# Chapter 1

# The geometry of Euclidean Space

We consider the basic operations on vectors in 3 and 3 dim. space: vector addition, scalar multiplication, dot product and cross product. In section 1.6 we generalize these notions to $n$ dim'l space.

## 1.1 Vectors in 2, 3 dim space

**Definition 1.1.1.** A vector in $\mathbf{R}^n, n = 2, 3$ is an ordered pair(triple) of real numbers, such as

$$(a_1, a_2),\ \text{or}\ (a_1, a_2, a_3).$$

$a_1$, $a_2$ are called $x$ **coordinate**, $y$ **coordinate** or $x$ **component**, $y$ **component** of $(a_1, a_2)$. The point $(0, 0)$ is called the **origin** and denoted by $O$.

$\mathbf{a} = (a_1, a_2)$ or $\mathbf{a} = (a_1, a_2, a_3)$ will be standard notation for vectors. A point $P$ is represented by ordered pairs of real numbers $(a_1, a_3)$ called **Cartesian coordinate)** of $P$.

(1) The set of all real numbers is denoted by $\mathbb{R}$.

(2) The set of all ordered pairs of real numbers $(x, y)$ is denoted by $\mathbb{R}^2$.

(3) The set of all ordered triples of real numbers $(x, y, z)$ is denoted by $\mathbb{R}^3$.

Vectors are identified with points in the plane or space.

$$\mathbb{R}^2 = \{(a_1, a_2) \mid a_1 \in \mathbb{R}, a_2 \in \mathbb{R}\}$$

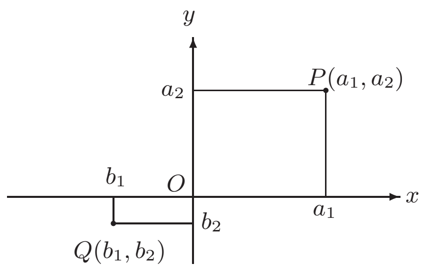



Figure 1.1: Coordinate plane

The planes in $\mathbb{R}^3$determined by $z = 0$, (resp. $x = 0$ and $y = 0$) are called

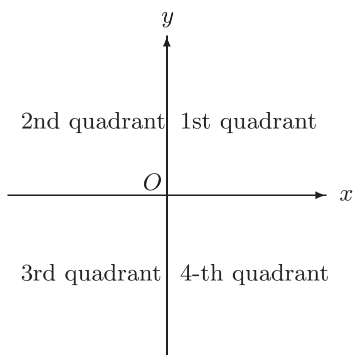



Figure 1.2: quadrant

$xy$**-plane, (resp.** $yz$**-plane,** $zx$**-plane)** These planes divides the space into eight parts: Each of them is called **octant**. If every component is positive, it is called **the first octant**.

**Example 1.1.2.** (1) The $xz$-plane is the set of all points with $y = 0$.

$$\{(x, y) \mid y = 0\}$$

(2) Similarly, the $xy$-plane is determined by $z = 0$.

$$\{(x, y, z) \mid z = 0\}$$

(3) $x$-axis is determined by

$$\begin{cases} y = 0 \\ z = 0 \end{cases}$$

or

$$\{(x, y, z) \mid y = 0, z = 0\}$$

### Vector addition and scalar multiplication-algebraic view

The operation of addition can be extended to $\mathbb{R}^3$. Given two triples, $\mathbf{a} = (a_1, a_2, a_3), \mathbf{b} = (b_1, b_2, b_3)$, we define

$$\mathbf{a} + \mathbf{b} = (a_1, a_2, a_3) + (b_1, b_2, b_3) = (a_1 + b_1, a_2 + b_2, a_3 + b_3) = \mathbf{b} + \mathbf{a}$$

to be the **sum** of $(a_1, a_2, a_3)$ and $(b_1, b_2, b_3)$. The vector $\mathbf{0} = (0, 0, 0)$ is the **zero element**. The vector $-(a_1, a_2, a_3) = (-a_1, -a_2, -a_3)$ is called the **additive inverse or negative** of $(a_1, a_2, a_3)$.

Commutative law and associate law for additions:

**(i)** $(x, y, z) + (u, v, w) = (u, v, w) + (x, y, z)$ (commutative law)

**(ii)** $((x, y, z) + (u, v, w)) + (l, m, n)$
$= (x, y, z) + ((u, v, w) + (l, m, n))$ (associate law)

The **difference** is defined as

$$(a_1, a_2, a_3) - (b_1, b_2, b_3) = (a_1 - b_1, a_2 - b_2, a_3 - b_3).$$

**Example 1.1.3.**

$$(6, 0, 2) + (-10, 3, 2) = (-4, 3, 4)$$
$$(3, 0, 3) - (5, 0, -2) = (-2, 0, 5)$$
$$(0, 0, 0) + (1, 3, 2) = (1, 3, 2)$$

For any real $\alpha$, and $(a_1, a_2, a_3)$ in $\mathbb{R}^3$, **scalar multiple** $\alpha(a_1, a_2, a_3)$ is defined as

$$\alpha(a_1, a_2, a_3) = (\alpha a_1, \alpha a_2, \alpha a_3).$$

Additions and scalar multiplication has the following properties:

**(i)** $(\alpha\beta)(x, y, z) = \alpha(\beta(x, y, z))$ (associate law)

**(ii)** $(\alpha + \beta)(x, y, z) = \alpha(x, y, z) + \beta(x, y, z)$ (distributive law)

**(iii)** $\alpha((x, y, z) + (u, v, w)) = \alpha(x, y, z) + \alpha(u, v, w)$ (distributive law)

**(iv)** $\alpha(0,0,0) = (0,0,0)$ (property of 0 )

**(v)** $0(x,y,z) = (0,0,0)$ (property of 0 )

**(vi)** $1(x,y,z) = (x,y,z)$ (property of 1 )

**Example 1.1.4.**

$$
\begin{aligned}
3\,(6,-3,2) &= (18,-9,6)\\
1\,(3,5,-2) &= (3,5,-2)\\
0\,(1,3,2) &= (0,0,0)\\
(-2)\,(-2,1,3) &= (4,-2,-6)
\end{aligned}
$$

$$
\begin{aligned}
(x,y) + (u,v) &= (x+u, y+v)\\
\alpha(x,y) &= (\alpha x, \alpha y)
\end{aligned}
$$

**Example 1.1.5.** Show

(1) $(\alpha+\beta)(x,y) = \alpha(x,y) + \beta(x,y)$

(2) $\alpha((x,y)+(u,v)) = \alpha(x,y) + \alpha(u,v)$

**sol.** (1) LHS is

$$
\begin{aligned}
(\alpha+\beta)(x,y) &= ((\alpha+\beta)x, (\alpha+\beta)y)\\
&= (\alpha x + \beta x, \alpha y + \beta y)\\
&= (\alpha x, \alpha y) + (\beta x, \beta y)\\
&= \alpha(x,y) + \beta(x,y)
\end{aligned}
$$

(2) LHS is

$$
\begin{aligned}
\alpha((x,y)+(u,v)) &= \alpha(x+u, y+v)\\
&= (\alpha(x+u), \alpha(y+v))\\
&= (\alpha x + \alpha u, \alpha y + \alpha v)\\
&= (\alpha x, \alpha y) + (\alpha u, \alpha v)\\
&= \alpha(x,y) + \alpha(u,v)
\end{aligned}
$$

□

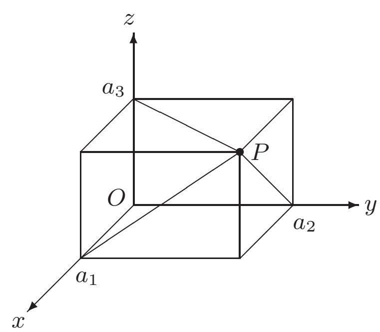



Figure 1.3: point $P(a_1, a_2, a_3)$

### Vectors-Geometric view

We associate a vector $\mathbf{a}$ with a point $(a_1, a_2, a_3)$ in the space. Thus, we can visualize it with a position vector $\mathbf{a} = (a_1, a_2, a_3)$. One can also interpret a **vector** as **directed line segment having an initial point** (Usually the initial point is at the origin), i.e, a line segment with specified *magnitude* and *direction*, and initial point at the origin. Vectors are usually denoted by boldface such as $\mathbf{a}$ or $\vec{a}$.

