Chapter 3

Vector valued functions

In this chapter we study two types of special functions:

(1) Continuous mapping of one variable(called a **path**)

(2) Mapping from a subset of $\mathbb{R}^n$ to itself(called **vector fields**)

3.1 Parametrized curves

Definition 3.1.1. A **path** is a continuous function $\mathbf{x} : I = [a, b] \to \mathbb{R}^n$. It is a parameterized curve. $\mathbf{x}(a)$ and $\mathbf{x}(b)$ are called the **endpoints** of the path. A

A parameterized curve $\mathbf{c}$ in $\mathbb{R}^2$ or $\mathbb{R}^3$ can be written as $\mathbf{c}(t) = (x(t), y(t), z(t))$. If $x(t), y(t), z(t)$ are differentiable, then $\mathbf{c}$ is said to be differentiable. If $x'(t)$, $y'(t)$, $z'(t)$ are continuous then we say $\mathbf{c}$ is C^1-curve. A path may have many **parametrization**.

Example 3.1.2. (1) $\mathbf{x}(t) = \mathbf{a} + t\mathbf{b}$ is a line

(2) $\mathbf{x}(t) = (\cos t, \sin t)$ on $[0, 2\pi]$ is path traveling a circle once. If the domain is $[0, 4\pi]$, it travels twice.

(3) $\mathbf{x}(t) = (a \cos t, a \sin t, bt)$ defines a circular helix.

We distinguish between a path $\mathbf{x}$ and its range (the image set $\mathbf{x}(I)$). The image of a path is called a **curve**, while the path is a function describing the curve. The **velocity** of the path $\mathbf{x}$ is the vector

$$\mathbf{v}(t) = \mathbf{x}'(t).$$

Definition 3.1.3. Let $\mathbf{x}$ be a differentiable path. Then the velocity $\mathbf{v}(t) = \mathbf{x}'(t)$ exists and define the speed of $\mathbf{x}$ to be

$$\|\mathbf{v}(t)\|.$$

Also, $\mathbf{a}(t) = \mathbf{v}'(t) = \mathbf{x}''(t)$ is called the **acceleration** vector.

Proposition 3.1.4. *Let* $\mathbf{x}$ *be a differentiable path and assume* $\mathbf{v}_0 = \mathbf{v}(t_0) \neq 0$. *The tangent line to the path is given by*

$$\ell(t) = \mathbf{x}_0 + (t - t_0)\mathbf{v}_0. \tag{3.1}$$

Example 3.1.5 (Throwing a ball)**.** Assume a baseball a player throws a ball with a speed 20 m/sec in the direction of $(\cos 30^o, \sin 30^o)$. Describe the trajectory.

sol. The acceleration is

$$\mathbf{a}(t) = \mathbf{x}''(t) = -g\mathbf{j}$$

where $g = 9.8m^2/sec$ is the gravity constant. Hence

$$\mathbf{v}(t) = -gt\mathbf{j} + \mathbf{c}$$

for some constant vector $\mathbf{c}$. Integrating, we obtain

$$\mathbf{x}(t) = -\frac{1}{2}gt^2\mathbf{j} + \mathbf{c}t + \mathbf{d}.$$

Since the initial velocity is $\mathbf{v}(0) = \mathbf{c} = 20(\cos 30^o, \sin 30^o)$, we have

$$\mathbf{x}(t) = -\frac{1}{2}gt^2\mathbf{j} + 10\sqrt{3}\mathbf{i} + 10\sqrt{3}\mathbf{j} + \mathbf{d},$$

where $\mathbf{d}$ is the initial position of the ball.

□

Differentiation Rues

(1) $\frac{d}{dt}[\mathbf{b}(t) + \mathbf{c}(t)] = \mathbf{b}'(t) + \mathbf{c}'(t)$

(2) $\frac{d}{dt}[p(t)\mathbf{c}(t)] = p'(t)\mathbf{c}(t) + p(t)\mathbf{c}'(t)$ for any differentiable scalar function $p(t)$

(3) $\frac{d}{dt}[\mathbf{b}(t)\cdot\mathbf{c}(t)] = \mathbf{b}'(t)\cdot\mathbf{c}(t) + \mathbf{b}(t)\cdot\mathbf{c}'(t)$

(4) $\frac{d}{dt}[\mathbf{b}(t)\times\mathbf{c}(t)] = \mathbf{b}'(t)\times\mathbf{c}(t) + \mathbf{b}(t)\times\mathbf{c}'(t)$

(5) $\frac{d}{dt}[\mathbf{c}(q(t))] = q'(t)\mathbf{c}'(q(t))$

Example 3.1.6. Show that if $\mathbf{c}(t)$ is a vector function such that $\|\mathbf{c}(t)\|$ is constant, then $\mathbf{c}'(t)$ is perpendicular to $\mathbf{c}(t)$ for all t.

Solution:
$\|\mathbf{c}(t)\|^2 = \mathbf{c}(t)\cdot\mathbf{c}(t)$. Derivative of constant is zero. Hence

$$0 = \frac{d}{dt}[\mathbf{c}(t)\cdot\mathbf{c}(t)] = \mathbf{c}'(t)\cdot\mathbf{c}(t) + \mathbf{c}(t)\cdot\mathbf{c}'(t) = 2\mathbf{c}(t)\cdot\mathbf{c}'(t)$$

Thus $\mathbf{c}'(t)$ is perpendicular to $\mathbf{c}(t)$.

□

Example 3.1.7. A particle moves with a constant acceleration $\mathbf{a}(t) = -\mathbf{k}$. When $t = 0$ is the position is $(0, 0, 1)$ and velocity is $\mathbf{i}+\mathbf{j}$. Describe the motion of the particle.

sol. Let $(x(t), y(t), z(t))$ represents the path traveled by the particle. Since the acceleration is $\mathbf{c}''(t) = -\mathbf{k}$ we see the velocity is

$$\mathbf{c}'(t) = C_1\mathbf{i} + C_2\mathbf{j} - t\mathbf{k} + C_3\mathbf{k}.$$

Hence by initial condition, $\mathbf{c}'(t) = \mathbf{i} + \mathbf{j} - t\mathbf{k}$ and so $\mathbf{c}(t) = t\mathbf{i} + t\mathbf{j} - \frac{t^2}{2}\mathbf{k} + Const\ vec$. The constant vector is $\mathbf{k}$. Hence $\mathbf{c}(t) = t\mathbf{i} + t\mathbf{j} + (1 - \frac{t^2}{2})\mathbf{k}$.

