Chapter 5

Double and Triple integrals

In this chapter we study the **double integral** and **triple integral**. First, we define how to compute volumes of a solid by two methods: One by Carvalieri principle and the other by double integral. Then we show the relation between them. In fact, one can interpret the Carvalieri principle as an iterated integral and show this equals the double integral.

The triple integral can be treated similarly.

5.1 Computing Volumes

For the convenience of presentation we assume the domain of a function on a rectangle R given by

$$R = \{(x, y)\colon a \le x \le b, c \le y \le d\}.$$

We also denote it by $[a, b] \times [c, d]$ and call it **Cartesian product** of $[a, b]$, $[c, d]$ If f is nonnegative, then the region under the graph of $z = f(x, y)$ defines a solid.

Cavalieri's Principle and iterated integral

See Fig 5.1. Let $A(x)$ be the volume of cross section of a solid by a plane perpendicular to the axis, then the volume is

$$V = \int_a^b A(x)\, dx. \tag{5.1}$$

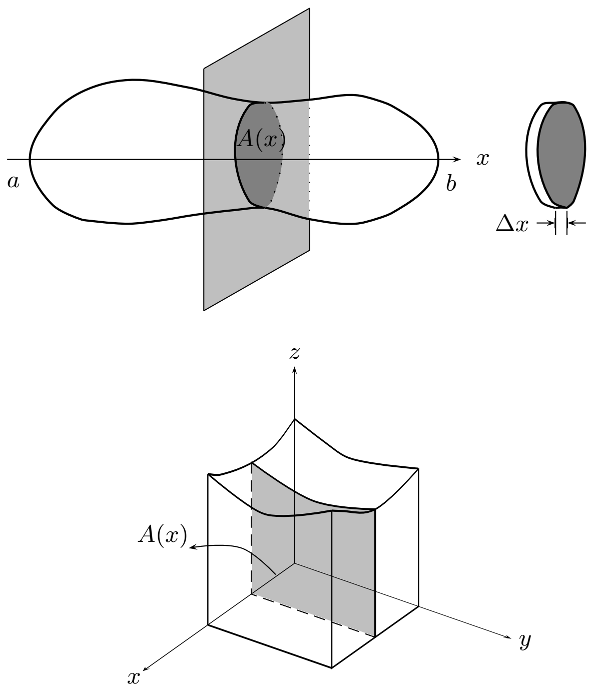


Figure 5.1: Cavalieri Principle, $V = \int_a^b A(x)\,dx$

This is **Cavalieri's Principle**. To understand it, let us divide the interval $[a, b]$ by n subintervals $a = x_0 < x_1 < \cdots < x_n = b$, $\Delta x_i = x_i - x_{i-1}$. Then $A(x)\Delta x_i$ is the approximate volume of each slice.(Fig 5.1) Now the Riemann sum

$$\mathcal{R}(A, n) = \sum_{i=1}^{n} A(x_i)\Delta x$$

is the approx volume of the whole solid. If $n \to \infty$, the limit becomes (5.1).

Reduction to iterated integrals

Consider the volume of a solid under f over $R = [a, b] \times [c, d]$ as in figure 5.2. The cross section along $x = x_0$ is the set given by $\{(x_0, y, z) | 0 \le z \le f(x_0, y),\ (c \le y \le d)\}$. The area of cross section is

$$A(x_0) = \int_c^d f(x_0, y)\,dy.$$

Hence by Cavalieri principle, the volume is

$$\int_a^b A(x)\,dx = \int_a^b \left[\int_c^d f(x,y)\,dy\right] dx.$$

The expression on the right hand side is called an **iterated integral**. On the other hand, if we cut it by the plane $y = y_0$, then the volume becomes

$$\int_a^b A(y)\,dy = \int_c^d \left[\int_a^b f(x,y)\,dx\right] dy.$$

Since these two values are equal,

$$\int_a^b \left[\int_c^d f(x,y)\,dy\right] dx = \int_c^d \left[\int_a^b f(x,y)\,dx\right] dy.$$

This is called **Fubini's theorem**.

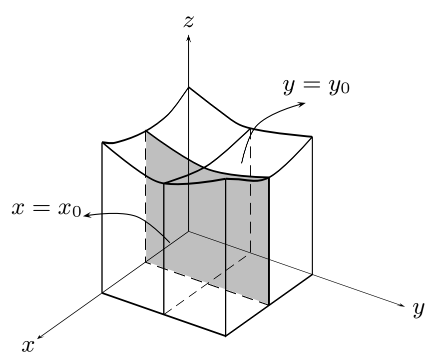


Figure 5.2: Fubini's theorem by Cavalieri Principle

Example 5.1.1. Evaluate

$$\iint_R (x^2+y^2)\,dxdy, \quad R = [-1,1]\times[0,1].$$

sol.

$$\int_0^1 \left[\int_{-1}^1 (x^2+y^2)dx\right] dy = \frac{4}{3}.$$

Now change the order to see the integrals are the same.

■

Example 5.1.2. Evaluate

$$\iint_S \cos x \sin y \, dx dy, \quad S = [0, \frac{\pi}{2}] \times [0, \frac{\pi}{2}]$$

sol.

$$\begin{aligned}
\iint_S \cos x \sin y \, dx dy &= \int_0^{\pi/2} \left[\int_0^{\pi/2} \cos x \sin y \, dx \right] dy \\
&= \int_0^{\pi/2} \sin y \left[\int_0^{\pi/2} \cos x \, dx \right] dy = \int_0^{\pi/2} \sin y \, dy = 1
\end{aligned}$$

Now change the order.

■

5.2 Double integral

Double Integral of a nonnegative function as a volume

When $f(x, y)$ is a nonnegative function over R, then the double integral of f is the volume of the region above R and under the graph of f. But the double integral of more general function (say continuous, or piecewise continuous) f can be similarly defined.

Definition 5.2.1. Assume $R = \{(x, y) \colon a \le x \le b, c \le y \le d\}$. Then we Subdivide two intervals $[a, b]$, $[c, d]$ into n -intervals

$$a = x_0 < x_1 < \cdots < x_n = b, \quad c = y_0 < y_1 < \cdots < y_n = d.$$

We call the subrectangles $R_{ij} = [x_{i-1}, x_i] \times [y_{j-1}, y_j]$ a **partition** of R and let

$$\Delta x_i = x_i - x_{i-1}, \quad \Delta y_j = y_j - y_{j-1}.$$

Definition 5.2.2. Given any function f defined on R, and for any point c_{ij} in R_{ij} consider the sum

$$S = \mathcal{R}(f) = \sum_{i,j=1}^{n} f(c_{ij}) \Delta A_{ij}, \tag{5.2}$$

where $\Delta A_{ij} = \Delta x_i \Delta y_j$ is the area of R_{ij}. It is called **Riemann sum** of f

corresponding to the partition. Here $\|\mathcal{R}\| = \max_{i,j}\{\Delta x_i, \Delta y_j\}$ is called the **norm** of the partition.

Definition 5.2.3 (Double integral)**.** If the sum S converge to the same limit regardless of the points c_{ij} and regardless of the partition, then f is called **integrable** over R and we write its limit by

$$\iint_R f(x,y)\,dA = \lim_{\|\mathcal{R}\|\to 0} \sum_{i,j=1}^{n} f(c_{ij})\Delta x_i \Delta y_j.$$

These are also written as $\int_R f\,dA$ or $\iint_R f(x,y)\,dxdy$.

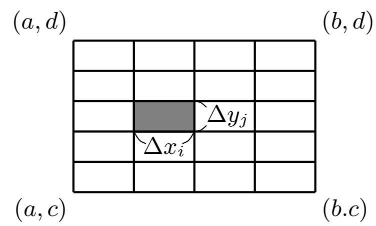


Figure 5.3: A partition of a rectangle

Now we define the integral of more general functions.

Integrability of bounded function

Definition 5.2.4. f is called **bounded** if there is M such that $-M \le f(x,y) \le M$ holds for all (x,y) in R.

Theorem 5.2.5. *If f is continuous function on a closed set R, then $\iint_R f\,dA$ exists.*

Theorem 5.2.6. *If f is bounded on R and the set of points where f is discontinuous lies on a finite union of graphs of continuous functions, then f is integrable over R.*

This result is useful to define the integral over general region.

Properties of integral

Theorem 5.2.7. *Let f, g be integrable over R, R_1, R_2. Then we have*

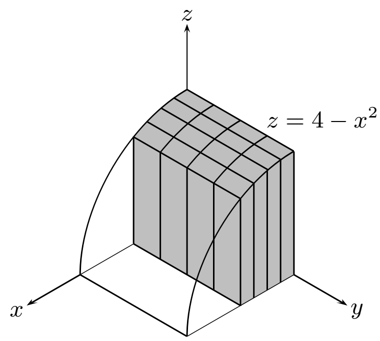


Figure 5.4: $z = 4 - x^2$, $0 \le x \le 1$, $0 \le y \le 2$

(1) $\displaystyle\iint_R cf(x,y)\,dxdy = c\iint_R f(x,y)\,dxdy$, *(c is constant).*

(2) $\displaystyle\iint_R (f(x,y)+g(x,y))\,dxdy$

$$= \iint_R f(x,y)\,dxdy + \iint_R g(x,y)\,dxdy.$$

(3) *If* $f(x,y) \ge 0$, $\displaystyle\iint_R f(x,y)\,dxdy \ge 0.$

(4) *If* $f(x,y) \ge g(x,y)$, $\displaystyle\iint_R f(x,y)\,dxdy \ge \iint_R g(x,y)\,dxdy.$

(5) If R_1 *and* R_2 *do not meet, then for* $R = R_1 \cup R_2$

$$\iint_R f(x,y)\,dxdy = \iint_{R_1} f(x,y)\,dxdy + \iint_{R_2} f(x,y)\,dxdy.$$

(6) $\displaystyle\left|\iint_R f dA\right| \le \iint_R |f| dA.$

Theorem 5.2.8 (Fubini Theorem 1)**.** *Let* f *be continuous on* $R = [a,b]\times[c,d]$*. Then* f *satisfies*

$$\int_a^b \left[\int_c^d f(x,y)\,dy\right] dx = \int_c^d \left[\int_a^b f(x,y)\,dx\right] dy = \iint_R f(x,y)\,dA. \tag{5.3}$$

This result holds when f *is discontinuous on the graph of continuous functions.*

Sketch of proof first.

Let $a = x_0 < x_1 < \cdots < x_n = b$, $\quad c = y_0 < y_1 < \cdots < y_n = d$ be the regular partition of $[a,b]$ and $[c,d]$. Riemann sum for iterated integral in this case is

$$\sum_{i,j=1}^{n} f(c_{ij})\Delta x \Delta y = \sum_{i=1}^{n} \left(\sum_{j=1}^{n} f(c_{ij})\Delta y \right) \Delta x = \sum_{j=1}^{n} \left(\sum_{i=1}^{n} f(c_{ij})\Delta x \right) \Delta y.$$

This can be proved as follows: Let $[a_{ij}]$ be $n \times n$ array of numbers. Then

$$\sum_{i,j=1}^{n} a_{ij} = \sum_{i=1}^{n} \left(\sum_{j=1}^{n} a_{ij} \right) = \sum_{j=1}^{n} \left(\sum_{i=1}^{n} a_{ij} \right)$$

This idea will be used.

