Chapter 7

Surface integrals and Vector Analysis

7.1 Parameterized Surfaces

Graphs are too restrictive.

See the following surface or simply a sphere or torus. Those are important examples of figures that arise often in real life. But those figures cannot be represented as the graphs of functions. Thus we need other ways of representing surfaces.

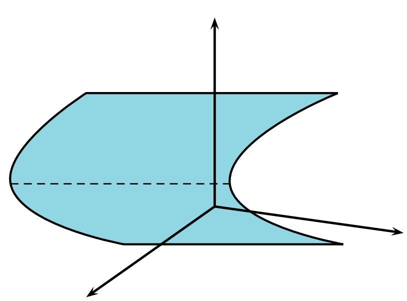


Figure 7.1: A surface that is not the graph of a function

Definition 7.1.1. A parameterized surface is a (one-to-one) function $\mathbf{X}\colon D \subset \mathbb{R}^2 \to \mathbb{R}^3$ where D is a domain in $\mathbb{R}^2$. The **underlying surface**

S is the image $\mathbf{X}(D)$ of $\mathbf{X}$. The function $\mathbf{X}$ is also called a **parametrization** of S. Usually, we write

$$\mathbf{X}(s,t) = (x(s,t), y(s,t), z(s,t)).$$

If $\mathbf{X}$ is differentiable or C^1, then we say S is differentiable or C^1-surface.

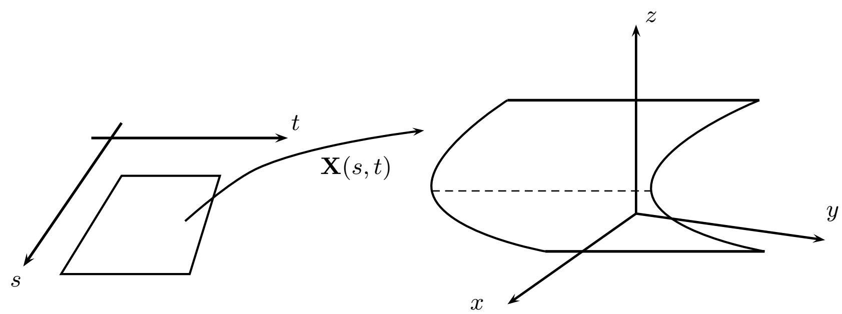


Figure 7.2: A parametrization

Example 7.1.2. The graph of a function is a special case. If $z = f(x,y)$ $(x,y) \in D$ then

$$\mathbf{X}(s,t) = (s,t,f(s,t))$$

is a parametrization of the surface.

Example 7.1.3. Let $D = [0,\pi] \times [0,2\pi)$ and

$$\mathbf{X}(s,t) = (a \sin s \cos t, a \sin s \sin t, a \cos t).$$

The parametric surface is a sphere of radius a.

How about the cylinder? We can set $\mathbf{X}(s,t) = (a\cos s, a\sin s, t)$, $0 \le s \le 2\pi$.

Example 7.1.4. Consider a parametrization of the surface.

$$\begin{cases} x = (a + b\cos t)\cos s, & 0 \le s,t \le 2\pi, \\ y = (a + b\cos t)\sin s, & a > b > 0, \\ z = b \sin t \end{cases}$$

Investigate it.

sol. The surface satisfies the equation

$$\left(\sqrt{x^2+y^2}-a\right)^2+z^2=b^2.$$

Let us fix $t=t_0$. Then it describes a circle of radius $(a+b\cos t_0)$ lying in the plane: $z=b\sin t_0$.

Let's fix $s=s_0$. Then

$$\begin{cases} x=(a+b\cos t)\cos s_0, & 0\le t\le 2\pi,\\ y=(a+b\cos t)\sin s_0, \\ z=b\sin t \end{cases} \Rightarrow \begin{cases} x-a\cos s_0=b\cos t\cos s_0, & 0\le t\le 2\pi,\\ y-a\sin s_0=b\cos t\sin s_0, \\ z=b\sin t \end{cases}$$

Since

$$(x-a\cos s_0,\ y-a\sin s_0,\ z)=b(\cos s_0\cos t,\ \sin s_0\cos t,\ \sin t)$$

has length b, the curve is a circle centered at $(a\cos s_0,\ a\sin s_0, 0)$. This surface is called a **torus.**

■

Coordinate Curves, Normal Vectors and Tangent Planes

Consider the mapping $\mathbf{X}\colon D\to\mathbb{R}^3$, where we write $\mathbf{X}=(x,y,z)$. First look at the case when the surface is the graph of $f\colon D\to\mathbb{R}$. Then we have

$$\mathbf{X}(x,y)=(x,y,f(x,y)).$$

To study the surface we look at the sections: First fix $y=y_0$ and then $x=x_0$. Then tangent vectors in the direction of x-axis and y-axis at $\mathbf{X}(x_0,y_0)=(x_0,y_0,f(x_0,y_0))$ are

$$\mathbf{X}_x(x_0,y_0)=\mathbf{i}+f_x(x_0,y_0)\mathbf{k},\quad \mathbf{X}_y(x_0,y_0)=\mathbf{j}+f_y(x_0,y_0)\mathbf{k}.$$

Hence the tangent plane is perpendicular to the normal vector given by the cross product

$$
\begin{aligned}
\mathbf{X}_x(x_0,y_0)\times\mathbf{X}_y(x_0,y_0) &= (\mathbf{i}+f_x(x_0,y_0)\mathbf{k})\times(\mathbf{j}+f_y(x_0,y_0)\mathbf{k})\\
&= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k}\\ 1 & 0 & f_x(x_0,y_0)\\ 0 & 1 & f_y(x_0,y_0)\end{vmatrix}\\
&= -f_x(x_0,y_0)\mathbf{i}-f_y(x_0,y_0)\mathbf{j}+\mathbf{k}.
\end{aligned}
$$

In general, we see two tangent vectors are

$$
\begin{aligned}
\mathbf{T}_s &= \frac{\partial\mathbf{X}}{\partial s} = \frac{\partial x}{\partial s}\mathbf{i}+\frac{\partial y}{\partial s}\mathbf{j}+\frac{\partial z}{\partial s}\mathbf{k}\bigg|_{(s_0,t_0)}\\
\mathbf{T}_t &= \frac{\partial\mathbf{X}}{\partial t} = \frac{\partial x}{\partial t}\mathbf{i}+\frac{\partial y}{\partial t}\mathbf{j}+\frac{\partial z}{\partial t}\mathbf{k}\bigg|_{(s_0,t_0)}
\end{aligned}
$$

These are obtained by considering the cross sections $t = t_0$ and $s = s_0$ respectively. If the normal vector

$$\mathbf{N} = \mathbf{T}_s\times\mathbf{T}_t = \frac{\partial\mathbf{X}}{\partial s}\times\frac{\partial\mathbf{X}}{\partial t}$$

is nonzero, then we say the surface is **smooth**.

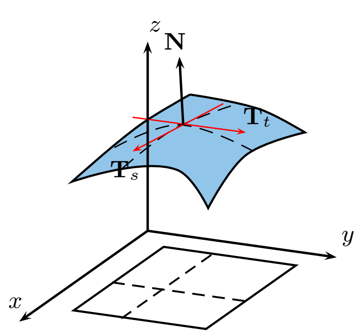


Figure 7.3: Coord. curves, Tangent vectors and normal vectors to a surface

Definition 7.1.5. When **N** is a normal vector to a surface, the **tangent**

plane at $\mathbf{X}(s_0, t_0) = (x_0, y_0, z_0)$ is defined by

$$\mathbf{N} \cdot (x - x_0, y - y_0, z - z_0) = 0.$$

Or if $\mathbf{N} = (n_1, n_2, n_3)$, then the equation of **tangent plane** is

$$n_1(x - x_0) + n_2(y - y_0) + n_3(z - z_0) = 0.$$

Example 7.1.6. Consider a surface given by

$$x = s\cos t, \quad y = s\sin t, \quad z = s^2 + t^2.$$

Find the tangent plane at $\mathbf{X}(1, 0)$.

sol. $\mathbf{X}(s, t) = (s\cos t, s\sin t, s^2 + t^2)$. So

$$\mathbf{T}_s = (\cos t, \sin t, 2s), \quad \mathbf{T}_t = (-s\sin t, s\cos t, 2t).$$

We see $\mathbf{T}_s \times \mathbf{T}_t = (-2s^2\cos t + 2t\sin t, -2s^2\sin t - 2t\cos t, s)$. Since $\mathbf{X}(1, 0) = (1, 0, 1)$ and $\mathbf{N} = \mathbf{T}_s \times \mathbf{T}_t(1, 0) = (-2, 0, 1)$, we have

$$-2(x - 1) + 0(y - 0) + 1(z - 1).$$

■

Example 7.1.7 (Cone)**.** Consider

$$\mathbf{X}(s, t) = (s\cos t, s\sin t, s), \quad s \geq 0.$$

Is it smooth(regular) ?

sol.

$$\mathbf{T}_s = (\cos t, \sin t, 1), \quad \mathbf{T}_t = (-s\sin t, s\cos t, 0).$$

Since $\mathbf{T}_s \times \mathbf{T}_t = 0$ at $(0, 0, 0)$, it is not regular.

■

Example 7.1.8. Find a parametrization of the following hyperboloid of one sheet

$$x^2 + y^2 - z^2 = 1.$$

sol. Since the graph is symmetric in x and y, it is natural to use polar coordinate

$$x = \cos\theta, \quad y = \sin\theta, \quad (0 \le \theta \le 2\pi)$$

to transform it to

$$r^2 - z^2 = 1.$$

Next we use the following parametrization

$$r = \cosh s, \quad z = \sinh s, \quad (-\infty < s < \infty)$$

to get

$$x = \cosh s \cos\theta, \quad y = \cosh s \sin\theta, \quad z = \sinh s.$$

So

$$\begin{aligned}\mathbf{X}(s,\theta) &= (x(s,\theta), y(s,\theta), z(s,\theta)) \\ &= (\cosh s\cos\theta, \cosh s\sin\theta, \sinh s),\ (-\infty < s < \infty, 0 \le \theta \le 2\pi).\end{aligned}$$

■

Definition 7.1.9. A **piecewise smooth** surface is a set which is defined as a union of finitely many surfaces $\mathbf{X}_i : D_i \to \mathbb{R}^3$, $i = 1, \cdots, m$, where each surface $\mathbf{X}_i$ is smooth.

Area of Parameterized Surface

Recall 2-D case: When : $D^* \to D$ is a transformation in $\mathbb{R}^2$, the two tangent vectors to the boundary of $D = T(D^*)$ at $T(u,v)$ are

$$T_u\Delta u, \quad T_v\Delta v.$$

These form a parallelogram approximating the region D(figure 7.4). The area of the parallelogram is

$$\left|\begin{array}{cc} \frac{\partial x}{\partial u}\Delta u & \frac{\partial x}{\partial v}\Delta v \\ \frac{\partial y}{\partial u}\Delta u & \frac{\partial y}{\partial v}\Delta v \end{array}\right| = \left|\begin{array}{cc} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{array}\right| \Delta u\Delta v = \frac{\partial(x,y)}{\partial(u,v)}\Delta u\Delta v.$$

$$\|T_u \times T_v\|\Delta u\Delta v = |J|\Delta u\Delta v.$$

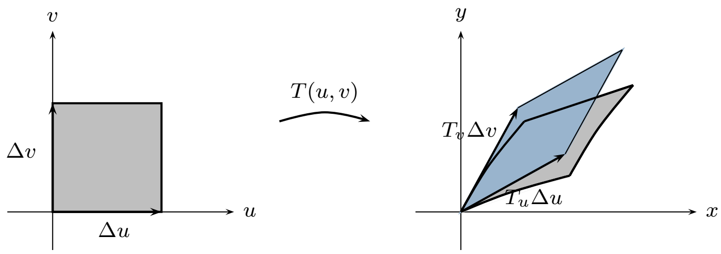


Figure 7.4: approximate $T(D^*)$

In this section, we show how to find the area of $U = \mathbf{X}(D)$ where $\mathbf{X}\colon D \to \mathbb{R}^3$ is a surface parametrization. First divide D into small rectangles. Consider a small rectangle $R = [s, s+\Delta s] \times [t, t+\Delta t]$. The image of R under $\mathbf{X}$ is a portion of the surface having four corners at

$$\mathbf{X}(s,t),\ \ \mathbf{X}(s+\Delta s, t),\ \ \mathbf{X}(s, t+\Delta t),\ \ \mathbf{X}(s+\Delta s, t+\Delta t).$$

This surface can be approximated by a parallelogram whose sides are given by(fig 7.5) $\mathbf{X}_s(s,t)\Delta s$ and $\mathbf{X}_t(s,t)\Delta t$, where

$$\begin{aligned} \mathbf{X}_s = \mathbf{T}_s &= \tfrac{\partial \mathbf{X}}{\partial s} = \tfrac{\partial x}{\partial s}\mathbf{i} + \tfrac{\partial y}{\partial s}\mathbf{j} + \tfrac{\partial z}{\partial s}\mathbf{k} \\ \mathbf{X}_t = \mathbf{T}_t &= \tfrac{\partial \mathbf{X}}{\partial t} = \tfrac{\partial x}{\partial t}\mathbf{i} + \tfrac{\partial y}{\partial t}\mathbf{j} + \tfrac{\partial z}{\partial t}\mathbf{k}. \end{aligned} \tag{7.1}$$

Hence the area of $\mathbf{X}(R)$ is approximated by

$$\|\mathbf{T}_s \times \mathbf{T}_t\| \Delta s \Delta t.$$

Hence the area of the surface is the limit of the following sum:

$$\sum \|\mathbf{T}_s \times \mathbf{T}_t\| \Delta s \Delta t.$$

Definition 7.1.10. We define the surface area $A(S)$ of a parameterized surface S by

$$A(S) = \iint_D \|\mathbf{T}_s \times \mathbf{T}_t\| ds dt.$$

We let

$$dS = \|\mathbf{X}_s \times \mathbf{X}_t\| ds dt,$$

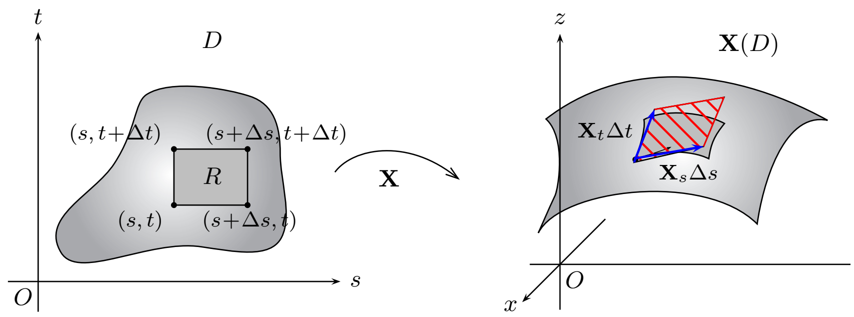


Figure 7.5: Approx. area of surface by a tangent plane

and call it the **surface element**. Then we see from (7.1) that [1]

$$\iint_{\mathbf{X}} dS = \iint_D \|\mathbf{X}_s \times \mathbf{X}_t\| dsdt$$
$$= \iint_D \sqrt{\left[\frac{\partial(y,z)}{\partial(s,t)}\right]^2 + \left[\frac{\partial(z,x)}{\partial(s,t)}\right]^2 + \left[\frac{\partial(x,y)}{\partial(s,t)}\right]^2}\, dsdt.$$

Remark 7.1.11. The area of a surface is independent of parametrization.