□

Remark 3.1.8. The image of C^1-curve is not necessarily "smooth". it may have sharp edges;

(1) Cycloid: $\mathbf{c}(t) = (t - \sin t, 1 - \cos t)$ has cusps when it touches x-axis. That is, when $\cos t = 1$ or when $t = 2\pi n, n = 1, 2, 3, \cdots$

(2) Hypocycloid: $\mathbf{c}(t) = (\cos^3 t, \sin^3 t)$ has cusps at four points when $\cos t = 0, \pm 1$

At all these points, we can check that $\mathbf{c}'(t) = 0$.(Roughly speaking, tangent vector has no direction or does not exist.)

Definition 3.1.9. A differentiable path $\mathbf{c}$ is said to be **regular** if $\mathbf{c}'(t) \neq 0$ at all t. In this case, the image curve looks smooth.

Circular Orbits

Consider a particle of mass m moving at constant speed s in a circular path of radius r_0. We can represent its motion (in the plane) as

$$\mathbf{r}(t) = (r_0 \cos Ct, r_0 \sin Ct)$$

Find C. Since speed is $\|\mathbf{r}'(t)\| = |C|r_0 = s$, we get $C = s/r_0$. So the motion is described as

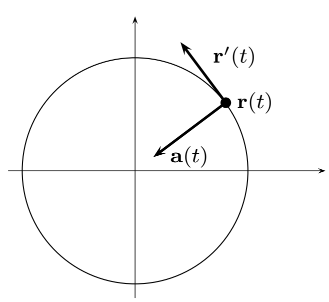
$$\mathbf{r}(t) = \left(r_0 \cos \frac{st}{r_0}, r_0 \sin \frac{st}{r_0}\right)$$

The quantity $\frac{s}{r_0}$ is called **frequency** denoted by ω. Thus

$$\mathbf{r}(t) = (r_0 \cos \omega t, r_0 \sin \omega t)$$

It's acceleration is

$$\mathbf{a}(t) = \mathbf{r}''(t) = -\frac{{}^2st}{r_0}\left(r_0 \cos \frac{st}{r_0}, r_0 \sin \frac{st}{r_0}\right) = -\frac{s^2t}{r_0^2}\mathbf{r}(t) = -\omega^2\mathbf{r}(t).$$



$\mathbf{a} \perp \mathbf{r}'(t)$

Figure 3.1: acceleration and centripetal force $m\mathbf{a}$,

Let us describe motion of a particle having circular motion. The centripetal force must equal to Gravitational force; By Newton's Law $\mathbf{F} = m\mathbf{a}$, we have

$$-\frac{s^2 m}{r_0^2}\mathbf{r}(t) = -\frac{GmM}{r_0^3}\mathbf{r}(t).$$

Hence

$$s^2 = \frac{GM}{r_0}.$$

If T denotes the period $s = 2\pi r_0/T$; then we obtain

Kepler's Law

$$T^2 = r_0^3\frac{(2\pi)^2}{GM}.$$

It means that the square of the period is proportional to the cube of the radius.

Example 3.1.10. Suppose a satellite is in circular motion about the earth over the equator. What is the radius of *geosynchronous* orbit? (It stays fixed over a point on equator) $M = 5.98 \times 10^{24}$ kg and $G = 6.67 \times 10^{-11}$ meter kg -sec.

Sol. Period must be one day: So $T = 60 \times 60 \times 24 = 86,400$ seconds. From Kepler's law,

$$r_0^3 = \frac{T^2 GM}{(2\pi)^2} \approx 7.54 \times 10^{22} m^3 \approx 42,300 km.$$

3.2 Arc Length

To find the length of a path, we divide the path into small pieces and approximate each piece by a line segment joining the end points; then summing the length of individual line segments we obtain an approximate length. The length is obtained by taking the limit. To define it precisely, we use the Riemann integral.

First the sum of the line segment is

$$\sum_{i=1}^{k} \Delta s_i = \sum_{i=1}^{k} \|\mathbf{c}(t_i) - \mathbf{c}(t_{i-1})\|.$$

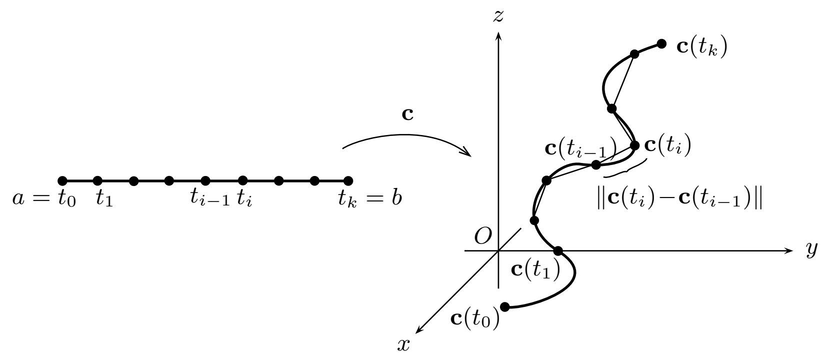


Figure 3.2: Riemann sum of the curve length

Then by mean value theorem, there are points $u_i \in [t_{i-1} - t_i]$ such that

$$\sum_{i=1}^{k} \|\mathbf{c}'(u_i)\|(t_i - t_{i-1}).$$

This is nothing but the Riemann sum of $\|\mathbf{c}'(t)\|$. As the norm of the partition $\|P\| \to 0$, [1] we see the sum $\sum_{i=1}^{k} \|\mathbf{c}'(u_i)\|(t_i - t_{i-1})$ approaches

$$\int_a^b \|\mathbf{c}'(t)\|\, dt.$$

Definition 3.2.1 (Arc Length)**.** Suppose a curve C has one-to-one differentiable parametrization $\mathbf{x}$. Then the **arc length** is defined by

$$L(\mathbf{x}) = \int_a^b \|\mathbf{x}'(t)\|\, dt = \int_a^b \sqrt{x'(t)^2 + y'(t)^2 + z'(t)^2} dt.$$

Example 3.2.2. Find the arclength of the helix $\mathbf{x} = (a\cos t, a\sin t, bt)$, $0 \le t \le 2\pi$.

Sol.

$$\|\mathbf{x}'(t)\| = \|a\cos t\mathbf{i} - a\sin t\mathbf{j} + b\mathbf{k}\| = \sqrt{a^2 + b^2}.$$

Hence

$$L(\mathbf{x}) = \int_0^{2\pi} \sqrt{a^2 + b^2} dt = 2\pi\sqrt{a^2 + b^2}.$$

[1]For a given partition $P = \{t_0, t_1, \ldots, t_n\}$, the norm of $\|P\| := \max_{1 \le i < n}(t_i - t_{i-1})$.

