Chapter 6

Line Integrals

6.1 Scalar and Vector line Integral

Path integral = scalar line integral

Let $\mathbf{x} = (x(t), y(t), z(t)) : [a, b] \to X$ be a C^1-path parameterized by t. Assume $f : X \subset \mathbb{R}^3 \to \mathbb{R}$ is a continuous function. Let $P\colon a = t_0 < t_1 < \cdots < t_k = b$ be the partition of $[a, b]$ and t_i^* be any point between t_{i-1} and t_i, for $i = 1, \cdots, n$. Then we consider the Riemann sum of $f(\mathbf{x}(t))$

$$\sum_{i=1}^{k} f(\mathbf{x}(t_i^*))\|\mathbf{x}(t_i) - \mathbf{x}(t_{i-1})\|$$

As $\|P\|$ approaches 0 the sum approaches

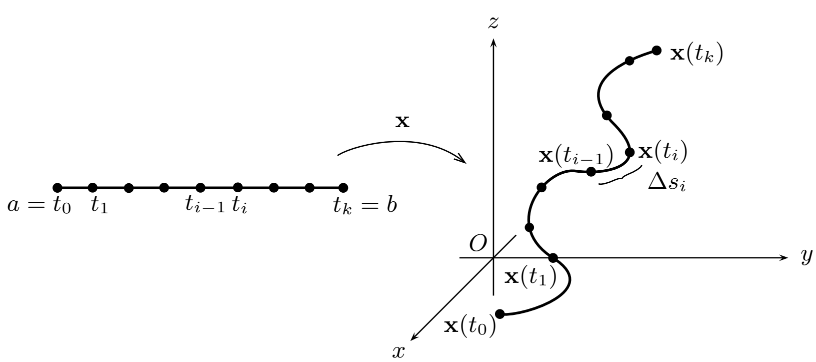


Figure 6.1: Riemann sum over a path

$$\sum_{i=1}^{k} f(\mathbf{x}(t_i^*))\Delta s_i$$

where $\Delta s_i = \int_{t_{i-1}}^{t_i} \|\mathbf{x}'(t)\| dt$ is the length of i-th segment of the curve $\mathbf{x}(t)$.

Example 6.1.1. The scalar function $f(\mathbf{x})$ may represent

(1) electric charge density along the wire $\mathbf{x}(t)$; Then the line integral is total charge along the wire.

(2) density of the wire $\mathbf{x}(t)$. Then the line integral is total mass of the wire

Definition 6.1.2. If $\mathbf{x}$ is a C^1-curve defined over $I = [a, b]$ having values in $\mathbb{R}^3$ and f is defined over a region containing the image of $\mathbf{x}$. Then $f \circ \mathbf{x}$ is real valued function defined on I. We define **the path integral-scalar line integral** of $\mathbf{x}$ as:

$$\int_a^b f(\mathbf{x}(t))\|\mathbf{x}'(t)\|\, dt$$

We denote it by $\int_{\mathbf{x}} f\, ds$ or $\int_{\mathbf{x}} f(x, y, z)\, ds$. If $f = 1$, then $\int_{\mathbf{x}} ds$ is the length of $\mathbf{x}$.

Example 6.1.3. Find path integral of $f(x, y, z) = x^2 + y^2 + z^2$ over $\mathbf{x}$.

$$\mathbf{x}(t) = (\cos t, \sin t, t), \quad t \in [0, 2\pi]$$

sol. Since $\mathbf{x}'(t) = (-\sin t, \cos t, 1)$, the line integral is

$$\begin{aligned}
\int_{\mathbf{x}} f\, ds &= \int_0^{2\pi} f(\mathbf{x}(t))\|\mathbf{x}'(t)\|\, dt \\
&= \int_0^{2\pi} (1 + t^2)\sqrt{2}\, dt \\
&= \sqrt{2}\left(2\pi + 8\pi^3/3\right).
\end{aligned}$$

■

Path integral over planar Curves

If $f(x, y)$ is a continuous function defined over a region containing the image of a path $\mathbf{x}$, then the path integral of f along $\mathbf{x}$ is given by

$$\int_{\mathbf{x}} f(x, y)\, ds = \int_a^b f(x(t), y(t))\sqrt{x'(t)^2 + y'(t)^2} dt$$

If $f = 1$, it is nothing but the arc -length.

Example 6.1.4 (Tom Sawyer's fence)**.** Find the area of fence along a parameterized curve $\mathbf{x} = \mathbf{x}(t) = (30\cos^3 t, 30\sin^3 t)$ in $\mathbb{R}^2$ and height is given by $f(x, y) = 1 + 3y$.

sol. $\mathbf{x}(t) = (30\cos^3 t, 30\sin^3 t)$ for $t \in [0, \pi/2]$. The area of one side is

$$\int_C f(x, y) ds$$

where $ds = \|\mathbf{x}'(t)\| dt = 90 \sin t \cos t \, dt$. So

$$\begin{aligned}\int_C f(x, y) ds &= \int_0^{\pi/2} \left(1 + 10\sin^3 t\right) 90 \sin t \cos t \, dt \\ &= 90 \int_0^{\pi/2} (\sin t + 10 \sin^4 t) \cos t \, dt = 225.\end{aligned}$$

This is half of the fence. Total area of fence(both sides) is 900 square ft. If he can get .05 dollar per square feet, he can make $900 \times 0.05 = 1.80$.

□

Example 6.1.5. Find path integral of $f(x, y, z) = x^2 + y^2 + z^2$ over C.

$$C = \{(\cos t, \sin t, t) \colon t \in [0, 2\pi]\} \cup \{(1, 0, t) \colon t \in [0, 2\pi]\}$$

sol. We write C as the union of C_1 and C_2, where

$$C_1 = \{(\cos t, \sin t, t) \colon t \in [0, 2\pi]\}, \quad C_2 = \{(1, 0, t) \colon t \in [0, 2\pi]\}.$$

We parameterize C_1 and C_2 as follows:

$$\mathbf{x}_1 = (\cos t, \sin t, t), \quad t \in [0, 1], \quad \mathbf{x}_2 = (1, 0, t), \quad t \in [0, 2\pi].$$

Then

$$\begin{aligned}\int_C f\,ds &= \int_{C_1} f\,ds + \int_{C_2} f\,ds \\ &= \int_{\mathbf{x}_1} f\,ds + \int_{\mathbf{x}_2} f\,ds \\ &= \int_0^{2\pi} (1+t^2)\sqrt{2}\,dt + \int_0^{2\pi} (1+t^2)\,dt \\ &= (1+\sqrt{2})\left(2\pi + 8\pi^3/3\right).\end{aligned}$$

Vector Line Integrals

As an example, consider the work done by a force field. Suppose a particle moves along a curve $\mathbf{x}$ while acted upon by a force $\mathbf{F}$. If a portion of $\mathbf{x}$ is a line segment given by the vector $\mathbf{d}$ and $\mathbf{F}$ is constant force, then the work done on the particle along $\mathbf{d}$ is, by definition

$Work = \mathbf{F}\cdot\mathbf{d} =$ magnitude of force $\times$ displacement in the direction of force.

