ we do similarly. Thus we have

$$\boxed{\begin{aligned}
\int \sec x\,dx &= \ln|\sec x + \tan x| + C \\
\int \csc x\,dx &= -\ln|\csc x + \cot x| + C.
\end{aligned}}$$

3.11 Linearization and differential

Definition 3.11.1. Given a differentiable function f, the linear function

$$L(x) = f(a) + f'(a)(x - a)$$

is called the **linearization** of f at a.

Example 3.11.2. (1) Find the linearization of $\cos x$ at $\pi/2$.

(2) Find an approx value of $\sqrt{1.003}$ using the linearization of $\sqrt{1+x}$ at $x = 0$.

(3) Find the linearization of $\frac{1}{\sqrt[3]{x^4+1}}$ at $x = 0$

(4) Find an approx value of $\sqrt{4.8}$

(5) Find the linearization of $\cos x$ at $\pi/2$. Ans $-x + \pi/2$.

(6) Find the linearization of $(1 + x)^k$. Ans $1 + kx$.

Differential

Definition 3.11.3. Let $y = f(x)$ be differentiable. We can treat dx(differential) like an independent variable. In this point of view, the quantity dy defined by

$$dy := f'(x)dx$$

is called the **differential** of f.

The geometric meaning of differential is given in Figure 3.11. We observe

$$\Delta y = f(a + dx) - f(a), \quad f(a + dx) = f(a) + \Delta y \approx f(a) + dy.$$

We see that dy is precisely the change of the tangent line as x changes by an amount of $dx = \Delta x$. In other words, dy is an approximation of exact change Δy.

Example 3.11.4. Find differential of

(1) $y = x^3 - \sin x$

(2) $y = \sin u(x)$

(3) $\tan(3x)$

(4) $d(\frac{x}{1+x})$.

Estimating with differentials

Radius of a circle is enlarged from 10 to 10.1. Use dA to estimate the increase in area. Compare with exact increase.

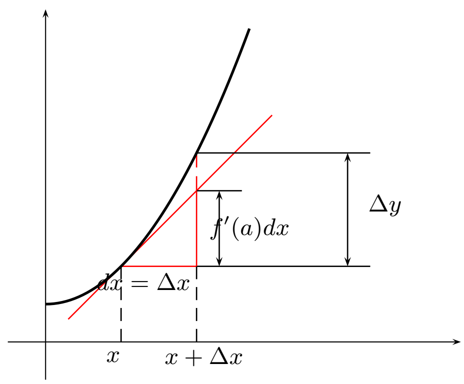

Figure 3.11: Differential $dy = f'(a)dx$ and Δy

$A = \pi r^2$,

$$dA = 2\pi r dr = 2\pi(10)(0.1) = 2\pi m^2.$$

Actual increase is $A(10.1) - A(10) = 2\pi((10.1)^2 - 100) = 2.01\pi$.

Error in differential approximation

We estimate the change in y in more detail.

Theorem 3.11.5. *We have*

$$\Delta f = f'(a)\Delta x + \epsilon \Delta x,$$

where $\epsilon \to 0$ *as* $\Delta x \to 0$.

Proof.

$$\begin{aligned}
\text{approximation error} &= \Delta f - df \\
&= \Delta f - f'(a)\Delta x \\
&= f(a+\Delta x) - f(a) - f'(a)\Delta x \\
&= \left(\frac{f(a+\Delta x) - f(a)}{\Delta x} - f'(a)\right)\Delta x \\
&= \epsilon \Delta x.
\end{aligned}$$

Since f is differentiable, we know $\epsilon := \left(\frac{f(a+\Delta x)-f(a)}{\Delta x} - f'(a)\right)$ approaches 0 as Δx approaches 0. Thus

$$\overset{\text{true change}}{\Delta f} = \overset{\text{estimated change}}{f'(a)\Delta x} + \overset{\text{error}}{\epsilon\Delta x}$$