The elements in $\mathbb{R}^3$ are not only ordered triple of numbers, but are also regarded as vectors. We call $a_1, a_2$ and $a_3$ the **components** of $\mathbf{a}$. The triple $(0, 0, 0)$ is called **(zero vector)** denoted by $\mathbf{0}$ or $\vec{0}$.

Two vectors $\mathbf{a} = (a_1, a_2, a_3)$ and $\mathbf{b} = (b_1, b_2, b_3)$ are equal if $a_1 = b_1, a_2 = b_2$ and $a_3 = b_3$. Geometrically, this means they have the same direction and magnitude.

### Geomteric representation of vectors

See Figure 1.4. The directed line segment $PQ$ from $P$ to $Q$ is denoted by $\overrightarrow{PQ}$. $P$ and $Q$ are called **tail** and **head** respectively. The vector with tail at origin is called **position vector**. If we move the vector in parallel, we regard it as the same vector. In other words, a vector is determined by direction and magnitude. Referring the parallelogram $ABDC$ in Figure 1.4, we see $\overrightarrow{AB} = \overrightarrow{CD}$ and $\overrightarrow{AC} = \overrightarrow{BD}$.

See figure 1.5 (1). If two vectors $\mathbf{u}$, $\mathbf{v}$ have same tail $P$, we define the sum of $\mathbf{u}$ and $\mathbf{v}$ as the vector $\mathbf{u} + \mathbf{v}$ having position at the opposite vertex of the parallelogram.

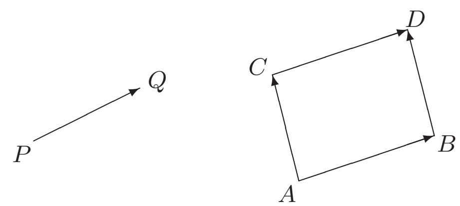



Figure 1.4: vector

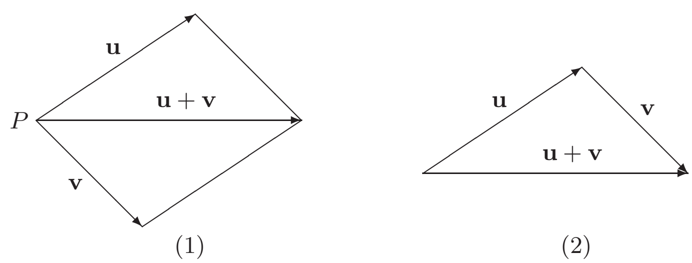



Figure 1.5: sum of two vectors

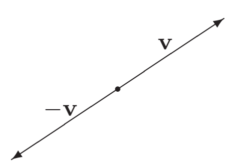



Figure 1.6: $\mathbf{v}$ and $-\mathbf{v}$

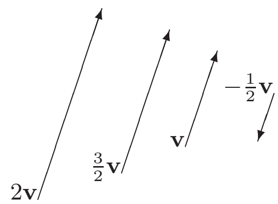



Figure 1.7: scalar multiple of $\mathbf{v}$

**(i)** $\mathbf{u}+\mathbf{v}=\mathbf{v}+\mathbf{u}$ (commutative law)

**(ii)** $(\mathbf{u}+\mathbf{v})+\mathbf{w}=\mathbf{u}+(\mathbf{v}+\mathbf{w})$ (associate law)

Scalar multiple of a vector. For real $s$ and vector $\mathbf{v}$, $s\mathbf{v}$ is the vector having magnitude $|s|$, having same direction as $\mathbf{v}$ when $s>0$, opposite direction when $s<0$. $s$ is called **scalar** $s\mathbf{v}$ is the **scalar multiple** of $\mathbf{v}$ (Fig 1.11). The following hold:

**(iii)** $(st)\mathbf{u}=s(t\mathbf{u})$ (associative law)

**(iv)** $(s+t)\mathbf{u}=s\mathbf{u}+t\mathbf{u}$ (distributive law)

**(v)** $s(\mathbf{u}+\mathbf{v})=s\mathbf{u}+s\mathbf{v}$ (distributive law)

**(vi)** $s\mathbf{0}=\mathbf{0}$ (0-vector)

**(vii)** $0\mathbf{u}=\mathbf{0}$ (0 )

**(viii)** $1\mathbf{u}=\mathbf{u}$ (1 )

**Example 1.1.6.** Show that $(-s)\mathbf{v}=-(s\mathbf{v})$ for any scalar $s$ and vector $\mathbf{v}$.

**Example 1.1.7** (3D)**.**

$$\mathbf{a}=(a_1,a_2,a_3)$$

$a_1,\ a_2,\ a_3$ are called $x$**-component,** $y$**- component,** $z$ **-component** of $\mathbf{a}$. As in figure 1.8 when $A=(a_1,a_2,a_3)$ shift the line segment $OA$ by $b_1$ along $x$-axis, by $b_2$ along $y$-axis, $b_3$ along $z$-axis is denoted by $BP$. Then the coordinate of $B$ is $(b_1,b_2,b_3)$, $P$ is $(a_1+b_1,a_2+b_2,a_3+b_3)$ and $OBPA$ is parallelogram. Hence

$$\overrightarrow{OA}+\overrightarrow{OB}=\overrightarrow{OP}$$

In vector notation, we have

$$(a_1,a_2,a_3)+(b_1,b_2,b_3)=(a_1+b_1,a_2+b_2,a_3+b_3)$$

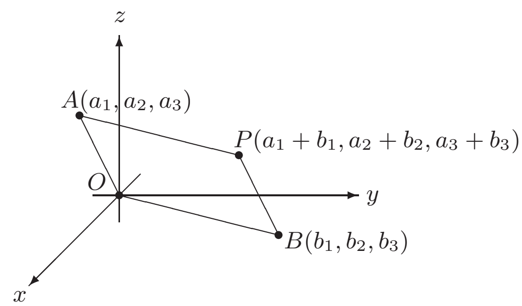



Figure 1.8: Addition

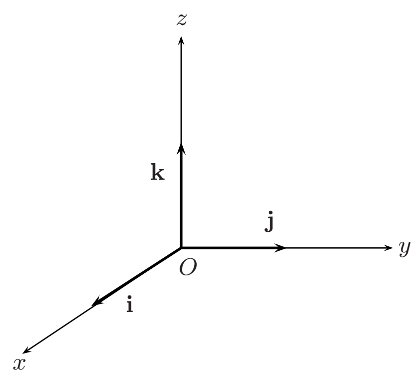



Figure 1.9: standard basis vector

similarly

$$\alpha(a_1, a_2, a_3) = (\alpha a_1, \alpha a_2, , \alpha a_3)$$

### Standard basis vectors

**Definition 1.1.8.** The following vectors **i**, **j**, **k** are called **(standard basis vector)** of $\mathbb{R}^3$ (Figure 1.13).

$$\mathbf{i} = (1, 0, 0),\ \mathbf{j} = (0, 1, 0),\ \mathbf{k} = (0, 0, 1)$$

**Remark 1.1.9.** (1) For a given $\mathbf{v} = (a_1, a_2, a_3)$

$$(a_1, a_2, a_3) = a_1(1, 0, 0) + a_2(0, 1, 0) + a_3(0, 0, 1)$$

we write $\mathbf{v} = a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k}$

(2)

**Example 1.1.10.** Write the following using standard basis vectors.

(1) $\mathbf{v} = (-1/2, 3, 5)$

(2) Express $3\mathbf{a} - 2\mathbf{b}$ when $\mathbf{a} = (3, 5, 0),\ \mathbf{b} = (-4, 1, 1)$