Example 7.1.12 (Cone)**.** Let D be the surface of a cone given by

$$x = r\cos\theta, \quad y = r\sin\theta, \quad z = r.$$

sol. Either use formula above or compute directly using $\|\mathbf{T}_r \times \mathbf{T}_\theta\| dr d\theta$. We can show that $\|\mathbf{T}_r \times \mathbf{T}_\theta\| = r\sqrt{2}$.

Example 7.1.13 (Helicoid-like surface)**.** Let S be the surface given by

$$x = r\cos\theta, \quad y = r\sin\theta, \quad z = \theta, (0 \le \theta \le 2\pi,\ 0 \le r \le 1).$$

Find its area.

sol. $\|\mathbf{T}_r \times \mathbf{T}_\theta\| = \sqrt{r^2+1}$. Need the table to see

$$\pi[\sqrt{2} + \log(1+\sqrt{2})].$$

[1] $\mathbf{X}$ is assumed to be 1-1.

Surface Area of a Graph

When a surface U is given by the graph of function $z = f(x,y)$ on D, we see U is parameterized by $\mathbf{X}(x,y) = (x, y, f(x,y))$. Find $\mathbf{X}_x$, $\mathbf{X}_y$ by

$$\mathbf{X}_x = \mathbf{i} + \frac{\partial f}{\partial x}\mathbf{k}, \qquad \mathbf{X}_y = \mathbf{j} + \frac{\partial f}{\partial y}\mathbf{k}.$$

Since

$$\mathbf{X}_x \times \mathbf{X}_y = (\mathbf{i} + \frac{\partial f}{\partial x}\mathbf{k}) \times (\mathbf{j} + \frac{\partial f}{\partial y}\mathbf{k}) = -\frac{\partial f}{\partial x}\mathbf{i} - \frac{\partial f}{\partial y}\mathbf{j} + \mathbf{k},$$

the area is

$$\iint_{\mathbf{X}} dS = \iint_D \Big((\partial f/\partial x)^2 + (\partial f/\partial y)^2 + 1\Big)^{1/2} dxdy.$$

The unit normal vector $\mathbf{N}(x,y,z)$ on U is

$$\mathbf{N}(x,y,z) = -\frac{\partial f}{\partial x}\mathbf{i} - \frac{\partial f}{\partial y}\mathbf{j} + \mathbf{k}.$$

We can find the formula using the angle between $\mathbf{N}$ and $\mathbf{k}$. Let φ be the angle between $\mathbf{N}$ and $\mathbf{k}$. Then $\cos\varphi$ satisfies

$$\cos\varphi = \frac{\mathbf{N}\cdot\mathbf{k}}{\|\mathbf{N}\|} = \frac{1}{\sqrt{(\partial f/\partial x)^2 + (\partial f/\partial y)^2 + 1}}.$$

Hence

$$dS = \sqrt{(\partial f/\partial x)^2 + (\partial f/\partial y)^2 + 1}dxdy = \frac{dxdy}{\cos\varphi},$$

and we get

$$\iint_{\mathbf{X}} dS = \iint_D \frac{dxdy}{\cos\varphi}.$$

Example 7.1.14. Find the surface area of a unit ball.

sol. From $x^2 + y^2 + z^2 = 1$, we let $z = f(x,y) = \sqrt{1 - x^2 - y^2}$.

$$\frac{\partial f}{\partial x} = \frac{-x}{\sqrt{1-x^2-y^2}}, \quad \frac{\partial f}{\partial y} = \frac{-y}{\sqrt{1-x^2-y^2}}.$$

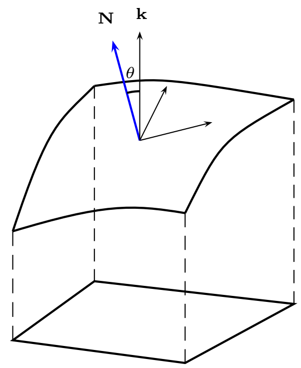


Figure 7.6: Ratio between two surface

Area of the half sphere is

$$\begin{aligned}\iint_{\mathbf{X}} dS &= \iint_D \frac{1}{\sqrt{1-x^2-y^2}}dxdy\\ &= \int_0^{2\pi}\int_0^1 \frac{r}{\sqrt{1-r^2}}\,drd\theta\\ &= 2\pi.\end{aligned}$$

Surface of revolution

The surface area generated by revolving the graph $y = f(x) \geq 0$ about x-axis is

$$A = 2\pi \int_a^b y\sqrt{1+(f'(x))^2}dx.$$

Example 7.1.15. Use a parametrization to express the area generated by revolving the graph $y = f(x)$ about x-axis. We can choose the parametrization

$$\mathbf{X}(s,t) = (x,y,z) = (s, f(s)\cos t, f(s)\sin t).$$

over the region

$$a \leq s \leq b, \quad 0 \leq t \leq 2\pi.$$

sol. We see

$$\frac{\partial(y,z)}{\partial(s,t)} = f(s)f'(s),\ \frac{\partial(z,x)}{\partial(s,t)} = f(s)\cos t,\ \frac{\partial(x,y)}{\partial(s,t)} = -f(s)\sin t.$$

Hence the area is

$$\begin{aligned}
\iint_{\mathbf{X}} dS &= \iint_D |f(s)| \sqrt{\left[\frac{\partial(y,z)}{\partial(s,t)}\right]^2 + \left[\frac{\partial(z,x)}{\partial(s,t)}\right]^2 + \left[\frac{\partial(x,y)}{\partial(s,t)}\right]^2}\, dsdt \\
&= \iint_D f(s)\sqrt{1+[f'(s)]^2}\, dsdt \\
&= \int_a^b \int_0^{2\pi} f(s)\sqrt{1+[f'(s)]^2}\, dsdt \\
&= 2\pi \int_a^b f(s)\sqrt{1+[f'(s)]^2}\, ds.
\end{aligned}$$

This formula coincide with earlier formula.

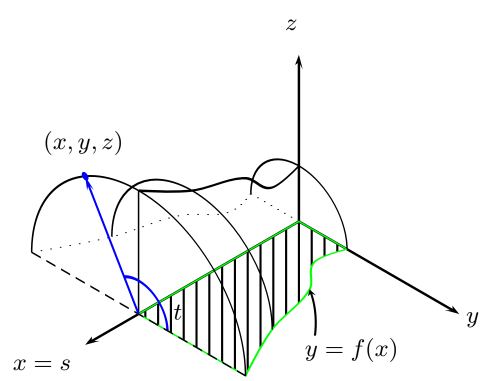


Figure 7.7: The surface of revolution of $z = f(x)$ about x-axis, $(x, y, z) = (s, f(s)\cos t, f(s)\sin t)$

7.2 Surface Integrals

Integrals of Scalar functions over Surface

Let $\mathbf{X}\colon D \to \mathbb{R}^3$ be a parameterized surface $S = \mathbf{X}(D)$ and let $f\colon S \to \mathbb{R}$ be a real valued function defined on $\mathbf{X}$. If $f = 1$, it represents the area:

$$\iint_S dS = \iint_D \|\mathbf{X}_s \times \mathbf{X}_s\| dsdt.$$

In general, we have

Definition 7.2.1. Let S be a surface parameterized by $\mathbf{X}(s,t) = (x(s,t), y(s,t), z(s,t))$, where $(s,t) \in D$. Then the surface integral of a scalar function f defined on S is

$$\iint_S f\,dS = \iint_D f(\mathbf{X}(s,t))\|\mathbf{X}_s \times \mathbf{X}_t\| dsdt.$$

Changing it to an integral of (s,t) variables, we see

$$\iint_S f\,dS = \iint_D f(\mathbf{X}(s,t))\|\mathbf{T}_s \times \mathbf{T}_t\|\,dsdt$$
$$= \iint_D f(x(s,t), y(s,t), z(s,t))\sqrt{\left[\frac{\partial(y,z)}{\partial(s,t)}\right]^2 + \left[\frac{\partial(z,x)}{\partial(s,t)}\right]^2 + \left[\frac{\partial(x,y)}{\partial(s,t)}\right]^2}\,dsdt.$$

When a surface consists of several pieces, the the parametrization $\mathbf{X}$ is the sum of $\mathbf{X}_1, \mathbf{X}_2, \ldots, \mathbf{X}_m$, and in this case we define

$$\iint_{\mathbf{X}} f\,dS = \iint_{\mathbf{X}_1} f\,dS + \iint_{\mathbf{X}_2} f\,dS + \cdots + \iint_{\mathbf{X}_m} f\,dS.$$

Surface integrals over graphs

Suppose S is the graph of a C^1 function $z = g(x,y)$. Then we parameterize it by

$$x = s, \quad y = t, \quad z = g(s,t)$$

and

$$\|\mathbf{T}_s \times \mathbf{T}_t\| = \sqrt{1 + (\frac{\partial g}{\partial s})^2 + (\frac{\partial g}{\partial t})^2}.$$

So the integral of f on S becomes

$$\iint_S f(x,y,z)\,dS = \iint_D f(x,y,g(x,y))\sqrt{1 + (\frac{\partial g}{\partial x})^2 + (\frac{\partial g}{\partial y})^2}\,dxdy.$$

Example 7.2.2. Let S be graph of $z = x^2 + y$, where D is $0 \le x \le 1, \ \ -1 \le y \le 1$. Find $\iint_S x\,dS$.

sol.

$$\iint_S x\,dS = \iint_D x\sqrt{1+(\frac{\partial g}{\partial x})^2+(\frac{\partial g}{\partial y})^2}\,dxdy = \int_{-1}^{1}\int_0^1 x\sqrt{1+4x^2+1}dxdy$$
$$= \frac{1}{8}\int_{-1}^{1}\left[\int_0^1 (2+4x^2)^{1/2}(8xdx)\right]dy = \frac{2}{3}\frac{1}{8}\int_{-1}^{1}\left[(2+4x^2)^{3/2}\right]\Big|_0^1 dy$$
$$= \sqrt{6}-\frac{\sqrt{2}}{3}.$$

Example 7.2.3. Evaluate $\iint_S z^2 dS$ when S is the unit sphere.

sol. The unit sphere is described by

$$\mathbf{X}(\phi,\theta) = (\sin\phi\cos\theta, \sin\phi\sin\theta, \cos\phi),\ (0\le\theta\le 2\pi,\ 0\le\phi\le\pi).$$

Since

$$\|\mathbf{T}_\phi\times\mathbf{T}_\theta\| = \sin\phi$$

and $z^2=\cos^2\phi$, we have

$$\begin{aligned}\iint_S z^2 dS &= \iint_D \cos^2\phi\|\mathbf{T}_\theta\times\mathbf{T}_\phi\|d\phi d\theta\\ &= \int_0^{2\pi}\int_0^{\pi}\cos^2\phi\sin\phi\,d\phi d\theta\\ &= \frac{4\pi}{3}.\end{aligned}$$

Geometric interpretation

We show

$$\iint_S f(x,y,z)dS = \iint_D f(\mathbf{X}(s,t))\|\mathbf{T}_s\times\mathbf{T}_t\|dsdt = \iint_D \frac{f(x,y,g(x,y))}{\cos\theta}dxdy,$$

where θ is the angle between normal vector and $\mathbf{k}$ vector. As a special case, when the surface is a plane, we see $\Delta S = \Delta A/\cos\theta$. See figures 7.6, 7.8. For general surface, we have

$$\cos\theta = \frac{\mathbf{N}\cdot\mathbf{k}}{\|\mathbf{N}\|} = \frac{1}{\sqrt{(\partial g/\partial x)^2+(\partial g/\partial y)^2+1}}$$

and the same relation holds by taking the limit in the Riemann sum.

Example 7.2.4. Compute $\iint_S x dS$ where S is a triangle with vertices $(1,0,0)$, $(0,1,0)$ and $(0,0,1)$.

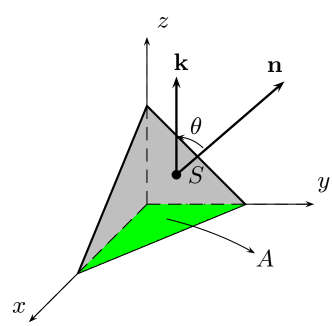


Figure 7.8: ratio between ΔS and ΔA

sol. The angle between the normal and the $\mathbf{k}$ vector satisfies $\cos\theta = \mathbf{n}\cdot\mathbf{k} = 1/\sqrt{3}$. Hence

$$\iint_S x dS = \sqrt{3}\iint_D x dx dy = \sqrt{3}\int_0^1\int_0^{1-x} x dy dx = \frac{\sqrt{3}}{6}.$$

Example 7.2.5. Let $\mathbf{X} = (r\cos\theta, r\sin\theta, \theta)$ be the parametrization of a helicoid-like surface S, where $0 \le r \le 1$, $0 \le \theta \le 2\pi$. Suppose S is covered with a metal of density m which equal to twice the distance to the central axis, i.e, $m = 2\sqrt{x^2+y^2} = 2r$. Find the total mass of metal covering the surface.

sol. First we can show $\|\mathbf{T}_r \times \mathbf{T}_\theta\| = \sqrt{1+r^2}$. Hence we have

$$\begin{aligned} M &= \iint_S 2r dS = 2\iint_D r\|\mathbf{T}_r \times \mathbf{T}_\theta\| dr d\theta \\ &= \int_0^{2\pi}\int_0^1 2r\sqrt{1+r^2} dr d\theta = \frac{4}{3}\pi(2^{3/2}-1). \end{aligned}$$

Surfaces Integrals of vector Fields

In this section we develop the notion of integral of a vector field over a surface.

Recall the line integral of a vector field has a physical interpretation: **Work**. Similarly, the notion of integral of a vector field over a surface can be interpreted as a **Flux**.

Assume the vector field $\mathbf{F}\colon V \to \mathbb{R}^3$ represents the velocity of a fluid and the parametrization $\mathbf{X}\colon D \to \mathbb{R}^3$ describes the shape of the net. Then the surface integral $\iint_{\mathbf{X}} \mathbf{F} \cdot d\mathbf{S}$ is the **amount of water that passes through your net(per unit time)**.

We now define the surface integral $\iint_{\mathbf{X}} \mathbf{F} \cdot d\mathbf{S}$:

Definition 7.2.6.

$$\iint_{\mathbf{X}} \mathbf{F} \cdot d\mathbf{S} = \iint_D \mathbf{F}(\mathbf{X}(s,t)) \cdot (\mathbf{X}_s \times \mathbf{X}_t)\, dsdt = \iint_D \mathbf{F}(\mathbf{X}(s,t)) \cdot \mathbf{N}\, dsdt.$$

Here $d\mathbf{S}$ is similar to the scalar surface element dS given in the definition 7.2.1, but different in that it is a vector (pointing into the direction of the normal vector.)