If the path is a curve, we break the curve into small pieces and add the work done on each piece then take the limit. So the work is defined by

$$\lim_n \sum_{i=0}^{n-1} \mathbf{F}(\mathbf{x}(t_i)) \cdot [\mathbf{x}(t+\Delta t) - \mathbf{x}(t)] = \int_a^b \mathbf{F}(\mathbf{x}(t)) \cdot \mathbf{x}'(t)dt$$

Here $\mathbf{x}(t+\Delta t) - \mathbf{x}(t)$ represents the line segment. We use the notation:

$$\Delta\mathbf{s} = \mathbf{x}(t+\Delta t) - \mathbf{x}(t)$$

Definition 6.1.6. Let $\mathbf{F}$ be a vector field on $\mathbb{R}^3$ that is continuous on the C^1- path defined on a set containing the image of $\mathbf{x} : [a,b] \to \mathbb{R}^3$. Define the **(vector) line integral**

$$\int_{\mathbf{c}} \mathbf{F}\cdot d\mathbf{s} = \int_a^b \mathbf{F}(\mathbf{x}(t)) \cdot \mathbf{x}'(t)dt.$$

An interpretation of (vector) line integral in terms of scalar integral is given as follows: For $\mathbf{x}'(t) \neq 0$, we see the vector $\mathbf{T}(t) = \mathbf{x}'(t)/\|\mathbf{x}'(t)\|$ is the unit tangent vector. Hence

$$\begin{aligned}\int_{\mathbf{x}} \mathbf{F}\cdot d\mathbf{s} &= \int_a^b \mathbf{F}(\mathbf{x}(t))\cdot \mathbf{x}'(t)dt \\ &= \int_a^b \left[\mathbf{F}(\mathbf{x}(t))\cdot \frac{\mathbf{x}'(t)}{\|\mathbf{x}'(t)\|}\right]\|\mathbf{x}'(t)\|dt \\ &= \int_a^b [\mathbf{F}(\mathbf{x}(t))\cdot \mathbf{T}(t)]\,\|\mathbf{x}'(t)\|dt \\ &= \int_{\mathbf{x}} (\mathbf{F}\cdot\mathbf{T})\,ds.\end{aligned}$$

So the (vector) line integral is the **path integral(scalar line integral) of the tangential component $\mathbf{F}\cdot\mathbf{T}$** along $\mathbf{x}$. When $\mathbf{x}$ is a closed curve, the integral $\int_{\mathbf{x}} \mathbf{F}\cdot d\mathbf{s}$ is called the **circulation** of $\mathbf{F}$ along $\mathbf{x}$.

Example 6.1.7. Suppose $\mathbf{F}(x,y,z) = x^3\mathbf{i} + y\mathbf{j} + z\mathbf{k}$ and $\mathbf{x}$ is a circle given by $x = 0, y^2 + z^2 = a^2$. Compute $\int_{\mathbf{x}} \mathbf{F}\cdot d\mathbf{s}$.

sol. We parameterize the circle

$$x = 0,\ y = a\cos\theta,\ z = a\sin\theta,\quad 0 \le \theta \le 2\pi$$

$$\mathbf{x}'(t) = (0, -a\sin\theta, a\cos\theta)$$

Since $\mathbf{F}(\mathbf{x}(\theta))\cdot\mathbf{x}'(\theta) = 0$, the work must be zero. You can verify by finding the value.

Another notation for line integral-differential form

Suppose $\mathbf{x}(t) = (x(t), y(t), z(t))$ and $\mathbf{F} = (F_1, F_2, F_3)$. Since $(dx, dy, dz) = (x'(t), y'(t), z'(t))dt$, we can write the line integral as

$$\begin{aligned}\int_{\mathbf{x}} \mathbf{F}\cdot d\mathbf{s} &= \int_{\mathbf{x}} (F_1, F_2, F_3)\cdot(x'(t), y'(t), z'(t))dt \\ &= \int_{\mathbf{x}} F_1dx + F_2dy + F_3dz.\end{aligned}$$

Example 6.1.8. Show

$$\int_C x^2dx + xydy + dz = \frac{11}{15}$$

where $\mathbf{x}(t) = (t, t^2, 1) = (x, y, z)$ on $[0, 1]$.

Example 6.1.9. Compute

$$\int_C \cos z dx + e^x dy + e^y dz$$

where $\mathbf{c}(t) = (1, t, e^t)$ on $[0, 2]$.

Example 6.1.10. Show

$$\int_C (\sin z dx + \cos z dy - (xy)^{1/3} dz) = -\frac{1}{2}$$

where $x = \cos^3 \theta, \quad y = \sin^3 \theta, z = \theta$ on $[0, 7\pi/2]$.

Draw the curve in $\mathbb{R}^3$.

Reparametrization

Definition 6.1.11. Let $\mathbf{x} : I_1 = [a_1, b_1] \to \mathbb{R}^3$ be a C^1 curve. If $u : I = [a, b] \to I_1 = [a_1, b_1]$ is a real valued C^1 curve that is one-to-one and onto, then the composition

$$\mathbf{y}(t) = (\mathbf{x} \circ u)(t) : I = [a, b] \to \mathbb{R}^3$$

is called a **reparametrization** of $\mathbf{x}$.