(3) For two points $P(1, 4, 3),\ Q(4, 1, 2)$, express $\overrightarrow{PQ}$

(4) For three points $A(0, -1, 4),\ B(2, 4, 1),\ C(3, 0, 2)$, express

$$\frac{1}{2}\overrightarrow{OA} + \frac{1}{3}\overrightarrow{OB} + \frac{1}{6}\overrightarrow{OC}$$

**sol.** (1) $\mathbf{v} = (-1/2)\mathbf{i} + 3\mathbf{j} + 5\mathbf{k}$

(2) $3\mathbf{a} - 2\mathbf{b} = 3(3\mathbf{i} + 5\mathbf{j}) - 2(-4\mathbf{i} + \mathbf{j} + \mathbf{k})$
$= (9 + 8)\mathbf{i} + (15 - 2)\mathbf{j} + (-2)\mathbf{k} = 17\mathbf{i} + 13\mathbf{j} - 2\mathbf{k}$

(3) $\overrightarrow{PQ} = (4 - 1)\mathbf{i} + (1 - 4)\mathbf{j} + (2 - 3)\mathbf{k} = 3\mathbf{i} - 3\mathbf{j} - \mathbf{k}$

(4) $(1/2)\overrightarrow{OA} + (1/3)\overrightarrow{OB} + (1/6)\overrightarrow{OC}$
$= (1/2)(-\mathbf{j} + 4\mathbf{k}) + (1/3)(2\mathbf{i} + 4\mathbf{j} + \mathbf{k}) + (1/6)(3\mathbf{i} + 2\mathbf{k})$
$= (7/6)\mathbf{i} + (5/6)\mathbf{j} + (8/3)\mathbf{k}$

□

## 1.2 More about vectors

### Parametric equation of lines(Point-Direction form)

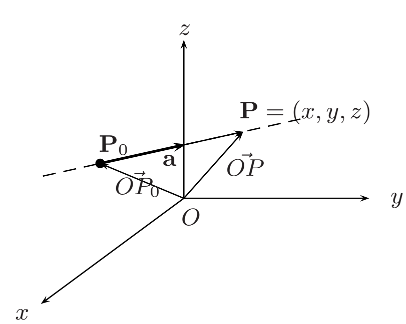



Figure 1.10: A line is determined by a point and a vector

$$\ell(t) = \vec{OP_0} + t\vec{P_0P} = \mathbf{b} + t\mathbf{a}$$

The equation of the line $\ell$ through the tip of $\vec{OP_0}$ and pointing in the direction of $\vec{P_0P}$ is $\ell(t) = \mathbf{b} + t\mathbf{a}$ where $t$ takes all real values. In coordinate form, we have

$$\begin{aligned} x &= x_1 + at, \\ y &= y_1 + bt, \\ z &= z_1 + ct, \end{aligned}$$

where $\mathbf{b} = (x_1, y_1, z_1)$ and $\mathbf{a} = (a, b, c)$.

**Example 1.2.1.** (1) Find equation of line through $(2, 1, 5)$ in the direction of $4\mathbf{i} - 2\mathbf{j} + 5\mathbf{k}$.

(2) In what direction, the the line $x = 3t - 2, y = t - 1, z = 7t + 4$ points ?

**sol.** (1) $\mathbf{v} = (2, 1, 5) + t(4, -2, 5)$
(2) $(3, 1, 7) = 3\mathbf{i} + \mathbf{j} + 7\mathbf{k}$.

□

**Example 1.2.2.** Does the two lines $(x, y, z) = (t, -6t + 1, 2t - 8)$ and $(3t + 1, 2t, 0)$ intersect ?

**sol.** If two line intersect, we must have

$$(t_1, -6t_1, 2t_1 - 8) = (3t_2 + 1, 2t_2, 0)$$

for some numbers $t_1, t_2$.(Note: we have used two different parameters $t_1$ and $t_2$). But since the system of equation

$$\begin{aligned} t_1 &= 3t_2 + 1 \\ -6t_1 &= 2t_2 \\ 2t_1 - 8 &= 0 \end{aligned}$$

has no solution, the lines do not meet.

□

## Two point form

We describe the equation of line through two points $\mathbf{a}, \mathbf{b}$.

The direction is given by $\mathbf{v} = \mathbf{b} - \mathbf{a}$. So by point -direction form

$$\ell(t) = \mathbf{a} + t(\mathbf{b} - \mathbf{a}).$$

If $P = (x_1, y_1, z_1)$ is the tip of $\mathbf{a}$ and $Q = (x_2, y_2, z_2)$ is the tip of $\mathbf{b}$, then $\mathbf{v} = (x_2 - x_1, y_2 - y_1, z_2 - z_1)$. Writing it componentwise, we see

$$\begin{aligned} x &= x_1 + (x_2 - x_1)t \\ y &= y_1 + (y_2 - y_1)t \\ z &= z_1 + (z_2 - z_1)t \end{aligned}$$

Solving these for $t$, we see

$$t = \frac{x - x_1}{x_2 - x_1} = \frac{y - y_1}{y_2 - y_1} = \frac{z - z_1}{z_2 - z_1}$$

This is another equation of line.

**Example 1.2.3.** Find eq. of a line through $(2, 1, -3)$ and $(6, -1, -5)$.

**Example 1.2.4.** Find eq. of line segment between $(1, 1, -3)$ and $(2, -1, 0)$

**sol.** $0 \le t \le 1$

□

The last two equations of (9) yield

$$t_2 = 5t_1 + 6 = -2t_1 - 1 \quad \Rightarrow \quad t_1 = -1.$$

Using $t_1 = -1$ in the second equation of (9), we find that $t_2 = 1$. Note that the values $t_1 = -1$ and $t_2 = 1$ also satisfy the first equation of (9); therefore, we have solved the system. Setting $t = -1$ in the set of parametric equations for the first line gives the desired intersection point, namely, $(0, 1, 2)$. ◆

## Parametric Equations in General

Vector geometry makes it relatively easy to find parametric equations for a variety of curves. We provide two examples.

**EXAMPLE 7** If a wheel rolls along a flat surface without slipping, a point on the rim of the wheel traces a curve called a **cycloid**, as shown in Figure 1.26.

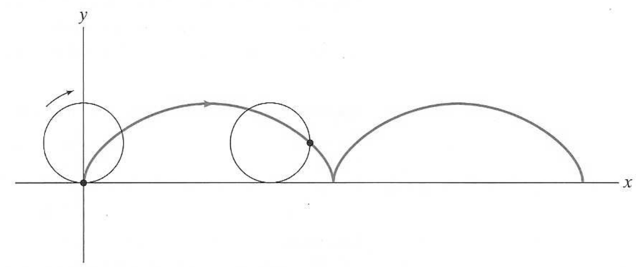



Figure 1.26 The graph of a cycloid.

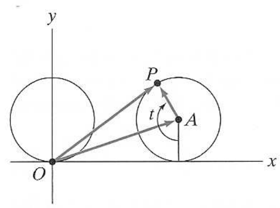



Figure 1.27 The result of the wheel in Figure 1.26 rolling through a central angle of $t$.

Suppose that the wheel has radius $a$ and that coordinates in $\mathbf{R}^2$ are chosen so that the point of interest on the wheel is initially at the origin. After the wheel has rolled through a central angle of $t$ radians, the situation is as shown in Figure 1.27. We seek the vector $\overrightarrow{OP}$, the position vector of $P$, in terms of the parameter $t$. Evidently, $\overrightarrow{OP} = \overrightarrow{OA} + \overrightarrow{AP}$, where the point $A$ is the center of the wheel. The vector $\overrightarrow{OA}$ is not difficult to determine. Its $\mathbf{j}$-component must be $a$, since the center of the wheel does not vary vertically. Its $\mathbf{i}$-component must equal the distance the wheel has rolled; if $t$ is measured in radians, then this distance is $at$, the length of the arc of the circle having central angle $t$. Hence, $\overrightarrow{OA} = at\mathbf{i} + a\mathbf{j}$.