Thus

$$L(\mathbf{c}) = \cdots$$

Example 3.2.3. Find the arclength of $(\cos t, \sin t, t^2)$ $0 \le t \le 2\pi$.

Sol.

$$\|\mathbf{v}\| = \sqrt{1+4t^2} = 2\sqrt{t^2+\frac{1}{4}}.$$

To evaluate this integral we need a table of integrals:

$$\int \sqrt{x^2+a^2}\,dx = \frac{1}{2}[x\sqrt{x^2+a^2} + a^2\log(x+\sqrt{x^2+a^2})] + C.$$

Thus

$$L(\mathbf{x}) = \cdots .$$

Example 3.2.4. Find the length of the cycloid

$$\mathbf{x}(t) = (t-\sin t, 1-\cos t).$$

Since

$$\|\mathbf{x}'(t)\| = \sqrt{(t-\sin t)^2 + (1-\cos t)^2} = \sqrt{2-2\cos t}$$

$$\begin{aligned} L(\mathbf{x}) &= \int_0^{2\pi} \sqrt{2-2\cos t}dt = 2\int_0^{2\pi}\sqrt{\frac{1-\cos t}{2}}\,dt \\ &= 2\int_0^{2\pi} \sin\frac{t}{2}\,dt \\ &= 4\left(-\cos\frac{t}{2}\right)\Big|_0^{2\pi} = 8. \end{aligned}$$

Example 3.2.5. Suppose a function $y = f(x)$ given. Then the graph is viewed as a curve parameterized by $t = x$ and $\mathbf{x}(x) = (x, f(x))$. So the length of the graph from a to b is

$$L(\mathbf{x}) = \int_a^b \sqrt{1+(f'(x))^2}dx.$$

Warning! A continuous path may not have finite length: Consider

$$\begin{cases} (t, t\sin\frac{1}{t}), & t \ne 0, \\ (0,0), & t = 0. \end{cases}$$

This function is continuous everywhere. But does not have finite length.

Arc-Length Parameter

Definition 3.2.6. Suppose $\mathbf{x} : [a, b] \to \mathbb{R}^n$ is a C^1-parametrization of a curve C. Then the **arc length** of C is defined by

$$L(\mathbf{x}) = \int_a^b \|\mathbf{x}'(t)\|\, dt = \int_a^b \sqrt{x'(t)^2 + y'(t)^2 + z'(t)^2} dt.$$

Now if we treat upper limit of the integral as a variable t, the arclength becomes a function **arc-length function** $s(t)$:

$$s(t) = \int_a^t \|\mathbf{x}'(t)\|\, dt.$$

The arc-length function satisfies

$$\frac{ds}{dt} = s'(t) = \|\mathbf{x}'(t)\| = \textit{speed}.$$

Assuming $\mathbf{x}'(t) \neq 0$, we see $\frac{ds}{dt}$ is always positive. Hence we can solve for s in terms of t(inverse function theorem). Hence we can use s as a new variable to represent the curve C.

Example 3.2.7. For the helix $\mathbf{x} = (a\cos t, a\sin t, bt)$, we can find a new parametrization by s as follows:

$$s(t) = \int_0^t \|\mathbf{x}'(\tau)\|\, d\tau = \int_0^t \sqrt{a^2 + b^2} d\tau = \sqrt{a^2 + b^2}\ t,$$

so that

$$s = \sqrt{a^2 + b^2}\ t, \text{ or } t = \frac{s}{\sqrt{a^2 + b^2}}.$$

Hence

$$\mathbf{x}(s) = \left(a\cos\left(\frac{s}{\sqrt{a^2 + b^2}}\right), a\sin\left(\frac{s}{\sqrt{a^2 + b^2}}\right), \frac{bs}{\sqrt{a^2 + b^2}}\right).$$

Example 3.2.8. In general finding a parametrization by arclength parameter s is not a simple task. However, it has important meaning: Assume $\mathbf{x}(s)$ be a parametrization by arclength parameter. Then by chain rule and property of arclength parameter,

$$\mathbf{x}'(t) = \mathbf{x}'(s)\frac{ds}{dt}$$
$$= \mathbf{x}'(s)\|\mathbf{x}'(t)\|.$$

Since $\|\mathbf{x}'(t)\| \neq 0$, we have

$$\mathbf{x}'(s) = \frac{\mathbf{x}'(t)}{\|\mathbf{x}'(t)\|}.$$

. Thus $\mathbf{x}(s)$ has always unit speed($\mathbf{x}'(s)$ always has a unit length). Note that the arc length parameter s depends on how the curve bends, not how fast the curve is traced.

Definition 3.2.9. The **unit tangent vector T** of the path $\mathbf{x}$ is the normalized velocity vector

$$\mathbf{T} == \frac{\mathbf{x}'(t)}{\|\mathbf{x}'(t)\|}.$$

Example 3.2.10. For the helix $\mathbf{x} = (a\cos t, a\sin t, bt)$, we have

$$\mathbf{T} = \frac{\mathbf{x}'(t)}{\|\mathbf{x}'(t)\|} = \frac{a\cos t\mathbf{i} - a\sin t\mathbf{j} + b\mathbf{k}}{\sqrt{a^2+b^2}}.$$

To measure how the curve bends we need observe the following:

Proposition 3.2.11. *Assume* $\mathbf{x}'(t)$ *is never zero. Then*

(1) $\frac{d\mathbf{T}}{dt}$ *is perpendicular to* $\mathbf{T}$*.*

(2) $\|\frac{d\mathbf{T}}{dt}\|_{t=t_0}$ *equals the angular rate of change of direction of* $\mathbf{T}$ *when* $t = t_0$*. i.e.,* $\|\frac{d\mathbf{T}}{dt}\|_{t=t_0} = \frac{d\theta}{dt}$*, where* θ *is the angle of the tangent line.(For example,* $\Delta\theta$ *is the angle between* $\mathbf{T}(t_0)$ *and* $\mathbf{T}(t_0+\Delta t)$*.)*

Proof. For (1), we see

$$\mathbf{T}(t)\cdot\mathbf{T}(t) = 1.$$

Hence by taking the derivative,

$$\frac{d}{dt}\mathbf{T}(t)\cdot\mathbf{T}(t) = \mathbf{T}(t)\cdot\frac{d\mathbf{T}(t)}{dt}\mathbf{T}(t) + \frac{d\mathbf{T}(t)}{dt}\cdot\mathbf{T}(t) = 0.$$

Hence $\frac{d\mathbf{T}}{dt} \perp \mathbf{T}$.