Proof of Chain rule

Assume $y = f(u)$ is a diff'ble function of u and $u = g(x)$ is a diff'ble function of x. Then the composite function $y = f(g(x))$ is diff'ble and by theorem there exist ϵ_1, ϵ_2 which approaches 0 as $\Delta u, \Delta x$ approaches 0 in such a way that

$$\begin{aligned} \Delta y &= f'(u_0)\Delta u + \epsilon_2 \Delta u \\ \Delta u &= g'(x_0)\Delta x + \epsilon_1 \Delta x. \end{aligned}$$

Hence

$$\Delta y = (f'(u_0) + \epsilon_2)(g'(x_0) + \epsilon_1)\Delta x$$

$$\frac{\Delta y}{\Delta x} = (f'(u_0) + \epsilon_2)(g'(x_0) + \epsilon_1).$$

Let $\Delta x \to 0$. Then we obtain the Chain rule.

Example 3.11.6. Converting mass to energy: The Newton's law

$$F = m\frac{dv}{dt} = ma$$

is not exactly true when an object is moving at very high speed, because the mass increases with velocity. In Einstein's correction, the mass is

$$m = \frac{m_0}{\sqrt{1 - v^2/c^2}} \approx m_0(1 + \frac{v^2}{2c^2}).$$

So the new mass is

$$m \approx m_0 + \frac{mv^2}{2c^2}.$$

By multiplying c^2

$$(m - m_0)c^2 \approx \frac{1}{2}mv^2 - \frac{1}{2}m0^2 = \Delta(KE).$$

Thus the change in the mass corresponds to the change in the Kinetic Energy.

Chapter 7

Integrals and transcendental function

In chapter 2 we introduced natural logarithmic function as the inverse of the exponential function e^x, where the number e was chosen to satisfy certain slope condition. In this chapter, we introduce an alternative theory for exponential and log. function.

7.1 Logarithm defined as integral

Definition 7.1.1.

$$\ln x = \int_1^x \frac{1}{t}\,dt,\ \ (x > 0)$$

Derivative of $\ln x$

$$\frac{d}{dx}\ln x = \frac{d}{dx}\int_1^x \frac{1}{t}\,dt = \frac{1}{x}.$$

Hence

$$\int \frac{1}{t}\,dt = \ln|x| + C.$$

By substitution

$$\int \frac{f'(x)}{f(x)}\,dx = \ln|f(x)| + C \text{ whenever } f(x) \neq 0.$$

Example 7.1.2. (1) $\int_0^2 \frac{2x}{x^2-5}dx = \ln|u|_{-5}^{-1}$.

(2) $\int_{-\pi/2}^{\pi/2} \frac{4\cos\theta}{3+2\sin\theta}d\theta = \int_1^5 \frac{2}{u}\,du$.

Exponential function

Definition 7.1.3. Define the (natural) exponential function $exp(x)$ as the inverse function of $\ln x$. Thus

$$y = exp(x) \Leftrightarrow x = \ln y.$$

Thus

$$exp(\ln x) = x, \ (x > 0) \tag{7.1}$$
$$\ln(exp(x)) = x, \forall x. \tag{7.2}$$

Definition 7.1.4. The number e is defined as the number satisfying

$$\ln e = 1, \ \text{or} \ \int_1^e \frac{1}{t}\,dt = 1$$

The number e is approximately

$$e = 2.718281828\cdots. \tag{7.3}$$

The function e^x

We can raise the number e to a rational power such as:

$$e^2 = e \cdot e, \ e^{1/2} = \sqrt{e}, ...$$

For rational number x, the value e^x is well defined. Let us consider

$$\ln e^x = x \ln e = x. \tag{7.4}$$

Thus the rational power e^x is the same as the $exp(x)$. Thus it is natural to define

Definition 7.1.5. For all real number x, we define

$$e^x = exp(x). \tag{7.5}$$

The exponential function $exp(x)$ satisfies the usual rule for exponentiation such as $e^{a+b} = e^a e^b$.