The value of vector methods becomes apparent when we determine $\overrightarrow{AP}$. Parallel translate the picture so that $\overrightarrow{AP}$ has its tail at the origin, as in Figure 1.28. From the parametric equations of a circle of radius $a$,

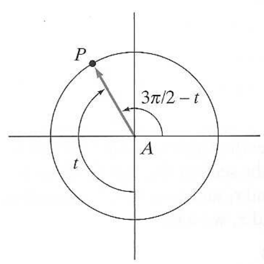



Figure 1.28 $\overrightarrow{AP}$ with its tail at the origin.

$$\overrightarrow{AP} = a\cos\left(\frac{3\pi}{2} - t\right)\mathbf{i} + a\sin\left(\frac{3\pi}{2} - t\right)\mathbf{j} = -a\sin t\,\mathbf{i} - a\cos t\,\mathbf{j},$$

from the addition formulas for sine and cosine. We conclude that

$$\begin{aligned}\overrightarrow{OP} = \overrightarrow{OA} + \overrightarrow{AP} &= (at\mathbf{i} + a\mathbf{j}) + (-a\sin t\mathbf{i} - a\cos t\mathbf{j}) \\ &= a(t - \sin t)\mathbf{i} + a(1 - \cos t)\mathbf{j},\end{aligned}$$

so the parametric equations are

$$\begin{cases} x = a(t - \sin t) \\ y = a(1 - \cos t) \end{cases}.$$

◆

**EXAMPLE 8** If you unwind adhesive tape from a nonrotating circular tape dispenser so that the unwound tape is held taut and tangent to the dispenser roll, then the end of the tape traces a curve called the **involute** of the circle. Let's find the parametric equations for this curve, assuming that the dispensing roll has constant radius $a$ and is centered at the origin. (As more and more tape is unwound, the radius of the roll will, of course, decrease. We'll assume that little enough tape is unwound so that the radius of the roll remains constant.)

Considering Figure 1.29, we see that the position vector $\overrightarrow{OP}$ of the desired point $P$ is the vector sum $\overrightarrow{OB} + \overrightarrow{BP}$. To determine $\overrightarrow{OB}$ and $\overrightarrow{BP}$, we use the angle $\theta$ between the positive $x$-axis and $\overrightarrow{OB}$ as our parameter. Since $B$ is a point on the circle,

$$\overrightarrow{OB} = a\cos\theta\,\mathbf{i} + a\sin\theta\,\mathbf{j}.$$

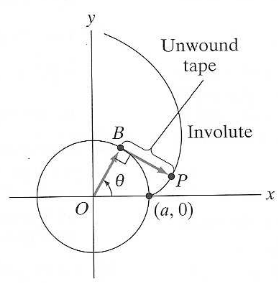



**Figure 1.29** Unwinding tape, as in Example 8. The point $P$ describes a curve known as the **involute** of the circle.

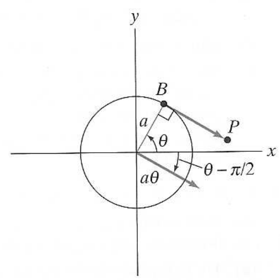



**Figure 1.30** The vector $\overrightarrow{BP}$ must make an angle of $\theta - \pi/2$ with the positive $x$-axis.

To find the vector $\overrightarrow{BP}$, parallel translate it so that its tail is at the origin. Figure 1.30 shows that $\overrightarrow{BP}$'s length must be $a\theta$, the amount of unwound tape, and its direction must be such that it makes an angle of $\theta - \pi/2$ with the positive $x$-axis. From our experience with circular geometry and, perhaps, polar coordinates, we see that $\overrightarrow{BP}$ is described by

$$\overrightarrow{BP} = a\theta\cos\left(\theta - \frac{\pi}{2}\right)\mathbf{i} + a\theta\sin\left(\theta - \frac{\pi}{2}\right)\mathbf{j} = a\theta\sin\theta\,\mathbf{i} - a\theta\cos\theta\,\mathbf{j}.$$

Hence,

$$\overrightarrow{OP} = \overrightarrow{OB} + \overrightarrow{BP} = a(\cos\theta + \theta\sin\theta)\,\mathbf{i} + a(\sin\theta - \theta\cos\theta)\,\mathbf{j}.$$

So

$$\begin{cases} x = a(\cos\theta + \theta\sin\theta) \\ y = a(\sin\theta - \theta\cos\theta) \end{cases}$$

are the parametric equations of the involute, whose graph is pictured in Figure 1.31. ◆

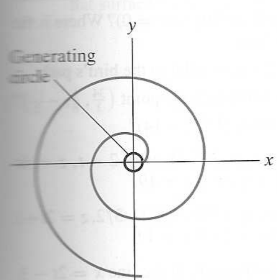



**Figure 1.31** The involute.

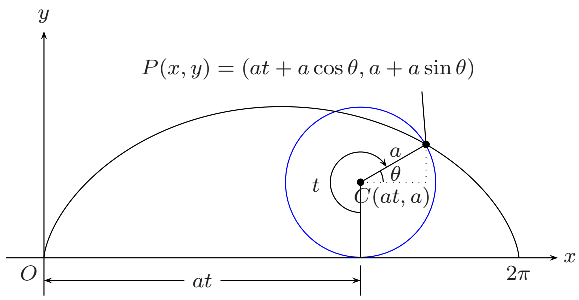



Figure 1.11: Cycloid

Assume a circle of radius $a$ is rolling on the $x$-axis. Let $P$ be a point on the circle located at the origin in the beginning. As the circle rolls, $P$ starts to move from the origin. (Fig 1.11) The trajectory of $P$ is called a *cycloid*. If circle rotates by $t$ radian, then $P = (x, y)$ is given by

$$x = at + a\cos\theta, \qquad y = a + a\sin\theta. \tag{1.1}$$

Since $\theta = \frac{3\pi}{2} - t$, $\cos\theta = -\sin t$, $\sin\theta = -\cos t$, we have

$$x = a(t - \sin t), \qquad y = a(1 - \cos t).$$

## 1.3 Inner product, length, distance

### Dot product-Inner product

**Definition 1.3.1.** Given two vectors $\mathbf{a} = a_1\mathbf{i}+a_2\mathbf{j}+a_3\mathbf{k}$ and $\mathbf{b} = b_1\mathbf{i}+b_2\mathbf{j}+b_3\mathbf{k}$ we define

$$a_1b_1 + a_2b_2 + a_3b_3$$

to be the **dot product** or **(inner product)** of $\mathbf{a}$ and $\mathbf{b}$ and write $\mathbf{a} \cdot \mathbf{b}$.

**Example 1.3.2.** Let $\mathbf{a} = 2\mathbf{i} - 3\mathbf{j} + \mathbf{k}$, $\mathbf{b} = \mathbf{i} + 2\mathbf{j} - \mathbf{k}$. Find

(1) $\mathbf{a} \cdot \mathbf{a}$

(2) $\mathbf{a} \cdot \mathbf{b}$

(3) $\mathbf{a} \cdot (\mathbf{a} - 3\mathbf{b})$

(4) $(3\mathbf{a} + 2\mathbf{b}) \cdot (\mathbf{a} - \mathbf{b})$

**sol.** (1) $\mathbf{a}\cdot\mathbf{a} = 4+9+1 = 14$
(2) $\mathbf{a}\cdot\mathbf{b} = 2-6-1 = -5$
(3) $\mathbf{a}\cdot(\mathbf{a}-3\mathbf{b}) = (2\mathbf{i}-3\mathbf{j}+\mathbf{k})\cdot(-\mathbf{i}-9\mathbf{j}+4\mathbf{k})$
$= -2+27+4 = 29$
(4) $(3\mathbf{a}+2\mathbf{b})\cdot(\mathbf{a}-\mathbf{b}) = (8\mathbf{i}-5\mathbf{j}+\mathbf{k})\cdot(\mathbf{i}-5\mathbf{j}+2\mathbf{k})$
$= 8+25+2 = 35$