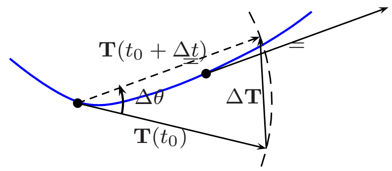


Figure 3.3: Change of angle as the point moves along curve

For part 2, we see the angular rate of change is $\lim_{\Delta t\to 0^+}\frac{\Delta\theta}{\Delta t}$. For the time being we shall assume the following. (Quite reasonable because the length of $\|\mathbf{T}\|$ is always 1.)

$$\lim_{\Delta t\to 0^+}\frac{\Delta\theta}{\|\Delta\mathbf{T}\|}=1. \tag{3.2}$$

Assuming (3.2) we have

$$\begin{aligned}\lim_{\Delta t\to 0^+}\frac{\Delta\theta}{\Delta t} &= \lim_{\Delta t\to 0^+}\frac{\Delta\theta}{\|\Delta\mathbf{T}\|}\frac{\|\Delta\mathbf{T}\|}{\Delta t}\\ &= 1\cdot\lim_{\Delta t\to 0^+}\frac{\|\Delta\mathbf{T}\|}{\Delta t}=\left\|\frac{d\mathbf{T}}{dt}\right\|.\end{aligned}$$

This is the desired result.

To show (3.2) we proceed as follows: Use law of cosines for figure 3.3 and the fact that $\|\mathbf{T}(t)\|\equiv 1$,

$$\begin{aligned}\|\Delta\mathbf{T}\|^2 &= \|\mathbf{T}(t+\Delta t)\|^2+\|\mathbf{T}(t)\|^2-2\|\mathbf{T}(t+\Delta t)\|\|\mathbf{T}(t)\|\cos\Delta\theta\\ &= 2-2\cos\Delta\theta.\end{aligned}$$

Hence

$$\begin{aligned}\lim_{\Delta\theta\to 0^+}\frac{\Delta\theta}{\|\Delta\mathbf{T}\|} &= \lim_{\Delta\theta\to 0^+}\frac{\Delta\theta}{\sqrt{2-2\cos\Delta\theta}}\\ &= \lim_{\Delta\theta\to 0^+}\frac{\Delta\theta}{2\sqrt{\sin^2(\Delta\theta/2)}}\\ &= \lim_{\Delta\theta\to 0^+}\frac{\Delta\theta/2}{\sin(\Delta\theta/2)}=1.\end{aligned}$$

□

From the proposition, it is natural to define

Definition 3.2.12. The **curvature** of a path $\mathbf{x}$ is the angular rate of change of unit tangent vector $\mathbf{T}$ per unit change along the path. In other words,

$$\kappa(t) = \frac{\|d\mathbf{T}/dt\|}{ds/dt} = \left\| \frac{d\mathbf{T}}{ds} \right\|.$$

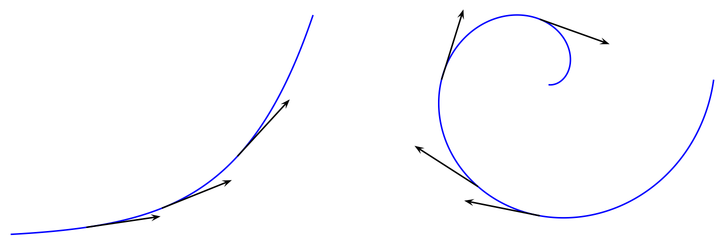


Figure 3.4: Rate of change of tangent vectors are different

3.3 Vector Fields

Definition 3.3.1. Let $X \subset \mathbb{R}^n$. A **vector field** on X is a mapping

$$\mathbf{F} : X \subset \mathbb{R}^n \to \mathbb{R}^n.$$

A vector field $\mathbf{F}$ having values in $\mathbb{R}^n$ is represented by n-real valued functions $F_1, F_2, \ldots, F_n$.

$$\mathbf{F}(\mathbf{x}) = (F_1(\mathbf{x}), F_2(\mathbf{x}), \ldots, F_n(\mathbf{x}))$$

If $n = 3$, $\mathbf{F}(P) = (F_1(P), F_2(P), F_3(P))$ is written as

$$\mathbf{F}(P) = F_1(P)\mathbf{i} + F_2(P)\mathbf{j} + F_3(P)\mathbf{k}.$$

Example 3.3.2. Let $\mathbf{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$. The gravitational force acting on a particle is given by

$$\mathbf{F}(x, y, z) = \frac{c}{\|\mathbf{r}\|^3}\mathbf{r}.$$

If it is acted on an object of mass m, then considering the direction into account, the gravity is (Figure 3.5)

$$\mathbf{F} = -\frac{GmM}{\|\mathbf{r}\|^3}\mathbf{r} = -\frac{GmM}{\|\mathbf{r}\|^2}\mathbf{u},$$

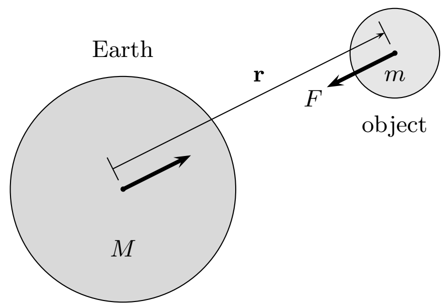


Figure 3.5: Gravitational force is represented by vectors

where M is the mass of earth and G is the gravitational constant. Here $\mathbf{u} = \mathbf{r}/\|\mathbf{r}\|$.

Gradient fields and potentials

Given real valued function $f(x_1, x_2, \dots, x_n)$ we recall the **gradient field**

$$\nabla f := (\frac{\partial f}{\partial x_1}, \frac{\partial f}{\partial x_2}, \dots, \frac{\partial f}{\partial x_n}).$$

If a vector field $\mathbf{F}$ is given by

$$\mathbf{F}(\mathbf{x}) = \nabla f(\mathbf{x})$$

for some scalar function f, then f is called the **potential function**.

Example 3.3.3. A gravitational force field has potential $f = \frac{GmM}{r}$.

$$\mathbf{F} = -\frac{GmM}{r^3}\mathbf{r} = \nabla f.$$

Example 3.3.4. Heat flux vector fields is

$$J = -k\nabla T,$$

where k is a constant for heat conduction and T is the temperature.

Example 3.3.5 (Coulomb's law)**.** The force acting on an electric charge e at position **r** due to a charge Q at the origin is

$$\mathbf{F} = \frac{\epsilon Q e}{r^3}\mathbf{r} = -\nabla V,$$

where $V = \epsilon Qe/r$. The level sets of V are called **equipotential surface or lines**. Note that the force field is orthogonal to the equipotential surfaces. We see $\mathbf{F} = -\nabla V$, where the potential V is given by

$$V = -\frac{GmM}{r}.$$

Note that **F** points to the direction of decreasing V.