The line integral depends not only on $\mathbf{F}$ but also depends on the path $\mathbf{x}$. If $\mathbf{x}_1$, $\mathbf{x}_2$ are two different parametrization of the same curve, we shall see

$$\int_{\mathbf{x}_1} \mathbf{F} \cdot d\mathbf{s} = \pm \int_{\mathbf{x}_2} \mathbf{F} \cdot d\mathbf{s}$$

Theorem 6.1.12. *Let $\mathbf{x}\colon [a_1, b_1] \to \mathbb{R}^3$ be a curve and $\mathbf{y}\colon [a, b] \to \mathbb{R}^3$ is given by $\mathbf{y} = \mathbf{x} \circ u$ where $u\colon [a, b] \to [a_1, b_1]$ satisfies*

$$u(a) = a_1, \quad u(b) = b_1 \textit{(Orientation preserving)}$$

or

$$u(a) = b_1, \quad u(b) = a_1 \textit{(Orientation reversing)}$$

Then we have

$$\int_{\mathbf{y}} \mathbf{F} \cdot d\mathbf{s} = \pm \int_{\mathbf{x}} \mathbf{F} \cdot d\mathbf{s}$$

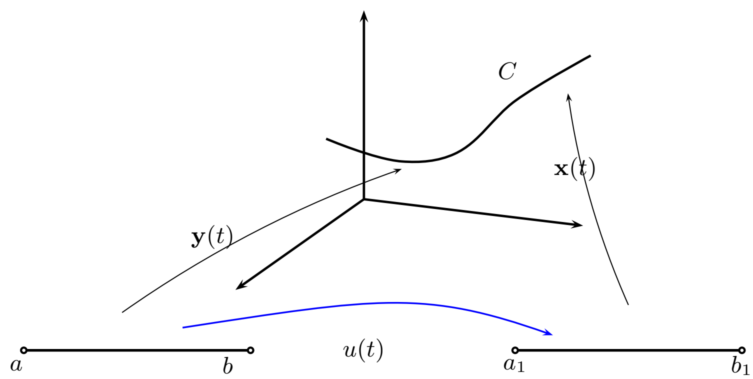


Figure 6.2: Reparametrization of a curve

Here we have + sign, if $\mathbf{y}$ *is orientation preserving, and − sign, if* $\mathbf{y}$ *is orientation reversing.*

Proof. If u is orientation preserving then $u(a) = a_1$, $u(b) = b_1$. In this case,

$$\begin{aligned}
\int_{\mathbf{y}} \mathbf{F}(\mathbf{x}(s)) \cdot d\mathbf{s} &= \int_a^b \mathbf{F}(\mathbf{x}(u(t))) \cdot \mathbf{x}'(u(t))u'(t)\,dt \\
&= \int_{u(a)}^{u(b)} \mathbf{F}(\mathbf{x}(s)) \cdot \mathbf{x}'(s)\,ds (s = u(t)) \\
&= \int_{a_1}^{b_1} \mathbf{F}(\mathbf{x}(s)) \cdot \mathbf{x}'(s)\,ds \qquad (6.1) \\
&= \int_{\mathbf{x}} \mathbf{F}(\mathbf{x}(s))\,d\mathbf{s}.
\end{aligned}$$

If $\mathbf{y}$ is orientation reversing, then from the third line 6.1 the integral becomes

$$\begin{aligned}
&= \int_{b_1}^{a_1} \mathbf{F}(\mathbf{x}(s)) \cdot \mathbf{x}'(s)\,ds \\
&= -\int_{a_1}^{b_1} \mathbf{F}(\mathbf{x}(s)) \cdot \mathbf{x}'(s)\,ds \\
&= -\int_{\mathbf{x}} \mathbf{F}(\mathbf{x}(s))\,ds.
\end{aligned}$$

□

Example 6.1.13. (1) Given $\mathbf{x}\colon [a, b] \to \mathbb{R}^n$. As a typical example, consider

$\mathbf{x}_{op}\colon [a,b] \to \mathbb{R}^n$ defined by

$$\mathbf{x}_{op} = \mathbf{x}(a+b-t)\colon [a,b] \to \mathbb{R}^n.$$

$\mathbf{x}_{op}$ is called opposite path. This is orientation reversing. We see

$$\begin{aligned}\int_{\mathbf{x}_{op}} \mathbf{F}\cdot d\mathbf{s} &= \int_a^b \mathbf{F}\cdot(\mathbf{x}_{op})'(t)dt\\ &= \int_a^b \mathbf{F}\cdot\mathbf{x}'(b+a-t)(-1)dt\\ &= \int_b^a \mathbf{F}\cdot\mathbf{x}'(u)du\,(u=b+a-t)\\ &= -\int_a^b \mathbf{F}\cdot\mathbf{x}'(u)du\\ &= -\int_{\mathbf{x}} \mathbf{F}\cdot d\mathbf{s}.\end{aligned}$$

(2) The path $\mathbf{p}; [0,1] \to \mathbb{R}^3$ given by $\mathbf{p}(t) = \mathbf{x}(a+(b-a)t)$ is an orientation preserving reparametrization.

The line integral is an *oriented integral*, in the sense that change of sign occurs if the orientation is reversed. The *path integral* does not have this property.

Theorem 6.1.14 (Path integral is independent of parametrization)**.** *If* $\mathbf{x}$ *and* $\mathbf{y}$ *are two parametrization of a piecewise* C^1*-curve* C*, and* f *is any real valued continuous function, then*

$$\int_{\mathbf{x}} f(x,y,z)\,ds = \int_{\mathbf{y}} f(x,y,z)\,ds.$$

Let $\mathbf{x}(t) : [a,b] \to \mathbb{R}^n$. As an example, let $\mathbf{y}$ defined by $\mathbf{y}(t) = \mathbf{x}(a+b-t)$.