Exponential function a^x

Since $a = e^{\ln a}$ for any positive number a, we can define a^x by

$$\begin{aligned} a^x &= e^{\ln a^x} \\ &= e^{x \ln a} \\ &= e^{(\ln a)x}. \end{aligned}$$

Definition 7.1.6.

$$\boxed{\text{If } a \text{ is a positive number and } x \text{ is any number, } a^x = e^{x \ln a}.} \tag{7.6}$$

Since $\ln e^n = x$ for all x, we have

$$\ln x^n = \ln(e^{n \ln x}) = n \ln x, \ x > 0.$$

Example 7.1.7 (Power rule)**.** The derivative of x^n for any number n:

$$\begin{aligned} \frac{d}{dx} x^n &= \frac{d}{dx} e^{n \ln x} \ (x > 0) \\ &= e^{n \ln x} \cdot \frac{d}{dx}(n \ln x) \\ &= x^n \cdot \frac{n}{x} \\ &= n x^{n-1}. \end{aligned}$$

Derivative of a^x

By definition, $a^x = e^{x \ln a}$. Thus

$$\frac{d}{dx} a^x = \frac{d}{dx} e^{x \ln a} = \ln a e^{x \ln a} = a^x \ln a.$$

$$\frac{d}{dx} a^u = a^u \ln a \frac{du}{dx}.$$

General logarithmic function $\log_a x$

$y = \log_a x$ is defined as the inverse function of $y = a^x (a > 0, a \neq 1)$. Thus

$$\log_a x = y \Leftrightarrow a^y = x$$

$$\log_a(a^x) = x, \text{ for all } x, a^{(\log_a x)} = x, (x > 0)$$

$\log_{10} x$ is written as $\log x$ and called the *common logarithmic function.*

Properties

(1) Product rule: $\log_a xy = \log_a x + \log_a y$

(2) Quotient rule: $\log_a \frac{x}{y} = \log_a x - \log_a y$

(3) Product rule: $\log_a \frac{1}{y} = -\log_a y$

(4) Power rule: $\log_a x^y = y \log_a x$

Inverse properties

(1) Base a: $a^{\log_a x} = x$, $\log_a(a^x) = x (a > 0, a \neq 1, x > 0)$

(2) Base e: $e^{\ln x} = x$, $\ln(e^x) = x (x > 0)$

Derivative of $\log_a x$

$$\log_a x = \frac{\ln x}{\ln a}. \tag{7.7}$$

Proof.

$$\begin{aligned} a^{(\log_a x)} &= x \\ \ln a^{(\log_a x)} &= \ln x \\ \log_a x \cdot \ln a &= \ln x \\ \log_a x &= \frac{\ln x}{\ln a}. \end{aligned}$$

So

$$\frac{d}{dx} \log_a x = \frac{1}{x \ln a}$$

and

$$\frac{d}{dx} \log_a u = \frac{1}{u \ln a} \frac{du}{dx}.$$

Logarithmic Differentiation

Find dy/dx if $y = \frac{(x^2+1)^{1/3}(x-3)^{1/2}}{x+5}$.

Derivatives of Power function

We prove the following theorem which was stated earlier.

Theorem 7.1.8. *For any real r, $\frac{d}{dx}u^r = u^{r-1}\frac{du}{dx}$.*

Proof. Since $u^r = e^{r\ln u}$ we have

$$\frac{d}{dx}u^r = ru^r\frac{d\ln u}{dx} = ur^r\frac{1}{u}\frac{du}{dx} = ru^{r-1}\frac{du}{dx}.$$

Example 7.1.9. Differentiate $f(x) = x^x, x > 0$

sol. Write $f(x) = x^x = e^{x\ln x}$. So

$$\begin{aligned} f'(x) &= \frac{d}{dx}(e^{x\ln x}) \\ &= (e^{x\ln x})\frac{d}{dx}(x\ln x) \\ &= e^{x\ln x}(\ln x + x\cdot\frac{1}{x}) \\ &= x^x(\ln x + 1). \end{aligned}$$

■

The number e as a limit

Theorem 7.1.10. *The number e satisfies*

$$e = \lim_{x\to 0}(1+x)^{1/x}.$$

Proof. If $f(x) = \ln x$. Then $f'(1) = \frac{1}{x}|_{x=1} = 1$. By definition,

$$1 = f'(1) = \lim_{x\to 0^+}\frac{\ln(1+x) - \ln 1}{x} = \lim_{x\to 0^+}\ln[(1+x)^{\frac{1}{x}}] = \ln[\lim_{x\to 0^+}(1+x)^{\frac{1}{x}}].$$

Now exponentiate.