Example 3.3.6. Show the vector field $\mathbf{V}(x,y) = y\mathbf{i} - x\mathbf{j}$ is not a gradient vector field. i.e, there is no C^1-function f such that

$$\mathbf{V} = \nabla f = \frac{\partial f}{\partial x}\mathbf{i} + \frac{\partial f}{\partial y}\mathbf{j}.$$

sol. Suppose there is such an f. Then $\frac{\partial f}{\partial x} = y$ and $\frac{\partial f}{\partial y} = -x$. Solving, $f(x,y) = xy + g(y)$. Then $\frac{\partial f}{\partial y} = x + g'(y) = -x$, which is impossible.

□

Conservation of energy

Consider a particle of mass m moving in a force field that is a potential field. ($\mathbf{F} = -\nabla V$)

$$m\mathbf{r}''(t) = -\nabla V(\mathbf{r}(t)).$$

A basic fact about such a motion is the *conservation of energy.* The energy E is defined to be the sum of kinetic energy and potential energy

$$E = \frac{1}{2}m\|\mathbf{r}'(t)\|^2 + V(\mathbf{r}(t)).$$

The principle of *Conservation of energy* says: E is independent of time. So $dE/dt = 0$. We can prove it simply:

$$\frac{dE}{dt} = m\mathbf{r}'(t)\cdot\mathbf{r}''(t) + (\nabla V)\cdot\mathbf{r}'(t) = \mathbf{r}'\cdot(-\nabla V + \nabla V) = 0.$$

Escape Velocity

As an application of conservation of energy, we compute the velocity of a rocket to escape the earth gravitational influence. The energy(kinetic energy +potential energy) is

$$E_0 = \frac{1}{2}mv_e^2 - \frac{mMG}{R_0}.$$

The escape velocity is obtained when this energy is zero. Thus

$$v_e = \sqrt{\frac{2MG}{R_0}}.$$

Now MG/R_0^2 is gravity g, thus

$$v_e = \sqrt{2gR_0}.$$

Flow lines

Assume we have a vector field $\mathbf{F}$. Where does it come from ? Think of a water flow (river). At each point of the river, we can think of a flow velocity at that point. Another view is as follows: one may imagine a small particle in the water flowing along the flow. This curve(line) is the concept of **flow line**.

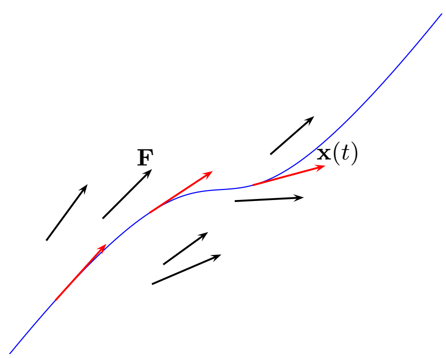


Figure 3.6: Flow lines of a vector fields

The precise definition is the following:

Definition 3.3.7. Given a vector field $\mathbf{F}$, a path $\mathbf{x}(t)$ satisfying

$$\mathbf{x}'(t) = \mathbf{F}(\mathbf{x}(t))$$

is called a **flow line** for $\mathbf{F}$. That is, $\mathbf{F}$ yields the velocity fields of the path $\mathbf{x}(t)$. A **flow line** is also called as **streamlines** or **integral curves**.

Example 3.3.8. Suppose water is flowing in a pipe as in fig 3.7. Suppose it does not depends on time. Then it is given by a vector field.

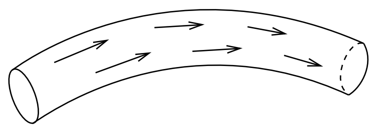


Figure 3.7: Water flow in a pipe

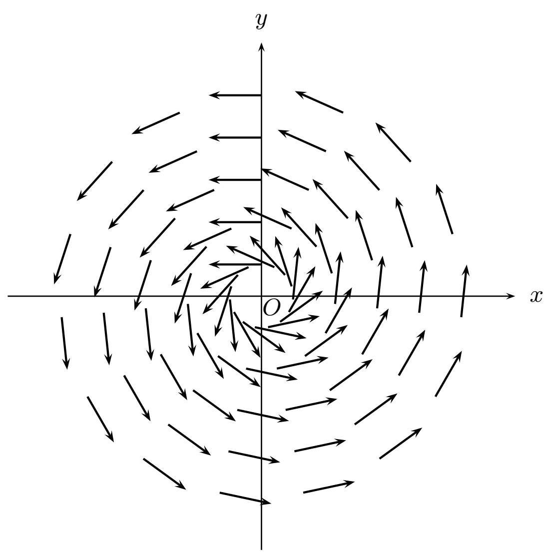


Figure 3.8: The vector field $\frac{-y\mathbf{i}+x\mathbf{j}}{\sqrt{x^2+y^2}}$ describes the drain of bathtub

Example 3.3.9. Find the flow line of $\mathbf{F} = -y\mathbf{i} + x\mathbf{j}$.

sol. Let $\mathbf{x}(t) = (x(t), y(t))$ be the flow line. Then

$$\begin{aligned} x'(t) &= -y(t) \\ y'(t) &= x(t). \end{aligned}$$

From this, one can easily verify $\mathbf{x}(t) = (a\cos t, a\sin t)$ satisfies

$$\mathbf{x}'(t) = \mathbf{F}(\mathbf{x}(t)).$$

Or one can obtain

$$x''(t) = -x(t).$$

Solving this we get the similar solution. Others may be

$$\mathbf{x}(t) = (r\cos(t - t_0), r\sin(t - t_0)).$$

□

Example 3.3.10. Find the flow line of the vector field $\mathbf{F}$. (Fig 3.8)

$$\mathbf{F}(x, y) = \frac{-y\mathbf{i} + x\mathbf{j}}{\sqrt{x^2 + y^2}}, (x, y) \neq (0, 0).$$

Let $\mathbf{x}(t) = (x(t), y(t))$ be the flow line. Then $\mathbf{x}'(t) = (x'(t), y'(t))$ must be $\mathbf{F}(\mathbf{x}(t))$. Hence

$$x'(t) = \frac{-y(t)}{\sqrt{x(t)^2 + y(t)^2}} \tag{3.3}$$

$$y'(t) = \frac{x(t)}{\sqrt{x(t)^2 + y(t)^2}}. \tag{3.4}$$

Multiply first by $x(t)$ and second by $y(t)$. Then adding we get

$$x'(t)x(t) + y'(t)y(t) = 0.$$

Integrating

$$x(t)^2 + y(t)^2 = C$$

for some constant C. Let $C = r^2$. This is equation for circle. So we can parameterize it by trig function.