Proof. We will first prove that

$$\int_a^b \left[\int_c^d f(x,y)\,dy \right] dx = \int_R f(x,y)\,dA.$$

Let

$$F(x) = \int_c^d f(x,y)\,dy.$$

Then

$$F(x) = \sum_{j=1}^{n} \int_{y_{j-1}}^{y_j} f(x,y)\,dy.$$

Then by mean value theorem for integrals(with fixed x), there exists some $Y_j(x)$ in $[y_{j-1}, y_j]$ such that

$$\int_{y_{j-1}}^{y_j} f(x,y)\,dy = f(x, Y_j(x))\Delta y.$$

Thus

$$F(x) = \sum_{j=1}^{n} f(x, Y_j(x))\Delta y.$$

By one variable integration theory, it holds that, for any choice of $p_i \in [x_{i-1}, x_i]$

$$\int_a^b F(x)\,dx = \lim_{n\to\infty} \sum_{i=1}^{n} F(p_i)\Delta x.$$

Let $c_{ij} = (p_i, Y_k(p_i))$. Then

$$F(p_i) = \sum_{j=1}^{n} f(p_i, Y_j(p_i))\Delta y = \sum_{j=1}^{n} f(c_{ij})\Delta y$$

holds. Hence

$$\begin{aligned}
\int_a^b \left[\int_c^d f(x,y)\,dy\right] dx &= \int_a^b F(x)\,dx \\
&= \lim_{n\to\infty} \sum_{i=1}^{n} F(p_i)\Delta x \\
&= \lim_{n\to\infty} \sum_{i=1}^{n}\sum_{j=1}^{n} f(c_{ij})\Delta y \Delta x \\
&= \iint_R f(x,y)\,dA.
\end{aligned}$$

By the same reasoning, we can show

$$\int_c^d \left[\int_a^b f(x,y)\,dx\right] dy = \iint_R f(x,y)\,dA.$$

□

Example 5.2.9. Find the volume of the region $0 \le x \le 1$, $0 \le y \le 1$, $0 \le z \le 2 - x - y$.

sol. First fix x. Then the area of cross section with a plane perpendicular to x-axis is

$$A(x) = \int_0^1 (2 - x - y)\,dy.$$

So the volume is

$$\begin{aligned}
V &= \int_0^1 A(x)\,dx = \int_{x=0}^{x=1}\int_{y=0}^{y=1} (2 - x - y)\,dydx \\
&= \int_0^1 \left[2y - xy - \frac{y^2}{2}\right]_0^1 dx \\
&= \int_0^1 \left(\frac{3}{2} - x\right) dx = \left[\frac{3x}{2} - \frac{x^2}{2}\right]_0^1 = 1.
\end{aligned}$$

Change the order: You can fix y. Then the area of cross section with a plane perpendicular to y-axis is

$$A(y) = \int_0^1 (2 - x - y)\, dx.$$

Hence the volume is

$$\begin{aligned} V &= \int_0^1 A(y)\, dy = \int_{y=0}^{y=1} \int_{x=0}^{x=1} (2 - x - y)\, dxdy \\ &= \int_0^1 \left[2x - \frac{x^2}{2} - xy\right]_0^1 dy \\ &= \int_0^1 \left(\frac{3}{2} - y\right) dy = \left[\frac{3y}{2} - \frac{y^2}{2}\right]_0^1 = 1. \end{aligned}$$

□

Example 5.2.10. Compute $\iint_R (x^2 + y) dA$, where $A = [0,1] \times [0,1]$.

sol.

$$\iint_R (x^2+y) dA = \int_0^1 \int_0^1 (x^2+y) dxdy = \int_0^1 [\int_0^1 (x^2+y) dx] dy = \int_0^1 (\frac{1}{3}+y) dy = \frac{5}{6}.$$

□

Example 5.2.11. Find $\iint_R f(x,y)\, dxdy$. Here the function $f = y(x^3 - 12x)$ takes both positive and negative values and R is given by $-2 \le x \le 1$, $0 \le y \le 1$.

sol.

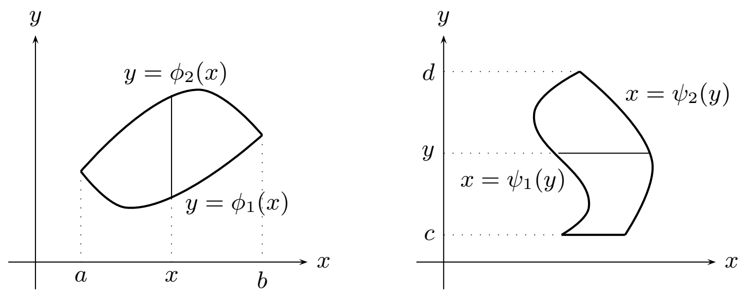
$$\iint_R y(x^3 - 12x) dxdy = \int_0^1 \left[\int_{-2}^1 y(x^3 - 12x) dx\right] dy = \frac{57}{4} \int_0^1 y dy = \frac{57}{8}.$$

□

Double integral over general regions

So far we have defined double integral over a rectangle. How can we define double integral on general domains? We begin with classifying the regions.

Definition 5.2.12. Elementary regions



(a) region of type 1 (b) region of type 2

Figure 5.5: region of type 1, region of type 2

There are three kind of elementary regions: Let $y = \phi_1(x)$, $y = \phi_2(x)$ be two continuous functions satisfying $\phi_1(x) \le \phi_2(x)$ for $x \in [a, b]$. Then the region

$$D = \{(x, y) \mid a \le x \le b,\ \phi_1(x) \le y \le \phi_2(x)\}$$

is called **region of type 1**.

Now change the role of x, y as in figure 5.5 (b). If $x = \psi_1(y)$, $x = \psi_2(y)$, satisfies $\psi_1(y) \le \psi_2(y)$ for $y \in [c, d]$, then the region determined by

$$D = \{(x, y) \mid c \le y \le d,\ \psi_1(y) \le x \le \psi_2(y)\}$$

is called **region of type 2**. The region that is both Type 1 and Type 2 is called **region of type 3**. These are called **elementary regions**.

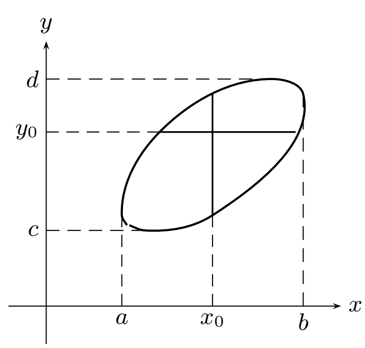


Figure 5.6: Region of type 3

Integrals over elementary regions(by extension to 0)

Now we are ready to define the integral of f defined on an elementary region. The idea is to extend the function to a rectangular domain. Given a continuous function f on D where D is an elementary region

$$D = \{(x, y) \mid \phi_1(x) \le y \le \phi_2(x), \quad a \le x \le b\},$$

we consider a rectangle which contains D and extend f to R outside D by zero:

$$f^{ext}(x, y) = \begin{cases} f(x, y), & (x, y) \in D \\ 0, & (x, y) \in R \setminus D. \end{cases}$$

Then f^{ext} has discontinuities on the graphs of $y = \phi_1(x)$, $y = \phi_2(x)$, $a \le x \le b$. Hence it is integrable by Theorem 5.2.6. Now we can define the integral of f over R.

Definition 5.2.13. The integral of f is defined as

$$\iint_D f(x, y)\, dA := \iint_R f^{ext}(x, y)\, dA.$$

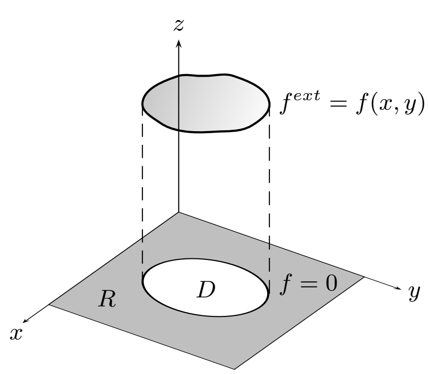


Figure 5.7: Extension of a function

From this definition we have an important result useful in the computation of double integral.

Theorem 5.2.14 (Reduction of a double integral to iterated integral)**.** *Let f be a continuous on an elementary region $D \subset R$.*

(1) If D is a domain of type 1, i.e, $D = \{(x,y) : \phi_1(x) \le y \le \phi_2(x),\ a \le x \le b\}$ for some continuous functions ϕ_1, ϕ_2, then f is integrable on D and

$$\iint_D f(x,y)\,dA = \int_a^b \left[\int_{\phi_1(x)}^{\phi_2(x)} f(x,y)\,dy\right] dx.$$

(2) Similarly if D is a domain of type 2, i.e, $D = \{(x,y) : \psi_1(y) \le x \le \psi_2(y),\ c \le y \le d\}$ for some continuous functions ψ_1, ψ_2, then

$$\iint_D f(x,y)\,dA = \int_c^d \left[\int_{\psi_1(y)}^{\psi_2(y)} f(x,y)\,dx\right] dy.$$

Proof. By Fubini theorem, we have

$$\begin{aligned}
\iint_D f(x,y)\,dA &= \iint_D f^{ext}(x,y)\,dA && (5.4)\\
&= \int_a^b \int_c^d f^{ext}(x,y)\,dydx && (5.5)\\
&= \int_c^d \int_a^b f^{ext}(x,y)\,dxdy. && (5.6)
\end{aligned}$$

For type 1 region, we see

$$\int_c^d f^{ext}(x,y)\,dy = \int_{\phi_1(x)}^{\phi_2(x)} f(x,y)\,dy.$$

Hence by (5.5) we obtain (1). For type 2 region, we see

$$\int_a^b f^{ext}(x,y)\,dx = \int_{\psi_1(y)}^{\psi_2(y)} f(x,y)\,dx.$$

Hence by (5.5) we obtain (2). □

Example 5.2.15. Find the following integral when $D : 0 \le x \le 1,\ x \le y \le 1$

$$\iint_D (x + y^2)\,dxdy$$

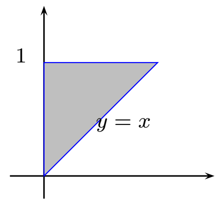


Figure 5.8: Region $0 \le x \le 1,\ x \le y \le 1$

sol. Use Fubini's theorem

$$\int_0^1 \int_x^1 (x+y^2)\,dydx = \int_0^1 \left[xy + \frac{y^3}{3}\right]_x^1 dx$$
$$= \int_0^1 \left(x + \frac{1}{3} - x^2 - \frac{x^3}{3}\right) dx$$
$$= \left[\frac{x^2}{2} + \frac{x}{3} - \frac{x^3}{3} - \frac{x^4}{12}\right]_0^1 = \frac{5}{12}$$

□

Example 5.2.16. Find $\iint_D x^2y\,dA$ where D is given by $0 \le x,\ 0 \le y \le 4-x^2$, $3x^2 \le y$.

sol. Two curve meet at $(1,3)$ hence

$$\int_0^1 \int_{3x^2}^{4-x^2} x^2y\,dydx = \int_0^1 \left(\frac{x^2y^2}{2}\right)\Bigg|_{y=3x^2}^{4-x^2} dx$$
$$= \int_0^1 \left(\frac{x^2}{2}((4-x^2)^2 - (3x^2)^2)\right) dx$$
$$= \frac{1}{2}\int_0^1 x^2(16 - 8x^2 + x^4 - 9x^4)\,dx = \frac{136}{105}.$$

□

Example 5.2.17. Find $\iint_D (x^3y + \cos x)\,dA$ where D is given by $0 \le x \le \pi/2$, $0 \le y \le x$.