7.2 Exponential change and separable differentiable equations

The decay of radioactive material or money earning interests in bank account, temperature between a cup of hot water and room air it sits, etc follows the law of *exponential change* Suppose $y(t)$ denotes some quantity which changes according to the exponential law: The rate of change of y is proportional to y.

$$\frac{dy}{dt} = ky$$

with I.C. Then $y = Ae^{kt}$.

Example 7.2.1. Assume a disease is spreading "Entero virus", "A.I" Let y be the number of people infected by disease. Assume we cure people as much as possible. Then dy/dt is proportional to y.(The more people, the more infected, the more cured) Suppose for each year the number is reduced by 20% and 10,000 people infected today, how many years will it take to reduce to $1,000$?

sol. $y = Ae^{kt}$, $A = 10,000$ Since it is reduced by 0.2 each year, we see

$$0.8 = e^{k\cdot 1} \to k = \ln 0.8 < 0$$

So we have $y = 10,000e^{(\ln 0.8)t}$ we want $10,000e^{(\ln 0.8)t} = 1,000$. So $e^{(\ln 0.8)t} = \frac{1}{10}$. $\ln(0.8)t = \ln(0.1)$. $t = \frac{\ln(0.1)}{\ln(0.8)} \approx 10.32$ yrs.

■

Example 7.2.2 (Half life of a radioactive material)**.** $y_0 e^{-kt} = \frac{1}{2}y_0$. so $t = \ln 2/k$.

Example 7.2.3 (Carbon 14)**.** It is estimated the half life of Carbon 14 is 5700 yrs. AS wooden artifact was found from an ancient site. This contains carbon 14 about 10% less than the living tree. How old is the site? $k = \ln 2/Half\ life = \ n2/5700$. $y = y_0 e^{-kt} = 0.9y_0$ So $e^{-kt} = 0.9$ or $t = -5700\frac{\ln 0.9}{\ln 2} = 866$ yrs.

Example 7.2.4 (Law of Cooling)**.** If H is the temperature of an object and H_s the surrounding temperature. Then the rate of change(cooling) is proportional to the temperature difference. Thus

$$\frac{dH}{dt} = -k(H - H_s).$$

Solving

$$H - H_s = (H_0 - H_s)e^{-kt}.$$

A boiled egg at 98^o is put in the sink of 18^o to cool down. In 5 min, the egg was 38^o. how much longer will it take to reach 20^o?

sol.

$$H - 18 = (98 - 18)e^{-kt}, \quad H = 18 + 80e^{-kt}.$$

Set $H = 38, t = 5$. Then $e^{-5k} = 1/4$ and

$$k = -\frac{\ln 1/4}{5} = 0.2 \ln 4 \approx 0.28.$$

$$H = 18 + 80e^{-(0.2 \ln 4)t}.$$

Solving $t \approx 13$ min.

Separable Differential Equations

A general differential equation is given in the form

$$\frac{dy}{dx} = f(x, y) \tag{7.8}$$

with certain initial condition such as $y(x_0) = y_0$. Such equation is called **separable** if f is expressed as a product of a function of x and a function of y, i.e,

$$\frac{dy}{dx} = g(x)H(y).$$

We rewrite it to

$$\frac{dy}{dx} = \frac{g(x)}{h(y)}$$

and

$$\int h(y)\,dy = \int g(x)\,dx. \tag{7.9}$$

Example 7.2.5. Solve

$$\frac{dy}{dx} = (1+y)e^x, \; y > -1. \tag{7.10}$$

$$\begin{aligned} \frac{dy}{dx} &= (1+y)e^x \\ \frac{dy}{1+y} &= e^x dx \\ \int \frac{dy}{1+y} &= \int e^x dx \\ \ln(1+y) &= e^x + C. \end{aligned}$$