Then

$$\begin{aligned}\int_{\mathbf{y}} f\,ds &= \int_a^b f(-\mathbf{x}(t))\|(-\mathbf{x})'(t)\|\,dt \\ &= \int_a^b f(\mathbf{x}(b+a-t))\|\mathbf{x}'(b+a-t)(-1)\|\,dt \\ &= \int_b^a f(\mathbf{x}(s))\|\mathbf{x}'(s)\|(-1)\,ds \ (s=b+a-t) \\ &= \int_{\mathbf{x}} f\,ds.\end{aligned}$$

6.2 Green's Theorem

Boundary and interior of a region

Let D be a region in $\mathbb{R}^2$. We denote its boundary by ∂D and assume its orientation is given in the counterclockwise direction, i.e, when one walks along the boundary, the region on his left is assumed to be **interior.**

Green's Theorem

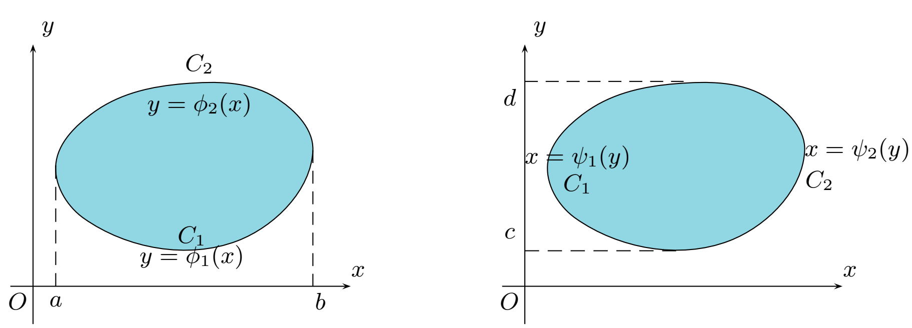


Figure 6.3: As type 1 region and boundary

Theorem 6.2.1. ***(Green's theorem)*** *Let D be a closed bounded, region in $\mathbb{R}^2$ with boundary $C = \partial D$ consisting of finitely many simple closed curve with positive orientation. (The region D is on the left side as one traverses C.)*

Suppose $\mathbf{F}(x,y) = M(x,y)\mathbf{i} + N(x,y)\mathbf{j}$ *be a vector field of class* $\mathcal{C}^1$*. Then*

$$\oint_{\partial D} M\,dx + N\,dy = \iint_D \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dxdy.$$

Here ∂D *denotes the boundary of* D *and* $\oint_{\partial D}$ *means that the integral is defined on a closed curve.(thus it is the same as* $\int_{\partial D}$*).*

Proof. Assume D is a region of type 1 given as follows:

$$D = \{(x,y)|\ a \le x \le b, \phi_1(x) \le y \le \phi_2(x)\}.$$

We decompose the boundary of D as $\partial D = C_1^+ + C_2^-$(fig 6.3). Using the Fubini's theorem, we can evaluate the double integral as an iterated integral

$$\begin{aligned}\iint_D -\frac{\partial M(x,y)}{\partial y} dxdy &= \int_a^b \int_{\phi_1(x)}^{\phi_2(x)} -\frac{\partial M(x,y)}{\partial y} dydx \\ &= \int_a^b [M(x,\phi_1(x)) - M(x,\phi_2(x))]dx.\end{aligned}$$

On the other hand, C_1^+ can be parameterized as $x \to (x,\phi_1(x)), a \le x \le b$ and C_2^+ can be parameterized as $x \to (x,\phi_2(x)), a \le x \le b$. Hence

$$\int_a^b M(x,\phi_i(x))dx = \int_{C_i^+} M(x,y)dx, \quad i = 1,2.$$

By reversing orientations

$$-\int_a^b M(x,\phi_2(x))dx = \int_{C_2^-} M(x,y)dx.$$

Hence

$$\iint_D -\frac{\partial M}{\partial y} dydx = \int_{C_1^+} Mdx + \int_{C_2^-} Mdx.$$

Hence

$$\iint_D -\frac{\partial M}{\partial y}\,dxdy = \int_{C_1^+} M\,dx + \int_{C_2^-} M\,dx = \int_{\partial D} M\,dx.$$

Similarly if D is a region of type 2, one can show that

$$\iint_D \frac{\partial N}{\partial x} dxdy = \int_{C_1^+} Ndy + \int_{C_2^-} Ndy = \int_{\partial D} N\,dy.$$

Here C_1 and C_2 are the curves defined by $x = \psi_1(y)$ and $x = \psi_2(y)$ for $c \le y \le d$. The proof is completed. □

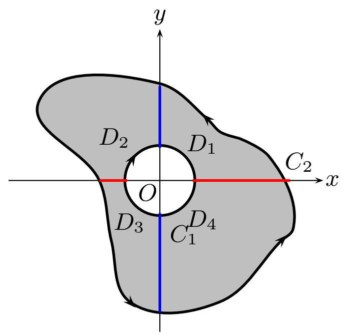


Figure 6.4: Region is divided into four regions to apply Green's theorem

Generalizing Green's theorem

In fact, Green's theorem holds for more general region. For example, Green's theorem can be used for a region with a hole. One cuts the region so that each region is type 3.

Theorem 6.2.2. ***(Green's theorem for general region)*** *Let D be a region which can be divided into a several pieces of regions where Green's theorem apply, and let ∂D be the boundary. Suppose M and $N : D \to \mathbb{R}$ are $\mathcal{C}^1$ functions, then*

$$\int_{\partial D} M dx + N dy = \iint_D \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \right) dxdy.$$

Proof. Assume D is the union of type 3 regions D_i, $i = 1, 2, \ldots, n$ whose boundary ∂D is the sum of ∂D_i, $i = 1, 2, \ldots, n$. In other words,

$$D = \sum_{i=1}^{n} D_i \ , \qquad \partial D = \sum_{i=1}^{n} \partial D_i.$$

So

$$\int_{\partial D} M dx + N dy = \sum_{i=1}^{n} \int_{\partial D_i} M dx + N dy.$$

and

$$\iint_D \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dxdy = \sum_{i=1}^{n} \iint_{D_i} \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dxdy.$$

Since each D_i is type 3, we can apply theorem 6.2.1 to have

$$\int_{\partial D_i} Mdx = \iint_{D_i} -\frac{\partial M}{\partial y} dxdy$$

and

$$\int_{\partial D_i} Ndy = \iint_{D_i} \frac{\partial N}{\partial x} dxdy.$$

We add all these terms to get the result. □

Example 6.2.3. Verify Green's theorem for

$$M(x, y) = \frac{-y}{x^2 + y^2}, \quad N(x, y) = \frac{x}{x^2 + y^2}$$

on $D = \{(x, y)|\ h^2 \le x^2 + y^2 \le 1\}$, $0 < h < 1$.