$$\mathbf{x}(t) = (x(t), y(t)) = (r\cos\theta(t), r\sin\theta(t)).$$

Hence

$$x'(t) = -r\theta'(t)\sin\theta(t) \tag{3.5}$$
$$y'(t) = r\theta'(t)\cos\theta(t) \tag{3.6}$$

From (3.3) - (3.6), $x'(t) = -y/r = -\sin\theta$,

$$-r\theta'(t)\sin\theta(t) = -\sin\theta(t).$$

Hence

$$\theta'(t) = \frac{1}{r}.$$

So the flow line $\mathbf{x}(t)$ is

$$\mathbf{x}(t) = (x(t), y(t)) = (r\cos\frac{t}{r}, r\sin\frac{t}{r}).$$

The period of $\mathbf{x}(t)$ is $2\pi r$.

Example 3.3.11. Show that $\mathbf{F}(x, y) = x\mathbf{i} - y\mathbf{j}$ is a gradient field and find flow line

sol. Suppose $\mathbf{F}$ is a gradient field of $f(x, y)$ then

$$\frac{\partial f}{\partial x}\mathbf{i} + \frac{\partial f}{\partial y}\mathbf{j} = x\mathbf{i} - y\mathbf{j}.$$

Find $f(x, y)$ such that

$$\frac{\partial f}{\partial x} = x, \quad \frac{\partial f}{\partial y} = -y.$$

Hence $f(x, y) = \frac{1}{2}(x^2 - y^2)$. The flow line is obtained by solving

$$x'(t) = x(t), \quad y'(t) = y(t).$$

□

3.4 Divergence and curl and del operator

For divergence and curl operations(PROCESS), we make use of the **del operator** defined by

$$\boxed{\nabla = \mathbf{i}\frac{\partial}{\partial x} + \mathbf{j}\frac{\partial}{\partial y} + \mathbf{k}\frac{\partial}{\partial z}} \tag{3.7}$$

It works like this: For scalar functions, it works as the gradient f:

$$\nabla f = \left(\mathbf{i}\frac{\partial}{\partial x} + \mathbf{j}\frac{\partial}{\partial y} + \mathbf{k}\frac{\partial}{\partial z}\right) f = \mathbf{i}\frac{\partial f}{\partial x} + \mathbf{j}\frac{\partial f}{\partial y} + \mathbf{k}\frac{\partial f}{\partial z}.$$

It can also act on vector functions. It is called 'divergence'.

Divergence

Definition 3.4.1 (Divergence)**.** If $\mathbf{F} = F_1\mathbf{i} + F_2\mathbf{j} + F_3\mathbf{k}$ is a vector field, then the **divergence** of $\mathbf{F}$ is the scalar field defined by

$$\begin{aligned}\operatorname{div}\mathbf{F} &= \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z}\\ &= \Big(\frac{\partial}{\partial x}\mathbf{i} + \frac{\partial}{\partial y}\mathbf{j} + \frac{\partial}{\partial z}\mathbf{k}\Big)\cdot\Big(F_1\mathbf{i} + F_2\mathbf{j} + F_3\mathbf{k}\Big)\\ &= \nabla\cdot\mathbf{F},\end{aligned}$$

where

$$\boxed{\nabla = \mathbf{e}_1\frac{\partial}{\partial x_1} + \mathbf{e}_2\frac{\partial}{\partial x_2} + \cdots + \mathbf{e}_n\frac{\partial}{\partial x_n}}$$

is the del operator. Similarly, for n-variable functions, we define

$$\operatorname{div}\mathbf{F} = \sum_{i=1}^{n}\frac{\partial F_i}{\partial x_i} = \frac{\partial F_1}{\partial x_1} + \frac{\partial F_2}{\partial x_2} + \cdots + \frac{\partial F_n}{\partial x_n}.$$

Example 3.4.2. Find the divergence of $\mathbf{F} = (e^x \sin y,\, e^x \cos y,\, yz^2)$.

sol.

$$\begin{aligned}\operatorname{div}\mathbf{F} &= \frac{\partial}{\partial x}(e^x\sin y) + \frac{\partial}{\partial y}(e^x\cos y) + \frac{\partial}{\partial z}(yz^2)\\ &= e^x\sin y + (-e^x\sin y) + 2yz = 2yz.\end{aligned}$$

□

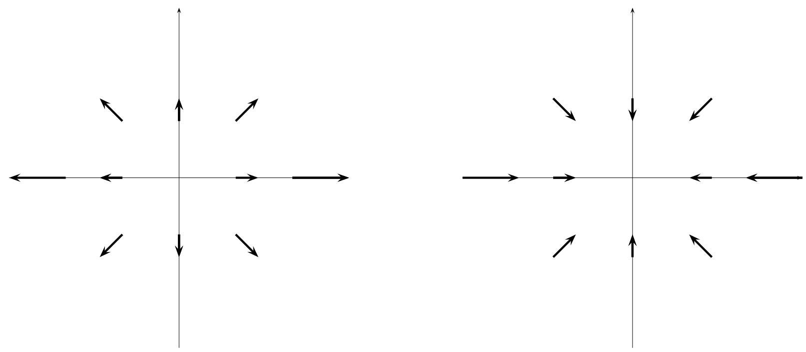


Figure 3.9: vector field (x, y) and $(-x, -y)$

Example 3.4.3.

$$\mathbf{F} = x^2 y\mathbf{i} + z\mathbf{j} + xyz\mathbf{k}.$$

Meaning of divergence

Suppose $\mathbf{F}$ represent the velocity of a gas or fluid. Then divergence represents **the rate of expansion per unit volume**: If $\operatorname{div} \mathbf{F}(P) > 0$ then it is expanding. If $(\operatorname{div} \mathbf{F}(P) < 0)$ then it is compressing. More precisely, if $V(t)$ represent the volume of a region occupied by the fluid at time t, then it can be shown that

$$\frac{1}{V(0)} \frac{d}{dt} V(t)\Big|_{t=0} \approx \operatorname{div} \mathbf{F}(\mathbf{x}_0).$$

If $\operatorname{div} \mathbf{F} \equiv 0$ everywhere, then we say the fluid is **incompressible(Solenoidal)**.

Example 3.4.4. Draw flow lines of the following vector fields:

$$(a)\, \mathbf{F} = x\mathbf{i} + y\mathbf{j}, \quad (b)\, \mathbf{F} = -x\mathbf{i} - y\mathbf{j}.$$

The divergence of the first one is positive, while that of second is negative(See Figure 3.9).