If we let $\mathbf{n} = \mathbf{X}_s \times \mathbf{X}_t / \|\mathbf{X}_s \times \mathbf{X}_t\|$ be the unit normal vector to the surface, then

$$\begin{aligned}
\iint_{\mathbf{X}} \mathbf{F} \cdot d\mathbf{S} &= \iint_D \mathbf{F} \cdot \frac{\mathbf{X}_s \times \mathbf{X}_t}{\|\mathbf{X}_s \times \mathbf{X}_t\|} \|\mathbf{X}_s \times \mathbf{X}_t\|\, dsdt \\
&= \iint_D (\mathbf{F} \cdot \mathbf{n}) \cdot \|\mathbf{X}_s \times \mathbf{X}_t\| dsdt \\
&= \iint_S \mathbf{F} \cdot \mathbf{n}\, dS.
\end{aligned}$$

Example 7.2.7 (Spherical coordinate)**.** Let S be the unit sphere parameterized by

$$\mathbf{X}(\phi, \theta) = (\sin\phi\cos\theta, \sin\phi\sin\theta, \cos\phi),\ \ (0 \le \theta \le 2\pi,\ \ 0 \le \phi \le \pi).$$

Compute $\iint_S \mathbf{r} \cdot d\mathbf{S}$, where $\mathbf{r} = x\mathbf{i} + y\mathbf{i} + z\mathbf{k}$ denotes the position vector.

sol. We see

$$\begin{aligned}
\mathbf{X}_\phi &= \cos\phi\cos\theta\mathbf{i} + \cos\phi\sin\theta\mathbf{j} - \sin\phi\mathbf{k} \\
\mathbf{X}_\theta &= -\sin\phi\sin\theta\mathbf{i} + \sin\phi\cos\theta\mathbf{j} \\
\mathbf{X}_\phi \times \mathbf{X}_\theta &= \sin\phi(\cos\theta\sin\phi\mathbf{i} + \sin\theta\sin\phi\mathbf{j} + \cos\phi\mathbf{k})
\end{aligned}$$

Hence $\mathbf{r} \cdot d\mathbf{S} = \mathbf{r} \cdot (\mathbf{X}_\phi \times \mathbf{X}_\theta) d\phi\, d\theta = \sin\phi\, d\phi\, d\theta$ and

$$\iint_{\mathbf{X}} \mathbf{r} \cdot d\mathbf{S} = \int_0^{2\pi} \int_0^{\pi} \sin\phi\, d\phi\, d\theta = 4\pi.$$

Orientation

As in the case of line integral, the surface integral also has the notion of direction. First we need to define the **orientation** of a surface S. It depends on the particular parametrization.

Definition 7.2.8 (Oriented Surface)**.** An **orientable surface** is a two sided surface with one side specified as outside(or positive side). For orientable surface, there are two possible normal vectors at each point, i.e, two unit normal vectors $\mathbf{n}_1$ and $\mathbf{n}_2$, where $\mathbf{n}_1 = -\mathbf{n}_2$. Each of these normal vector can be associated with an **orientation**. There are nonorientable surfaces.(Example: Möbius strip)

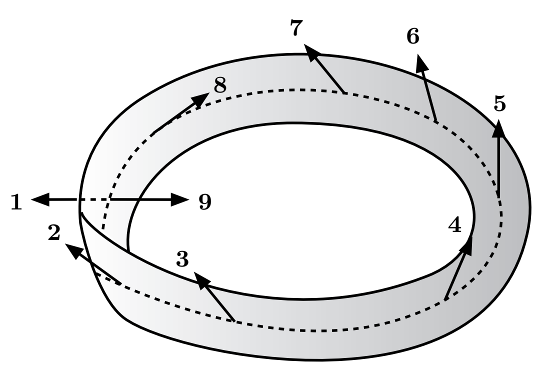


Figure 7.9: Möbius strip

Let $\mathbf{X}\colon D \to \mathbb{R}^3$ represent an oriented surface. If $\mathbf{n}(\mathbf{X})$ is the unit normal to S, then

$$\mathbf{n}(\mathbf{X}) = \pm \frac{\mathbf{X}_s \times \mathbf{X}_t}{\|\mathbf{X}_s \times \mathbf{X}_t\|}.$$

A parametrization is called **orientation-preserving** if

$$\frac{\mathbf{T}_s \times \mathbf{T}_t}{\|\mathbf{T}_s \times \mathbf{T}_t\|} = +\mathbf{n}(\mathbf{X}).$$

Otherwise, it is **orientation-reversing**.

Example 7.2.9. The parametrization of sphere by spherical coordinate by (ρ, ϕ, θ) is orientation-preserving. By changing the order of θ and ϕ, we can get orientation-reversing parametrization.

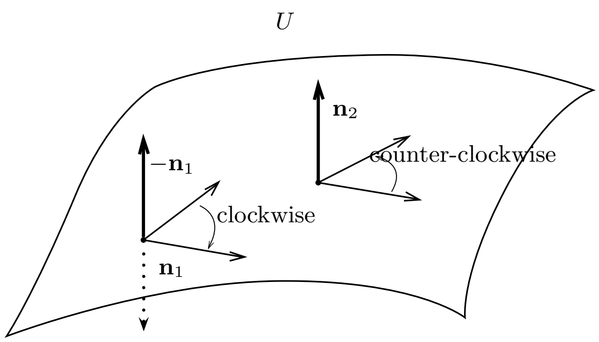


Figure 7.10: clockwise, counter-clockwise ($\mathbf{n}_1$ and $\mathbf{n}_2$ are normals the orientation points)

Example 7.2.10 (Möbius strip)**.**

$$\begin{cases} x = \left(1 + t\cos\frac{s}{2}\right)\cos s \\ y = \left(1 + t\cos\frac{s}{2}\right)\sin s, \ 0 \le s \le 2\pi, \ -\frac{1}{2} \le t \le \frac{1}{2} \\ z = t\sin\frac{s}{2} \end{cases}$$

Let $s = s_0$. Then

$$\begin{cases} x = \left(\cos s_0 \cos\frac{s_0}{2}\right) t + \cos s_0 \\ y = \left(\sin s_0 \cos\frac{s_0}{2}\right) t + \sin s_0, \ -\frac{1}{2} \le t \le \frac{1}{2} \\ z = \left(\sin\frac{s_0}{2}\right) t \end{cases}$$

Orientation of a graph

Example 7.2.11. Let S be the graph of a function $z = g(x, y)$. Usually, we give the orientation of such surface by taking the positive side to be the side away from which $\mathbf{n}$ points, where the unit normal is given by

$$\mathbf{n} = \frac{-\frac{\partial g}{\partial x}\mathbf{i} - \frac{\partial g}{\partial y}\mathbf{j} + \mathbf{k}}{\sqrt{1 + \left(\frac{\partial g}{\partial x}\right)^2 + \left(\frac{\partial g}{\partial y}\right)^2}}\, dxdy.$$

Independence of parametrization

Theorem 7.2.12. *Let S be an oriented surface and let $\mathbf{X}_1$, $\mathbf{X}_2$ be two regular orientation preserving parametrizations, then for any continuous vector field $\mathbf{F}$ defined on S, we have*

$$\iint_{\mathbf{X}_1} \mathbf{F} \cdot d\mathbf{S} = \iint_{\mathbf{X}_2} \mathbf{F} \cdot d\mathbf{S}.$$

If one of them is orientation reversing, then

$$\iint_{\mathbf{X}_1} \mathbf{F} \cdot d\mathbf{S} = -\iint_{\mathbf{X}_2} \mathbf{F} \cdot d\mathbf{S}.$$

For scalar f, the integral is independent of orientation: we have for any parametrization $\mathbf{X}_1, \mathbf{X}_2$,

$$\iint_{\mathbf{X}_1} f dS = \iint_{\mathbf{X}_2} f dS.$$

If a surface $\mathbf{X}$ consists of several pieces which is parametrized by $\mathbf{X}_1$, $\mathbf{X}_2$, $\ldots$, $\mathbf{X}_m$, then

$$\iint_{\mathbf{X}} \mathbf{F} \cdot d\mathbf{S} = \iint_{\mathbf{X}_1} \mathbf{F} \cdot d\mathbf{S} + \iint_{\mathbf{X}_2} \mathbf{F} \cdot d\mathbf{S} + \cdots + \iint_{\mathbf{X}_m} \mathbf{F} \cdot d\mathbf{S}.$$

Hence we can define the sum of surfaces as

$$\mathbf{X} = \mathbf{X}_1 + \mathbf{X}_2 + \cdots + \mathbf{X}_m.$$

Example 7.2.13. Let S be the surface of the cylinder bounded by $x^2+y^2 = 4$, $z = 0, z = 0$ with positive orientation. Evaluate $\iint_S (x^3\mathbf{i} + y^3\mathbf{j}) \cdot d\mathbf{S}$.

Surface Integral over Graphs

Suppose S is the graph of $z = g(x, y)$. We parameterize the surface S by $\mathbf{X}(x, y) = (x, y, g(x, y))$ and compute

$$\mathbf{T}_x = \mathbf{i} + \frac{\partial g}{\partial x}\mathbf{k}, \quad \mathbf{T}_y = \mathbf{j} + \frac{\partial g}{\partial y}\mathbf{k}.$$

Hence

$$\mathbf{T}_x \times \mathbf{T}_y = -(\frac{\partial g}{\partial x})\mathbf{i} - (\frac{\partial g}{\partial y})\mathbf{j} + \mathbf{k}$$

and we proved

$$\iint_S \mathbf{F}\cdot d\mathbf{S} = \iint_D \mathbf{F}\cdot(\mathbf{T}_x \times \mathbf{T}_y)dxdy = \iint_D \left[F_1(-\frac{\partial g}{\partial x}) + F_1(-\frac{\partial g}{\partial y}) + F_3\right] dxdy.$$

Relation with scalar integrals

Recall the definition:

$$\iint_{\mathbf{X}} \mathbf{F} \cdot d\mathbf{S} = \iint_D \mathbf{F}(\mathbf{X}(s, t)) \cdot (\mathbf{X}_s \times \mathbf{X}_t)\, dsdt.$$

If we write

$$\mathbf{n} = (\mathbf{X}_s \times \mathbf{X}_t)/\|\mathbf{X}_s \times \mathbf{X}_t\|, \quad d\mathbf{S} = (\mathbf{X}_s \times \mathbf{X}_t)\, dsdt, \quad d\mathbf{S} = \mathbf{n}\, dS,$$

then we see

$$\iint_{\mathbf{X}} \mathbf{F} \cdot d\mathbf{S} = \iint_{\mathbf{X}} \mathbf{F} \cdot \mathbf{n}\, dS.$$

Physical Interpretation of Surface Integrals

Consider the parallelepiped determined by three vectors $\mathbf{F}$, $\mathbf{T}_s\Delta s$ and $\mathbf{T}_t\Delta t$. (See figure 7.11.) Its volume is

$$\mathbf{F} \cdot (\mathbf{T}_s\Delta s \times \mathbf{T}_t\Delta t) = \mathbf{F} \cdot (\mathbf{T}_s \times \mathbf{T}_t)\Delta s\Delta t.$$

If $\mathbf{F}$ is the velocity of a fluid, the volume is *the amount of fluid to flow out of the surface per unit time.* Hence

$$\iint_S \mathbf{F} \cdot d\mathbf{S}$$

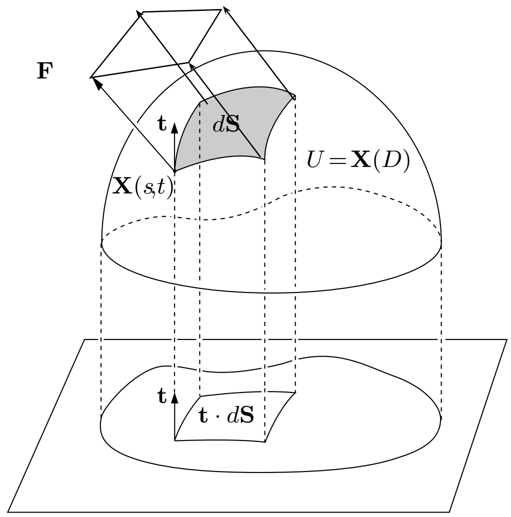


Figure 7.11: Area of shadow region and flux across S

is the *net quantity of fluid to flow across the surface per unit time*, i.e, **the rate of fluid flow**. It is also called **flux of F across** S.

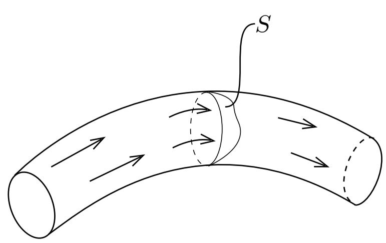


Figure 7.12: Water through a pipe and a surface S

Example 7.2.14 (Heat flow)**.** Let T denote the temperature at a point. Then

$$\nabla T = \frac{\partial T}{\partial x}\mathbf{i} + \frac{\partial T}{\partial y}\mathbf{j} + \frac{\partial T}{\partial z}\mathbf{k}$$

represents the temperature gradient and heat "flows" with the vector field $-k\nabla T$.

Example 7.2.15. Suppose temperature on a sphere $S\colon x^2+y^2+z^2=1$ is $T=x^2+y^2+z^2$. Find the total heat flux across S if $k=1$.

sol. We have $\mathbf{F}=-k\nabla T=-2\mathbf{r}$ and the unit normal vector to S is $\mathbf{n}=(x,y,z)=\mathbf{r}$. Hence $\mathbf{r}\cdot\mathbf{n}=-2$. So

$$\iint_S \mathbf{F}\cdot d\mathbf{S}=-2\iint_S dS=-8\pi.$$

Example 7.2.16 (Gauss Law)**.** The flux of an electric field $\mathbf{E}$ over a closed surface S is the net charge Q contained in the surface. Namely,

$$\iint_S \mathbf{E}\cdot d\mathbf{S}=Q.$$

Suppose $\mathbf{E}=E\mathbf{n}$(constant multiple of the unit normal vector) then

$$\iint_S \mathbf{E}\cdot d\mathbf{S}=\iint_S E dS=Q=E\cdot A(S).$$

So $E=\frac{Q}{A(S)}$ and if S is sphere of radius R then

$$E=\frac{Q}{4\pi R^2}. \tag{7.2}$$

Example 7.2.17. Given a disk lying on the plane $z=12$ described by

$$z=12,\quad x^2+y^2\le 25,$$

compute $\iint_S \mathbf{r}\cdot d\mathbf{S}$ where $\mathbf{r}=x\mathbf{i}+y\mathbf{j}+z\mathbf{k}$.

sol. We see

$$\mathbf{T}_x\times\mathbf{T}_y=\mathbf{i}\cdot\mathbf{j}=\mathbf{k}$$

So $\mathbf{r}\cdot(\mathbf{T}_x\times\mathbf{T}_y)=z$ and

$$\iint_S \mathbf{r}\cdot d\mathbf{S}=\iint_D z dxdy=12A(D)=300\pi.$$

Summary

(1) Given a parameterized surface $\mathbf{X}(s,t)$

(a) Integral of a scalar function f:

$$\iint_{\mathbf{X}} f\,dS = \iint_D f(\mathbf{X}(s,t))\|\mathbf{T}_s \times \mathbf{T}_t\| dsdt$$

(b) Scalar surface element:

$$dS = \|\mathbf{T}_s \times \mathbf{T}_t\| dsdt$$

(c) Integral of a vector field:

$$\iint_{\mathbf{X}} \mathbf{F}\cdot d\mathbf{S} = \iint_D \mathbf{F}(\mathbf{X}(s,t))\cdot(\mathbf{T}_s \times \mathbf{T}_t)\,dsdt = \iint_D (\mathbf{F}\cdot\mathbf{n})\,dS$$

(d) Vector surface element:

$$d\mathbf{S} = (\mathbf{T}_s \times \mathbf{T}_t)\,dsdt = \mathbf{n}\,dS$$

(2) When the surface is given by a graph $z = g(x,y)$

(a) Integral of a scalar f:

$$\iint_S f\,dS = \iint_D \frac{f(x,y,g(x,y))}{\cos\theta} dxdy$$

(b) Scalar surface element:

$$dS = \frac{dx\,dy}{\cos\theta} = \sqrt{\left(\frac{\partial g}{\partial x}\right)^2 + \left(\frac{\partial g}{\partial y}\right)^2 + 1}\ dxdy$$

(c) Integral of a vector field:

$$\iint_{\mathbf{S}} \mathbf{F}\cdot d\mathbf{S} = \iint_D \left(-F_1\frac{\partial g}{\partial x} - F_2\frac{\partial g}{\partial y} + F_3\right) dxdy$$

(d) Vector surface element:

$$d\mathbf{S} = \mathbf{n}\,dS = \left(-\frac{\partial g}{\partial x}\mathbf{i} - \frac{\partial g}{\partial y}\mathbf{j} + \mathbf{k}\right) dxdy$$

(3) On the sphere $x^2 + y^2 + z^2 = R^2$

(a) Scalar surface element:

$$dS = R^2 \sin\phi \, d\phi d\theta$$

(b) Vector surface element:

$$d\mathbf{S} = (x\mathbf{i} + y\mathbf{j} + z\mathbf{k})R \sin\phi \, d\phi d\theta = \mathbf{r}R \sin\phi \, d\phi d\theta = \mathbf{n}R^2 \sin\phi \, d\phi d\theta$$

7.3 Stokes' Theorem

In $\mathbb{R}^2$, the vector form of Green's theorem gives the relation between the line integral of a vector field on a simple closed curve to the integral of the curl of the vector on the domain having the curve as boundary.