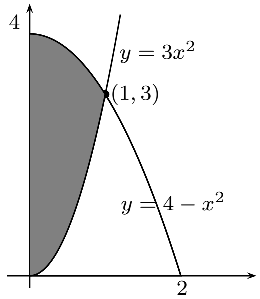


Figure 5.9: Domain of integration of example 5.2.16

sol.

$$\begin{aligned}
&\iint_D (x^3 y + \cos x)\, dA \\
&= \int_0^{\pi/2} \int_0^x (x^3 y + \cos x) dy\, dx \\
&= \int_0^{\pi/2} \left[\frac{x^3 y^2}{2} + y \cos x \right]_{y=0}^x dx = \int_0^{\pi/2} \left(\frac{x^5}{2} + x \cos x \right) dx \\
&= \frac{\pi^6}{768} + \frac{\pi}{2} - 1.
\end{aligned}$$

■

Example 5.2.18. Find volume of tetrahedron bounded by the planes $y = 0, x = 0, y - x + z = 1$.

sol. We let $z = f(x, y) = 1 - y + x$. Then the volume of tetrahedra is the volume under the graph of f. Hence

$$\begin{aligned}
\iint_D (1 - y + x) dA &= \int_{-1}^0 \int_0^{1+x} (1 - y + x) dy dx \\
&= \int_{-1}^0 \left[(1 + x)y - \frac{y^2}{2} \right]_{y=0}^{1+x} dx = \frac{1}{6}.
\end{aligned}$$

■

Example 5.2.19. Let D be given by $D = \{(x, y) | 0 \le x \le \ln 2, \quad 0 \le y \le e^x - 1\}$. Express the double integral

$$\iint_D f(x, y)\, dA$$

in two iterated integrals.

sol. See figure 5.10. To view it as a region of type 1, the points of intersection is $y = 0$, $y = e^x - 1 (0 \le x \le \ln 2)$. Hence

$$\int_0^{\ln 2} \int_0^{e^x - 1} f(x, y)\, dydx$$

As a y-simple region, the points of intersection is $x = \ln(y+1)$, $x = \ln 2 (0 \le y \le 2)$ So the integral is

$$\int_0^1 \int_{\ln(y+1)}^{\ln 2} f(x, y)\, dxdy$$

□

Example 5.2.20. Given domain D (fig. 5.11) by

$$4 - 2x \le y \le 4 - x^2, \quad 0 \le x \le 2$$

Find

$$\iint_D (1 + x)\, dA$$

sol. This region is of third kind.

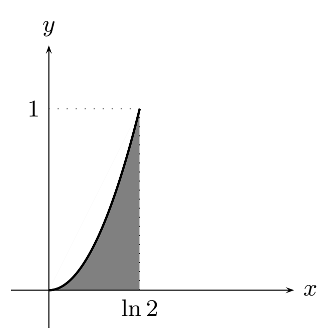


Figure 5.10: $0 \le y \le e^x - 1, \quad 0 \le x \le \ln 2$

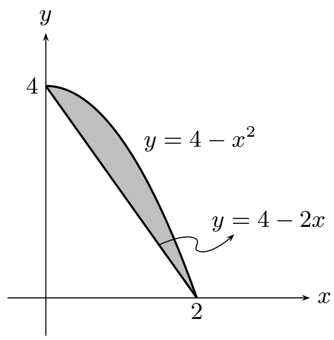


Figure 5.11: $4-2x \le y \le 4-x^2$

$$\begin{aligned}\int_0^2 \int_{4-2x}^{4-x^2} (1+x)\, dydx &= \int_0^2 [(1+x)y]_{y=4-2x}^{y=4-x^2}\, dx\\ &= \int_0^2 (-x^3+x^2+2x)\, dx\\ &= \left[-\frac{x^4}{4}+\frac{x^3}{3}+x^2\right]_0^2 = \frac{8}{3}.\end{aligned}$$

On the other hand, as a function of y $x=(4-y)/2$, $x=\sqrt{4-y}$. So

$$\begin{aligned}\int_0^4 \int_{(4-y)/2}^{\sqrt{4-y}} (1+x)\, dxdy &= \int_0^4 \left[x+\frac{x^2}{2}\right]_{x=(4-y)/2}^{x=\sqrt{4-y}} dy\\ &= \int_0^4 \left(\sqrt{4-y}-\frac{(4-y)^2}{8}\right) dy\\ &= \left[-\frac{2}{3}(4-y)^{3/2}+\frac{(4-y)^3}{24}\right]_0^4\\ &= \frac{2}{3}4^{3/2}-\frac{4^3}{24}=\frac{8}{3}.\end{aligned}$$

■

Example 5.2.21 (Breaking into several pieces)**.**

$$\int_D f\, dA = \int_{D_1} f\, dA + \int_{D_2} f\, dA + \int_{D_3} f\, dA + \int_{D_4} f\, dA.$$

See Figure 5.12

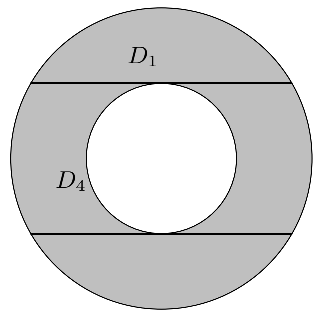


Figure 5.12: Region can be divided

5.3 Change order of integration

Suppose D is of type 3. Then it is given by two ways:

$$\phi_1(x) \le y \le \phi_2(x),\ a \le x \le b$$

and

$$\psi_1(y) \le x \le \psi_2(y),\ c \le y \le d.$$

Thus by Theorem 5.2.14

$$\iint_D f(x,y)dA = \int_a^b \int_{\phi_1(x)}^{\phi_2(x)} f(x,y)dydx = \int_c^d \int_{\psi_1(y)}^{\psi_2(y)} f(x,y)\,dxdy.$$

Example 5.3.1. Compute by change of order of integration

$$\int_0^a \int_0^{(a^2-x^2)^{1/2}} (a^2-y^2)^{1/2}\,dydx.$$

sol.

$$\begin{aligned}\int_0^a \int_0^{(a^2-x^2)^{1/2}} (a^2-y^2)^{1/2}\,dydx &= \int_0^a \int_0^{(a^2-y^2)^{1/2}} (a^2-y^2)^{1/2}\,dxdy \\ &= \int_0^a [x(a^2-y^2)^{1/2}]_0^{(a^2-y^2)^{1/2}} (a^2-y^2)^{1/2}\,dy \\ &= \int_0^a (a^2-y^2)\,dy = \frac{2a^3}{3}.\end{aligned}$$

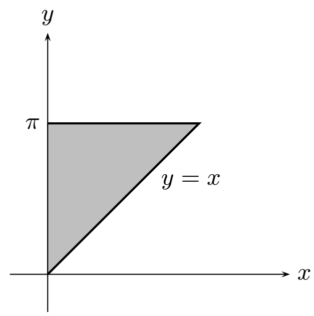


Figure 5.13: $0 \le x \le \pi$, $x \le y \le \pi$

There are cases when the given integral is almost impossible to find, but if we change the order the integral can be found.

Example 5.3.2. Find

$$\int_0^\pi \int_x^\pi \frac{\sin y}{y}\, dydx.$$

sol. It is not easy to find the integral as the given form. But if we change the order of integration (fig 5.13)

$$\begin{aligned}\int_0^\pi \int_x^\pi \frac{\sin y}{y}\, dydx &= \int_0^\pi \int_0^y \frac{\sin y}{y}\, dxdy \\ &= \int_0^\pi \left[\frac{\sin y}{y} x\right]_{x=0}^{x=y} dy \\ &= \int_0^\pi \sin y\, dy = [-\cos y]_0^\pi = 2.\end{aligned}$$

Example 5.3.3. Find

$$\int_0^1 \int_y^1 \frac{e^x - 1}{x}\, dx\, dy.$$

Example 5.3.4. Find

$$\int_0^2 \int_{x^2}^4 y\cos(x^2)\, dx\, dy.$$

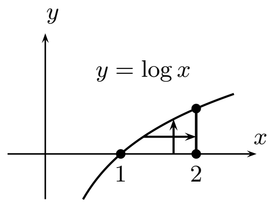


Figure 5.14: order of integration

sol. It is very difficult to find $\int_{y^2}^4 \cos(x^2)\,dx$. However, if we change the order of integration to have

$$\begin{aligned}\int_0^2 \int_{y^2}^4 \cos(x^2)\,dx\,dy &= \int_0^4 \int_0^{\sqrt{x}} y\cos(x^2)\,dy\,dx \\ &= \int_0^4 \frac{y^2}{2}\cos(x^2)\Big|_0^{\sqrt{x}}\,dx \\ &= \int_0^4 \frac{x}{2}\cos(x^2)\,dx \\ &= \frac{1}{4}\int_0^{16} \cos u\,du = \frac{1}{4}\sin 16.\end{aligned}$$

□

5.3.1 Mean value inequality

Theorem 5.3.5. *Suppose $f : D \to \mathbb{R}$ is continuous on an elementary region D and $m = \min_D f(x,y) \le f(x,y) \le M = \max_D f(x,y)$. Then we have*

$$mA(D) \le \iint_D f\,dA \le MA(D). \tag{5.7}$$

Example 5.3.6. Estimate

$$\int_D \frac{1}{\sqrt{1+x^6+y^7}} dxdy$$

where D is the unit square. Then we can easily see the following holds.

$$\frac{1}{\sqrt{3}} \le \frac{1}{\sqrt{1+x^6+y^7}} \le 1.$$

Theorem 5.3.7. *If f is continuous over D then there is a point (x_0, y_0) such* *that*

$$f(x_0, y_0) = \frac{1}{A(D)} \iint_D f.$$

Proof. Divide (**??**) by $A(D)$ to get

$$m \leq \frac{1}{A(D)} \iint_D f\, dA \leq M.$$

Since f assume the values m,M and f is continuous, (by intermediate value theorem) there is a point (x_0, y_0) such that

$$f(x_0, y_0) = \frac{1}{A(D)} \iint_D f$$

which is precisely the conclusion. □

5.4 Triple integrals

Assume $f(x, y, z)$ is defined on a box $B = [a, b] \times [c, d] \times [p, q]$.

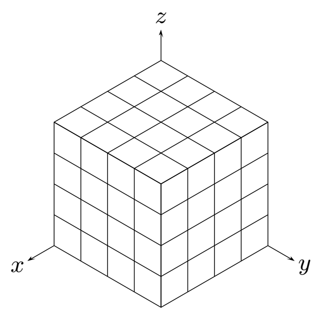


Figure 5.15: partition of box

Definition 5.4.1. Assume $R = [a, b] \times [c, d] \times [p, q]$ be a box. Then we subdivide intervals $[a, b]$, $[c, d]$ and $[p, q]$ into n -intervals

$$\boxed{\begin{aligned} &a = x_0 < x_1 < \cdots < x_n = b,\\ &c = y_0 < y_1 < \cdots < y_n = d,\\ &p = z_0 < y_1 < \cdots < z_n = q, \end{aligned}}$$

and call the resulting subboxes $B_{jk} = [x_{i-1}, x_i] \times [y_{j-1}, y_j] \times [z_{k-1}, z_k]$ a **partition** of R.

Definition 5.4.2. We let

$$\Delta x_i = x_i - x_{i-1}, \quad \Delta y_j = y_j - y_{j-1} \text{ and } \Delta z_k = z_k - z_{k-1}.$$

We partition the box into small n^3- boxes as in Fig 5.15, and denote the volume of each subbox as ΔV_{ijk} $(i, j, k = 1, \dots, n)$ and let $\|\mathcal{P}\| = \max_{i,j,k}\{\Delta x_i, \Delta y_j, \Delta z_k\}$. Then the Riemann sum becomes

$$\mathcal{R}(f, n) = S_n = \sum_{i,j,k=1}^{n} f(c_{ijk})\Delta V_{ijk}.$$

Here c_{ijk} is any point in the subbox B_{ijk}.