□

**Proposition 1.3.3** (Properties of Inner Product)**.** *For vectors* $\mathbf{a}$*,* $\mathbf{b}$*,* $\mathbf{c}$ *and scalar* $\alpha$*, the following hold:*

*(1)* $\mathbf{a}\cdot\mathbf{a} \geq 0$ *(equality holds only when* $\mathbf{a} = \mathbf{0}$*)*

*(2)* $\mathbf{a}\cdot\mathbf{b} = \mathbf{b}\cdot\mathbf{a}$

*(3)* $(\mathbf{a}+\mathbf{b})\cdot\mathbf{c} = \mathbf{a}\cdot\mathbf{c}+\mathbf{b}\cdot\mathbf{c}$

*(4)* $(\alpha\mathbf{a})\cdot\mathbf{b} = \alpha(\mathbf{a}\cdot\mathbf{b})$

*(5)* $\|\mathbf{a}\| = \sqrt{\mathbf{a}\cdot\mathbf{a}}$

*Proof.* These can be proved easily. □

**Example 1.3.4.** For $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$ Show the following.

(1) $(\mathbf{a}-\mathbf{b})\cdot\mathbf{c} = \mathbf{a}\cdot\mathbf{c}-\mathbf{b}\cdot\mathbf{c}$

(2) $\mathbf{a}\cdot(\mathbf{b}+\mathbf{c}) = \mathbf{a}\cdot\mathbf{b}+\mathbf{a}\cdot\mathbf{c}$

(3) $\mathbf{a}\cdot(\mathbf{b}-\mathbf{c}) = \mathbf{a}\cdot\mathbf{b}-\mathbf{a}\cdot\mathbf{c}$

(4) $\mathbf{a}\cdot\mathbf{b} = \frac{1}{2}(\|\mathbf{a}\|^2+\|\mathbf{b}\|^2-\|\mathbf{a}-\mathbf{b}\|^2)$

**sol.** We see
(1) $(\mathbf{a}-\mathbf{b})\cdot\mathbf{c} = (\mathbf{a}+(-1)\mathbf{b}))\cdot\mathbf{c} = \mathbf{a}\cdot\mathbf{c}+((-1)\mathbf{b})\cdot\mathbf{w}$
$= \mathbf{a}\cdot\mathbf{c}+(-1)\mathbf{b}\cdot\mathbf{c} = \mathbf{a}\cdot\mathbf{c}-\mathbf{b}\cdot\mathbf{c}$
(2) $\mathbf{a}\cdot(\mathbf{b}+\mathbf{c}) = (\mathbf{b}+\mathbf{c})\cdot\mathbf{a} = \mathbf{b}\cdot\mathbf{a}+\mathbf{c}\cdot\mathbf{a} = \mathbf{a}\cdot\mathbf{b}+\mathbf{a}\cdot\mathbf{c}$
(3) $\mathbf{a}\cdot(\mathbf{b}-\mathbf{c}) = (\mathbf{b}-\mathbf{c})\cdot\mathbf{a} = \mathbf{b}\cdot\mathbf{a}-\mathbf{c}\cdot\mathbf{a} = \mathbf{a}\cdot\mathbf{b}-\mathbf{a}\cdot\mathbf{c}$
(4) $\|\mathbf{a}-\mathbf{b}\|^2 = (\mathbf{a}-\mathbf{b})\cdot(\mathbf{a}-\mathbf{b}) = \mathbf{a}\cdot(\mathbf{a}-\mathbf{b})-\mathbf{b}\cdot(\mathbf{a}-\mathbf{b})$
$= \mathbf{a}\cdot\mathbf{a}-\mathbf{a}\cdot\mathbf{b}-\mathbf{b}\cdot\mathbf{a}+\mathbf{b}\cdot\mathbf{b} = \|\mathbf{a}\|^2-2\,\mathbf{a}\cdot\mathbf{b}+\|\mathbf{b}\|^2$

□

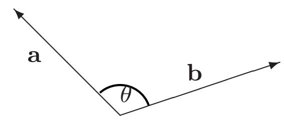



Figure 1.12: Angle between two vectors

## Length of vectors

The **length, norm** of a vector $\mathbf{a} = (a_1, a_2, a_3)$ is

$$\sqrt{(a_1 - 0)^2 + (a_2 - 0)^2 + (a_3 - 0)^2} = \sqrt{a_1^2 + a_2^2 + a_3^2}$$

denoted by $\|\mathbf{a}\|$. Also we note that

$$\|\mathbf{a}\| = (\mathbf{a} \cdot \mathbf{a})^{1/2}.$$

**Example 1.3.5.** Find the lengths of the following vectors.

(1) $\mathbf{a} = (3, 2, 1)$

(2) $3\mathbf{i} - 4\mathbf{j} + \mathbf{k}$

(3) $\overrightarrow{AB}$ when $A(2, -1/3, -1)$, $B(8/3, 0, 1)$.

**sol.** (1) $\|\mathbf{a}\| = \sqrt{9 + 4 + 1} = \sqrt{14}$
(2) $\|3\mathbf{i} - 4\mathbf{j} + \mathbf{k}\| = \sqrt{9 + 16 + 1} = \sqrt{26}$
(3) $\|\overrightarrow{AB}\| = \sqrt{(8/3 - 2)^2 + (0 - (-1/3))^2 + (1 - (-1))^2}$
$= \sqrt{4/9 + 1/9 + 4} = \sqrt{41}/3$

□

**Definition 1.3.6.** A vector with norm 1 is called a **unit vector**. Any nonzero vector $\mathbf{a}$ can be made into a unit vector by setting $\mathbf{a}/\|\mathbf{a}\|$. This process is called a **normalization**.

**Example 1.3.7.** Normalize the followings.

(1) $\mathbf{i} + \mathbf{j} + \mathbf{k}$

(2) $3\mathbf{i} + 4\mathbf{k}$

(3) $a\mathbf{i} - \mathbf{j} + c\mathbf{k}$

**sol.** (1) $(1/\sqrt{3})\,\mathbf{i} + (1/\sqrt{3})\,\mathbf{j} + (1/\sqrt{3})\,\mathbf{k}$
(2) $(3/5)\,\mathbf{i} + (4/5)\,\mathbf{k}$
(3) $(a/\sqrt{1+a^2+c^2})\,\mathbf{i} - (1/\sqrt{1+a^2+c^2})\,\mathbf{j} + (c/\sqrt{1+a^2+c^2})\,\mathbf{k}$

□

## Angle between two vectors

**Proposition 1.3.8.** *Let* $\mathbf{a}, \mathbf{b}$ *be two nonzero vectors in* $\mathbb{R}^2$ *or* $\mathbb{R}^3$ *and let* $\theta$ *be the angle between them. Then*

$$\mathbf{a} \cdot \mathbf{b} = \|\mathbf{a}\|\,\|\mathbf{b}\|\,\cos\theta.$$

*Hence*

$$\theta = \cos^{-1}\frac{\mathbf{a}\cdot\mathbf{b}}{\|\mathbf{a}\|\,\|\mathbf{b}\|}$$

*Proof.* Let $\mathbf{a} = \overrightarrow{AB}$, $\mathbf{b} = \overrightarrow{AC}$. Then $\mathbf{a} - \mathbf{b} = \overrightarrow{CB}$. Let$\angle CAB = \theta$. Then by

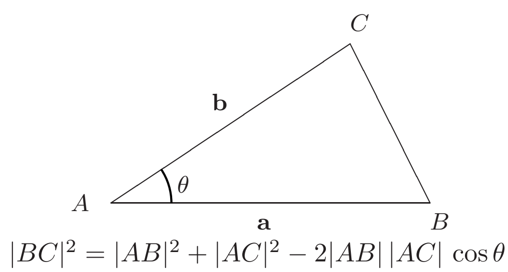



Figure 1.13: law of cosine

the law of cosine (figure 1.13) we have

$$\|\mathbf{b} - \mathbf{a}\|^2 = \|\mathbf{a}\|^2 + \|\mathbf{b}\|^2 - 2\|\mathbf{a}\|\,\|\mathbf{b}\|\,\cos\theta.$$

The left hand side is

$$\begin{aligned}\|\mathbf{a}-\mathbf{b}\|^2 &= (\mathbf{a}-\mathbf{b})\cdot(\mathbf{a}-\mathbf{b})\\ &= \mathbf{a}\cdot\mathbf{a} - \mathbf{a}\cdot\mathbf{b} - \mathbf{b}\cdot\mathbf{a} + \mathbf{b}\cdot\mathbf{b}\\ &= \|\mathbf{a}\|^2 - 2\,\mathbf{a}\cdot\mathbf{b} + \|\mathbf{b}\|^2.\end{aligned}$$

Hence we obtain

$$\|\mathbf{a}\|\,\|\mathbf{b}\|\,\cos\theta = \mathbf{a}\cdot\mathbf{b}.$$

□

**Corollary 1.3.9.** *Two nonzero vector* $\mathbf{a}$ *and* $\mathbf{b}$ *are* ***perpendicular, orthogonal*** *if and only if* $\mathbf{a} \cdot \mathbf{b} = 0$.