Stokes' theorem is the generalization of Green's theorem to the surface lying in $\mathbb{R}^3$: Consider a simple closed curve lying in $\mathbb{R}^3$ and a surface having the curve as boundary: A caution: there are many surfaces having the same curve as boundary. But as long as the vector fields are C^1 in a large region containing the curve and the surface, any surface play the same role.

Theorem 7.3.1 (Stokes' theorem)**.** *Let S be a piecewise smooth oriented surface. Suppose the boundary ∂S consists of finitely many piecewise C^1 curve with the same orientation with S. Let $\mathbf{F}$ be a C^1-vector field defined on S. Then*

$$\iint_S (\nabla \times \mathbf{F}) \cdot d\mathbf{S} = \int_{\partial S} \mathbf{F} \cdot d\mathbf{s}.$$

Proof. First assume S is defined by C^1-function $z = f(x, y)$ on D, a region to which Green's theorem holds. Then it can be parameterized by

$$\begin{cases} x = x \\ y = y \\ z = f(x, y), \end{cases}$$

for (x, y) in D. Recall the integral of a vector field $\mathbf{F} = F_1\,\mathbf{i} + F_2\,\mathbf{j} + F_3\,\mathbf{k}$ over S is defined by

$$\iint_S \mathbf{F} \cdot d\mathbf{S} = \iint_D \left[F_1 \left(-\frac{\partial z}{\partial x} \right) + F_2 \left(-\frac{\partial z}{\partial y} \right) + F_3 \right] dxdy. \tag{7.3}$$

If $\mathbf{F} = F_1\mathbf{i} + F_2\mathbf{j} + F_3\mathbf{k}$, then

$$\begin{aligned}\nabla \times \mathbf{F} = \operatorname{curl}\mathbf{F} &= \Big(\frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z}\Big)\mathbf{i} + \Big(\frac{\partial F_1}{\partial z} - \frac{\partial F_3}{\partial x}\Big)\mathbf{j} + \Big(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\Big)\mathbf{k} \\ &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ F_1 & F_2 & F_3 \end{vmatrix}.\end{aligned}$$

Hence by (7.3)

$$\begin{aligned}\iint_S \operatorname{curl}\mathbf{F}\cdot d\mathbf{S} &= \iint_D \Big[\Big(\frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z}\Big)\left(-\frac{\partial z}{\partial x}\right) \\ &\quad + \Big(\frac{\partial F_1}{\partial z} - \frac{\partial F_3}{\partial x}\Big)\left(-\frac{\partial z}{\partial y}\right) + \Big(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\Big)\Big]\,dxdy.\end{aligned}$$

On the other hand

$$\int_{\partial S} \mathbf{F}\cdot d\mathbf{s} = \int_{\mathbf{p}} \mathbf{F}\cdot d\mathbf{s} = \int_{\mathbf{p}} F_1dx + F_2dy + F_3dz.$$

Here $\mathbf{p} = \mathbf{F}\circ\mathbf{x}$ is a parametrization of **boundary curve** ∂S obtained from a parametrization of ∂D in positive direction. Assume ∂D has the orientation induced by $\mathbf{c}$. Then

$$\int_{\partial S} \mathbf{F}\cdot d\mathbf{s} = \int_a^b \left(F_1\frac{dx}{dt} + F_2\frac{dy}{dt} + F_3\frac{dz}{dt}\right)dt. \tag{7.4}$$

By the chain rule

$$\frac{dz}{dt} = \frac{\partial z}{\partial x}\frac{dx}{dt} + \frac{\partial z}{\partial y}\frac{dy}{dt}.$$

Substituting this into above

$$\begin{aligned}\int_{\partial S} \mathbf{F}\cdot d\mathbf{s} &= \int_a^b \left[\left(F_1 + F_3\frac{\partial z}{\partial x}\right)\frac{dx}{dt} + \left(F_2 + F_3\frac{\partial z}{\partial y}\right)\frac{dy}{dt}\right]dt \\ &= \int_{\mathbf{c}} \left(F_1 + F_3\frac{\partial z}{\partial x}\right)dx + \left(F_2 + F_3\frac{\partial z}{\partial y}\right)dy \qquad (7.5) \\ &= \int_{\partial D} \left(F_1 + F_3\frac{\partial z}{\partial x}\right)dx + \left(F_2 + F_3\frac{\partial z}{\partial y}\right)dy.\end{aligned}$$

Applying Green's theorem to (7.5) yields

$$\iint_D \left[\left(\frac{\partial (F_2 + F_3 \frac{\partial z}{\partial y})}{\partial x} - \frac{\partial (F_1 + F_3 \frac{\partial z}{\partial x})}{\partial y} \right) \right] dxdy.$$

Now use chain rule keeping in mind that F_1, F_2, F_3 are functions of x, y and z, while z is again a function of x, y. (Here $\frac{\partial F_2}{\partial x}$ has to be interpreted carefully. For example, we let $G(x, y) = F_2(x, y, f(x, y))$, and $\frac{\partial F_2}{\partial x}$ is understood as $\frac{\partial G}{\partial x}$. In other words, treat x, y as independent variables, while regarding z as dependent variable.) Thus by chain rule, above integral becomes

$$\begin{aligned} &\iint_D \left[\left(\frac{\partial F_2}{\partial x} + \frac{\partial F_2}{\partial z}\frac{\partial z}{\partial x} + \frac{\partial F_3}{\partial x}\frac{\partial z}{\partial y} + \frac{\partial F_3}{\partial z}\frac{\partial z}{\partial x}\frac{\partial z}{\partial y} + F_3 \frac{\partial^2 z}{\partial x \partial y} \right) \right. \\ &\left. - \left(\frac{\partial F_1}{\partial y} + \frac{\partial F_1}{\partial z}\frac{\partial z}{\partial y} + \frac{\partial F_3}{\partial y}\frac{\partial z}{\partial x} + \frac{\partial F_3}{\partial z}\frac{\partial z}{\partial y}\frac{\partial z}{\partial x} + F_3 \frac{\partial^2 z}{\partial x \partial y} \right) \right] dA. \end{aligned}$$

Because mixed partials are equal, the last two integrals cancel each other and we obtain

$$\begin{aligned} &\iint_D \left[\left(\frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z} \right) \left(-\frac{\partial z}{\partial x} \right) + \left(\frac{\partial F_1}{\partial z} - \frac{\partial F_3}{\partial x} \right) \left(-\frac{\partial z}{\partial y} \right) + \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} \right) \right] dxdy \\ = &\iint_S \operatorname{curl} \mathbf{F} \cdot d\mathbf{S}. \end{aligned}$$

□

Example 7.3.2. Let S be smooth surface having an oriented simple closed curve C as boundary and let $\mathbf{F} = ye^z\mathbf{i} + xe^z\mathbf{j} + xye^z\mathbf{k}$. Compute $\int_C \mathbf{F} \cdot d\mathbf{s}$.

$$\operatorname{curl} \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ ye^z & xe^z & xye^z \end{vmatrix} = 0.$$

By Stoke's theorem,

$$\int_C \mathbf{F} \cdot d\mathbf{s} = \iint_S \operatorname{curl} \mathbf{F} \cdot d\mathbf{S} = 0.$$

Example 7.3.3. Show that $\int_C \mathbf{F} \cdot d\mathbf{s} = -4\pi$ when $\mathbf{F} = (x^2 + y)\mathbf{i} + (x^2 + 2y)\mathbf{j} + 2z^3\mathbf{k}$ and $C : x^2 + y^2 = 4,\ z = 2$.

sol. Let S be the disk $D = \{(x, y, z) : x^2 + y^2 = 4, z = 2\}$. If $\mathbf{n}$ is the unit normal to S, then $\mathbf{n} = \mathbf{k}$ and

$$\begin{aligned}\nabla \times \mathbf{F} &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ x^2+y & x^2+2y & 2z^3 \end{vmatrix} \\ &= (0-0)\mathbf{i} - (0-0)\mathbf{j} + (2x-1)\mathbf{k} = (2x-1)\mathbf{k}.\end{aligned}$$

Hence Stokes' theorem

$$\begin{aligned}\int_C \mathbf{F}\cdot d\mathbf{s} &= \iint_S (\nabla\times\mathbf{F})\cdot d\mathbf{S} = \iint_S (\nabla\times\mathbf{F})\cdot \mathbf{n}dS \\ &= \iint_S (2x-1)\mathbf{k}\cdot\mathbf{k}dS = \int_{-2}^{2}\int_{-\sqrt{4-y^2}}^{\sqrt{4-y^2}} (2x-1)dxdy \\ &= -2\int_{-2}^{2}\sqrt{4-y^2}dy = -4\pi.\end{aligned}$$

■

Example 7.3.4. Evaluate

$$\int_C -y^3dx + x^3dy - z^3dz$$

where C is the intersection of the cylinder $x^2+y^2=1$ and plane $x+y+z=1$.

sol. Let $\mathbf{F} = -y^3\mathbf{i} + x^3\mathbf{j} - z^3\mathbf{k}$. Then above integral is $\int_C \mathbf{F}\cdot d\mathbf{s}$. If we consider any reasonable surface S having C as boundary, we can use Stokes' theorem with $\operatorname{curl}\mathbf{F} = 3(x^2+y^2)\mathbf{k}$. Let us assume S is the surface defined by $x+y+z=1$, $x^2+y^2\le 1$. A parametrization of S is given by $(s,t,1-s-t)$. We need to compute

$$d\mathbf{S} = \mathbf{T}_s\times\mathbf{T}_t = (\mathbf{i}-\mathbf{k})\times(\mathbf{j}-\mathbf{k}) = \mathbf{i}+\mathbf{j}+\mathbf{k}.$$

Hence

$$\int_C \mathbf{F}\cdot d\mathbf{s} = \iint_S \operatorname{curl}\mathbf{F}\cdot d\mathbf{S} = \iint_D 3(x^2+y^2)dxdy = \frac{3\pi}{2}.$$

Here the domain D is the set $\{(x,y)|x^2+y^2\le 1\}$.

■

Example 7.3.5. A surface S is defined by $z = e^{-(x^2+y^2)}$ for $z \ge 1/e$. Let

$$\mathbf{F} = (e^{y+z} - 2y)\mathbf{i} + (xe^{y+z} + y)\mathbf{j} + e^{x+y}\mathbf{k}.$$

Evaluate $\iint_S \nabla \times \mathbf{F} \cdot d\mathbf{S}$.

sol. We see

$$\nabla \times \mathbf{F} = (e^{x+y} - xe^{y+z})\mathbf{i} + (e^{y+z} - e^{x+y})\mathbf{j} + 2\mathbf{k}$$

and

$$\mathbf{N} = 2xe^{-(x^2+y^2)}\mathbf{i} + 2ye^{-(x^2+y^2)}\mathbf{j} + \mathbf{k}.$$

So direct computation of $\int_S \nabla \times \mathbf{F} \cdot d\mathbf{S}$ seems almost impossible. Now try to use Stoke's theorem. First parameterize the boundary by

$$x = \cos t, y = \sin t, z = 1/e.$$

Then

$$\int_C \mathbf{F} \cdot d\mathbf{s} = \int_C (e^{\sin t + 1/e} - 2\sin t, \cdots, e^{\cos t + \sin t}) \cdot (-\sin t, \cos t, 0)\, dt$$

This again is very difficult! Now think of another way. Think of another surface S' which has the same boundary as S., i.e, let S' be the unit disk $x^2 + y^2 \leq 1, z = 1/e$. Then $\mathbf{n} = 1$ and hence

$$\iint_S \nabla \times \cdot d\mathbf{S} = \iint_S \nabla \times \cdot \mathbf{n} dS = \iint_S 2dS = 2\pi.$$

Curl as Circulation per Unit area

Let F_T denote the tangential component of $\mathbf{F}$, we have

$$\iint_S (\operatorname{curl} \mathbf{F}) \cdot \mathbf{n} dS = \iint_S \operatorname{curl} \mathbf{F} \cdot d\mathbf{S} = \int_{\partial S} \mathbf{F} \cdot d\mathbf{s} = \int_{\partial S} F_{\mathbf{T}} ds.$$

Suppose $\mathbf{V}$ represent the velocity of a fluid. Consider a point P and unit normal vector $\mathbf{n}$. If S_ρ is a disk centered at P with radius ρ perpendicular to

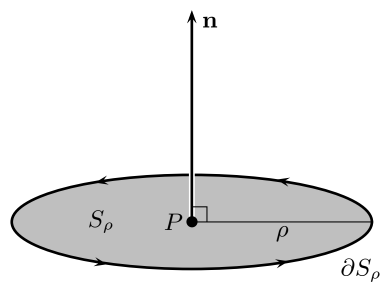


Figure 7.13: $\mathbf{n}$ unit normal to S_ρ

$\mathbf{n}$ (fig 7.13)then by Stokes' theorem,

$$\iint_{S_\rho} \operatorname{curl} \mathbf{V} \cdot d\mathbf{S} = \iint_{S_\rho} (\operatorname{curl} \mathbf{V}) \cdot \mathbf{n} dS = \int_{\partial S_\rho} \mathbf{V} \cdot d\mathbf{s}.$$

holds. Here ∂S_ρ has the orientation according to $\mathbf{n}$. If $A(S_\rho) = \pi\rho^2$ denote the area of S_ρ, $\operatorname{curl} \mathbf{V}(Q)$ is an average $\operatorname{curl} \mathbf{V}$ on Q , $\mathbf{n}(Q)$, we have by MVT

$$\iint_{S_\rho} (\operatorname{curl} \mathbf{V}) \cdot \mathbf{n} dS = [\operatorname{curl} \mathbf{V}(Q) \cdot \mathbf{n}(Q)] A(S_\rho)$$

for some point Q in S_ρ. Hence

$$\begin{aligned} \lim_{\rho \to 0} \frac{1}{A(S_\rho)} \int_{\partial S_\rho} \mathbf{V} \cdot d\mathbf{s} &= \lim_{\rho \to 0} \frac{1}{A(S_\rho)} \iint_{S_\rho} (\operatorname{curl} \mathbf{V}) \cdot \mathbf{n} dS \\ &= \lim_{\rho \to 0} (\operatorname{curl} \mathbf{V}(Q)) \cdot \mathbf{n}(Q) \\ &= (\operatorname{curl} \mathbf{V}(P)) \cdot \mathbf{n}(P). \end{aligned}$$

$$(\operatorname{curl} \mathbf{V}(P)) \cdot \mathbf{n}(P) = \lim_{\rho \to 0} \frac{1}{A(S_\rho)} \int_{\partial S_\rho} \mathbf{V} \cdot d\mathbf{s}. \tag{7.6}$$

Now consider physical meaning of $\int_C \mathbf{V} \cdot d\mathbf{s}$ (fig 7.14).