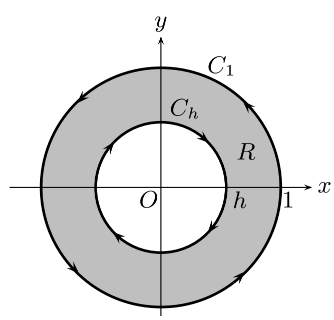


Figure 6.5: Concentric region for Green's theorem

sol. The boundary of D consists of two circles(fig 6.9)

$$C_1 : x = \cos t, \quad y = \sin t, \quad 0 \le t \le 2\pi$$
$$C_h : x = h\cos t, \quad y = h\sin t, \quad 0 \le t \le 2\pi.$$

In the curve $\partial D = C_h \cup C_1$, C_1 is oriented counterclockwise while C_h is oriented clockwise. Since M, N are class $\mathcal{C}^1$ in the annuls D, we can use

Green's theorem. Since

$$\frac{\partial M}{\partial y} = \frac{(x^2+y^2)(-1)+2(2y)}{(x^2+y^2)^2} = \frac{y^2-x^2}{(x^2+y^2)^2} = \frac{\partial N}{\partial x}$$

we have

$$\iint_D \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dxdy = \int_D 0\ dxdy = 0.$$

On the other hand,

$$\begin{aligned}\int_{\partial D} Mdx + Ndy &= \int_{C_1} \frac{xdy - ydx}{x^2+y^2} + \int_{C_h} \frac{xdy-ydx}{x^2+y^2} \\ &= \int_0^{2\pi} (\cos^2 t + \sin^2 t)dt + \int_{2\pi}^0 \frac{h^2(\cos^2 t + \sin^2 t)}{h^2} dt \\ &= 2\pi - 2\pi = 0.\end{aligned}$$

Hence

$$\int_{\partial D} Mdx + Ndy = 0 = \iint_D \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dxdy.$$

■

Example 6.2.4. Evaluate $\int_C \frac{xdy-ydx}{x^2+y^2}$ where C is the unit circle.

sol. Since the integrand is not continuous at $(0,0)$, we cannot use Green's theorem. But we can use the Green's theorem on the region between two two curves (fig 6.9) as in the previous example to see

$$\int_C Mdx + Ndy = -\int_{C_h} Mdx + Ndy.$$

Now the integral $-\int_{C_h}(Mdx + Ndy)$ can be computed by polar coordinate: From

$$\begin{aligned} x &= h\cos\theta, \qquad y = h\sin\theta, \\ dx &= -h\sin\theta d\theta, \\ dy &= h\cos\theta d\theta, \end{aligned}$$

we see

$$\frac{xdy-ydx}{x^2+y^2} = \frac{h^2(\cos^2\theta + \sin^2\theta)}{h^2} d\theta = d\theta.$$

Hence

$$\int_C \frac{xdy - ydx}{x^2 + y^2} = 2\pi.$$

Area

Theorem 6.2.5. *If C is a simple closed curve bounding a region D, then the area A is*

$$A = \frac{1}{2}\int_{\partial D} xdy - ydx.$$

Proof. Let $M(x,y) = -y, N(x,y) = x$. Then

$$\begin{aligned}\frac{1}{2}\int_{\partial D} xdy - ydx &= \frac{1}{2}\iint_D \left(\frac{\partial x}{\partial x} - \frac{\partial(-y)}{\partial y}\right) dxdy \\ &= \frac{1}{2}\iint_D (1+1)dxdy = \iint_D dxdy = A.\end{aligned}$$

□

Example 6.2.6. Find the area of the region enclosed by $x^{2/3} + y^{2/3} = a^{2/3}$.

sol. Let $x = a\cos^3\theta$, $y = a\sin^3\theta$, $(0 \le \theta \le 2\pi)$. Then

$$\begin{aligned}A &= \frac{1}{2}\int_{\partial D} xdy - ydx \\ &= \frac{1}{2}\int_0^{2\pi} [(a\cos^3\theta)(3a\sin^2\theta\cos\theta) - (a\sin^3\theta)(-3a\cos^2\theta\sin\theta)]d\theta \\ &= \frac{3}{2}a^2\int_0^{2\pi} (\sin^2\theta\cos^4\theta + \cos^2\theta\sin^4\theta)d\theta \\ &= \frac{3}{8}a^2\int_0^{2\pi} \sin^2 2\theta d\theta = \frac{3}{8}\pi a^2.\end{aligned}$$

Hence area is $3\pi a^2/8$. (fig 6.6).

Vector Form using the Curl

Any vector field in $\mathbb{R}^2$ can be treated as a vector field in $\mathbb{R}^3$. For example, the vector field $\mathbf{F} = P\mathbf{i} + Q\mathbf{j}$ on $\mathbb{R}^2$ can be viewed as $\mathbf{F} = P\mathbf{i} + Q\mathbf{j} + 0\mathbf{k}$.

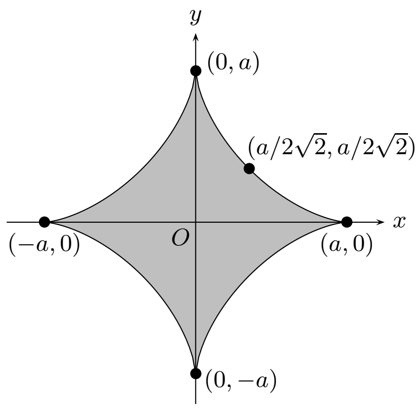


Figure 6.6: $x^{2/3} + y^{2/3} = a^{2/3}$

Then we can define its curl and it can be shown that the curl is (compute!) $(\partial Q/\partial x - \partial P/\partial y)\mathbf{k}$. Then we obtain

$$(\operatorname{curl}\mathbf{F})\cdot\mathbf{k} = \left[\Big(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\Big)\mathbf{k}\right]\cdot\mathbf{k} = \Big(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\Big).$$

Hence by Green's theorem,

$$\int_{\partial D}\mathbf{F}\cdot d\mathbf{s} = \int_{\partial D} Pdx + Qdy = \iint_D \Big(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\Big)dxdy = \iint_D (\nabla\times\mathbf{F})\cdot\mathbf{k}\,dxdy.$$

This is a vector form of Green's theorem.