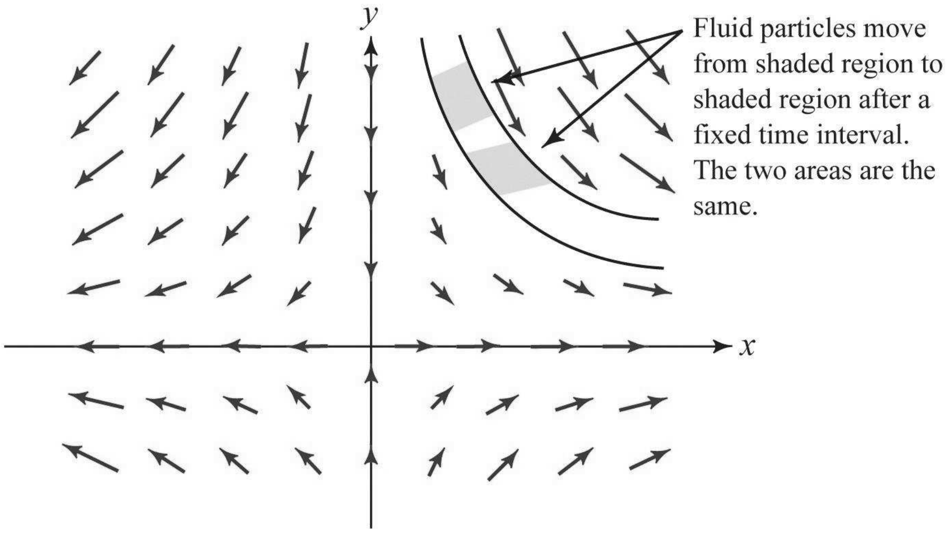


Figure 3.10: $\mathbf{F} = x\mathbf{i} - y\mathbf{j}$ is volume preserving(incompressible)

Example 3.4.5. The vector field $F = x\mathbf{i} - y\mathbf{j}$ is divergence free. The flow lines are as in figure 3.10.

Curl operator

We define the curl of a vector field $\mathbf{F} : X \subset \mathbb{R}^3 \to \mathbb{R}^3$ using the symbol ∇ and cross product:

$$\begin{aligned}\operatorname{curl}\mathbf{F} \equiv \nabla \times \mathbf{F} &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ F_1 & F_2 & F_3 \end{vmatrix} \\ &= \Big(\frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z}\Big)\mathbf{i} + \Big(\frac{\partial F_1}{\partial z} - \frac{\partial F_3}{\partial x}\Big)\mathbf{j} + \Big(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\Big)\mathbf{k}.\end{aligned}$$

Example 3.4.6. Let $\mathbf{F} = x\mathbf{i} + xy\mathbf{j} + \mathbf{k}$. Find $\nabla \times \mathbf{F}$.

$$\nabla \times \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ x & xy & 1 \end{vmatrix} = 0\mathbf{i} - 0\mathbf{j} + y\mathbf{k}.$$

Example 3.4.7. Let $\mathbf{F} = xy\mathbf{i} - \sin z\mathbf{j} + \mathbf{k}$. Find $\nabla \times \mathbf{F}$.

$$\begin{aligned}
\nabla \times \mathbf{F} &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ xy & -\sin z & 1 \end{vmatrix} \\
&= \begin{vmatrix} \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ -\sin z & 1 \end{vmatrix}\mathbf{i} - \begin{vmatrix} \frac{\partial}{\partial z} & \frac{\partial}{\partial z} \\ xy & 1 \end{vmatrix}\mathbf{j} + \begin{vmatrix} \frac{\partial}{\partial x} & \frac{\partial}{\partial y} \\ xy & \sin z \end{vmatrix}\mathbf{k} \\
&= \cos z\mathbf{i} - x\mathbf{k}.
\end{aligned}$$

Meaning of curl

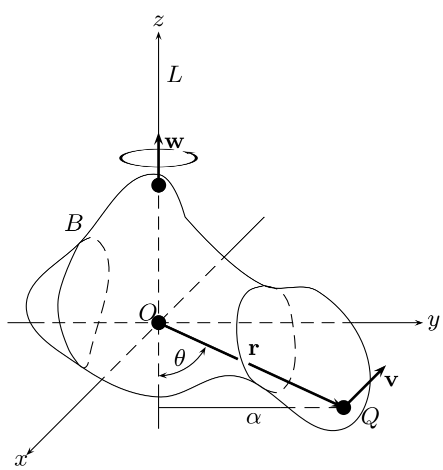


Figure 3.11: velocity $\mathbf{v}$ and angular velocity $\mathbf{w}$ has relation $\mathbf{v} = \mathbf{w} \times \mathbf{r}$.

Consider a rigid body B rotating about an axis L. (Fig 3.11). **The rotational motion of B can be described by a vector along axis of rotation w**. Let $\mathbf{w}$ the vector along z-axis s.t. $\omega = ||\mathbf{w}||$. The vector $\mathbf{w}$ is called the **angular velocity vector** and ω is angular speed.

Assume L is z-axis Q is any point on the body B, α is distance from Q to L. Then $\alpha = ||\mathbf{r}|| \sin\theta$ ($\mathbf{r}$ points to Q). Consider the tangent vector $\mathbf{v}$ at Q. Since Q moves around a circle of radius α and parallel to xy-plane (counterclockwise), we see,

$$||\mathbf{v}|| = \omega\alpha = \omega||\mathbf{r}|| \sin\theta = ||\mathbf{w}||||\mathbf{r}|| \sin\theta,$$

Then by definition of cross product,

$$\mathbf{v} = \mathbf{w} \times \mathbf{r}.$$

Since $\mathbf{w} = \omega\mathbf{k}$, $\mathbf{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$ we see from the property of cross product,

$$\mathbf{v} = \mathbf{w} \times \mathbf{r} = -\omega y\mathbf{i} + \omega x\mathbf{j}.$$

So $\operatorname{curl}\mathbf{v} = 2\omega\mathbf{k} = 2\mathbf{w}$. Hence for the rotation of a rigid body, the curl is a vector field whose direction is along the axis of rotation and magnitude is twice the angular speed.

Curl and rotational flow

$\nabla \times \mathbf{F}$ represents twice the angular velocity: So if it is 0, then we have irrotational fluid.