Assume $\mathbf{V}$ is tangent to C and $\int_C \mathbf{V} \cdot d\mathbf{s} > 0$ then an object on C rotates along the direction of C. If $\int_C \mathbf{V} \cdot d\mathbf{s} < 0$, it rotates counter-clockwise on C.

Also, if $\mathbf{V} \perp C$ then object on C does not rotate and

$$\int_C \mathbf{V} \cdot d\mathbf{s} = 0.$$

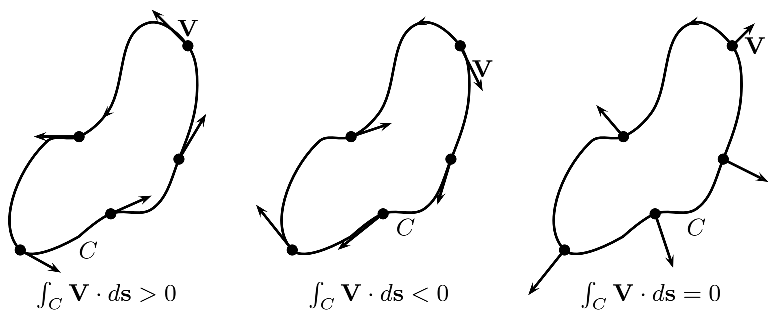


Figure 7.14: Meaning of $\int_C \mathbf{V}\cdot d\mathbf{s}$

In general the integral of tangential component of a fluid vector field $\mathbf{V}$ $\int_C \mathbf{V}\cdot d\mathbf{s}$ represent **the net amount of turning** around C. Thus,

$$\int_C \mathbf{V}\cdot d\mathbf{s}$$

is called the **circulation** of $\mathbf{V}$ around C. (fig 7.15).

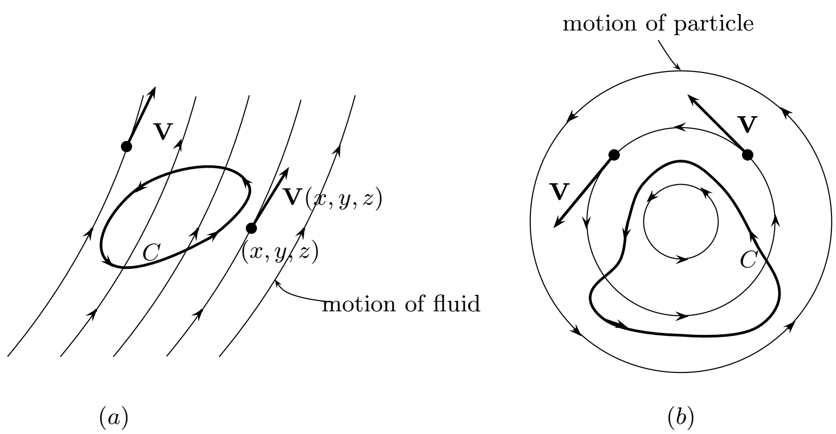


Figure 7.15: Circulation of a vector field; (a) 0 circulation (b) nonzero circulation

The circulation $\int_{\partial S_\rho} \mathbf{V}\cdot d\mathbf{s}$ is the net velocity of a fluid around ∂S_ρ, and $(\operatorname{curl}\mathbf{V})\cdot\mathbf{n}$ is the circulation of V per unit area on a surface perpendicular to $\mathbf{n}$

Observe that $(\operatorname{curl}\mathbf{V})\cdot\mathbf{n}$ is maximized when $\mathbf{n} = \operatorname{curl}\mathbf{V}/||\operatorname{curl}\mathbf{V}||$. So the rotating effect is maximized about an axis parallel to $\operatorname{curl}\mathbf{V}/||\operatorname{curl}\mathbf{V}||$.

Example 7.3.6. p 541. Use physical interpretation to compute $\nabla\times\mathbf{F}\cdot\mathbf{e}_r$. Let

$\mathbf{e}_r$, $\mathbf{e}_\theta$, $\mathbf{e}_z$ associate to cylindrical coordinates as fig 7.17. Let $\mathbf{F} = F_r\mathbf{e}_r + F_\theta\mathbf{e}_\theta + F_z\mathbf{e}_z$. Find a formula for $\mathbf{e}_r$ component of $\nabla \times \mathbf{F}$ in cylindrical coordinate.

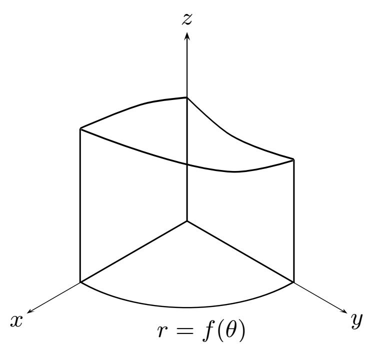


Figure 7.16: cylindrical coordinate

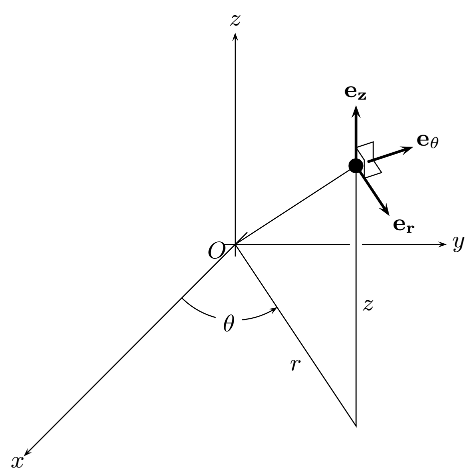


Figure 7.17: unit orthogonal vectors in cylindrical coordinate $\mathbf{e}_r, \mathbf{e}_\theta, \mathbf{e}_z$

sol. The $\mathbf{e}_r$ component of $\nabla \times \mathbf{F}$ is $\nabla \times \mathbf{F} \cdot \mathbf{e}_r$ which is also

$$\lim_{|S|\to 0} \frac{1}{|S|} \iint_S \nabla \times \mathbf{F} \cdot \mathbf{e}_r \, dS. \tag{7.7}$$

For convenience, just take S to be a rectangular part of lateral surface of the cylinder(See figure 7.18.) The area of S is $rd\theta\, dz$. By Stokes' theorem,

$$\lim_{|S|\to 0} \frac{1}{|S|} \iint_S \nabla \times \mathbf{F} \cdot \mathbf{e}_r \, dS = \lim_{|S|\to 0} \frac{1}{|S|} \int_{\partial S} \mathbf{F} \cdot d\mathbf{s}.$$

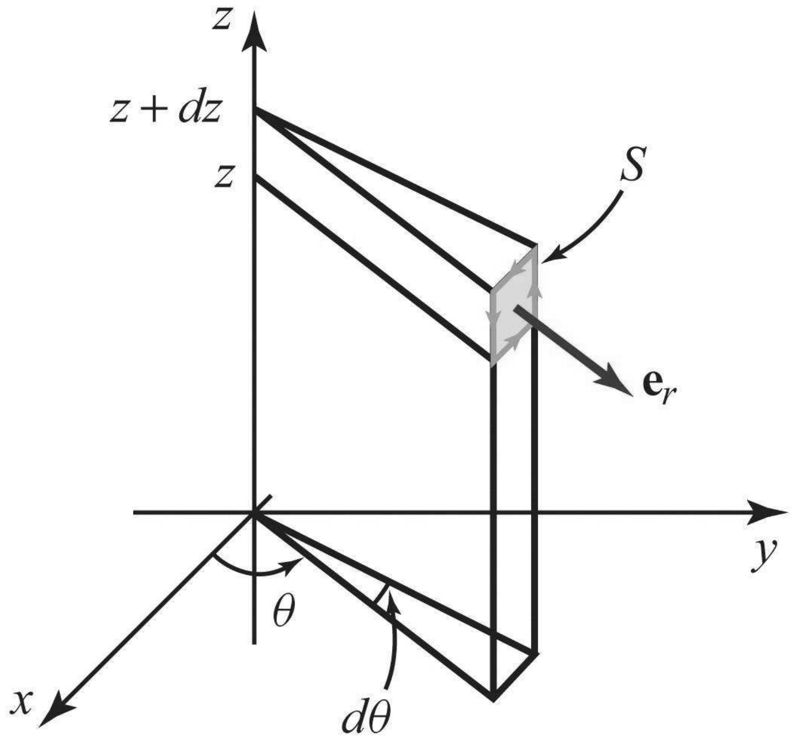


Figure 7.18: Vertical Surface-2

Thus it suffices to compute (7.7): The integral of $\mathbf{F}$ around the edges of S is

$$[F_\theta(r,\theta,z) - F_\theta(r,\theta,z+dz)]rd\theta + [F_z(r,\theta+d\theta,z) - F_z(r,\theta,z)]dz$$
$$\approx -\frac{\partial F_\theta}{\partial z} dz\, r d\theta + \frac{\partial F_z}{\partial \theta} d\theta\, dz.$$

Divide by the area and take the limit, we get

$$\nabla \times \mathbf{F} \cdot \mathbf{e}_r = \lim_{|S|\to 0} \frac{1}{|S|} \iint_S \nabla \times \mathbf{F} \cdot \mathbf{e}_r\, dS = \lim_{|S|\to 0} \frac{1}{|S|} \int_{\partial S} \mathbf{F} \cdot d\mathbf{s} = \frac{1}{r}\frac{\partial F_z}{\partial \theta} - \frac{\partial F_\theta}{\partial z}.$$

This is the circulation at points of S per unit area.

Example 7.3.7. Let $\mathbf{F} = F_r\mathbf{e}_r + F_\theta\mathbf{e}_\theta + F_z\mathbf{e}_z$. Find $\mathbf{e}_z$ component of $\nabla \times \mathbf{F}$ as follows: Consider S(fig. 7.19) which is perpendicular to the normal vector

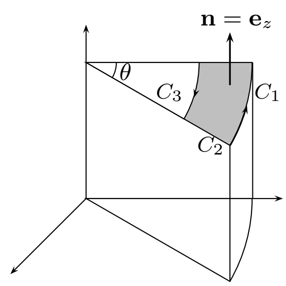


Figure 7.19: Surface in cylinder

$\mathbf{e}_z$. Thus $\mathbf{e}_z = \mathbf{n}$ on S and hence $\iint_S \nabla \times \mathbf{F} \cdot \mathbf{e}_z dS$ equals

$$\begin{aligned}
&= \iint_S \nabla \times \mathbf{F} \cdot d\mathbf{S} = \int_{\partial S} \mathbf{F} \cdot d\mathbf{s} \\
&= \int_{C_1 - C_3} F_\theta r d\theta + \int_{-C_2 + C_4} F_r dr \\
&= [F_\theta(r + dr, \theta, z) - F_\theta(r, \theta, z)] r d\theta + [F_r(r, \theta, z) - F_r(r, \theta + d\theta, z)] dr \\
&\approx \frac{\partial F_\theta}{\partial r} r dr d\theta - \frac{\partial F_r}{\partial \theta} dr d\theta.
\end{aligned}$$

Dividing by $A(S) = r\, dr d\theta$, we see

$$\nabla \times \mathbf{F} \cdot \mathbf{e}_z = \lim_{|S| \to 0} \frac{1}{|S|} \iint_S \nabla \times \mathbf{F} \cdot \mathbf{e}_z dS = \frac{\partial F_\theta}{\partial r} - \frac{1}{r} \frac{\partial F_r}{\partial \theta}.$$

Example 7.3.8. Find $\mathbf{e}_\theta$ component of $\nabla \times \mathbf{F}$ referring to the figure 7.20. Note $\mathbf{n} = -\mathbf{e}_\theta$ on S.

$$\begin{aligned}
-\nabla \times \mathbf{F} \cdot \mathbf{e}_\theta A(S) &\approx \iint_S \nabla \times \mathbf{F} \cdot d\mathbf{S} = \int_{\partial S} \mathbf{F} \cdot d\mathbf{s} \\
&= \int_{C_1 - C_3} F_z dz + \int_{-C_2 + C_4} F_r dr \\
&= [F_z(r + dr, \theta, z) - F_z(r, \theta, z)] dz + [F_r(r, \theta, z) - F_r(r, \theta, z + dz)] dr \\
&\approx \frac{\partial F_z}{\partial r} dr dz - \frac{\partial F_r}{\partial z} dr dz.
\end{aligned}$$

Dividing by area $drdz$, we see

$$\nabla \times \mathbf{F} \cdot \mathbf{e}_\theta = \frac{\partial F_r}{\partial z} - \frac{\partial F_z}{\partial r}.$$

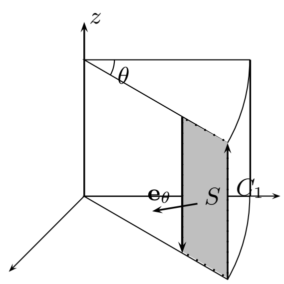


Figure 7.20: Vertical Surface

7.5 Gauss' Theorem

The flux of a vector field $\mathbf{F}$ across Ω is equal to the sum of $\operatorname{div}\mathbf{F}$ in Ω.