Theorem 6.2.7. ***(Vector form of Green's theorem)*** *Let $D \subset \mathbb{R}^2$ be region with ∂D. If $\mathbf{F} = P\mathbf{i} + Q\mathbf{j}$ is a $\mathcal{C}^1$-vector field on D then*

$$\int_{\partial D}\mathbf{F}\cdot d\mathbf{s} = \iint_D (\operatorname{curl}\mathbf{F})\cdot\mathbf{k}\,dxdy = \iint_D (\nabla\times\mathbf{F})\cdot\mathbf{k}\,dxdy.$$

Divergence Theorem

Theorem 6.2.8. ***Divergence form of Green's theorem.*** *If $\mathbf{F} = P\mathbf{i} + Q\mathbf{j}$ is a $\mathcal{C}^1$-vector field on D then*

$$\int_{\partial D}(\mathbf{F}\cdot\mathbf{n})ds = \iint_D \operatorname{div}\mathbf{F}\,dxdy.$$

Proof. Let $\mathbf{x}(t)$ be a parametrization of the boundary of D. Since $\mathbf{x}'(t) =$

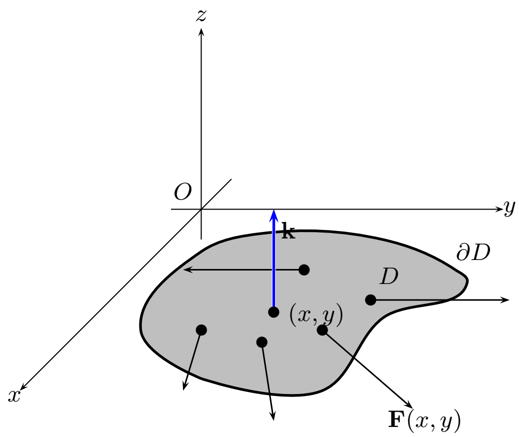


Figure 6.7: vector in Green's Theorem

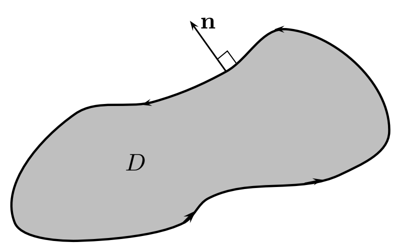


Figure 6.8: $\mathbf{n}$ is the unit outward normal vector to ∂D

$(x'(t), y'(t))$ is tangent to ∂D we see $\mathbf{n} \cdot \mathbf{x}'(t) = 0$. i.e, $\mathbf{n}$ is perpendicular to the boundary. Choosing the proper sign of $\mathbf{n}$, we see

$$\mathbf{n} = \frac{(y'(t), -x'(t))}{\|\mathbf{x}'(t)\|}.$$

Hence

$$\begin{aligned}
\int_{\partial D} (\mathbf{F} \cdot \mathbf{n}) ds &= \int_a^b \left(\frac{P(x,y)y'(t) - Q(x,y)x'(t)}{\sqrt{[x'(t)]^2 + [y'(t)]^2}} \right) \sqrt{[x'(t)]^2 + [y'(t)]^2} dt \\
&= \int_a^b [P(x,y)y'(t) - Q(x,y)x'(t)] dt \\
&= \int_{\partial D} P dy - Q dx.
\end{aligned}$$

By Green's theorem,

$$\int_{\partial D} (\mathbf{F}\cdot\mathbf{n})ds = \int_{\partial D} P dy - Q dx$$
$$= \iint_D \left(\frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y}\right) dxdy$$
$$= \iint_D \operatorname{div}\mathbf{F}\, dxdy.$$

□

6.3 Conservative vector fields

Definition 6.3.1. A line integral a vector field $\mathbf{F}$ is called **path independent** if

$$\int_{C_1} \mathbf{F}\cdot d\mathbf{s} = \int_{C_2} \mathbf{F}\cdot d\mathbf{s} \tag{6.2}$$

for any two oriented curves C_1, C_2 lying in the domain of $\mathbf{F}$ having same end points.

Definition 6.3.2. A region R in $\mathbb{R}^2$ or $\mathbb{R}^3$ is called **simply connected** if it consists of s single piece and every simple closed curve C in R can be continuously shrunk to a point while remaining in R throughout the deformation.

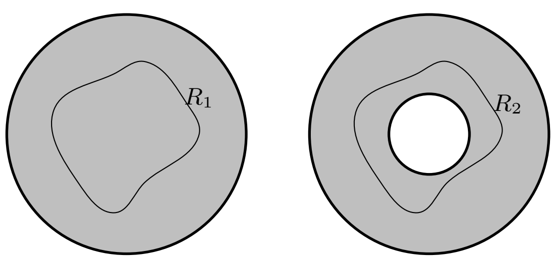


Figure 6.9: 'Simply connected' region and 'not simply connected' region in $\mathbb{R}^2$

Example 6.3.3. Let $\mathbf{F} = y\mathbf{i} - x\mathbf{j}$ and consider two paths C_1 and C_2 connecting $(0,0)$ and $(1,1)$. We compare $\int_{C_1} \mathbf{F}\cdot d\mathbf{s}$ and $\int_{C_2} \mathbf{F}\cdot d\mathbf{s}$. These curves may be parameterized as

$$C_1 : \begin{cases} x = t \\ y = t \end{cases} \quad (0 \leq t \leq 1) \text{ and } C_2 : \begin{cases} x = t \\ y = t^2 \end{cases} \quad (0 \leq t \leq 1).$$

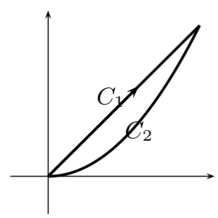


Figure 6.10: Two path connecting $(0, 0)$ and $(1, 1)$

Theorem 6.3.4. ***(Conservative Field)*** *Let* $\mathbf{F}$ *be a* $\mathcal{C}^1$*-vector field in* $\mathbb{R}^3$ *except finite number of points. Then the following conditions are equivalent:*

(1) For any oriented simple closed curve C, $\int_C \mathbf{F} \cdot d\mathbf{s} = 0$.

(2) For any two oriented simple curve C_1, C_2 *having same end points,*

$$\int_{C_1} \mathbf{F} \cdot d\mathbf{s} = \int_{C_2} \mathbf{F} \cdot d\mathbf{s}.$$

(3) $\mathbf{F}$ *is the gradient of some function* f*, i.e,* $\mathbf{F} = \nabla f$.

(4) $\nabla \times \mathbf{F} = \mathbf{0}$.