Definition 5.4.3. If $\lim_n S_n = S$ exists independently of the choice of c_{ijk}, then we say f is integrable in B and call S the **triple integral** and we write

$$\iiint_D f dV, \quad \iiint_D f(x, y, z) dV, \text{ or } \quad \iiint_D f(x, y, z) dxdydz.$$

Reduction to iterated integral

Theorem 5.4.4 (Fubini's theorem)**.** *Suppose f is continuous on $D = [a, b] \times [c, d] \times [p, q]$. Then the triple integral $\iiint_D f(x, y, z)dxdydz$ equals with any of the following integrals.*

$$\int_p^q \int_c^d \int_a^b f(x, y, z)\, dxdydz, \quad \int_p^q \int_a^b \int_c^d f(x, y, z)\, dydxdz$$
$$\int_a^b \int_c^d \int_p^q f(x, y, z)\, dzdydx, \text{ etc.}$$

Example 5.4.5. $B = [0, 1] \times [-\frac{1}{2}, 0] \times [0, \frac{1}{3}]$.

$$\iiint_B (x + 2y + 3z)^2 dxdydz = \frac{1}{12}.$$

Change the order and compute again to see the value does not change.

Example 5.4.6.

$$\int_B e^{x+y+z} dV,$$

where B is the unit cube at origin.

Elementary regions

Definition 5.4.7. A region D is **elementary regions** if the points lie between graph of continuous functions of two variables, and the domain of these functions is elementary. If f is continuous on D, then we extend f on a box E containing D

$$f^{ext}(x,y,z) = \begin{cases} f(x,y,z), & (x,y,z) \in D \\ 0, & (x,y,z) \in E \setminus D \end{cases}$$

and define

$$\int_D f dV = \int_E f^{ext} dV.$$

Suppose R is an elementary region in xy-plane and there are continuous functions $\gamma_1(x,y)$, $\gamma_2(x,y)$ such that

$$D = \{(x,y,z) \mid \gamma_1(x,y) \le z \le \gamma_2(x,y), \quad (x,y) \in R\}. \tag{5.8}$$

Then this is called an **elementary region of type 1**.

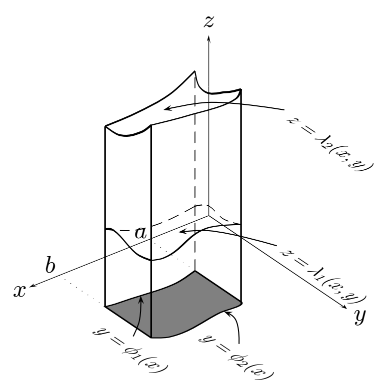


Figure 5.16: elementary region of type 1

If roles of x, z are interchanged, i.e,

$$D = \{(x,y,z) \mid \gamma_1(y,z) \le x \le \gamma_2(y,z), \quad (y,z) \in R\} \tag{5.9}$$

for some elementary region R in (y,z)-plane, then it is called an **elementary region of type 2**.

Similarly, we can define an **elementary region of type 3** and an **ele-**

mentary region of type 4.

Example 5.4.8. Describe the unit ball as an elementary region.

sol. The domain of defining function is described by

$$-\sqrt{1-x^2} \le y \le \sqrt{1-x^2}, \quad -1 \le x \le 1$$

while the functions are

$$-\sqrt{1-x^2-y^2} \le z \le \sqrt{1-x^2-y^2}, \quad \text{on unit disk.}$$

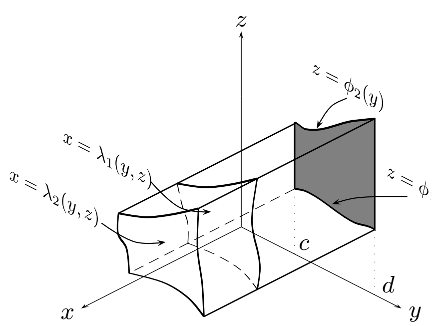


Figure 5.17: elementary region of 2

Integrals over elementary regions

Suppose D is defined by

$$D = \{(x, y, z) \mid \gamma_1(x, y) \le z \le \gamma_2(x, y), \quad (x, y) \in R\},$$

where R is a type 1 region in xy-plane

$$R = \{(x, y) \mid \phi_1(x) \le y \le \phi_2(x), \quad a \le x \le b\}.$$

Then the integral is given by

$$\iiint_D f\,dV = \int_R \int f(x,y,z)\,dzdA$$
$$= \int_a^b \int_{\phi_1(x)}^{\phi_2(x)} \int_{\gamma_1(x,y)}^{\gamma_2(x,y)} f(x,y,z)\,dzdydx.$$

Example 5.4.9. Find the volume of radius 1.

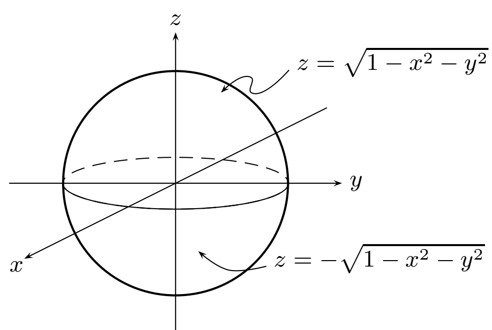


Figure 5.18: $x^2+y^2+z^2=1$

sol. Unit ball is described by $x^2+y^2+z^2 \le 1$. The volume is (fig 5.18)

$$\int_D 1\,dV, \quad D = \{(x,y,z) \mid x^2+y^2+z^2 \le 1\}$$

Here we can take $R = \{(x,y) \mid x^2+y^2 \le 1\}$ and $D = \{-\sqrt{1-x^2-y^2} \le z \le \sqrt{1-x^2-y^2}, (x,y) \in R\}$. Hence

$$\int_R \int dzdydx = \int_R \int_{z=-\sqrt{1-x^2-y^2}}^{z=\sqrt{1-x^2-y^2}} 1\,dzdydx$$
$$= 2\int_R \sqrt{1-x^2-y^2}\,dydx$$
$$= 2\int_{-1}^{1} \int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} \sqrt{1-x^2-y^2}\,dydx.$$

Let $\sqrt{1-x^2} = a$. The inner integral is area of semi circle or radius a

$$2\int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} \sqrt{1-x^2-y^2}\,dy = 2\int_{-a}^{a} \sqrt{a^2-y^2}dy = a^2\pi = (1-x^2)\pi.$$

Hence

$$2\int_{-1}^{1}\int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} \sqrt{1-x^2-y^2}\,dydx = \int_{-1}^{1}(1-x^2)\pi\,dx$$
$$= \left[(x-\frac{x^3}{3})\pi\right]_{-1}^{1} = 2(1-\frac{1}{3})\pi = \frac{4}{3}\pi.$$

Other type of elementary regions can be described similarly. If a region can be described in all three ways we call these regions **symmetric elementary regions**.

Example 5.4.10. Let D be the region bounded by $x+y+z=1$, $x=0$, $y=0$, $z=0$. Find

$$\iiint_D (1+2z)dxdydz.$$

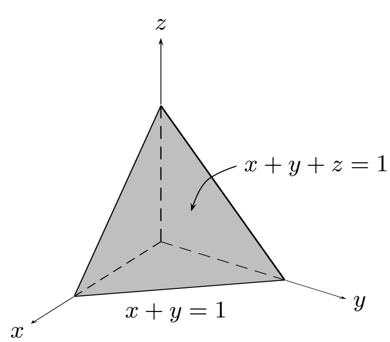


Figure 5.19: $x+y+z=1$

sol. Let $R = \{(x,y) \mid 0 \le y \le 1-x, \quad 0 \le x \le 1\}$. Then D is described by

$$D = \{(x,y,z) \mid 0 \le z \le 1-x-y, \quad (x,y) \in R\}$$

and integrate along z direction.

$$\begin{aligned}\iiint_D (1+2z)\,dxdydz &= \iint_R \left[z+z^2\right]_0^{1-x-y} dxdy \\ &= \int_0^1 \int_{y=0}^{y=1-x} (1-x-y+(1-x-y)^2)\,dydx \\ &= \int_0^1 \left[-\frac{(1-x-y)^2}{2} - \frac{(1-x-y)^3}{3}\right]_{y=0}^{y=1-x} dx \\ &= \int_0^1 \left(\frac{(1-x)^2}{2} + \frac{(1-x)^3}{3}\right) dx = \frac{1}{4}.\end{aligned}$$

■

Example 5.4.11. Let W be bounded by $x=0, y=0, z=2$ and the surface $z=x^2+y^2$ where $x \geq 0, y \geq 0$. Find $\iiint_W x\,dxdydz$.

sol. Method1. We describe the region by type 1.

$$0 \leq x \leq \sqrt{2}, \quad 0 \leq y \leq \sqrt{2-x^2}, \quad x^2+y^2 \leq z \leq 2.$$

$$\begin{aligned}\iiint_W x\,dxdydz &= \int_0^{\sqrt{2}} \left[\int_0^{\sqrt{2-x^2}} (\int_{x^2+y^2}^{2} x\,dz)dy\right] dx \\ &= \frac{8\sqrt{2}}{15}.\end{aligned}$$

Method2. We describe the region by type 2: Solving for x, i.e, $0 \leq x \leq (z-y^2)^{1/2}$, $(y,z) \in R$ where R is given by the relation

$$0 \leq z \leq 2, \quad 0 \leq y \leq z^{1/2}.$$

Then

$$\begin{aligned}\iiint_W x\,dxdydz &= \iint_R \left(\int_0^{(z-y^2)^{1/2}} xdx\right) dydz \\ &= \frac{8\sqrt{2}}{15}.\end{aligned}$$

■

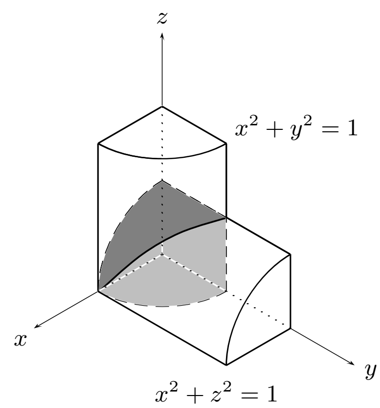


Figure 5.20: common region of two cylinders

Example 5.4.12. Evaluate

$$\int_0^1 \int_0^x \int_{x^2+y^2}^2 dzdydx.$$

Sketch region first.

sol. Sketch

■

Example 5.4.13. Find the common region of two cylinders (figure 5.20) $x^2 + y^2 \le 1$, $x^2 + z^2 \le 1$ ($z \ge 0$).

sol.

$$\begin{aligned}\iint_{x^2+y^2\le 1} \int_0^{\sqrt{1-x^2}} dzdxdy &= \int_{-1}^1 \int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} \sqrt{1-x^2}dydx \\ &= 2\int_{-1}^1 (1-x^2)dx \\ &= 2\left[x - \frac{x^3}{3}\right]_{-1}^1 = 4(1-\frac{1}{3}) = \frac{8}{3}.\end{aligned}$$