Example 3.4.8. Find $\operatorname{curl}\mathbf{F}$ when $\mathbf{F}(x, y, z) = (y\mathbf{i} - x\mathbf{j})/(x^2 + y^2)$ in $\mathbb{R}^3$.

sol. Write $\mathbf{F}(x, y, z) = \dfrac{y}{x^2 + y^2}\mathbf{i} + \dfrac{-x}{x^2 + y^2}\mathbf{j} + 0\mathbf{k}$. Then we see

$$\begin{aligned}
\nabla \times \mathbf{F} &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \frac{y}{x^2+y^2} & \frac{-x}{x^2+y^2} & 0 \end{vmatrix} \\
&= \left[\frac{\partial}{\partial x}\left(\frac{-x}{x^2+y^2}\right) - \frac{\partial}{\partial y}\left(\frac{y}{x^2+y^2}\right)\right]\mathbf{k} \\
&= \left[\frac{-(x^2+y^2) - (-x)(2x)}{(x^2+y^2)^2} - \frac{(x^2+y^2) - (y)(2y)}{(x^2+y^2)^2}\right]\mathbf{k} \\
&= \left[\frac{x^2-y^2}{(x^2+y^2)^2} - \frac{x^2-y^2}{(x^2+y^2)^2}\right]\mathbf{k} \\
&= \mathbf{0}.
\end{aligned}$$

□

Gradients are curl Free

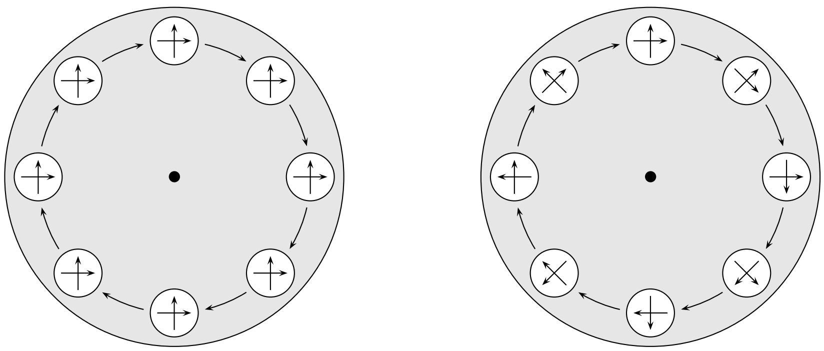
Theorem 3.4.9. *For any C^2 function*

$$\nabla \times (\nabla f) = 0.$$

Proof.

$$
\begin{aligned}
\nabla \times \nabla f &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} & \frac{\partial f}{\partial z} \end{vmatrix} \\
&= \left(\frac{\partial^2 f}{\partial y \partial z} - \frac{\partial^2 f}{\partial z \partial y} \right) \mathbf{i} + \left(\frac{\partial^2 f}{\partial z \partial z x} - \frac{\partial^2 f}{\partial x \partial z} \right) \mathbf{j} + \left(\frac{\partial^2 f}{\partial x \partial y} - \frac{\partial^2 f}{\partial y \partial x} \right) \mathbf{k}.
\end{aligned}
$$

□



(a) $\mathbf{F} = (y\mathbf{i} - x\mathbf{j})/(x^2 + y^2)$ (b) $\mathbf{F} = y\mathbf{i} - x\mathbf{j}$

Figure 3.12: Movement of small paddle in vector fields

Remark 3.4.10. Vector field $\mathbf{F}(x, y, z) = (y\mathbf{i} - x\mathbf{j})/(x^2 + y^2)$(It describes flow in a tub) does not rotate about any point except z-axis. When small paddle is placed in the fluid, it will follow the flow line(a circle in this case), but it does not rotate about its own axis. Such a field is called **irrotational**.

But the vector field $\mathbf{F}(x, y, z) = y\mathbf{i} - x\mathbf{j}$ has nonzero rotation. (fig 3.12(b)).

Curls are divergence free

Theorem 3.4.11. *For any C^2 vector field* $\mathbf{F}$

$$\operatorname{div} \operatorname{curl} \mathbf{F} = \nabla \cdot (\nabla \times \mathbf{F}) = 0.$$

Example 3.4.12. Curl of earth or any planet is nonzero, except one. What is the exception?

Physical meaning of divergence

Let

$$\mathbf{F}(x,y,z) = (F_1, F_2, F_3) = F_1\mathbf{i} + F_2\mathbf{j} + F_3\mathbf{k}$$

be a velocity vector field of some fluid in $\mathbb{R}^3$.

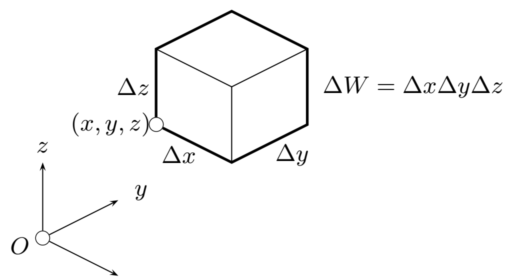


Figure 3.13: Geometric meaning of divergence

Fig 3.13. Consider a box W with dimension $\Delta x, \Delta y, \Delta z$ Then volume of W is $\Delta W = \Delta x \Delta y \Delta z$. Consider the loss of fluid across W per unit time. First consider fluid loss through left side of W whose area is $\Delta x \Delta z$. (Consider F_2 only). The outflux is

$$\mathbf{F}(x,y,z)\cdot(-\mathbf{j})\Delta x \Delta z = -F_2(x,y,z)\Delta x \Delta z.$$

And the influx is

$$\mathbf{F}(x,y+\Delta y,z)\cdot\mathbf{j}\Delta x \Delta z = F_2(x,y+\Delta y,z)\Delta x \Delta z.$$

$$\Big(F_2(x,y+\Delta y,z) - F_2(x,y,z)\Big)\Delta x \Delta z \approx \Big(\frac{\partial F_2}{\partial y}\Delta y\Big)\Delta x \Delta z.$$

Considering all the direction, the change in fluid across W per unit time is(total flux)

$$\Big(\frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z}\Big)\Delta x \Delta y \Delta z.$$

Now divide by volume ΔW

$$\text{density of flux/time} = \frac{\text{Flux across boundary}}{\text{vol}} \approx \Big(\frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z}\Big).$$

Let $\Delta x, \Delta y, \Delta z \to 0$ Then fluid density of $\mathbf{F}$ is $\operatorname{div}\mathbf{F}$. If $\mathbf{F}$ is gas, then $\operatorname{div}\mathbf{F}$ represents the rate of expansion of gas per unit time per unit volume. If

$\mathbf{F}(x, y, z) = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$, then $\operatorname{div} \mathbf{F} = 3$ and this means the gas is expanding three times per unit time.