**Example 1.3.10.** Find the angle between $\mathbf{i} + \mathbf{j} + 2\mathbf{k}$ and $-\mathbf{i} + 2\mathbf{j} + \mathbf{k}$.

**sol.** By proposition 1.2.10,

$$\frac{(\mathbf{i} + \mathbf{j} + 2\mathbf{k}) \cdot (-\mathbf{i} + 2\mathbf{j} + \mathbf{k})}{\|\mathbf{i} + \mathbf{j} + 2\mathbf{k}\| \, \| - \mathbf{i} + 2\mathbf{j} + \mathbf{k}\|} = \frac{-1 + 2 + 2}{\sqrt{1 + 1 + 4}\sqrt{1 + 4 + 1}} = \frac{3}{6} = \frac{1}{2}$$

Hence the angle is $\cos^{-1}(1/2) = \pi/3$.

□

**Corollary 1.3.11.** *Given two points* $A(a_1, a_2, a_3)$, $B(b_1, b_2, b_3)$, *the area of the triangle* $OAB$ *is*

$$\frac{1}{2}\sqrt{(a_2b_3 - a_3b_2)^2 + (a_3b_1 - a_1b_3)^2 + (a_1b_2 - a_2b_1)^2}$$

*Proof.* Let $\overrightarrow{OA} = \mathbf{a}$, $\overrightarrow{OB} = \mathbf{b}$, $\angle BOA = \theta$. Then the area of $\triangle OAB$ is

$$\begin{aligned}
&\frac{1}{2}|OA|\,|OB|\,\sin\theta \\
&= \frac{1}{2}\|\mathbf{a}\|\,\|\mathbf{b}\|\,\sqrt{1 - \cos^2\theta} \\
&= \frac{1}{2}\sqrt{\|\mathbf{a}\|^2\,\|\mathbf{b}\|^2 - (\mathbf{a} \cdot \mathbf{b})^2} \\
&= \frac{1}{2}\sqrt{(a_1^2 + a_2^2 + a_3^2)(b_1^2 + b_2^2 + b_3^2) - (a_1b_1 + a_2b_2 + a_3b_3)^2} \\
&= \frac{1}{2}\sqrt{(a_2b_3 - a_3b_2)^2 + (a_3b_1 - a_1b_3)^2 + (a_1b_2 - a_2b_1)^2}
\end{aligned}$$

□

**Example 1.3.12.** Find the area of the triangle with vertices $A(a, 0, 0), B(0, b, 0), C(0, 0, c)$.

**sol.** Shift(translate) $A$ to the origin, then the points $B, C$ are moved to the points $(-a, b, 0)$ and $(-a, 0, c)$. Hence

$$\frac{1}{2}\sqrt{(bc - 0)^2 + (0 + ac)^2 + (0 + ab)^2} = \frac{1}{2}\sqrt{b^2c^2 + c^2a^2 + a^2b^2}$$

□

**Theorem 1.3.13** (Cauchy-Schwarz inequality). *For any two vectors* $\mathbf{a}$, $\mathbf{b}$

$$|\mathbf{a}\cdot\mathbf{b}| \le \|\mathbf{a}\|\,\|\mathbf{b}\|$$

*holds, and the equality holds iff* $\mathbf{a}$ *and* $\mathbf{b}$ *are parallel.*

*Proof.* We may assume $\mathbf{a}$, $\mathbf{b}$ are nonzero. Let $\theta$ be the angle between $\mathbf{a}$ and $\mathbf{b}$. Then by prop 1.3.8

$$|\mathbf{a}\cdot\mathbf{b}| = \|\mathbf{a}\|\,\|\mathbf{b}\|\,|\cos\theta| \le \|\mathbf{a}\|\,\|\mathbf{b}\|$$

holds. Since $\|\mathbf{a}\|\,\|\mathbf{b}\| \ne 0$, if equality holds $|\cos\theta| = 1$ i.e, $\theta = 0$ or $\pi$. Hence $\mathbf{a}$ and $\mathbf{b}$ are parallel. □

**Remark 1.3.14.** The Cauchy-Schwarz inequality reads, componentwise, as

$$(ax+by+cz)^2 \le (a^2+b^2+c^2)(x^2+y^2+z^2)$$

**Example 1.3.15.** Show Cauchy-Schwarz inequality for $\mathbf{i}+3\mathbf{j}+2\mathbf{k}$, $-\mathbf{i}+\mathbf{j}$.

**sol.** Since the inner product and lengths are

$$\begin{aligned}
&(\mathbf{i}+3\mathbf{j}+2\mathbf{k})\cdot(-\mathbf{i}+\mathbf{j}) = -1+3 = 2,\\
&\|\mathbf{i}+3\mathbf{j}+2\mathbf{k}\|\,\|-\mathbf{i}+\mathbf{j}\| = \sqrt{1+9+4}\,\sqrt{1+1} = \sqrt{28} = 2\sqrt{7}
\end{aligned}$$

we have

$$|(\mathbf{i}+3\mathbf{j}+2\mathbf{k})\cdot(-\mathbf{i}+\mathbf{j})| \le \|\mathbf{i}+3\mathbf{j}+2\mathbf{k}\|\,\|-\mathbf{i}+\mathbf{j}\|$$

□

**Theorem 1.3.16** (Triangle inequality). *For any two vector* $\mathbf{a}$, $\mathbf{b}$ *it holds that*

$$\|\mathbf{a}+\mathbf{b}\| \le \|\mathbf{a}\| + \|\mathbf{b}\|$$

*and equality holds when* $\mathbf{a}$, $\mathbf{b}$ *are parallel and having same direction.*

*Proof.*

$$\|\mathbf{a}+\mathbf{b}\|^2 = (\mathbf{a}+\mathbf{b})\cdot(\mathbf{a}+\mathbf{b}) = \|\mathbf{a}\|^2 + 2\mathbf{a}\cdot\mathbf{b} + \|\mathbf{b}\|^2$$

By C-S

$$\|\mathbf{a}+\mathbf{b}\|^2 \le \|\mathbf{a}\|^2 + 2\|\mathbf{a}\|\,\|\mathbf{b}\| + \|\mathbf{b}\|^2 = (\|\mathbf{a}\| + \|\mathbf{b}\|)^2$$