7.3 Hyperbolic function

hyperbolic function

Any $f(x)$ can be written as even part and odd part

$$f(x) = \underbrace{\frac{f(x)+f(-x)}{2}}_{\text{even part}} + \underbrace{\frac{f(x)-f(-x)}{2}}_{\text{odd part}}.$$

Hence e^x can be written as

$$e^x = \frac{e^x + e^{-x}}{2} + \frac{e^x - e^{-x}}{2} \tag{7.11}$$

Definition 7.3.1 (hyperbolic function). [1]

$$
\begin{aligned}
&\textbf{hyperbolic cosine } \cosh x = \frac{e^x + e^{-x}}{2},\\
&\textbf{hyperbolic sine } \sinh x = \frac{e^x - e^{-x}}{2},\\
&\textbf{hyperbolic tangent } \tanh x = \frac{\sinh x}{\cosh x} = \frac{e^x - e^{-x}}{e^x + e^{-x}},\\
&\textbf{hyperbolic cotangent } \coth x = \frac{1}{\tanh x} = \frac{e^x + e^{-x}}{e^x - e^{-x}},\\
&\textbf{hyperbolic secant } \operatorname{sech} x = \frac{1}{\cosh x} = \frac{2}{e^x + e^{-x}},\\
&\textbf{hyperbolic cosecant } \operatorname{csch} x = \frac{1}{\sinh x} = \frac{2}{e^x - e^{-x}}.
\end{aligned}
$$

Some identities of hyperbolic functions:

Proposition 7.3.2.

(1) $\sinh 2x = 2 \sinh x \cosh x$

(2) $\cosh 2x = \cosh^2 x + \sinh^2 x$

(3) $\sinh^2 x = \dfrac{\cosh 2x - 1}{2}$

(4) $\cosh^2 x = \dfrac{\cosh 2x + 1}{2}$

(5) $\cosh^2 x - \sinh^2 x = 1$

(6) $\tanh^2 x = 1 - \operatorname{sech}^2 x$

(7) $\coth^2 x = 1 + \operatorname{csch}^2 x$

Proposition 7.3.3.

(1) $\dfrac{d}{dx}(\sinh u) = \cosh u \dfrac{du}{dx}$

(2) $\dfrac{d}{dx}(\cosh u) = \sinh u \dfrac{du}{dx}$

[1]hyperbolic functions have many things in common with trig. functions. We can define trig. functions $\sin x$ and $\cos x$ using complex numbers. We define $e^{i\theta} = \cos\theta + i \sin\theta$. Then $e^{-i\theta} = \cos\theta - i\sin\theta$ and hence

$$\cos\theta = \frac{e^{i\theta} + e^{-i\theta}}{2}, \quad \sin\theta = \frac{e^{i\theta} - e^{-i\theta}}{2i}$$