Remark 6.3.5. For $\mathbb{R}^2$, we need the concept of **simply connected domain.**

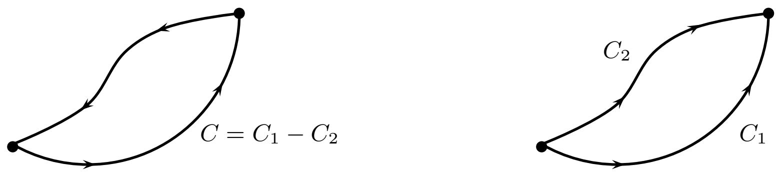
Proof. We use the sequence of implication: (i)⇒(2)⇒(3)⇒(4)⇒(1).

(1)⇒(2): Suppose we have two curves C_1, C_2 having same end points, we can form a closed curve by $C = C_1 - C_2$(fig 6.11). Hence by (1)

$$0 = \int_C \mathbf{F} \cdot d\mathbf{s} = \int_{C_1 - C_2} \mathbf{F} \cdot d\mathbf{s} = \int_{C_1} \mathbf{F} \cdot d\mathbf{s} - \int_{C_2} \mathbf{F} \cdot d\mathbf{s}.$$

(2)⇒(3): Fix a point (x_0, y_0, z_0). Given any point (x, y, z), choose any curve C connecting two points we define. Given $\mathbf{F} = (F_1, F_2, F_3)$ define

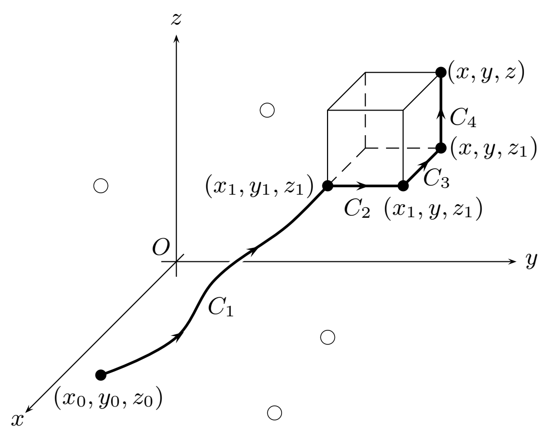
$$f(x, y, z) = \int_C \mathbf{F} \cdot d\mathbf{s} = \int_C F_1 dx + F_2 dy + F_3 dz.$$



(a) oriented simple closed curve $C = C_1 - C_2$ (b) two oriented simple curves C_1

Figure 6.11: Constructing oriented simple closed curve from two oriented simple curve C_1, C_2

Here f is well-defined, since it is defined independent of the choice of C. So we choose C consisting of edges of rectangular pipe.



Path avoid points where vector field is not de

Figure 6.12: A path from (x_0, y_0, z_0) to (x, y, z) is $C = C_1 + C_2 + C_3 + C_4$

In particular, choose $C = C_1 + C_2 + C_3 + C_4$ (fig 6.12). Then

$$\begin{aligned}
f(x, y, z) &= \int_C \mathbf{F} \cdot d\mathbf{s} \\
&= \int_{C_1} \mathbf{F} \cdot d\mathbf{s} + \int_{C_2} \mathbf{F} \cdot d\mathbf{s} + \int_{C_3} \mathbf{F} \cdot d\mathbf{s} + \int_{C_4} \mathbf{F} \cdot d\mathbf{s} \\
&= \int_{C_1} \mathbf{F} \cdot d\mathbf{s} + \int_{y_1}^{y} F_2(x_1, t, z_1)dt \\
&\qquad + \int_{x_1}^{x} F_1(t, y, z_1)dt + \int_{z_1}^{z} F_3(x, y, t)dt.
\end{aligned}$$

From this we see $\partial f/\partial z = F_3$. Similarly by choosing different path(i.e, choosing a path whose last path is along x-direction) we have

$$\begin{aligned} f(x,y,z) &= \int_{C_1} \mathbf{F}\cdot d\mathbf{s} + \int_{y_1}^{y} F_2(x_1,t,z_1)dt \\ &+ \int_{z_1}^{z} F_3(x_1,y,t)dt + \int_{x_1}^{x} F_1(t,y,z)dt \end{aligned}$$

so $\partial f/\partial x = F_1$. Similarly, we have $\partial f/\partial y = F_2$. Thus $\mathbf{F} = \nabla f$.

(3)⇒(4): By theorem 5.1.2, we have $\nabla \times \nabla f = 0$. So $\operatorname{curl}\mathbf{F} = \nabla \times \mathbf{F} = 0$.

(4)⇒(1): Let S be a surface having C as boundary. Then

$$\int_C \mathbf{F}\cdot d\mathbf{s} = \int_S \operatorname{curl}\mathbf{F}\cdot d\mathbf{S}.$$

Here in 2-D, this is nothing but the Green's theorem with $d\mathbf{S} = \mathbf{k}dxdy$. Since $\operatorname{curl}\mathbf{F} = \mathbf{0}$ the integral $\int_C \mathbf{F}\cdot d\mathbf{s} = 0$. □

Line integrals of Gradient Fields

A vector field $\mathbf{F}$ is called a **gradient vector field** if $\mathbf{F} = \nabla f$ for some real valued function f. Thus

$$\mathbf{F} = \frac{\partial f}{\partial x}\mathbf{i} + \frac{\partial f}{\partial y}\mathbf{j} + \frac{\partial f}{\partial z}\mathbf{k}.$$

This f is called **potential** of $\mathbf{F}$.

Theorem 6.3.6. *Suppose $f : \mathbb{R}^3 \to \mathbb{R}$ is class C^1 and $\mathbf{x} : [a,b] \to \mathbb{R}^3$ is smooth. Then*

$$\int_{\mathbf{x}} \nabla f\cdot d\mathbf{s} = f(\mathbf{x}(b)) - f(\mathbf{x}(a)).$$

Proof. By the chain rule, we get

$$(f\circ\mathbf{x})'(t) = \nabla f(\mathbf{x}(t))\cdot\mathbf{x}'(t).$$

So

$$\int_{\mathbf{x}} \nabla f\cdot d\mathbf{s} = \int_a^b \nabla f(\mathbf{x}(t))\cdot\mathbf{x}'(t)dt = \int_a^b f'(u)du = f(\mathbf{x}(b)) - f(\mathbf{x}(a)).$$

So the line integral is independent of parametrization. □

A criterion for conservative Vector Fields

Definition 6.3.7. A region R is called **simply connected** if it consists of a single connected piece and if every closed curve C in R can be shrunk to a point while remaining in R.