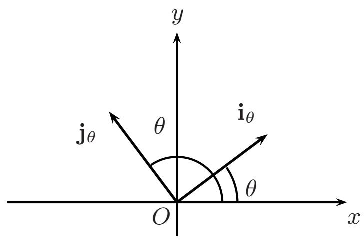



Figure 1.14: $\mathbf{i}_\theta$ and $\mathbf{j}_\theta$

Equality holds iff

$$\mathbf{a}\cdot\mathbf{b} = \|\mathbf{a}\|\,\|\mathbf{b}\|$$

i.e, the angle is 0. □

**Example 1.3.17.** Show triangle inequality for $-\mathbf{i}+\mathbf{j}$ and $\mathbf{i}+3\mathbf{j}+2\mathbf{k}$.

**sol.** Sum and difference is

$$\begin{aligned}\|(\mathbf{i}+3\mathbf{j}+2\mathbf{k})+(-\mathbf{i}+\mathbf{j})\| &= \|4\mathbf{j}+2\mathbf{k}\| \;=\; \sqrt{16+4}\\ &= 2\sqrt{5} \;=\; 4.4721\ldots\\ \|\mathbf{i}+3\mathbf{j}+2\mathbf{k}\|+\|-\mathbf{i}+\mathbf{j}\| &= \sqrt{1+9+4}+\sqrt{1+1}\\ &= \sqrt{14}+\sqrt{2} \;=\; 5.1558\ldots\end{aligned}$$

Hence

$$\|(\mathbf{i}+3\mathbf{j}+2\mathbf{k})+(-\mathbf{i}+\mathbf{j})\| \le \|\mathbf{i}+3\mathbf{j}+2\mathbf{k}\|+\|-\mathbf{i}+\mathbf{j}\|$$

□

**Definition 1.3.18.** If two vectors $\mathbf{a}$, $\mathbf{b}$ satisfy $\mathbf{a}\cdot\mathbf{b}=0$ then we say they are **orthogonal**(perpendicular).

**Example 1.3.19.** for any real $\theta$ two vectors are $\mathbf{i}_\theta = (\cos\theta)\mathbf{i}+(\sin\theta)\mathbf{j}$, $\mathbf{j}_\theta = -(\sin\theta)\mathbf{i}+(\cos\theta)\mathbf{j}$ orthogonal.

**Example 1.3.20.** Find a unit vector orthogonal to $2\mathbf{i}-\mathbf{j}+3\mathbf{k}$ and $\mathbf{i}+2\mathbf{j}+9\mathbf{k}$.

**sol.** Let $a\mathbf{i}+b\mathbf{j}+c\mathbf{k}$ be the desired vector. Then $a$, $b$, $c$ are determined by

$$\begin{aligned}2a-b+3c&=0 \text{ orthogonality}\\ a+2b+9c&=0 \text{ orthogonality}\\ a^2+b^2+c^2&=1 \text{ unicity}\end{aligned}$$

Hence the desired vector is

$$\pm\frac{1}{\sqrt{19}}\left(3\mathbf{i}+3\mathbf{j}-\mathbf{k}\right).$$

□

## Orthogonal projection

Given two nonzero vectors **a** and **b**, we may define the **orthogonal projection** of **b** onto **a** to be the vector **p** given in the figure 1.15. Since **p** is a scalar multiple of **a**, there is a constant $c$ such that $\mathbf{p} = c\mathbf{a}$. We let

$$\mathbf{b} = c\mathbf{a} + \mathbf{q}$$

where **q** is a vector orthogonal to **a**. Taking inner product with **a**, we have

$$\mathbf{a}\cdot\mathbf{b} = c\mathbf{a}\cdot\mathbf{a}.$$

Hence we obtain $c = (\mathbf{a}\cdot\mathbf{b})/(\mathbf{a}\cdot\mathbf{a})$. Thus the orthogonal projection is

$$\mathbf{p} = \frac{\mathbf{a}\cdot\mathbf{b}}{\|\mathbf{a}\|^2}\mathbf{a}.$$

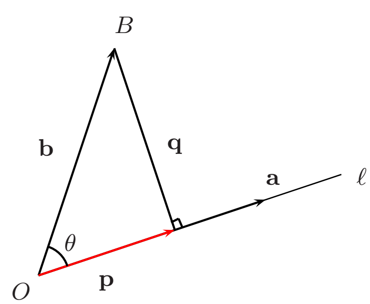



Figure 1.15: Projection of **b** onto **a**

The length of **p** is

$$\|\mathbf{p}\| = \frac{|\mathbf{a}\cdot\mathbf{b}|}{\|\mathbf{a}\|^2}\|\mathbf{a}\| = \frac{|\mathbf{a}\cdot\mathbf{b}|}{\|\mathbf{a}\|} = \|\mathbf{b}\|\cos\theta.$$

This agrees with the geometric interpretation.

**Definition 1.3.21.** For nonzero vector $\mathbf{b}$ and any vector $\mathbf{a}$, we define

$$proj_{\mathbf{a}}\mathbf{b} = \mathbf{p} = \frac{\mathbf{a}\cdot\mathbf{b}}{\|\mathbf{a}\|^2}\mathbf{a}.$$

We call it **orthogonal projection of b onto a**.

**Example 1.3.22.** $\mathbf{a} = 3\mathbf{i}+2\mathbf{j}-\mathbf{k}$, $\mathbf{b} = \mathbf{i}+\mathbf{j}+2\mathbf{k}$. Find orthogonal projection of $\mathbf{b}$ onto $\mathbf{a}$.

**sol.** The orthogonal projection is

$$\begin{aligned}\frac{\mathbf{a}\cdot\mathbf{b}}{\|\mathbf{a}\|^2}\mathbf{a} &= \frac{3\cdot 1+2\cdot 1+(-1)\cdot 2}{9+4+1}(3\mathbf{i}+2\mathbf{j}-\mathbf{k})\\ &= \frac{9}{14}\mathbf{i}+\frac{6}{14}\mathbf{j}-\frac{3}{14}\mathbf{k}.\end{aligned}$$

□

**Theorem 1.3.23.** *For any two nonzero* $\mathbf{u}$ *and* $\mathbf{v}$*, we can write* $\mathbf{v}$ *as the sum of two orthogonal vectors* $\mathbf{a}+\mathbf{b}$*, where* $\mathbf{a}$ *is the projection of* $\mathbf{v}$ *onto* $\mathbf{u}$ *and* $\mathbf{b}$ *is orthogonal to* $\mathbf{u}$*. This decomposition is unique.*

*Proof.* Denote by $\mathbf{a}$ the projection of $\mathbf{v}$ onto $\mathbf{u}$ and let $\mathbf{b} = \mathbf{v}-\mathbf{a}$. Then

$$\begin{aligned}\mathbf{a} &= \frac{\mathbf{u}\cdot\mathbf{v}}{\|\mathbf{u}\|^2}\mathbf{u}\\ \mathbf{b} &= \mathbf{v}-\frac{\mathbf{u}\cdot\mathbf{v}}{\|\mathbf{u}\|^2}\mathbf{u}.\end{aligned}$$

We can check $\mathbf{b}$ is orthogonal to $\mathbf{u}$:

$$\begin{aligned}\mathbf{u}\cdot\mathbf{b} &= \mathbf{u}\cdot\left(\mathbf{v}-\frac{\mathbf{u}\cdot\mathbf{v}}{\|\mathbf{u}\|^2}\mathbf{u}\right)\\ &= \mathbf{u}\cdot\mathbf{v}-\frac{\mathbf{u}\cdot\mathbf{v}}{\|\mathbf{u}\|^2}\,\mathbf{u}\cdot\mathbf{u}\\ &= \mathbf{u}\cdot\mathbf{v}-\mathbf{u}\cdot\mathbf{v} = 0\end{aligned}$$

This is an orthogonal decomposition. To see the uniqueness, assume there is real number $\alpha$ s.t. $\mathbf{v} = \alpha\mathbf{u}+\mathbf{c}$, with $\mathbf{u}\cdot\mathbf{c} = 0$. Then

$$\mathbf{u}\cdot\mathbf{v} = \mathbf{u}\cdot(\alpha\mathbf{u}+\mathbf{c}) = \alpha\mathbf{u}\cdot\mathbf{u}+\mathbf{u}\cdot\mathbf{c} = \alpha\|\mathbf{u}\|^2$$

we see

$$\alpha\mathbf{u} = \frac{\mathbf{u}\cdot\mathbf{v}}{\|\mathbf{u}\|^2}\mathbf{u} \;=\; \mathbf{a}$$
$$\mathbf{c} = \mathbf{v} - \alpha\mathbf{u} \;=\; \mathbf{v} - \mathbf{a} \;=\; \mathbf{b}.$$

Thus the decomposition of **v** along **u** and its orthogonal component is unique.