■

Example 5.4.14. Find the region bounded by two paraboloids $z = x^2 + y^2$ and $z = 2 - 3x^2 - y^2$.(figure 5.21)

sol. The intersection is the curve $x^2+y^2 = 2-3x^2-y^2$, i.e, $2x^2+y^2 = 1$. If we let $R = \{(x,y)\colon 2x^2+y^2 \le 1\}$ this region is 1st kind on R. Hence

$$\begin{aligned}\iiint_D dxdydz &= \iint_{2x^2+y^2\le 1} (2-3x^2-y^2)-(x^2+y^2)\,dxdy \\ &= \iint_{2x^2+y^2\le 1} (2-4x^2-2y^2)\,dxdy.\end{aligned}$$

Now use polar coordinate $x = r/\sqrt{2}\cos\theta, y = r/\sin\theta$. Then $dxdy = r/\sqrt{2}drd\theta$. Hence

$$\begin{aligned}&\frac{1}{\sqrt{2}}\int_0^{2\pi}\int_{r\le 1}(2-2r^2)r\,drd\theta \\ &= \frac{1}{\sqrt{2}}\int_0^{2\pi}\left[r^2-\frac{2r^4}{4}\right]_0^1 d\theta = \frac{\pi}{\sqrt{2}}.\end{aligned}$$

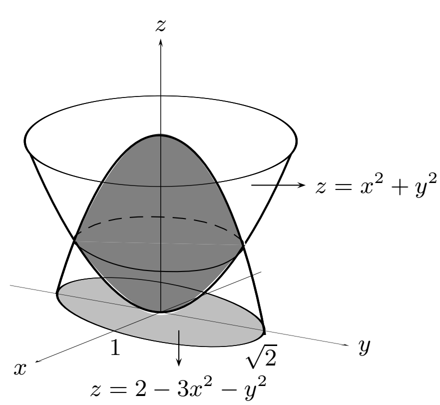


Figure 5.21: $z = x^2+y^2$, $z = 2-3x^2-y^2$

5.5 Change of variables

We recall one variable case: If $x : [a,b] \to [c,d]$ is $\mathcal{C}^1$ function and $f : [c,d] \to \mathbb{R}$ is integrable, then the integral of f on $[c,d]$ can be moved to an integral over

$[a, b]$ by

$$\int_c^d f(x)dx = \int_a^b f(x(t))x'(t)dt. \tag{5.10}$$

Here the change in the integrand is just the linear scaling factor $x'(t)$ and the change in the domain is again linear scaling to $[a, b]$. But for functions with two or more variables, the situation is not so simple, because the shape of domain change nontrivially. For example a simple domain like rectangle can be very complicated after mapping.

Coordinate transformations

Let D^* be a region in $\mathbb{R}^2$. Suppose T is C^1-map $D^* \to \mathbb{R}^2$. We denote the image by $D = T(D^*)$.(Fig 5.22)

$$T(D^*) = \{(x, y) \mid (x, y) = T(u, v), \quad (x^*, y^*) \in D^*\}.$$

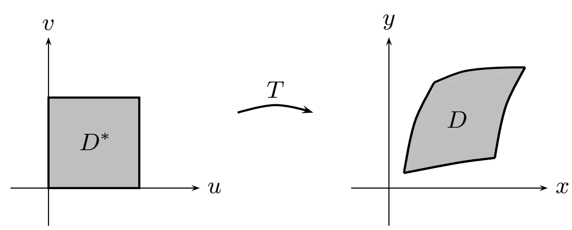


Figure 5.22: The transformation T maps D^* to D

Example 5.5.1. Let D^* be the rectangle $D^* = [0, 1] \times [0, 1]$ in (u, v) plane. Find the image of D^* under $T = T(u, v) = (2u + 1, 3v - 1)$.

Example 5.5.2. Let T be defined by $T(u, v) = ((u + v)/2, (u - v)/2)$ and $D^* = [-1, 1] \times [-1, 1]$. Determine the image $T(D^*)$.

sol. Let $D = T(D^*)$(see fig 5.24). Consider the effect of T on the line $\mathbf{c}_1(t) = (t, 1)$, $-1 \le t \le 1$.

$$T(\mathbf{c}(t)) = ((t + 1)/2, (t - 1)/2), \quad -1 \le t \le 1.$$

The image is the line segment $y = x - 1$, $-1 \le x \le 1$.

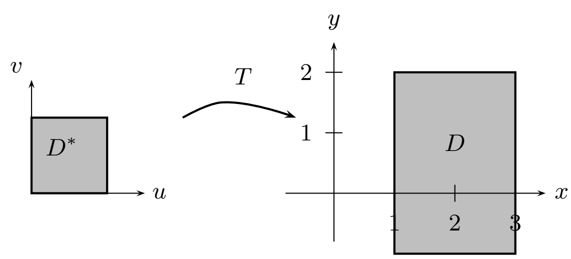


Figure 5.23: Simple Map

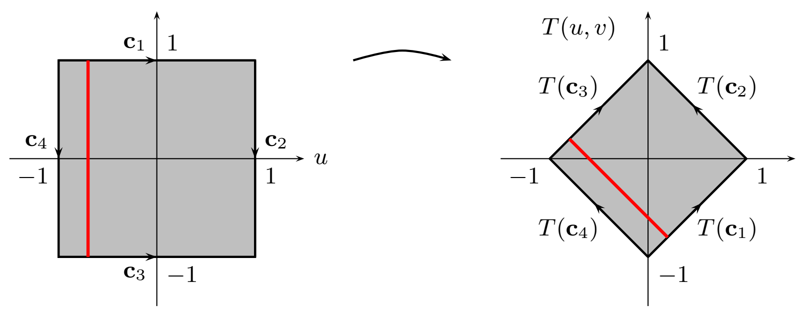


Figure 5.24: Effect of linear transform

Similarly, try to see the image of

$$\begin{aligned} \mathbf{c}_2(t) &= (1, t), \quad -1 \le t \le 1, \\ \mathbf{c}_3(t) &= (t, -1), \quad -1 \le t \le 1, \\ \mathbf{c}_4(t) &= (-1, t), \quad -1 \le t \le 1. \end{aligned}$$

The image is obtained by rotating and reducing by a factor of $\sqrt{2}$. To see indeed this is the case, we consider the image of the *red* line

$$\mathbf{c}(t) = (\alpha, t) - 1 \le t \le 1.$$

The image is

$$T(\mathbf{c}(t)) = ((\alpha + 1)/2, (\alpha - 1)/2), \quad -1 \le t \le 1.$$

Hence the image satisfies $x + y = \alpha$. This is the *red* line passing through inside the diamond shape.

■

Remark 5.5.3. Let $T(u,v) = (x,y) = ((u+v)/2, (u-v)/2)$. Then we have

$$x = (u+v)/2, \quad y = (u-v)/2.$$

Hence the map $T : \mathbb{R}^2 \to \mathbb{R}^2$ in Example 5.5.2 can be represented by a matrix

$$A = \frac{1}{2}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$$

$$T(u,v) := \begin{bmatrix} x \\ y \end{bmatrix} = A \begin{bmatrix} u \\ v \end{bmatrix}.$$

As for the area change, we have

Theorem 5.5.4. *Let A be a 2×2 matrix with non zero determinant. Let T be a linear transformation given by $T(\mathbf{x}) = A\mathbf{x}$. Then T maps a parallelogram D^* onto the parallelogram $D = T(D^*)$ and*

$$\text{Area of } D = |\det A| \cdot (\text{Area of } D^*).$$

Remark 5.5.5. (1) There is a similar statement about linear map from $\mathbb{R}^3$ to $\mathbb{R}^3$.

Example 5.5.6. Let T be $((x+y)/2, (x-y)/2)$ and let D be the square whose vertices are $(1,0), (0,1), (-1,0), (0,-1)$. Find a D^* such that $D = T(D^*)$.

sol. Since T is linear $T(\mathbf{x}) = A\mathbf{x}$ where A is 2×2 matrix whose determinant is nonzero. T^{-1} is also a linear transform. Hence by Theorem 5.5.4, D^* must be a parallelogram. So, to find D^*, it suffices to find the inverse image of vertices. It turns out that

$$D^* = [-1,1] \times [-1,1].$$

Now

$$A(D) = (\sqrt{2})^2 = 2, \ |\det A| = \frac{1}{2}, \ A(D^*) = 4, .$$

■

This idea can be generalized to non-linear mappings.

Change of variable in the definite integrals

Given two regions D and D^*, a differentiable mapping T on D^* with image $D = T(D^*)$, we would like to express the integral $\iint_D f(x,y)dxdy$ as an integral over D^* of the composite function $f \circ T$. Suppose we have a differentiable map

$$T(u,v) = (x(u,v), y(u,v)) \text{ for } (u,v) \in D^*.$$

As a special case, consider the case when $f = 1$. Then, in general

$$\iint_D dxdy = A(D) = |det\, A| \cdot A(D^*) \neq A(D^*) = \iint_{D^*} dudv.$$

Example 5.5.7. Let D^* be the rectangle $D^* = [0,1] \times [0, \pi/3]$ in (r,θ) plane. Find the image of D^* under $T = T(r,\theta) = (r\cos\theta, r\sin\theta)$.

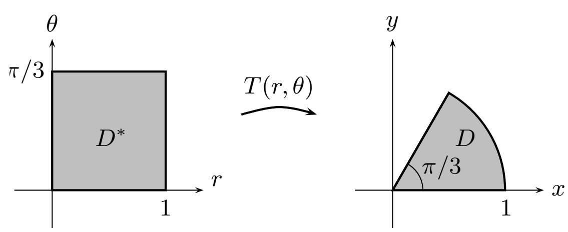


Figure 5.25: Map by Polar coordinate

sol. Let $T(r,\theta) = (x,y)$. Then $x^2 + y^2 = r^2$, $0 \le r \le 1$. Thus D is a circular sector $0 \le r \le 1$, $0 \le \theta \le \pi/3$. Figure 5.25.

□

One-to-one map

Definition 5.5.8. A map is called **one to one** on D^*, if for (u,v) and $(u',v') \in D^*$, $T(u,v) = T(u',v')$ implies $(u,v) = (u',v')$.

Example 5.5.9. Show the polar coordinate map $T = T(r,\theta) = (r\cos\theta, r\sin\theta)$ is not one-to-one. But the linear map in example 5.5.2 is one-to-one.

Onto map

Definition 5.5.10. A map T is called **onto** D, if for every point $(x,y) \in D$ there exists at least a point $(u,v) \in D$ such that $T(u,v) = (x,y)$.

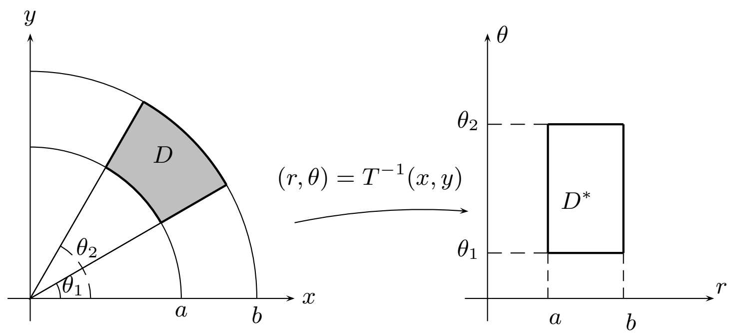


Figure 5.26: Inverse image of a polar rectangle

Thus if T is onto then we can solve the equation $T(u, v) = (x, y)$. If, in addition, T is one-to-one, the solution is unique.

Example 5.5.11. A linear transform from $\mathbb{R}^n$ to $\mathbb{R}^n$ given by a matrix A is **one to one** and **onto** if $\det A \neq 0$.