Theorem 7.5.1. ***[Gauss' Divergence Theorem]***

Let Ω be an elementary region in $\mathbb{R}^3$ and $\partial\Omega$ consists of finitely many oriented piecewise smooth closed surfaces. Let $\mathbf{F}$ be a $\mathcal{C}^1$ vector field on a region containing Ω. Then

$$\iiint_\Omega \operatorname{div}\mathbf{F}dV = \iint_{\partial\Omega} \mathbf{F}\cdot d\mathbf{S}.$$

Proof. Suppose Ω is an elementary region of type 4 and $\mathbf{F} = P\mathbf{i} + Q\mathbf{j} + R\mathbf{k}$. Then

$$\operatorname{div}\mathbf{F} = \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z}$$

and

$$\iiint_\Omega \operatorname{div}\mathbf{F}dV = \iiint_\Omega \frac{\partial P}{\partial x}dV + \iiint_\Omega \frac{\partial Q}{\partial y}dV + \iiint_\Omega \frac{\partial R}{\partial z}dV.$$

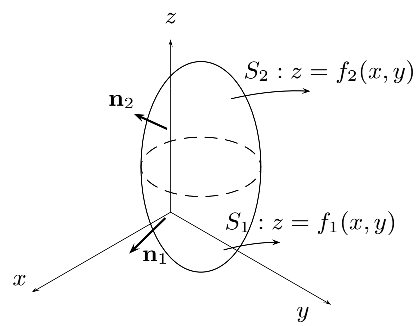


Figure 7.21: Region of type 1

On the other hand, the surface integral is

$$\begin{aligned}\iint_{\partial\Omega}(\mathbf{F}\cdot\mathbf{n})dS &= \iint_{\partial\Omega}(P\mathbf{i}+Q\mathbf{j}+R\mathbf{k})\cdot\mathbf{n}dS\\ &= \iint_{\partial\Omega} P\mathbf{i}\cdot\mathbf{n}dS + \iint_{\partial\Omega} Q\mathbf{j}\cdot\mathbf{n}dS + \iint_{\partial\Omega} R\mathbf{k}\cdot\mathbf{n}dS.\end{aligned}$$

If we show the following

$$\iint_{\partial\Omega} P\mathbf{i}\cdot\mathbf{n}dS = \iiint_\Omega \frac{\partial P}{\partial x}dV, \tag{7.8}$$

$$\iint_{\partial\Omega} Q\mathbf{j}\cdot\mathbf{n}dS = \iiint_\Omega \frac{\partial Q}{\partial y}dV, \tag{7.9}$$

$$\iint_{\partial\Omega} R\mathbf{k}\cdot\mathbf{n}dS = \iiint_\Omega \frac{\partial R}{\partial z}dV, \tag{7.10}$$

then the proof will be complete. First we shall prove (7.10). Suppose there exist two functions $z = f_1(x, y)$, $z = f_2(x, y)$ defined on a region D in xy-plane such that $\Omega = \{(x, y, z)|f_1(x, y) \le z \le f_2(x, y), (x, y) \in D\}$ (see fig 7.5). Then

$$\iiint_\Omega \frac{\partial R}{\partial z}dV = \iint_D \left(\int_{z=f_1(x,y)}^{z=f_2(x,y)} \frac{\partial R}{\partial z}dz\right) dxdy \tag{7.11}$$

$$= \iint_D [R(x, y, f_2(x, y)) - R(x, y, f_1(x, y))]dxdy. \tag{7.12}$$

The boundary of Ω consists of two surface S_1, S_2, where S_2 is the graph of

$z = f_2(x, y)$ and S_1 is the graph of $z = f_1(x, y)$, $(x, y) \in D$. Hence

$$\iint_{\partial\Omega} R\mathbf{k}\cdot\mathbf{n}dS = \iint_{S_1} R\mathbf{k}\cdot\mathbf{n}_1 dS + \iint_{S_2} R\mathbf{k}\cdot\mathbf{n}_2 dS. \tag{7.13}$$

Bottom surface S_1 is given by $z = f_1(x, y)$ where the unit normal is given by

$$\mathbf{n}_1 = \frac{\dfrac{\partial f_1}{\partial x}\mathbf{i} + \dfrac{\partial f_1}{\partial y}\mathbf{j} - \mathbf{k}}{\sqrt{\left(\dfrac{\partial f_1}{\partial x}\right)^2 + \left(\dfrac{\partial f_1}{\partial y}\right)^2 + 1}}.$$

Hence $\mathbf{k}\cdot\mathbf{n}_1 = -1/\sqrt{\left(\frac{\partial f_1}{\partial x}\right)^2 + \left(\frac{\partial f_1}{\partial y}\right)^2 + 1}$, and so

$$\iint_{S_1} R\mathbf{k}\cdot\mathbf{n}_1 dS = -\iint_D R(x, y, f_1(x, y))dxdy. \tag{7.14}$$

Similarly on S_2, $\mathbf{k}\cdot\mathbf{n}_2 = -1/\sqrt{\left(\frac{\partial f_2}{\partial x}\right)^2 + \left(\frac{\partial f_2}{\partial y}\right)^2 + 1}$. Hence

$$\iint_{S_2} R\mathbf{k}\cdot\mathbf{n}_2 dS = \iint_D R(x, y, f_2(x, y))dxdy. \tag{7.15}$$

Substitute (7.14), (7.15) into (7.13). Then by (7.11), (7.12), we obtain

$$\iint_{\partial\Omega} R\mathbf{k}\cdot\mathbf{n}dS = \iiint_\Omega \frac{\partial R}{\partial z}dV.$$

The identities (7.8) and (7.9) can be similarly shown. □

Example 7.5.2. S is the unit sphere $x^2+y^2+z^2 = 1$ and $\mathbf{F} = 2x\mathbf{i}+y^2\mathbf{j}+z^2\mathbf{k}$. Find $\iint_S \mathbf{F}\cdot\mathbf{n}dS$.

sol. Let Ω be the region inside S. By Gauss theorem, it holds that

$$\iint_S \mathbf{F}\cdot\mathbf{n}dS = \iiint_\Omega \operatorname{div}\mathbf{F}dV.$$

Since $\operatorname{div}\mathbf{F} = \nabla\cdot(2x\mathbf{i} + y^2\mathbf{j} + z^2\mathbf{k}) = 2(1 + y + z)$, the rhs is

$$2\iiint_\Omega (1 + y + z)dV = 2\iiint_\Omega 1dV + 2\iiint_\Omega ydV + 2\iiint_\Omega zdV.$$

By symmetry, we have

$$\iiint_\Omega y dV = \iiint_\Omega z dV = 0.$$

Hence

$$\iint_S \mathbf{F}\cdot\mathbf{n} dS = 2\iiint_\Omega (1+y+z) dV = 2\iiint_\Omega 1 dV = \frac{8}{3}\pi.$$

■

Generalizing Gauss' theorem

Example 7.5.3. Show Gauss' theorem holds for $\mathbf{F} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$ in Ω : $x^2+y^2+z^2 \le a^2$.

sol. First compute $\operatorname{div}\mathbf{F} = \nabla\cdot\mathbf{F}$,

$$\operatorname{div}\mathbf{F} = \frac{\partial x}{\partial x} + \frac{\partial y}{\partial y} + \frac{\partial z}{\partial z} = 3.$$

So

$$\iiint_\Omega (\operatorname{div}\mathbf{F}) dV = \iiint_\Omega 3\, dV = 3\Big(\frac{4}{3}\pi a^3\Big) = 4\pi a^3.$$

To compute the surface integral, we need to find the unit normal $\mathbf{n}$ on $\partial\Omega$. Since $\partial\Omega$ is the level set of $f(x,y,z) = x^2+y^2+z^2-a^2$, we see the unit normal vector to $\partial\Omega$ is

$$\mathbf{n} = \pm\frac{\nabla f}{||\nabla f||} = \mathbf{n} = \frac{2(x\mathbf{i}+y\mathbf{j}+z\mathbf{k})}{\sqrt{4(x^2+y^2+z^2)}} = \frac{x\mathbf{i}+y\mathbf{j}+z\mathbf{k}}{a}.$$

So when $(x,y,z)\in\partial\Omega$,

$$\mathbf{F}\cdot\mathbf{n} = \frac{x^2+y^2+z^2}{a} = \frac{a^2}{a} = a$$

and

$$\iint_{\partial\Omega} \mathbf{F}\cdot\mathbf{n} dS = \iint_{\partial\Omega} a\, dS = a(4\pi a^2) = 4\pi a^3.$$

Hence

$$\iiint_\Omega (\operatorname{div}\mathbf{F}) dV = 4\pi a^3 = \iint_{\partial\Omega} \mathbf{F}\cdot\mathbf{n} dS.$$

and Gauss' theorem holds.

Example 7.5.4. Let Ω be the region given by $x^2+y^2+z^2 \le 1$. Find $\iint_{\partial\Omega}(x^2+4y-5z)dS$ by Gauss' theorem.

sol. To use Gauss' theorem, we need a vector field $\mathbf{F} = F_1\mathbf{i}+F_2\mathbf{j}+F_3\mathbf{k}$ such that $\mathbf{F}\cdot\mathbf{n} = x^2+4y-5z$. Since the unit normal vector is $\mathbf{n} = x\mathbf{i}+y\mathbf{j}+z\mathbf{k}$, one such obvious choice is $\mathbf{F} = x\mathbf{i}+4\mathbf{j}-5\mathbf{k}$. Hence we have $\operatorname{div}\mathbf{F} = 1+0+(-0) = 1$. Now by Gauss theorem

$$\begin{aligned}\iint_{\partial\Omega}(x^2+4y-5z)dS &= \iint_{\partial\Omega}(x\mathbf{i}+4\mathbf{j}-5\mathbf{k})\cdot\mathbf{n}dS \\ &= \iint_{\partial\Omega}\mathbf{F}\cdot\mathbf{n}dS = \iiint_{\Omega}\operatorname{div}\mathbf{F}dV \\ &= \iiint_{\Omega} 1\ dV = \frac{4}{3}\pi.\end{aligned}$$

Hence $\iint_{\partial\Omega}(x^2+4y-5z)dS = 4\pi/3$.

Divergence as flux per unit Volume

As we have seen before that $\operatorname{div}\mathbf{F}(P)$ is the *rate of change of total flux* at P per unite volume. Let Ω_ρ be a ball of radius ρ center at P. Then for some Q in Ω_ρ,

$$\iint_{\partial\Omega_\rho}\mathbf{F}\cdot\mathbf{n}dS = \iiint_{\Omega_\rho}\operatorname{div}\mathbf{F}dV = \operatorname{div}\mathbf{F}(Q)\cdot\operatorname{Vol}(\Omega_\rho).$$

Dividing by the volume we get

$$\operatorname{div}\mathbf{F}(Q) = \frac{1}{\operatorname{Vol}(\Omega_\rho)}\iint_{\partial\Omega_\rho}\mathbf{F}\cdot\mathbf{n}dS. \tag{7.16}$$

Taking the limit, we see

$$\lim_{\rho\to 0}\frac{1}{\operatorname{Vol}(\Omega_\rho)}\iint_{\partial\Omega_\rho}\mathbf{F}\cdot\mathbf{n}dS = \operatorname{div}\mathbf{F}(P). \tag{7.17}$$

Now we can give a physical interpretation: If $\mathbf{F}$ is the velocity of a fluid, then

$\boxed{\operatorname{div}\mathbf{F}(P) \text{ is the rate at which the fluid flows out per unit volume.}}$

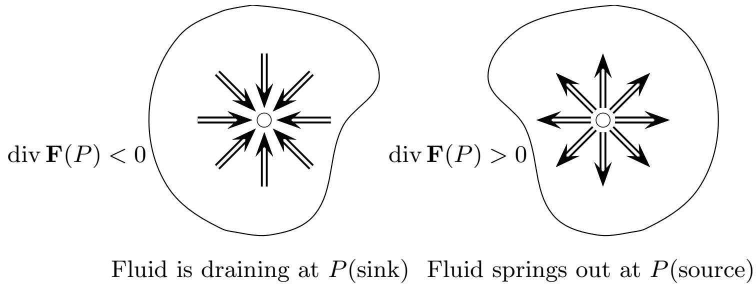


Figure 7.22: Physical meaning of divergence

If div $\mathbf{F}(P) > 0$, we say P is a **source** and if div $\mathbf{F}(P) < 0$, it is called **sink** of $\mathbf{F}$(fig 7.22).

If div $\mathbf{F} = 0$ then by Gauss theorem, the total flux of $\mathbf{F}$ through any closed surface S is $\int_S \mathbf{F} \cdot d\mathbf{S}$, which is zero. Thus we call this vector field **incompressible**.

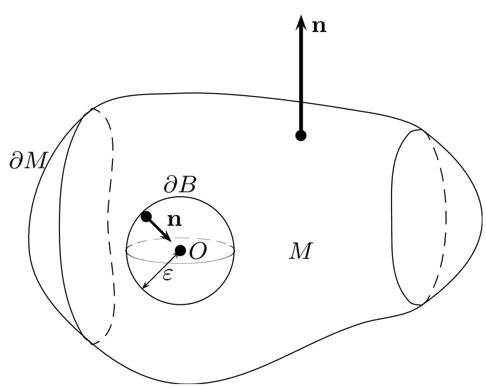


Figure 7.23: Unit outward normal vector $\mathbf{n}$ to M and Gauss' Law

Example 7.5.5. Find $\iint_S \mathbf{f} \cdot d\mathbf{S}$, where $\mathbf{F} = xy^2\mathbf{i} + x^2y\mathbf{j} + y\mathbf{k}$ and S is the surface of the the cylindrical region $x^2 + y^2 = 1$ bounded by the planes $z = 1$ and $z = -1$.

$\boxed{\textbf{sol.}}$ Let W denote the solid region given above. By divergence theorem,

$$\begin{aligned}\iiint_W \operatorname{div}\mathbf{F}\,dV &= \iiint_W (x^2+y^2)dxdydz \\ &= \int_{-1}^{1}\left(\iint_{x^2+y^2\le 1}(x^2+y^2)dxdy\right)dz \\ &= 2\iint_{x^2+y^2\le 1}(x^2+y^2)dxdy.\end{aligned}$$

Now by polar coordinate,

$$2\iint_{x^2+y^2\le 1}(x^2+y^2)dxdy = 2\int_0^{2\pi}\int_0^1 r^3 dr d\theta = \pi.$$

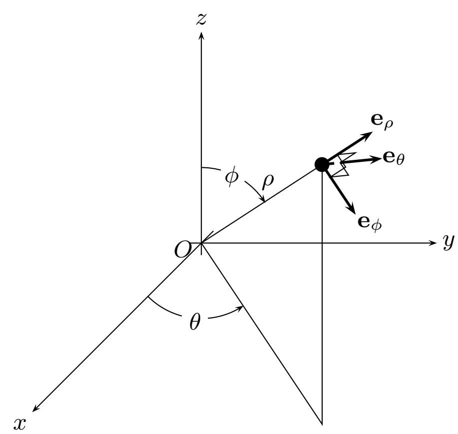


Figure 7.24: Standard basis vectors in spherical coordinate; $\mathbf{e}_\rho, \mathbf{e}_\phi, \mathbf{e}_\theta$

Gauss' Law

Now apply Gauss' theorem to a region with a hole and get an important result in physics:

Theorem 7.5.6. ***(Gauss' Law)*** *Let M be a region in $\mathbb{R}^3$ and $O \notin \partial M$. Then*

$$\iint_{\partial M}\frac{\mathbf{r}\cdot\mathbf{n}}{r^3}dS = \begin{cases} 0 & \text{if } O \notin M\ , \\ 4\pi & \text{if } O \in M. \end{cases}$$

Here $\mathbf{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$ *and* $r = \sqrt{x^2 + y^2 + z^2}$.

Proof. First suppose $O \notin M$. Then $\mathbf{r}/r^3$ is a C^1-vector field on M and ∂M. One can easily show $\nabla \cdot (\mathbf{r}/r^3) = 0$ for $r \neq 0$. Hence

$$\iint_{\partial M} \frac{\mathbf{r} \cdot \mathbf{n}}{r^3} dS = \iiint_M \nabla \cdot \left(\frac{\mathbf{r}}{r^3}\right) dV = 0.$$

Thus we have the result.

Next, if $O \in M$, $\mathbf{r}/r^3$ is not continuous on M. Then we remove small ball B of radius ε near O(fig 7.23). Let W be the region $M \backslash B$. Then the boundary of W is $S = \partial B \cup \partial M$, where the normal vector to B is opposite to the usual direction. Again we see in $\nabla \cdot (\mathbf{r}/r^3) = 0$ in W. Hence by Gauss theorem

$$\iint_S \frac{\mathbf{r} \cdot \mathbf{n}}{r^3} dS = \iiint_W \nabla \cdot \left(\frac{\mathbf{r}}{r^3}\right) dV = 0.$$

Since
$$\iint_S \frac{\mathbf{r} \cdot \mathbf{n}}{r^3} dS = \iint_{\partial M} \frac{\mathbf{r} \cdot \mathbf{n}}{r^3} dS + \iint_{\partial B} \frac{\mathbf{r} \cdot \mathbf{n}}{r^3} dS,$$
we have
$$\iint_{\partial M} \frac{\mathbf{r} \cdot \mathbf{n}}{r^3} dS = -\iint_{\partial B} \frac{\mathbf{r} \cdot \mathbf{n}}{r^3} dS.$$
Now on ∂B(a sphere of radius ε), we know $\mathbf{n} = -\mathbf{r}/r$ and $r = \varepsilon$. Hence

$$-\iint_{\partial B} \frac{\mathbf{r} \cdot \mathbf{n}}{r^3} dS = \iint_{\partial B} \frac{\varepsilon^2}{\varepsilon^4} dS = \frac{1}{\varepsilon^2} \iint_{\partial B} dS.$$

Since $\iint_{\partial B} dS = 4\pi\varepsilon^2$, we have $\iint_{\partial M} \mathbf{r} \cdot \mathbf{n}/r^3 dS = 4\pi$. □

Physical Interpretation of Gauss' Law

Let
$$\phi(x, y, z) = \frac{Q}{4\pi r} = \frac{Q}{4\pi\sqrt{x^2 + y^2 + z^2}}$$
be the potential to a point charge O at $(0, 0, 0)$. Then the electric field is

$$\mathbf{E} = -\nabla\phi = \frac{Q}{4\pi}\left(\frac{\mathbf{r}}{r^3}\right).$$

The total electric flux $\int_{\partial M} \mathbf{E} \cdot d\mathbf{S}$ is Q if the charge lies in M, and 0 if the charge lies outside of M.