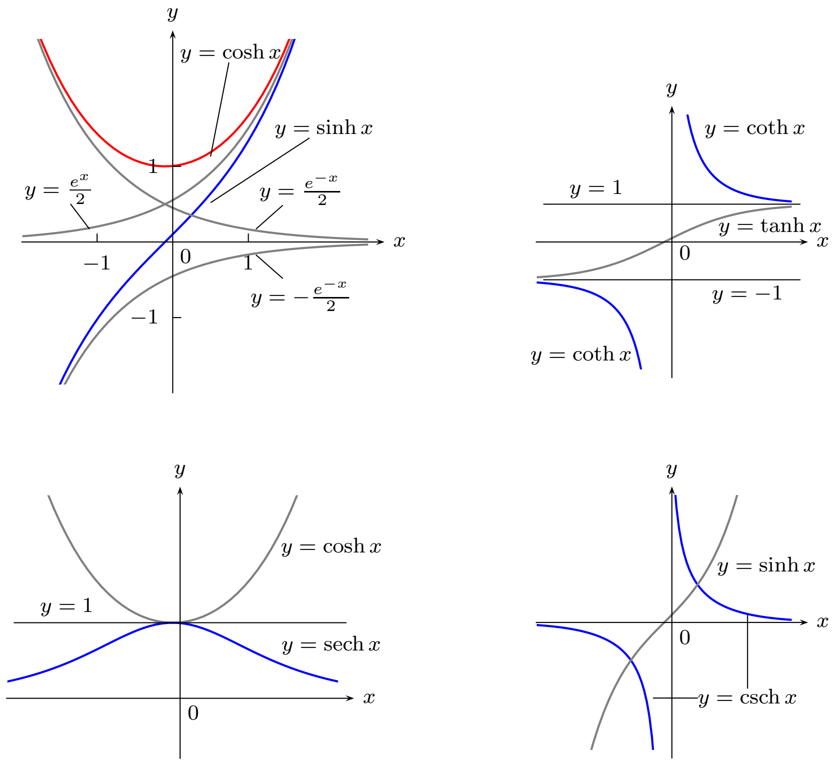

Figure 7.1: hyperbolic functions

(3) $\dfrac{d}{dx}(\tanh u) = \operatorname{sech}^2 u \dfrac{du}{dx}$

(4) $\dfrac{d}{dx}(\coth u) = -\operatorname{csch}^2 u \dfrac{du}{dx}$

(5) $\dfrac{d}{dx}(\operatorname{sech} u) = -\operatorname{sech} u \tanh u \dfrac{du}{dx}$

(6) $\dfrac{d}{dx}(\operatorname{csch} u) = -\operatorname{csch} u \coth u \dfrac{du}{dx}$

Proposition 7.3.4.

(1) $\displaystyle\int \sinh u \, du = \cosh u + C$

(2) $\displaystyle\int \cosh u \, du = \sinh u + C$

(3) $\displaystyle\int \operatorname{sech}^2 u \, du = \tanh u + C$

(4) $\displaystyle\int \operatorname{csch}^2 u \, du = -\coth u + C$

(5) $\displaystyle\int \operatorname{sech} u \tanh u \, du = -\operatorname{sech} u + C$

(6) $\displaystyle\int \operatorname{csch} u \coth u du = -\operatorname{csch} u + C$

Example 7.3.5. (1) The indefinite integral of $\sinh^2 x$ can be computed just as that of $\sin^2 x$.

$$\begin{aligned}\int_0^1 \sinh^2 x \, dx &= \int_0^1 \frac{\cosh 2x - 1}{2} dx \\ &= \frac{1}{2}\left[\frac{\sinh 2x}{2} - x\right]_0^1 \\ &= \frac{\sinh 2}{4} - \frac{1}{2}.\end{aligned}$$

(2) Using the definition of $\sinh x$

$$\begin{aligned}\int_0^{\ln 2} 4e^x \sinh x \, dx &= \int_0^{\ln 2} 4e^x \frac{e^x - e^{-x}}{2} dx = \int_0^{\ln 2} (2e^{2x} - 2) \, dx \\ &= \left[e^{2x} - 2x\right]_0^{\ln 2} \\ &= 3 - 2\ln 2.\end{aligned}$$

Inverse hyperbolic function

The function $y = \sinh x$ is defined on $(-\infty, \infty)$ having values in $(-\infty, \infty)$. So (inverse hyperbolic sine) $y = \sinh^{-1} x$ is defined on $(-\infty, \infty)$.

The function $y = \cosh x$ restricted to $x \geq 0$ is 1-1 to $[1, \infty)$. So inverse $y = \cosh^{-1}$ inverse hyperbolic cosine is defined on $[1, \infty)$.