Example 5.5.12. Let D be the region in the first quadrant lying between concentric circles $r = a, r = b$ and $\theta_1 \leq \theta \leq \theta_2$. Let

$$T(r, \theta) = (r\cos\theta, r\sin\theta)$$

be the polar coordinate map. Find a region D^* in (r, θ) coordinate plane such that $D = T(D^*)$.

sol. In D, we see

$$a^2 \leq r^2 \leq b^2, \quad \theta_1 \leq \theta \leq \theta_2.$$

Hence

$$D^* = [a, b] \times [\theta_1, \theta_2].$$

■

Jacobian Determinant-measures change of area

We first see how the area of a region changes under a linear map. (Theorem 5.5.4.) First let us see an example. Let $D^* = [0, 1] \times [0, 1]$, and construct a map T that maps D^* onto D. Consider the vector $\mathbf{c}_1 := \mathbf{a}_2 - \mathbf{a}_1$, $\mathbf{c}_2 := \mathbf{a}_4 - \mathbf{a}_1$, and set

$$T(u, v) = \mathbf{c}_1 u + \mathbf{c}_2 v + \mathbf{c}_1.$$

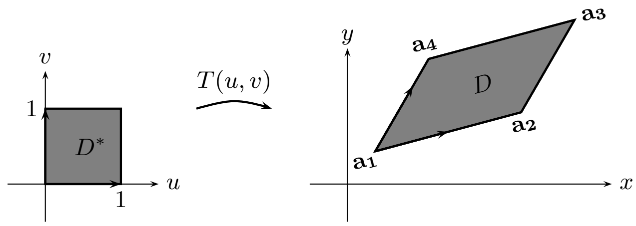


Figure 5.27: The image of a rectangle under a linear transform T

Then we can check $T(u,0)$ maps the line segment $\{0 \le u \le 1, v = 0\}$ to the side $\overline{\mathbf{a}_1\mathbf{a}_2}$. Similarly, $T(0,v)$ maps the line segment $\{0 \le v \le 1, u = 0\}$ to the side $\overline{\mathbf{a}_1\mathbf{a}_4}$. Hence we conclude T is the desired map. Now we see

$$\begin{aligned} T_u &= \mathbf{a}_2 - \mathbf{a}_1 \\ T_v &= \mathbf{a}_4 - \mathbf{a}_1. \end{aligned}$$

The area of the parallelogram D is $\|(\mathbf{a}_2 - \mathbf{a}_1) \times (\mathbf{a}_4 - \mathbf{a}_1)\|$.(viewed as three dimensional vectors) But this is nothing but the absolute value of the determinant of the derivative of T(at $(0,0)$). Thus

$$Area(D) = |J|,$$

where

$$J = \frac{\partial(x,y)}{\partial(u,v)} := det \begin{bmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{bmatrix} = |DT|.$$

J is called the **Jacobian**. Hence we see the rectangle of dimensions Δu, Δv along u, v direction is mapped to a parallelogram with area $|J|\Delta u \Delta v$.

Change of variable

Above idea of computing area of $D = T(D^*)$ can used when T is a differentiable (nonlinear) mapping from a subset of $\mathbb{R}^2$ to $\mathbb{R}^2$ by using the linear(tangent plane) approximation of T. Let $D^* = [u_0, u_0 + \Delta u] \times [v_0, v_0 + \Delta v]$ and D be the image of D^* under T. Consider

$$T(u,v) = \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x(u_0,v_0) + \frac{\partial x}{\partial u}(u_0,v_0)\Delta u + \frac{\partial x}{\partial v}(u_0,v_0)\Delta v + h.o.t \\ y(u_0,v_0) + \frac{\partial y}{\partial u}(u_0,v_0)\Delta u + \frac{\partial y}{\partial v}(u_0,v_0)\Delta v + h.o.t \end{bmatrix} \tag{5.11}$$

In vector form, we have

$$T\begin{bmatrix} u \\ v \end{bmatrix} = \mathbf{X} = \mathbf{X}_0 + DT\begin{bmatrix} \Delta u \\ \Delta v \end{bmatrix} + h.o.t$$

and replace the map T by its linear part DT.

Geometric meaning of DT

Let

$$T_u := DT(u,v)\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{bmatrix}\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} \frac{\partial x}{\partial u} \\ \frac{\partial y}{\partial u} \end{bmatrix}$$

and

$$T_v := DT(u,v)\begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{bmatrix}\begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial v} \end{bmatrix}.$$

First we note that the two curves $T(\cdot, v)$ and $T(u, \cdot)$ describes the boundary of $D = T(D^*)$ at $T(u,v)$. Fix a v. Then T_u is a tangent vector to the curve $T(u,v)$(as a function of u). Similarly for each fixed u, $T(u,v)$ represents a curve with v a parameter. Hence T_v us a tangent vector to the curve $T(u,v)$.

Now the tangent vectors

$$T_u\Delta u, \quad T_v\Delta v$$

form a parallelogram approximating the region D(figure 5.28). Hence the area of the parallelogram is

$$\begin{vmatrix} \frac{\partial x}{\partial u}\Delta u & \frac{\partial x}{\partial v}\Delta v \\ \frac{\partial y}{\partial u}\Delta u & \frac{\partial y}{\partial v}\Delta v \end{vmatrix} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix}\Delta u\Delta v = \frac{\partial(x,y)}{\partial(u,v)}\Delta u\Delta v.$$

$$\|T_u \times T_v\|\Delta u\Delta v = |J|\Delta u\Delta v.$$

Hence we obtain

Theorem 5.5.13. *If T is $D^* \to D \subset \mathbb{R}^2$, $\mathcal{C}^1$-map which is one to one, onto function, then the area of D is*

$$\iint_D dxdy = \iint_{D^*} |J|\, dudv. \tag{5.12}$$

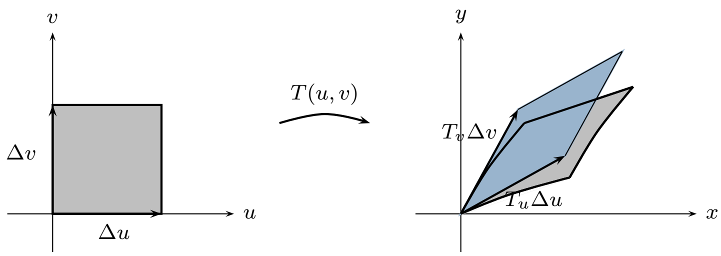


Figure 5.28: approximate $T(D^*)$

If f is a continuous function on D, the integral of f is given by

$$\iint_D f(x,y)\,dxdy = \iint_{D^*} f(x(u,v),y(u,v)) \left|\frac{\partial(x,y)}{\partial(u,v)}\right| dudv. \tag{5.13}$$

Remark 5.5.14. Since $|\det\, DT| = \|T_u \times T_v\|$, equation (5.12) becomes

$$\iint_D dxdy = \iint_{D^*} \|T_u \times T_v\| dudv. \tag{5.14}$$

As a special case, we may consider a polar coordinate. The shaded region of $D^* = \Delta r \times \Delta\theta$ is mapped by T to a part of circular sector between r and $r + \Delta r$, θ to $\theta + \Delta\theta$. The area of this sector is $r\Delta r\Delta\theta$. So the area under polar coordinate change is

$$\iint_{T(D^*)} dxdy = \iint_D r\,drd\theta.$$

Example 5.5.15. Change the integral $\iint f(x,y)\,dxdy$ to polar coordinate.

sol. Since $x = r\cos\theta$, $y = r\sin\theta$, we can let $T(r,\theta) = (r\cos\theta, r\sin\theta)$. Then Jacobian is

$$\begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} \end{vmatrix} = \begin{vmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{vmatrix} = r.$$

Hence

$$\iint f(x,y)\,dxdy = \iint f(r\cos\theta, r\sin\theta)\,rdrd\theta.$$

■

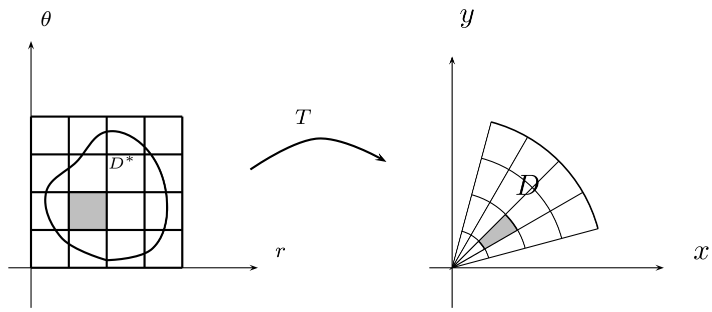


Figure 5.29: approximate $T(D^*)$

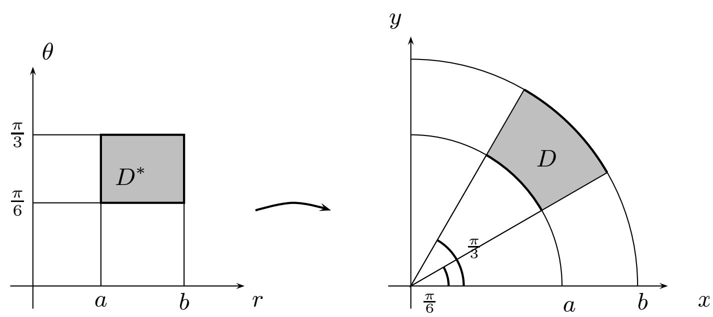


Figure 5.30: polar Coordinate

Example 5.5.16. D is between two concentric circles: $x^2 + y^2 = 4$, $x^2 + y^2 = 1 (x, y \geq 0)$. Find the integral

$$\iint_D \sqrt{x^2 + y^2 + 1}\, dxdy.$$

Here D is the quater disk $0 \leq y \leq \sqrt{4 - x^2}$.

sol. Use polar coordinate. The domain is

$$D^* = \{(r, \theta) | 0 \leq r \leq 2, 0 \leq \theta \leq \pi/2\}.$$

$$\begin{aligned}
\iint_D \sqrt{x^2+y^2+1}\,dxdy &= \iint_{D^*} \sqrt{r^2+1}r\,drd\theta \\
&= \int_0^{\pi/2}\int_1^2 2\sqrt{r^2+1}(2r)drd\theta \\
&= \int_0^{\pi/2} \frac{1}{3}(r^2+1)^{3/2}|_1^2 d\theta \\
&= \int_0^{\pi/2} \frac{1}{3}(5^{3/2}-2^{3/2})d\theta = \frac{\pi}{6}(5^{3/2}-2^{3/2}).
\end{aligned}$$

□

Example 5.5.17. D is the region between two concentric circles in the first quadrant: $1 \le x^2+y^2 \le 4$, $(x, y \ge 0)$. Find the integral

$$\iint_D \log(x^2+y^2)dxdy.$$

sol. Use polar coordinate. Since the boundary of the region are described by $r = 1, 2$, $0 \le \theta \le \pi/2$, we let $D^* = [1,2]\times[0,\pi/2]$ and $T(r,\theta) = (r\cos\theta, \sin\theta)$. Then $T(D^*) = D$ and

$$\begin{aligned}
\iint_D \log(x^2+y^2)dxdy &= \iint_{D^*} (\log r^2) r\,drd\theta \\
&= \int_1^2\int_0^{\pi/2} 2r\log r\,d\theta dr \\
&= \int_1^2 \pi r\log r\,dr \\
&= \pi\left[\frac{r^2}{2}\log r - \frac{r^2}{4}\right]_1^2 \\
&= \pi(2\log 2 - \frac{3}{4}).
\end{aligned}$$

□

Example 5.5.18. $D^* = \{(u,v)\colon 1 \le u^2+v^2 \le 4\}$ and T is given by

$$T(u,v) = \left(\frac{u}{u^2+v^2}, -\frac{v}{u^2+v^2}\right).$$

Find the area of $D = T(D^*)$.

sol. First compute T_u, T_v

$$T_u = \left(\frac{-u^2+v^2}{(u^2+v^2)^2}, \frac{2uv}{(u^2+v^2)^2} \right),$$
$$T_v = \left(\frac{-2uv}{(u^2+v^2)^2}, \frac{-u^2+v^2}{(u^2+v^2)^2} \right).$$

So

$$|J| = \|T_u \times T_v\| = \frac{1}{(u^2+v^2)^2}.$$

Hence area is

$$\begin{aligned}\iint_{T(D^*)} dxdy &= \iint_{D^*} |J|\, dudv \\ &= \iint_{D^*} \frac{dudv}{(u^2+v^2)^2} \\ &= \int_0^{2\pi} \int_1^2 \frac{1}{r^3}\, drd\theta \\ &= \frac{3\pi}{4}.\end{aligned}$$

The circle of radius r in D^* is $u^2+v^2 = r^2$. Hence the image satisfies

$$x^2+y^2 = \left(\frac{u}{u^2+v^2} \right)^2 + \left(\frac{-v}{u^2+v^2} \right)^2 = \frac{1}{u^2+v^2} = \frac{1}{r^2}.$$

It is circle of radius $1/r$ If $u = \alpha v$, $\alpha > 0$ represent a line through origin, the image is $x = -\alpha y$.