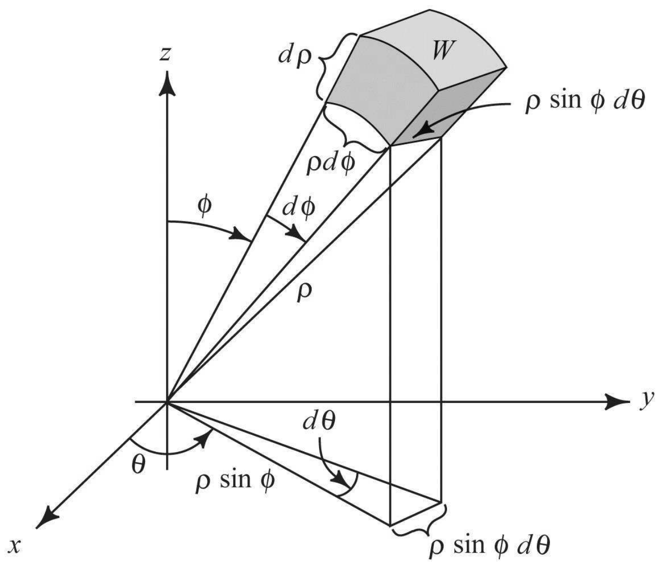


Figure 7.25:

For a charge with density ρ, the field $\mathbf{E}$ is related by

$$\operatorname{div}\mathbf{E} = \nabla \cdot \mathbf{E} = \rho.$$

Thus by Gauss' theorem

$$\int_S \mathbf{E} \cdot d\mathbf{S} = \int_\Omega \rho dV = \int_\Omega \frac{Q}{V} dV = Q.$$

In other words, total flux through a closed surface equals total charge inside the region enclosed by the surface.

Divergence in Spherical Coordinate

Using spherical coordinate, we can write $\mathbf{F} = F_\rho \mathbf{e}_\rho + F_\phi \mathbf{e}_\phi + F_\theta \mathbf{e}_\theta$. See figure 7.25. We use Gauss' theorem to derive

$$\mathrm{d}iv\, \mathbf{F} = \frac{1}{\rho^2}\frac{\partial}{\partial \rho}(\rho^2 F_\rho) + \frac{1}{\rho \sin\phi}\frac{\partial}{\partial \phi}(\sin\phi F_\phi) + \frac{1}{\rho\sin\phi}\frac{\partial F_\theta}{\partial \theta}.$$

Example 7.5.7. We prove the case $\mathbf{F} = F_\rho \mathbf{e}_\rho$.

Since

$$\mathrm{d}iv\, \mathbf{F} = \lim_{|W| \to 0} \frac{1}{|W|} \iiint_W \mathrm{d}iv\, \mathbf{F}\, dV = \lim_{|W| \to 0} \frac{1}{|W|} \iint_{\partial W} \mathbf{F} \cdot \mathbf{n}\, dS,$$

we compute the surface integral by inspection. Let W be the infinitesimal (shaded) region

$$\rho_0 \le \rho \le \rho_0 + d\rho, \quad \phi_0 \le \phi \le \phi_0 + d\phi, \quad \theta_0 \le \theta \le \theta_0 + d\theta$$

and let S be the two faces of W perpendicular to the radial direction.(other surface do not contribute because $\mathbf{F}$ is perpendicular to them) The integral of $\mathbf{F}$ around the edges of S is

$$\begin{aligned} &F_\rho(\rho + d\rho, \phi, \theta) \cdot \text{ A(outer face) } - F_\rho(\rho, \phi, \theta) \cdot \text{ A(inner face)} \\ &\approx F_\rho(\rho + d\rho, \phi, \theta)(\rho + d\rho)^2 \sin\phi\, d\phi d\theta - F_\rho(\rho, \phi, \theta)(\rho)^2 \sin\phi\, d\phi d\theta \\ &\approx \left[\rho^2 \frac{\partial F_\rho}{\partial \rho} d\rho + 2\rho F_\rho(\rho + d\rho)\right] \sin\phi\, d\rho d\phi d\theta. \end{aligned}$$

Dividing by the volume $\rho^2 \sin\phi\, d\rho d\phi d\theta$, and take the limit, we get

$$\frac{2}{\rho} F_\rho + \frac{\partial F_\rho}{\partial \rho} = \frac{1}{\rho^2} \frac{\partial}{\partial \rho}(\rho^2 F_\rho).$$

Example 7.5.8. Consider the vector field $\mathbf{F}(x, y) = F_\theta \mathbf{e}_\theta$. Let P be a fixed point in W. Also, let S_1 and S_2 be the two flat part of the boundary of W determined by the plane $\theta = const$ and $\theta + \Delta\theta = const$. Let $S = S_1 \cup S_2$. The volume $V(W) \approx \rho^2 \sin\phi d\rho\, d\phi\, d\theta$ and $A(S_1) = A(S_2) = \rho\, d\rho\, d\phi\, d\theta$.

$$\begin{aligned} \iiint_W \operatorname{div} \mathbf{F} dV &= \iint_{\partial W} \mathbf{F} \cdot d\mathbf{S} (\text{ use orthogonality of surfaces to } \mathbf{F}) \\ \iint_S F_\theta \mathbf{e}_\theta \cdot d\mathbf{S} &= \iint_{S_2} F_\theta \cdot dS - \iint_{S_1} F_\theta \cdot dS (\mathbf{n} = \pm \mathbf{e}_\theta \text{ on } S) \\ &\approx F_\theta(\rho, \phi, \theta + d\theta) A(S_2) - F_\theta(\rho, \phi, \theta) A(S_1) \\ &\approx \frac{\partial F_\theta}{\partial \theta} \rho\, d\rho\, d\phi\, d\theta. \end{aligned}$$

Hence

$$\operatorname{div} \mathbf{F}(P) = \lim_{W \to P} \frac{1}{V(W)} \iint_S \mathbf{F} \cdot d\mathbf{S} = \frac{1}{\rho \sin\phi} \frac{\partial F_\theta}{\partial \theta}.$$

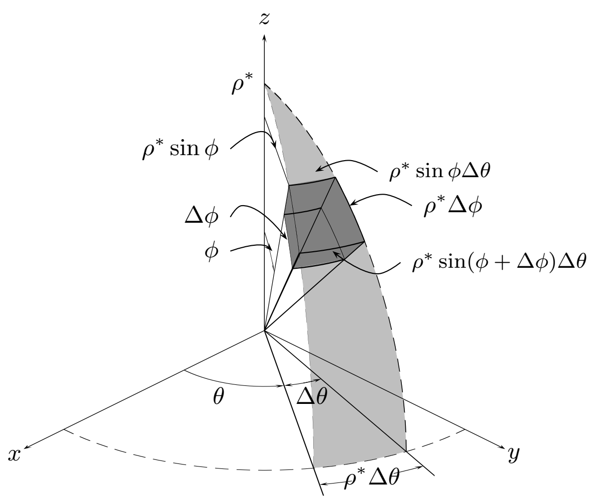


Figure 7.26: Partition in spherical coordinate, $\rho^* = \rho + d\rho$

Example 7.5.9. Finally, when $\mathbf{F} = F_\phi \mathbf{e}_\phi$, let S_1 and S_2 be the two flat part of the boundary of W determined by the plane $\phi = const$ and $\phi + \Delta\phi = const$ and $S = S_1 \cup S_2$. Then $A(S_1) = \rho \sin\phi d\theta$ and $A(S_2) = \rho \sin(\phi + d\phi) d\theta$.

$$
\begin{aligned}
\iiint_W \operatorname{div} \mathbf{F} dV &= \iint_{\partial W} \mathbf{F} \cdot d\mathbf{S} (\text{use orthogonality of surfaces to } \mathbf{F}) \\
\iint_S F_\phi \mathbf{e}_\phi \cdot d\mathbf{S} &= \iint_{S_2} F_\phi \, dS - \iint_{S_1} F_\phi \, dS (\mathbf{n} = \pm \mathbf{e}_\phi \text{ on } S) \\
&\approx F_\phi(\rho, \phi + d\phi, \theta) A(S_2) - F_\phi \ (\rho, \phi, \theta) A(S_1) \\
&\approx F_\phi(\rho, \phi + d\phi, \theta) \rho \sin(\phi + d\phi) d\theta - F_\phi(\rho, \phi, \theta) \rho \sin\phi d\theta
\end{aligned}
$$

Finally, we have

$$
\begin{aligned}
&\approx F_\phi(\rho, \phi + d\phi, \theta) \rho \sin(\phi + d\phi) d\theta - F_\phi(\rho, \phi, \theta) \rho \sin\phi \, d\theta \\
&\approx F_\phi(\phi + d\phi) \rho \, d\theta [\sin(\phi + d\phi) - \sin\phi] + [F_\phi(\phi + d\phi) - F_\phi(\phi)] \rho \sin\phi \, d\theta \\
&\approx F_\phi(\phi + d\phi) \rho \, d\theta [\sin(\phi + d\phi) - \sin\phi] + [F_\phi(\phi + d\phi) - F_\phi(\phi)] \rho \sin\phi \, d\theta \\
&\approx F_\phi(\phi + d\phi) \rho \, d\theta \cos\phi \, d\phi + \frac{\partial F_\phi}{\partial \phi} d\phi \rho \sin\phi d\theta \\
&\approx (\rho^2 \sin\phi \, d\rho \, d\phi \, d\theta) \left[\frac{\cos\phi}{\rho \sin\phi} F_\phi(\phi) + \frac{1}{\phi} \frac{\partial F_\phi}{\partial \phi} \right].
\end{aligned}
$$

Dividing by $V = \rho^2 \sin\phi d\rho\, d\phi\, d\theta$, we obtain

$$\frac{1}{\rho\sin\phi}\frac{\partial}{\partial\phi}(\sin\phi F_\phi(\phi)).$$

Example 7.5.10. Compute $\nabla\times\mathbf{F}$ with $\mathbf{F} = F_\rho\mathbf{e}_\rho + F_\phi\mathbf{e}_\phi + F_\theta\mathbf{e}_\theta$.

sol. First compute F_ρ. Let S be the inner surface in example 7.5.7:

$$\rho = \rho_0, \quad \phi_0 \le \phi \le \phi_0 + d\phi, \quad \theta_0 \le \theta \le \theta_0 + d\theta.$$

The area of S is $\rho^2 \sin\phi\, d\phi\, d\theta$. To see the $\mathbf{e}_\rho$ component, we do as follows: By Stokes' theorem,

$$\frac{1}{|S|}\iint_S \nabla\times\mathbf{F}\cdot\mathbf{e}_\rho\, dS = \frac{1}{|S|}\int_{\partial S}\mathbf{F}\cdot d\mathbf{s}.$$

The integral of $\mathbf{F}$ around the edges of S is

$$\begin{aligned}
& F_\theta(\rho,\phi+d\phi,\theta)\rho\sin(\phi+d\phi)\,d\theta - F_\theta(\rho,\phi,\theta)\rho\sin\phi\,d\theta(\text{horizontal})\\
& -F_\phi(\rho,\phi,\theta+d\theta)\rho\,d\phi + F_\phi(\rho,\phi,\theta)\rho\,d\phi(\text{vertical})\\
= \; & [F_\theta(\rho,\phi+d\phi,\theta) - F_\theta(\rho,\phi,\theta)]\rho\sin(\phi+d\phi)\,d\theta + F_\theta(\rho,\phi,\theta)[\sin(\phi+d\phi-\sin\phi]\rho\,d\theta\\
& -F_\phi(\rho,\phi,\theta+d\theta)\rho\,d\phi + F_\phi(\rho,\phi,\theta)\rho\,d\phi(\text{vertical})\\
\approx \; & \frac{\partial F_\theta}{\partial\phi}\rho\sin(\phi+d\phi)\,d\phi d\,\theta + F_\phi(\rho,\phi,\theta)\rho\cos\phi\,d\phi\,d\theta - \frac{\partial F_\phi}{\partial\theta}\rho\,d\phi\,d\theta.
\end{aligned}$$

Dividing by $\rho^2\sin\phi\,d\phi\,d\theta$ and taking the limit we get

$$\frac{1}{\rho\sin\phi}\left[\frac{\partial}{\partial\phi}(\sin\phi F_\theta) - \frac{\partial F_\phi}{\partial\theta}\right].$$

Hence

$$\nabla\times\mathbf{F}\cdot\mathbf{e}_\rho = \lim_{|S|\to 0}\frac{1}{|S|}\iint_S\nabla\times\mathbf{F}\cdot\mathbf{e}_\rho\,dS = \lim_{|S|\to 0}\frac{1}{|S|}\int_{\partial S}F_\rho ds = \frac{1}{\rho\sin\phi}\left[\frac{\partial}{\partial\phi}(\sin\phi F_\theta) - \frac{\partial F_\phi}{\partial\theta}\right].$$

To compute F_ϕ and F_θ components, we consider the surface integral with $\mathbf{e}_\phi$ and $\mathbf{e}_\theta$ on the surfaces perpendicular to each of them, i.e, S_ϕ is

$$\rho_0 \le \rho \le \rho_0 + d\rho, \quad \phi = \phi_0, \qquad \theta_0 \le \theta \le \theta_0 + d\theta.$$

and S_θ is

$$\rho_0 \le \rho \le \rho_0 + d\rho, \quad \phi_0 \le \phi \le \phi_0 + d\phi, \quad \theta = \theta_0.$$

$\square$

7.6 Conservative Field and Fundamental Theorem of Calculus

Let us summarize theorems so far.

- Fundamental Theorem of Calculus:

$$\int_\sigma \nabla f \cdot d\mathbf{s} = f(\sigma(b)) - f(\sigma(a))$$

- Green's Theorem :

$$\int_{\partial D} P dx + Q dy = \int_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dxdy$$

- Divergence Theorem for Plane:

$$\int_{\partial D} (\mathbf{F} \cdot \mathbf{n}) ds = \int_D \operatorname{div} \mathbf{F}\, dxdy$$

- Stokes' Theorem:

$$\iint_S \operatorname{curl} \mathbf{F} \cdot d\mathbf{S} = \int_{\partial S} \mathbf{F} \cdot d\mathbf{s}$$

- Gauss' Divergence Theorem:

$$\iiint_\Omega \operatorname{div} \mathbf{F} dV = \iint_{\partial\Omega} \mathbf{F} \cdot d\mathbf{S}$$

Look at (1):

$$\int_\sigma \nabla f \cdot d\mathbf{s} = f(\sigma(b)) - f(\sigma(a)) = f(A) - f(B)$$

The line integral of a gradient is independent of path:

For example, if $V = -f$ represents a potential energy(Gravitational, electrical) the $\mathbf{F} = \nabla f$ is a force. For $f = \frac{GmM}{r}$, the force $\mathbf{F} = -\frac{GmM}{r^2}\mathbf{r} = -\frac{GmM}{r^2}\mathbf{n}$ is the gravitational force.