Theorem 6.3.8. *Suppose* $\mathbf{F}$ *is a* $\mathcal{C}^1$*-vector field in a simply connected region in* $\mathbb{R}^2$ *or* $\mathbb{R}^3$*. Then* $\mathbf{F} = \nabla f$ *for some scalar function of* C^2 *if and only if*

$$\nabla \times \mathbf{F} = \mathbf{0}.$$

Proof. Suppose $\mathbf{F} = \nabla f$ for some scalar function of C^2. Then $\nabla \times \mathbf{F} = \nabla \times (\nabla f) = \mathbf{0}$. Conversely, suppose $\nabla \times \mathbf{F} = \mathbf{0}$. Then $\oint_C \mathbf{F} \cdot d\mathbf{s} = 0$ for any closed curve C. Then by Theorem 6.3.4 the line integral of $\mathbf{F}$ is path independent. Hence $\mathbf{F} = \nabla f$ for some f. □

Hence

$$\text{curl}\,\mathbf{F} = \nabla \times \mathbf{F} = \mathbf{0}$$

is equivalent to

$$\partial Q/\partial x = \partial P/\partial y.$$

Using this we can study conservative field $\mathbb{R}^2$. In $\mathbb{R}^3$ vector field may have a few points where function is undefined. But in planar case, the vector field must be defined everywhere.

Example 6.3.9. Find the potential of the vector field if it is conservative.

$$\mathbf{F}(x, y) = (2xy + \cos 2y)\mathbf{i} + (x^2 - 2x \sin 2y)\mathbf{j}.$$

sol.

First we check that $\nabla \times \mathbf{F} = \mathbf{0}$. Hence it is conservative. Let f be the potential function. Then it satisfies $\nabla f = \mathbf{F}$, i.e.,

$$\frac{\partial f}{\partial x} = 2xy + \cos 2y, \quad \frac{\partial f}{\partial y} = x^2 - 2x \sin 2y. \tag{6.3}$$

Thus we proceed as follows:

(1) Integrate: $f(x, y) = \int \frac{\partial f}{\partial x}\, dx = \int 2xy + \cos 2y\, dx = x^2y + x \cos 2yxy + g(y)$

(2) Set $\frac{\partial f}{\partial y} = x^2 - 2x \sin 2y + g'(y)$

(3) Show $g(x,y) = C$.

Thus we see $f(x,y) = x^2 - 2x\sin 2y + C$.

Example 6.3.10. Consider the vector field

$$\mathbf{F}(x,y,z) = (\frac{x}{x^2+y^2+z^2} - 6x)\mathbf{i} + \frac{y}{x^2+y^2+z^2}\mathbf{j} + \frac{z}{x^2+y^2+z^2}\mathbf{k}.$$

$\mathbf{F}$ is class C^1 except the origin. But $\mathbb{R}^3 - \mathbf{0}$ is simply connected. Also, we can check $\nabla \times \mathbf{F} = 0$. Hence $\mathbf{F}$ is conservative. $\mathbf{x} : [0,1] \to \mathbb{R}^3$ is given by $\mathbf{x}(t) = (1-t, \sin\pi t, t)$. Compute $\int_{\mathbf{x}} \mathbf{F}\cdot d\mathbf{s}$.

sol. Evaluating this integral is tricky. Instead, we choose different path(This is allowed, since the field is conservative), or find the potential. Noting that $\mathbf{x}(0) = (1,0,0)$ and $\mathbf{x}(1) = (0,0,1)$ we can choose a path on the unit sphere: $\mathbf{y}(t) = (\cos t, 0, \sin t), 0 \le t \le \pi/2$. This choice makes the integrand very simple.

$$\begin{aligned}
\int_{\mathbf{y}} \mathbf{F}\cdot d\mathbf{s} = \int_0^{\pi/2} \mathbf{F}\cdot d\mathbf{s} &= \int_0^{\pi/2} (\cos t - 6\cos t, 0, \sin t)\cdot(-\sin t, 0, \cos t)\,dt \\
&= \int_0^{\pi/2} 6\cos t \sin t\,dt \\
&= -\frac{3}{2}\cos 2t.
\end{aligned}$$

Example 6.3.11. Show that the vector field is irrotational and find its potential.

$$\mathbf{F}(x,y,z) = (e^x \sin y - yz)\mathbf{i} + (e^x\cos y - xz)\mathbf{j} + (z - xy)\mathbf{k}.$$

sol.

$$\nabla \times \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ e^x \sin y - yz & e^x \cos y - xz & z - xy \end{vmatrix}$$

$$= (\frac{\partial}{\partial y}(z - xy) - \frac{\partial}{\partial z}(e^x \cos y - xz))\mathbf{i} + (\frac{\partial}{\partial z}(e^x \sin y - yz) - \frac{\partial}{\partial x}(z - xy))\mathbf{j}$$
$$+ (\frac{\partial}{\partial x}(e^x \cos y - xz) - \frac{\partial}{\partial y}(e^x \sin y - yz))\mathbf{k} = \mathbf{0}.$$

So $\mathbf{F}$ is irrotational. To find a potential

$$\frac{\partial f}{\partial x} = e^x \sin y - yz, \quad \frac{\partial f}{\partial y} = e^x \cos y - xz, \quad \frac{\partial f}{\partial z} = z - xy. \qquad (6.4)$$

Thus we have

(1) $f(x, y, z) = \int (e^x \sin y - yz)dx = e^x \sin y - xyz + g(y, z)$

(2) $\frac{\partial f}{\partial y} = e^x \cos y - xz + \frac{\partial g}{\partial y} = e^x \cos y - xz$. Thus $g(y, z)$ is a function of z only. Taking derivative w.r.t z, we have

(3) $\frac{\partial f}{\partial z} = -xy + g'(z) = z - xy$. Thus $g(z) = \frac{1}{2}z^2 + C$.

(4) $f(x, y, z) = e^x \sin y - xyz + h(z)$.

Hence

$$f(x, y, z) = e^x \sin y - xyz + \frac{1}{2}z^2 + C.$$

□