■

Example 5.5.19 (The Gaussian integral)**.** Show that

$$\int_{-\infty}^{\infty} e^{-x^2} dx = \sqrt{\pi}.$$

To compute this, let us first observe

$$\begin{aligned}\left(\int_{-\infty}^{\infty} e^{-x^2} dx \right)^2 &= \int_{-\infty}^{\infty} e^{-x^2} dx \int_{-\infty}^{\infty} e^{-y^2} dy \\ &= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{-(x^2+y^2)} dxdy \\ &= \lim_{a\to\infty} \iint_{D_a} e^{-(x^2+y^2)} dxdy.\end{aligned}$$

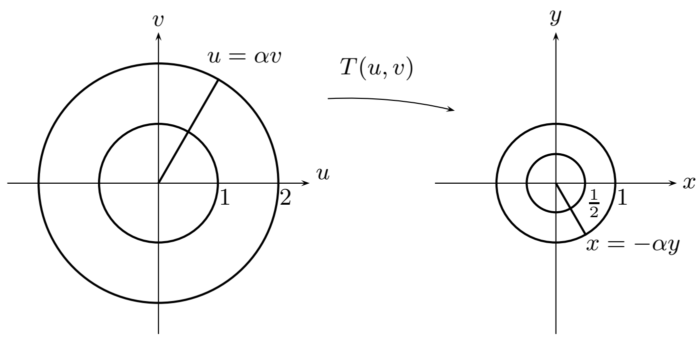


Figure 5.31: $T(u,v) = (u/(u^2+v^2), -v/(u^2+v^2))$

Thus it is necessary to compute

$$\iint_{D_a} e^{-(x^2+y^2)} dxdy.$$

By

$$\begin{aligned}\iint_{D_a} e^{-(x^2+y^2)} dxdy &= \int_0^{2\pi}\int_0^a e^{-r^2} r\,drd\theta = \int_0^{2\pi}\left(-\frac{1}{2}e^{-r^2}\right)\Big|_0^a \\ &= -\frac{1}{2}\int_0^{2\pi}(e^{-a^2}-1)d\theta = \pi(1-e^{-a^2}).\end{aligned}$$

Let $a \to \infty$. Then we obtain the result.

Change of Variables in Triple Integrals

Definition 5.5.20. Let $T : \mathbb{R}^3 \to \mathbb{R}^3$ be given by

$$T(u,v,w) = (x(u,v,w), y(u,v,w), z(u,v,w)).$$

The the determinant of the derivative DT is called **Jacobian** and denoted by J.

$$J = \frac{\partial(x,y,z)}{\partial(u,v,w)} = \det\begin{bmatrix}\frac{\partial x}{\partial u}, & \frac{\partial x}{\partial v}, & \frac{\partial x}{\partial w} \\ \frac{\partial y}{\partial u}, & \frac{\partial y}{\partial v}, & \frac{\partial y}{\partial w} \\ \frac{\partial z}{\partial u}, & \frac{\partial z}{\partial v}, & \frac{\partial z}{\partial w}\end{bmatrix}.$$

The absolute value of this determinant is equal to the volume of parallelepiped determ'd by the following vectors

$$\begin{aligned}\mathbf{T}_u &= \frac{\partial x}{\partial u}\mathbf{i} + \frac{\partial y}{\partial u}\mathbf{j} + \frac{\partial z}{\partial u}\mathbf{k}\\ \mathbf{T}_v &= \frac{\partial x}{\partial v}\mathbf{i} + + \frac{\partial y}{\partial v}\mathbf{j} + \frac{\partial z}{\partial v}\mathbf{k}\\ \mathbf{T}_w &= \frac{\partial x}{\partial w}\mathbf{i} + \frac{\partial y}{\partial w}\mathbf{j} + \frac{\partial z}{\partial w}\mathbf{k}.\end{aligned}$$

which is the absolute value of the triple product

$$|(\mathbf{T}_u \times \mathbf{T}_v) \cdot \mathbf{T}_w| = |J|.$$

Caution: Three vectors $\mathbf{T}_u, \mathbf{T}_v, \mathbf{T}_w$ are column vectors of DT, i.e.,

$$J = \frac{\partial(x,y,z)}{\partial(u,v,w)} = \det \begin{bmatrix} \mathbf{T}_u & \mathbf{T}_v & \mathbf{T}_w \end{bmatrix}.$$

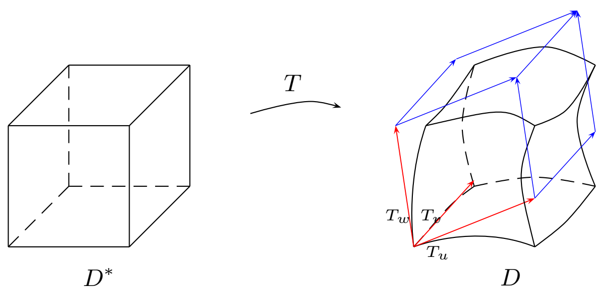


Figure 5.32: Deformed box and parallelepiped generated by tangent vectors.

Theorem 5.5.21. *If T is a $\mathcal{C}^1$- map from D^* onto D in $\mathbb{R}^3$ and $f : D \subset \mathbb{R}^3 \to \mathbb{R}$ is continuous, then*

$$\iiint_D dxdydz = \iiint_{D^*} |J|\, dudvdw, \tag{5.15}$$

$$\iiint_D f(x,y,z)\, dxdydz = \iiint_{D^*} f(T(u,v,w))|J|\, dudvdw, \tag{5.16}$$

where

$$J = \frac{\partial(x,y,z)}{\partial(u,v,w)}.$$

Cylindrical Coordinate

Let

$$x = r\cos\theta, \quad y = r\sin\theta, \quad z = z.$$

Then Jacobian is (compute)

$$J = \frac{\partial(x,y,z)}{\partial(r,\theta,z)} = r.$$

So

$$\iiint_D f(x,y,z)\,dxdydz \quad = \quad \iiint_{D^*} f(r\cos\theta, r\sin\theta, z)r\,drd\theta dz. \tag{5.17}$$

Spherical coordinate

sol. Spherical coordinate is given by

$$x = \rho\sin\phi\cos\theta, \quad y = \rho\sin\phi\sin\theta, \quad z = \rho\cos\phi.$$

The Jacobian of the mapping $(\rho,\phi,\theta) \to (x,y,z)$ is

$$\begin{aligned}
\frac{\partial(x,y,z)}{\partial(\rho,\phi,\theta)} &= \begin{vmatrix} \frac{\partial x}{\partial\rho} & \frac{\partial x}{\partial\phi} & \frac{\partial x}{\partial\theta} \\ \frac{\partial y}{\partial\rho} & \frac{\partial y}{\partial\phi} & \frac{\partial y}{\partial\theta} \\ \frac{\partial z}{\partial\rho} & \frac{\partial z}{\partial\phi} & \frac{\partial z}{\partial\theta} \end{vmatrix} \\
&= \begin{vmatrix} \sin\phi\cos\theta & \rho\cos\phi\cos\theta & -\rho\sin\phi\sin\theta \\ \sin\phi\sin\theta & \rho\cos\phi\sin\theta & \rho\sin\phi\cos\theta \\ \cos\phi & -\rho\sin\phi & 0 \end{vmatrix} \\
&= \cos\phi \begin{vmatrix} \rho\cos\phi\cos\theta & -\rho\sin\phi\sin\theta \\ \rho\cos\phi\sin\theta & \rho\sin\phi\cos\theta \end{vmatrix} \\
&+ \rho\sin\phi \begin{vmatrix} \sin\phi\cos\theta & -\rho\sin\phi\sin\theta \\ \sin\phi\sin\theta & \rho\sin\phi\cos\theta \end{vmatrix} \\
&= \rho^2\sin\phi(\cos^2\phi + \sin^2\phi) = \rho^2\sin\phi.
\end{aligned}$$

Hence

$$\iiint_D f(x,y,z)\,dxdydz = \iiint_{D^*} F(\rho,\phi,\theta)\rho^2\sin\phi\,d\rho\,d\phi\,d\theta.$$

Here $F(\rho,\phi,\theta)$ means $f(x(\rho,\phi,\theta), y(\rho,\phi,\theta), z(\rho,\phi,\theta))$. This agrees with earlier formula derived by geometric insight(below).

■

Spherical Coordinate-Geometric Derivation

Note that the surface given by (figure 5.33, 5.34)

$$\rho = c_1 \text{ is a sphere,} \tag{5.18}$$

$$\phi = c_3 \text{ is a cone,} \tag{5.19}$$

$$\theta = c_2 \text{ is a vertical plane.} \tag{5.20}$$

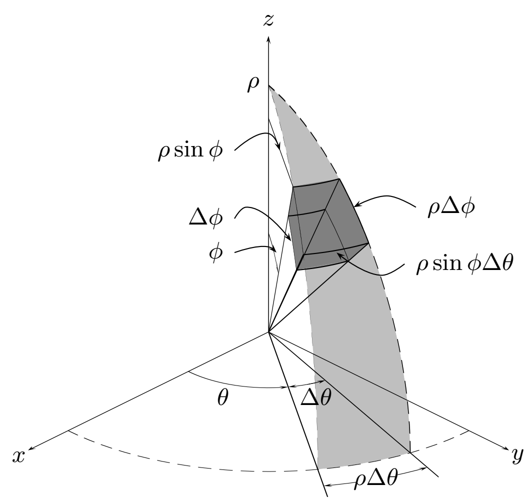


Figure 5.33: Partition in spherical coordinate

Consider the small region bounded by the following conditions:

$$\rho_0 \le \rho \le \rho_0 + \Delta\rho, \quad \theta_0 \le \theta \le \theta_0 + \Delta\theta, \quad \phi_0 \le \phi \le \phi_0 + \Delta\phi.$$

The region is the intersection of the region between two spheres of radius ρ, $\rho+\Delta\rho$, two cones $\phi = \phi_0$, $\phi = \phi_0 + \Delta\phi$ and two planes $\theta = \theta_0$, $\theta = \theta_0 + \Delta\theta$.