What vectors are Gradient?

Theorem 7.6.1. ***(Conservative Field*** *Let* $\mathbf{F}$ *be a* $\mathcal{C}^1$*-vector field in* $\mathbb{R}^3$ *except finite number of points. Then the following conditions are equivalent:*

(1)

(2) For any oriented simple closed curve C, $\int_C \mathbf{F} \cdot d\mathbf{s} = 0$.

(3) For any two oriented simple curve C_1, C_2 *having same end points,*

$$\int_{C_1} \mathbf{F} \cdot d\mathbf{s} = \int_{C_2} \mathbf{F} \cdot d\mathbf{s}.$$

(4) $\mathbf{F}$ *is the gradient of some function* f*, i.e,* $\mathbf{F} = \nabla f$.

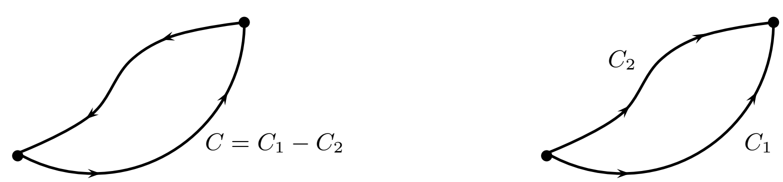
(5) $\operatorname{curl} \mathbf{F} = \mathbf{0}$.

If a vector field $\mathbf{F}$ *satisfy one of these conditions we say* ***conservative field****.*

Proof. We use the sequence of implication: (i)⇒(2)⇒(3)⇒(4)⇒(1).

(1)⇒(2): Suppose we have two curves C_1, C_2 having same end points, we can form a closed curve by $C = C_1 - C_2$(fig 7.27) Hence by (1)

$$0 = \int_C \mathbf{F} \cdot d\mathbf{s} = \int_{C_1 - C_2} \mathbf{F} \cdot d\mathbf{s} = \int_{C_1} \mathbf{F} \cdot d\mathbf{s} - \int_{C_2} \mathbf{F} \cdot d\mathbf{s}$$



(a) oriented simple closed curve $C = C_1 - C_2$ (b) two oriented simple curves C_1

Figure 7.27: Constructing oriented simple closed curve from two oriented simple curve C_1, C_2

(2)⇒(3): Fix a point (x_0, y_0, z_0). Given any point (x, y, z), choose any curve C connecting two points we define. Given $\mathbf{F} = (F_1, F_2, F_3)$ define

$$f(x, y, z) = \int_C \mathbf{F} \cdot d\mathbf{s} = \int_C F_1 dx + F_2 dy + F_3 dz$$

Here f is well-defined, since it is defined independent of the choice of C. So we choose C consisting of edges of rectangular pipe.

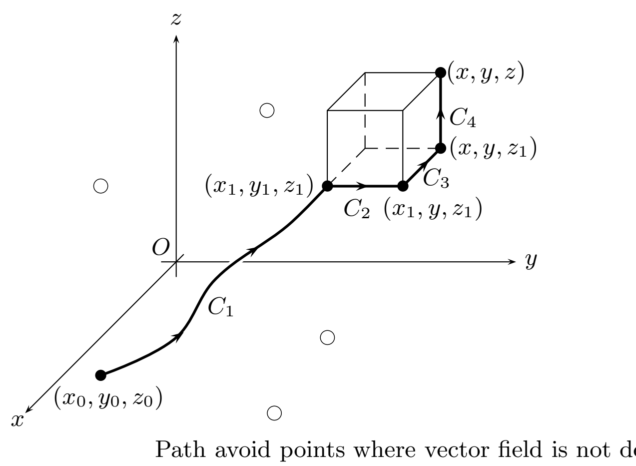


Figure 7.28: A path from (x_0, y_0, z_0) to (x, y, z) is $C = C_1 + C_2 + C_3 + C_4$

In particular, choose $C = C_1 + C_2 + C_3 + C_4$ (fig 7.28). Then

$$\begin{aligned}
f(x, y, z) &= \int_C \mathbf{F} \cdot d\mathbf{s} \\
&= \int_{C_1} \mathbf{F} \cdot d\mathbf{s} + \int_{C_2} \mathbf{F} \cdot d\mathbf{s} + \int_{C_3} \mathbf{F} \cdot d\mathbf{s} + \int_{C_4} \mathbf{F} \cdot d\mathbf{s} \\
&= \int_{C_1} \mathbf{F} \cdot d\mathbf{s} + \int_{y_1}^{y} F_2(x_1, t, z_1) dt \\
&\qquad + \int_{x_1}^{x} F_1(t, y, z_1) dt + \int_{z_1}^{z} F_3(x, y, t) dt.
\end{aligned}$$

From this we see $\partial f / \partial z = F_3$. Similarly by choosing different path(i.e, choosing a path whose last path is along x-direction) we have

$$\begin{aligned}
f(x, y, z) &= \int_{C_1} \mathbf{F} \cdot d\mathbf{s} + \int_{y_1}^{y} F_2(x_1, t, z_1) dt \\
&+ \int_{z_1}^{z} F_3(x_1, y, t) dt + \int_{x_1}^{x} F_1(t, y, z) dt
\end{aligned}$$

so $\partial f / \partial x = F_1$. Similarly, we have $\partial f / \partial y = F_2$. Thus $\mathbf{F} = \nabla f$.

(3)⇒(4): By theorem 5.1.2, we have $\nabla \times \nabla f = 0$. So $\operatorname{curl} \mathbf{F} = \nabla \times \mathbf{F} = 0$.

(4)⇒(1): Let S be a surface having C as boundary. Then

$$\int_C \mathbf{F} \cdot d\mathbf{s} = \iint_S \operatorname{curl} \mathbf{F} \cdot d\mathbf{S}$$

Since $\operatorname{curl}\mathbf{F} = \mathbf{0}$ the integral $\int_C \mathbf{F}\cdot d\mathbf{s} = 0$. □

Physical Interpretation of $\int_C \mathbf{F}\cdot d\mathbf{s}$

(1) $\int_C \mathbf{F}\cdot d\mathbf{s}$ is work done be $\mathbf{F}$ along C.

(2) Circulation: $\mathbf{F}\cdot\Delta\mathbf{s}$ is tangential component $\dot{\Delta}\mathbf{s}$. Thus $\int_C \mathbf{F}\cdot d\mathbf{s}$ is net tangential component of $\mathbf{F}$ along C.

Example 7.6.2.

$$\mathbf{F}(x,y,z) = y\mathbf{i} + (z\cos yz)\mathbf{j} + (y\cos yz)\mathbf{k}$$

Show $\mathbf{F}$ is irrotational and find a scalar potential.

Example 7.6.3.

$$\mathbf{F}(x,y) = -y\mathbf{i} + x\mathbf{j}$$

We see $\mathbf{F}(0,0) = 0$ but $\operatorname{curl}\mathbf{F} = 2$. So if a paddle is place at this point, it rotates even if it stay there. Show $\mathbf{F}$ is irrotational and find a scalar potential.

sol.

$$\begin{aligned}\nabla\times\mathbf{F} &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ y & z\cos yz & y\cos yz \end{vmatrix} \\ &= (\cos yz - yz\sin yz - \cos yz + yz\sin yz)\mathbf{i} - (0-0)\mathbf{j} + (1-1)\mathbf{k} = 0\end{aligned}$$

So $\mathbf{F}$ is irrotational. To find a potential

Method 1:

$$\begin{aligned} f(x,y,z) &= +\int_0^x F_1(t,0,0)dt + \int_0^y F_2(x,t,0)dt + \int_0^y F_3(x,t,0)dt \\ &= \int_0^x 0dt + \int_0^y x\,dt + \int_0^y y\cos yt\,dt \\ &= 0 + xy + \sin yz \end{aligned}$$

One easily check that $\nabla f = \mathbf{F}$.

■

sol.

Method 2: If such f exists, it satisfies

$$\frac{\partial f}{\partial x} = y, \quad \frac{\partial f}{\partial y} = x + z\cos yz, \quad \frac{\partial f}{\partial z} = y\cos yz. \tag{7.18}$$

Thus we obtain

(1) $f(x, y, z) = xy + h_1(y, z)$

(2) $f(x, y, z) = \sin yz + xy + h_2(x, z)$

(3) $f(x, y, z) = \sin yz + h_3(x, y)$

Substitute this into (7.18) we find

$$\frac{\partial h_1(y, z)}{\partial z} = y\cos yz$$

or

$$h_1(y, z) = \int y\cos yz\, dz + g(y) = \sin yz + g(y).$$

Substituting back to (1) (2) we see $g(y) = h_2(x, z)$. Then this must be constant.

Potential function

If $\mathbf{F}$ satisfies curl $\mathbf{F} = \nabla \times \mathbf{F} = \mathbf{0}$ then it is given by $\mathbf{F} = \nabla f$ for some f¿ This f is called **potential function** of $\mathbf{F}$.

Theorem 7.6.4. *If* $\mathbf{F}$ *is a* C^1 *vector field with* $div\, \mathbf{F} = 0$ *then there is a* C^1 *field* $\mathbf{G}$ *with* $\mathbf{F} = \mathrm{curl}\, \mathbf{G}$.

Example 7.6.5. By Newton's law, the force acting to an object of mass M at $\mathbf{r} = (x, y, z)$ is

$$\mathbf{F}(x, y, z) = -GM\mathbf{r}/r^3$$

Show $\mathbf{F}$ is irroataitonal and find potential for it.

sol. First show $\nabla \times \mathbf{F} = \mathbf{0}$.

$$\mathbf{r} \times \mathbf{r} = \mathbf{0}, \nabla\left(\frac{1}{r^3}\right) = -3\mathbf{r}/r^5,$$

$$\nabla \times \mathbf{r} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ x & y & z \end{vmatrix} = \mathbf{0},$$

$$\nabla \times \mathbf{F} = -GM\left\{\nabla\left(\frac{1}{r^3}\right) \times \mathbf{r} + \frac{1}{r^3}\nabla \times \mathbf{r}\right\} = \mathbf{0}.$$

From exercise we can show $\nabla(r^n) = nr^{n-2}\mathbf{r}$ $\mathbf{F} = -\nabla\phi$, The function $\phi(x, y, z) = -GM/r$ is the gravitational potential function.

Planar Case

Suppose $\mathbf{F}$ is a $\mathcal{C}^1$-vector field of the form $\mathbf{F} = P\mathbf{i} + Q\mathbf{j}$ then we have

$$\nabla \times \mathbf{F} = \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)\mathbf{k}$$

Hence

$$\operatorname{curl}\mathbf{F} = \nabla \times \mathbf{F} = \mathbf{0}$$

is equivalent to

$$\partial Q/\partial x = \partial P/\partial y$$

Using this we can study conservative field $\mathbb{R}^2$. In $\mathbb{R}^3$ vector field may have a few points where function is undefined. But in planar case, the vector field must be defined everywhere.

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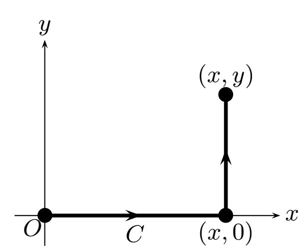

Figure 7.29: A path from $(0, 0)$ to (x, y)

Example 7.6.6. The vector field $\mathbf{F}(x, y, z) = (y\mathbf{i} - x\mathbf{j})/(x^2 + y^2)$ satisfies (a) $\nabla \times \mathbf{F} = \mathbf{0}$ but (b) $\mathbf{F}$ is not conservative.

sol. (a) We have seen $\nabla \times \mathbf{F} = \mathbf{0}$ before.

(b) To show $\mathbf{F}$ is not conservative we need to show the line integral $\int_C \mathbf{F} \cdot d\mathbf{s}$ is nonzero for some closed curve C. Let C be

$$x = \cos t, \quad y = \sin t, \quad 0 \le t \le 2\pi$$

Then

$$\begin{aligned}
\int_C \mathbf{F} \cdot d\mathbf{s} &= \int_C \frac{y}{x^2 + y^2} dx + \frac{-x}{x^2 + y^2} dy \\
&= \int_0^{2\pi} \left[\frac{\sin t}{\cos^2 t + \sin^2 t} (-\sin t dt) + \frac{-\cos t}{\cos^2 t + \sin^2 t} (\cos t dt) \right] \\
&= -\int_0^{2\pi} \frac{\sin^2 t + \cos^2 t}{\cos^2 t + \sin^2 t} dt = -\int_0^{2\pi} dt = -2\pi \ne 0.
\end{aligned}$$

Hence $\mathbf{F}$ is not conservative.

Example 7.6.7. (a) If $\mathbf{F} = e^{xy}\mathbf{i} + e^{x+y}\mathbf{j}$ then $P(x, y) = e^{xy}, Q(x, y) = e^{x+y}$, $\partial P/\partial y = xe^{xy}$, $\partial Q/\partial x = e^{x+y}$. So no potential exists.

(b) For $\mathbf{F} = (2x \cos y)\mathbf{i} - (x^2 \sin y)\mathbf{j}$ we see $\partial P/\partial y = -2x \sin y = \partial Q/\partial x$. Hence $\mathbf{F}$ has a potential f. To find it f we see

$$\frac{\partial f}{\partial x} = 2x \sin y, \quad \frac{\partial f}{\partial y} = -x \sin y$$

$$\begin{aligned}
f(x, y) &= x^2 \cos y + h_1(y) \\
f(x, y) &= x^2 \cos y + h_2(x).
\end{aligned}$$

we can set $h_1 = h_2 = 0$. So $f(x, y) = x^2 \cos y$.

Example 7.6.8. For a path $\sigma \colon [1, 2] \to \mathbb{R}^2$ $x = e^{t-1}, y = \sin(\pi/t)$ find the line integral of $\mathbf{F} = 2x \cos y\mathbf{i} - x^2 \sin y\mathbf{j}$.

$$\int_\sigma \mathbf{F} \cdot d\mathbf{s} = \int_\sigma 2x \cos y dx - x^2 \sin y dy$$

sol. Since $\sigma(1) = (1,0), \sigma(2) = (e,1), \partial(2x\cos y)/\partial y = \partial(-x^2 \sin y)/\partial x$ $\mathbf{F}$ is irrotataional. Hence we can replace the path by another $\mathcal{C}^1$ having same end points. Choose from $(1,0)$ to $(e,0)$. Next from $(e,0)$ to $(e,1)$. Then the integral is

$$\begin{aligned}\int_\sigma \mathbf{F}\cdot d\mathbf{s} &= \int_1^e 2t\cos 0dt + \int_0^1 -e^2 \sin tdt \\ &= (e^2-1) + e^2(\cos 1 - 1) \\ &= e^2\cos 1 - 1.\end{aligned}$$

On the other hand $f(x,y) = x^2\cos y$ is a potential of $\mathbf{F}$. Hence

$$\begin{aligned}\int_\sigma 2x\cos ydx - x^2\sin ydy &= \int_\sigma \nabla f\cdot d\mathbf{s} \\ &= f(\sigma(2)) - f(\sigma(1)) = e^2\cos 1 - 1.\end{aligned}$$

The latter integral is easier.

$\blacksquare$