$y = \operatorname{sech} x$ restricted to $x \geq 0$ is one-to-one. Hence its inverse $y = \operatorname{sech}^{-1} x$ is defined on $(0, 1]$. Meanwhile $y = \tanh x$, $y = \coth x$, $y = \operatorname{csch} x$ are one-to-one on $(-\infty, \infty)$. Hence their inverses $y = \tanh^{-1} x$, $y = \coth^{-1} x$, $y = \operatorname{csch}^{-1} x$ are defined accordingly. The graphs are as in figure 7.2

Proposition 7.3.6. *Inverse hyperbolic functions can be represented by log functions.*

(1) $\sinh^{-1} x = \ln\left(x + \sqrt{x^2 + 1}\right), \quad -\infty < x < \infty$

(2) $\cosh^{-1} x = \ln\left(x + \sqrt{x^2 - 1}\right), \quad x \geq 1$

(3) $\tanh^{-1} x = \dfrac{1}{2} \ln \dfrac{1 + x}{1 - x}, \quad |x| < 1$

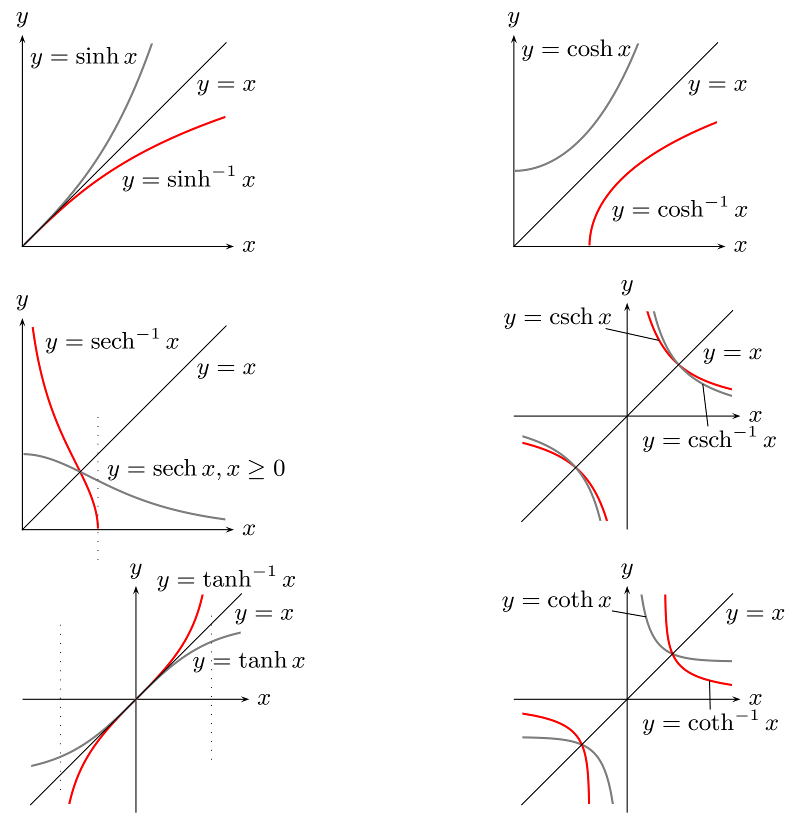

Figure 7.2: Inverse hyperbolic functions

(4) $\operatorname{sech}^{-1} x = \ln\left(\dfrac{1+\sqrt{1-x^2}}{x}\right), \quad 0 < x \leq 1$

(5) $\operatorname{csch}^{-1} x = \ln\left(\dfrac{1}{x} + \dfrac{\sqrt{1+x^2}}{|x|}\right), \quad x \neq 0$

(6) $\coth^{-1} x = \dfrac{1}{2} \ln \dfrac{x+1}{x-1}, \quad |x| > 1.$

Proof. We prove the formula for $\sinh^{-1} x$.

$$y = \sinh x = \frac{e^x - e^{-x}}{2},$$
$$e^x - e^{-x} = 2y,$$
$$e^{2x} - 2ye^x - 1 = 0,$$
$$e^x = y + \sqrt{y^2+1}. \quad \text{(Since } y - \sqrt{y^2+1} \text{ is negative, we drop it.)}$$

Hence $x = \ln(y + \sqrt{y^2+1})$, or $y = \ln(x + \sqrt{x^2+1})$ is the $\sinh^{-1} x$.

Proposition 7.3.7.