First let us find the area of the region bounded by $\theta_0 \le \theta \le \theta_0 + \Delta\theta$, $\phi_0 \le \phi \le \phi_0 + \Delta\phi$ on the sphere ρ. The distance from a point on the surface to the z-axis is $\rho \sin\phi$. And when $\Delta\rho$, $\Delta\theta$ are small, this region can

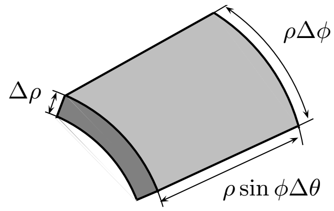


Figure 5.34: A piece

be approximated by rectangle whose area is (base is $\rho\sin\phi\Delta\theta$, height is $\rho\Delta\phi$)

$$\rho^2\sin\phi\Delta\phi\Delta\theta.$$

Now consider a solid with some thickness $\Delta\rho$. Then the volume is

$$\rho^2\sin\phi\Delta\rho\Delta\phi\Delta\theta.$$

Hence the volume of D is

$$\iiint_D dV = \int\int\int \rho^2\sin\phi\, d\rho\, d\phi\, d\theta \tag{5.21}$$

and if a continuous function f is given, the integral is defined as

$$\iiint_D f dV = \int\int\int f(\rho,\phi,\theta)\rho^2\sin\phi\, d\rho\, d\phi\, d\theta. \tag{5.22}$$

Example 5.5.22. The region D is given by

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}\le 1.$$

Find

$$\iiint_D |xyz|dxdydz.$$

sol. Let $T(u,v,w) = (au,bv,cw)$. Then T maps the unit ball $D^* = \{(u,v,w) \mid u^2+v^2+w^2\le 1\}$ to D one-to-one, onto fashion. Since $J(T) = abc$

we have

$$\begin{aligned}\iiint_D |xyz|dxdydz &= \iiint_{D^*} (abc)^2|uvw|\,du\,dv\,dw \\ &= 8\iiint_{D^*_+} (abc)^2 uvw\,du\,dv\,dw.\end{aligned}$$

Here D^*_+ denotes the region among D^* $u \geq 0, v \geq 0, w \geq 0$. Now use spherical coordinate,

$$\begin{aligned}&8\iiint_{D^*_+} (abc)^2 uvw\,du\,dv\,dw \\ =& 8(abc)^2 \int_0^{\pi/2}\int_0^{\pi/2}\int_0^1 \rho^5 \sin^3\phi\cos\phi\sin\theta\cos\theta\,d\rho\,d\phi\,d\theta \\ =& 8(abc)^2 \int_0^{\pi/2}\int_0^{\pi/2} \left[\frac{\rho^6}{6}\right]_0^1 \sin^3\phi\cos\phi\sin\theta\cos\theta\,d\phi\,d\theta \\ =& \frac{4}{3}(abc)^2 \int_0^{\pi/2} \left[\frac{\sin^4\phi}{4}\right]_0^{\pi/2} \sin\theta\cos\theta\,d\theta \\ =& \frac{1}{3}(abc)^2 \int_0^{\pi/2} \sin\theta\cos\theta d\theta \\ =& \frac{1}{3}(abc)^2 \left[\frac{\sin^2\theta}{2}\right]_0^{\pi/2} = \frac{1}{6}(abc)^2.\end{aligned}$$

■

Example 5.5.23. Compute

$$\iiint_W \exp(x^2+y^2+z^2)^{3/2} dV$$

where W is unit ball.

sol. By spherical coordinate,

$$\iiint_W \exp(x^2+y^2+z^2)^{3/2} dV = \iiint_{W^*} \rho^2 e^{\rho^3} \sin\phi d\theta\,d\phi\,d\rho.$$

Changing it to an iterated integral, we have

$$\begin{aligned} & \int_0^1 \int_0^\pi \int_0^{2\pi} \rho^2 e^{\rho^3} \sin\phi d\theta\, d\phi\, d\rho \\ = \;& 2\pi \int_0^1 \int_0^\pi \rho^2 e^{\rho^3} \sin\phi d\phi\, d\rho \\ = \;& 4\pi \int_0^1 \int_0^\pi \rho^2 e^{\rho^3} d\rho = \frac{4}{3}\pi(e-1). \end{aligned}$$

■

5.6 Application

Average

In $\mathbb{R}^n (n = 1, 2, 3)$, the average of a function f defined on $I($ D or $W)$is defined as

$$f_{av} = \frac{\int_a^b f(x)dx}{\int_a^b dx} = \frac{\int_a^b f(x)dx}{\text{length of } [a,b]}, \tag{5.23}$$

$$f_{av} = \frac{\iint_D f(x,y)dxdy}{\iint_D dA} = \frac{\iint_D f(x,y)dxdy}{\text{area of } D}, \tag{5.24}$$

$$f_{av} = \frac{\iiint_W f(x)dx}{\iiint_W dV} = \frac{\iiint_W f(x)dx}{\text{volume of } W}. \tag{5.25}$$

Example 5.6.1. Find average of $f(x, y) = x \sin^2(xy)$ over $D = [0, \pi] \times [0, \pi]$.

Example 5.6.2. The temperature at points in the cube is proportional to the square of distance from the origin.

(1) Find average temperature

(2) At which point is the temperature equal to the average temperature?

sol. $T = c(x^2 + y^2 + z^2)$. So $[T]_{av} = \frac{1}{8} \iiint_W T dV = c$.

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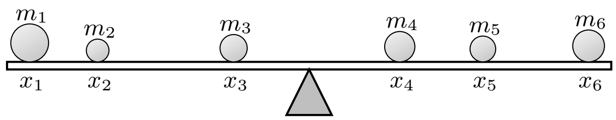

Figure 5.35: Moment

Center of Mass-discrete mass

If masses $m_1, \cdots, m_n$ are placed at points $x_1, \cdots, x_n$ on x-axis, the center of mass is defined to be

$$\bar{x} = \frac{\sum_{i=1}^n m_i x_i}{\sum_{i=1}^n m_i}.$$

The center of mass is a point where the total moment w.r.t that point is zero. i.e, $\sum_{i=1}^n m_i(x_i - \bar{x}) = 0$.

Moment in 2D

In 2 D, there are two kind of moments, i.e, Moment w.r.t x-axis, and the moment w.r.t y-axis.

Let Δm_i be the masses whose distance from the axis of our interest is x_i. Then the moment is

$$\sum x_i \Delta m_i.$$

In the limit,

$$\iint x dm.$$

Definition 5.6.3. Given n- masses $m_1, \cdots, m_n$ lying at points

$$(x_1, y_1), (x_2, y_2), \cdots, (x_n, y_n) \text{ in } \mathbb{R}^2,$$

The center of mass is the point $(\bar{x}, \bar{y})$, where

$$\bar{x} = \frac{\sum_{i=1}^n m_i x_i}{\sum_{i=1}^n m_i} \text{ and } \bar{y} = \frac{\sum_{i=1}^n m_i y_i}{\sum_{i=1}^n m_i}.$$

Center of Mass-continuous mass

When material is placed continuously on an axis, density is $\delta(x)$, the mass on $[x, x+\Delta x]$ is $\delta(x)\Delta x$ and moment is

$$\int x\delta(x)\,dx$$

We choose a point $\bar{x}$ so that the moment w.r.t $\bar{x}$ is zero.

$$\int (x-\bar{x})\delta(x)\,dx = 0 \Rightarrow \bar{x} = \frac{\int x\delta(x)\,dx}{\int \delta(x)\,dx}.$$

For 2-D, we have

Definition 5.6.4 (Moment, center of mass)**.** Let $\delta(x, y)$ be the density of some material we are interested in. The mass of this material occupying the place $[x, x+\Delta x] \times [y, y+\Delta y]$ is

$$\begin{aligned}
\text{The mass is } M &= \iint \delta(x,y)dxdy \\
\text{The moment w.r.t } x\text{-axis is } M_x &= \iint_R y\delta(x,y)\,dxdy \\
\text{The moment w.r.t } y\text{-axis is } M_y &= \iint_R x\delta(x,y)\,dxdy.
\end{aligned}$$

The **center of mass** is defined as

$$\begin{aligned}
\bar{x} = \frac{M_{\bar{y}}}{M} &= \frac{\iint_R x\delta(x,y)\,dxdy}{\iint_R \delta(x,y)\,dxdy}, \\
\bar{y} = \frac{M_{\bar{x}}}{M} &= \frac{\iint_R y\delta(x,y)\,dxdy}{\iint_R \delta(x,y)\,dxdy}.
\end{aligned}$$

The center of mass is defined so that it satisfies

$$\begin{aligned}
M_{\bar{x}} &= \iint_R (x-\bar{x})\delta(x,y)\,dxdy = 0, \\
M_{\bar{y}} &= \iint_R (x-\bar{y})\delta(x,y)\,dxdy = 0.
\end{aligned}$$

Example 5.6.5. A solid body occupies the region between $y = x$, $y = x^2$. The density is given by $\delta(x, y) = x$. Find the mass and M_x.

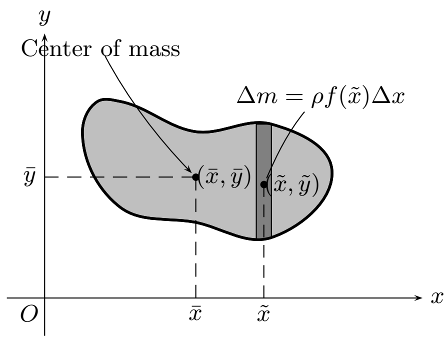

Figure 5.36: Vertical strip of mass Δm

sol. Mass is

$$M = \int_0^1 \int_{x^2}^x x\,dydx = \int_0^1 x\,[y]_{y=x^2}^{y=x}\,dx$$
$$= \int_0^1 (x^2 - x^3)dx = \left[\frac{x^3}{3} - \frac{x^4}{4}\right]_0^1 = \frac{1}{12}$$

and M_x is

$$M_x = \int_0^1 \int_{x^2}^x yxdydx = \int_0^1 x\left[\frac{y^2}{2}\right]_{y=x^2}^{y=x} dx$$
$$= \int_0^1 \frac{x}{2}(x^2 - x^4)dx = \left[\frac{x^4}{8} - \frac{x^6}{12}\right]_0^1 = \frac{1}{24}.$$

When the density $\delta = 1$, the center of mass is also called the **centroid**.

Example 5.6.6. Find the centroid of the region bounded by $y = x$, $y = x^2$.

sol.

$$M = \int_0^1 \int_{x^2}^x 1dydx = \int_0^1 [y]_{x^2}^x\,dx = \int_0^1 (x - x^2)dx = \frac{1}{6},$$
$$M_x = \int_0^1 \int_{x^2}^x ydydx = \int_0^1 \left[\frac{y^2}{2}\right]_{x^2}^x dx = \int_0^1 (\frac{x^2}{2} - \frac{x^4}{2})dx = \frac{1}{15},$$
$$M_y = \int_0^1 \int_{x^2}^x xdydx = \int_0^1 x\,[y]_{x^2}^x\,dx = \int_0^1 (x^2 - x^3)dx = \frac{1}{12}.$$

Hence

$$\bar{x} = \frac{1/12}{1/6} = \frac{1}{2}, \quad \bar{y} = \frac{1/15}{1/6} = \frac{2}{5}.$$