(1) $\operatorname{sech}^{-1} x = \cosh^{-1} \dfrac{1}{x}$

(2) $\operatorname{csch}^{-1} x = \sinh^{-1} \dfrac{1}{x}$

(3) $\coth^{-1} x = \tanh^{-1} \dfrac{1}{x}$

Derivatives of inverse hyperbolic functions

Proposition 7.3.8.

(1) $\dfrac{d(\sinh^{-1} u)}{dx} = \dfrac{1}{\sqrt{1+u^2}} \dfrac{du}{dx}$

(2) $\dfrac{d(\cosh^{-1} u)}{dx} = \dfrac{1}{\sqrt{u^2-1}} \dfrac{du}{dx}, \quad u > 1$

(3) $\dfrac{d(\tanh^{-1} u)}{dx} = \dfrac{1}{1-u^2} \dfrac{du}{dx}, \quad |u| < 1$

(4) $\dfrac{d(\coth^{-1} u)}{dx} = \dfrac{1}{1-u^2} \dfrac{du}{dx}, \quad |u| > 1$

(5) $\dfrac{d(\operatorname{sech}^{-1} u)}{dx} = \dfrac{-du/dx}{u\sqrt{1-u^2}}, \quad 0 < u < 1$

(6) $\dfrac{d(\operatorname{csch}^{-1} u)}{dx} = \dfrac{-du/dx}{|u|\sqrt{1+u^2}}, \quad u \neq 0$

Proposition 7.3.9.

(1) $\displaystyle\int \frac{du}{\sqrt{1+u^2}} = \sinh^{-1} u + C$

(2) $\displaystyle\int \frac{du}{\sqrt{u^2-1}} = \cosh^{-1} u + C, \quad u > 1$

(3) $\displaystyle\int \frac{du}{1-u^2} = \begin{cases} \tanh^{-1} u + C, & \text{if } |u| < 1, \\ \coth^{-1} u + C, & \text{if } |u| > 1 \end{cases}$

(4) $\displaystyle\int \frac{du}{u\sqrt{1-u^2}} = -\operatorname{sech}^{-1} |u| + C = -\cosh^{-1}\left(\frac{1}{|u|}\right) + C$

(5) $\displaystyle\int \frac{du}{u\sqrt{1+u^2}} = -\operatorname{csch}^{-1} |u| + C = -\sinh^{-1}\left(\frac{1}{|u|}\right) + C$

Example 7.3.10.

$$\begin{aligned} \int \frac{2\,dx}{\sqrt{3+4x^2}} &= \int \frac{du}{\sqrt{a^2+u^2}}, u = 2x, a = \sqrt{3} \\ &= \sinh^{-1}(\frac{u}{a}) + C \\ &= \sinh^{-1}(\frac{2x}{\sqrt{3}}) + C \end{aligned}$$

$$\frac{d^2y}{dx^2} = a\sqrt{1 + (\frac{dy}{dx})^2} \tag{7.12}$$

7.4 Relative Rate of Growth

Definition 7.4.1. Suppose $f(x), g(x)$ are positive for sufficiently large x.

(1) f **grows faster than** g as $x \to \infty$ if

$$\lim_{x\to\infty} \frac{f(x)}{g(x)} = \infty$$

(2) $f(x)$ grows at a **smaller order than** $g(x)$ as $x \to \infty$ if

$$\lim_{x\to\infty} \frac{f(x)}{g(x)} = 0.$$

In this case we write $f = o(g)$.

(3) f **grows at the same rate** as g as $x \to \infty$ if

$$\lim_{x\to\infty} \frac{f(x)}{g(x)} = L, \text{ for some postive finite number } L.$$

In this case we write $f = O(g)$.

Example 7.4.2. (1) e^x grows faster than x^3 as $x \to \infty$

(2) 3^x grows faster than 2^x as $x \to \infty$

(3) x grows faster than $\ln x$ as $x \to \infty$

Definition 7.4.3.

Example 7.4.4. (1) $\ln x = o(x)$ as $x \to \infty$

(2) $x^2 = o(x^3)$ as $x \to \infty$

(3) $x + \sin x = O(x)$

(4) $x = o(e^x)$