□

**Definition 1.3.24.** The vector **a** is called the **component parallel to u** and **b** is the **component orthogonal to u**.(**orthogonal complement**).

**Example 1.3.25.** Find the orthogonal decomposition of $\mathbf{v} = 3\mathbf{i}+5\mathbf{j}+\mathbf{k}$ w.r.t. $\mathbf{u} = \mathbf{i} + 2\mathbf{j} - \mathbf{k}$.

**sol.** Let **a** be the projection of **v** onto **u** and $\mathbf{b} = \mathbf{v} - \mathbf{a}$. Then

$$\begin{aligned}
\mathbf{a} &= \frac{\mathbf{u}\cdot\mathbf{v}}{\|\mathbf{u}\|^2}\mathbf{u} \\
&= \frac{1\cdot 3 + 2\cdot 5 + (-1)\cdot 1}{1+4+1}(\mathbf{i} + 2\mathbf{j} - \mathbf{k}) \\
&= 2\mathbf{i} + 4\mathbf{j} - 2\mathbf{k} \\
\mathbf{b} &= (3\mathbf{i} + 5\mathbf{j} + \mathbf{k}) - (2\mathbf{i} + 4\mathbf{j} - 2\mathbf{k}) \\
&= \mathbf{i} + \mathbf{j} + 3\mathbf{k}
\end{aligned}$$

Here **a** is parallel to **u**, **b** is orthogonal to **u** and $\mathbf{v} = \mathbf{a} + \mathbf{b}$.

□

## Triangle inequality

**Theorem 1.3.26.** *For any vectors* $\mathbf{a}, \mathbf{b}$*, we have*

$$\|\mathbf{a} + \mathbf{b}\| \le \|\mathbf{a}\| + \|\mathbf{b}\|.$$

Use C-S.

## Physical applications

Displacement : If an object has moved from $P$ to $Q$, then $\vec{PQ}$ is the displacement.

**Example 1.3.27.** A ship is running on the sea at the speed of $20km$ to north. but the current is flowing at the speed of $20km$ to the east, then in one hr, the displacement of the ship is $(20\sqrt{2}, 20\sqrt{2})$.

## 1.4 Matrices and Cross product

### Cross product

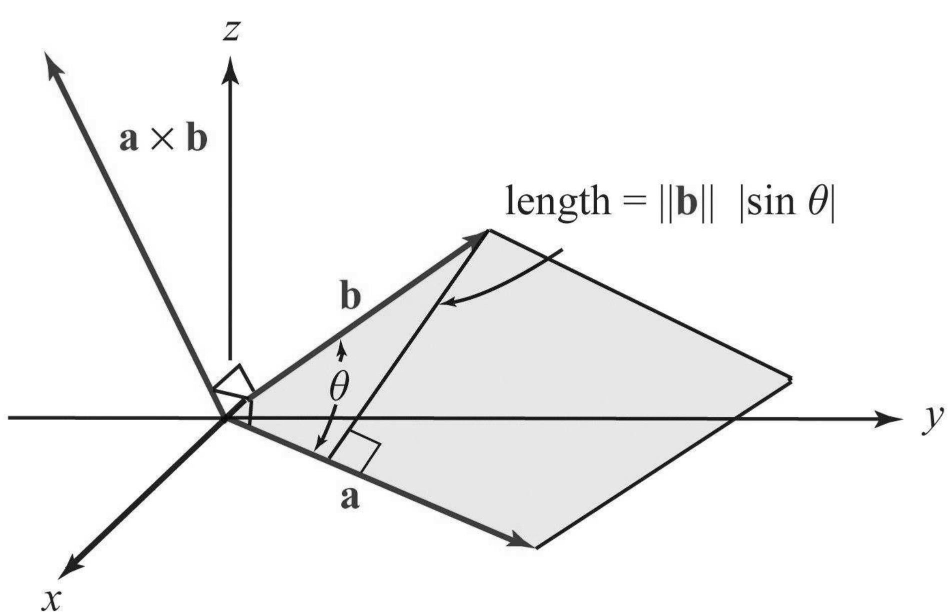
**Definition 1.4.1.** Let $\mathbf{a}, \mathbf{b}$ be two vectors in $\mathbb{R}^3$(not $\mathbb{R}^2$). The cross product of $\mathbf{a}, \mathbf{b}$, denoted by $\mathbf{a} \times \mathbf{b}$ is the vector whose length and direction are given as follows:

(1) The length is the area of the parallelogram spanned by $\mathbf{a}$ and $\mathbf{b}$.(zero if $\mathbf{a}, \mathbf{b}$ are parallel). Alternatively,

$$\|\mathbf{a} \times \mathbf{b}\| = \|\mathbf{a}\|\|\mathbf{b}\| \sin\theta,$$

where $\theta$ is the angle between $\mathbf{a}$ and $\mathbf{b}$.

(2) The direction of $\mathbf{a}\times\mathbf{b}$ is perpendicular to $\mathbf{a}$ and $\mathbf{b}$, and the triple $(\mathbf{a}, \mathbf{b}, \mathbf{a}\times\mathbf{b})$ form a right-handed set of vectors.




Algebraic rules:

(1) $\mathbf{a} \times \mathbf{b} = 0$, if $\mathbf{a}$, $\mathbf{b}$ are parallel or one of them is zero.

(2) $\mathbf{a} \times \mathbf{b} = -(\mathbf{b} \times \mathbf{a})$

(3) $\mathbf{a} \times (\mathbf{b} + \mathbf{c}) = \mathbf{a} \times \mathbf{b} + \mathbf{a} \times \mathbf{c}$

(4) $(\mathbf{a} + \mathbf{b}) \times \mathbf{c} = \mathbf{a} \times \mathbf{c} + \mathbf{b} \times \mathbf{c}$

(5) $(\alpha\mathbf{a}) \times \mathbf{b} = \alpha(\mathbf{a} \times \mathbf{b})$ for scalar $\alpha$.

Multiplication rules:

(1) $\mathbf{i} \times \mathbf{j} = \mathbf{k}, \quad \mathbf{j} \times \mathbf{k} = \mathbf{i}, \quad \mathbf{k} \times \mathbf{i} = \mathbf{j}$.

(2) $\mathbf{u} \times \mathbf{v} = -(\mathbf{v} \times \mathbf{u})$

(3) $\mathbf{i} \times \mathbf{i} = \mathbf{j} \times \mathbf{j} = \mathbf{k} \times \mathbf{k} = 0$

Note that

$$\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) \neq (\mathbf{a} \times \mathbf{b}) \times \mathbf{c}.$$

For example

$$\mathbf{i} \times (\mathbf{i} \times \mathbf{j}) = \mathbf{i} \times \mathbf{k} = -\mathbf{j} \neq (\mathbf{i} \times \mathbf{i}) \times \mathbf{j} = 0.$$

### $2 \times 2$ matrix

The array of numbers $a_{11},\ a_{12},\ a_{21},\ a_{22}$ in the form

$$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$$

is called $2 \times 2$ **matrix** and

$$[a_{11}\ a_{12}], \qquad \begin{bmatrix} a_{12} \\ a_{22} \end{bmatrix}$$

are the first row and second column. The real number $a_{11}a_{22} - a_{12}a_{21}$ is **determinant** and denoted by

$$det(A) = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}$$

**Example 1.4.2.** Find determinant of $2 \times 2$ matrices.

$$\begin{vmatrix} 3 & 4 \\ 2 & 1 \end{vmatrix} = 3 - 8 = -5, \quad \begin{vmatrix} 0 & 3 \\ -1 & 1 \end{vmatrix} = 0 - (-3) = 3, \quad \begin{vmatrix} 1 & 2 \\ -2 & 1 \end{vmatrix} = 1 - (-4) = 5$$

**Proposition 1.4.3.** *The area of parallelogram determined by the two vectors* $a\mathbf{i} + b\mathbf{j}$ *and* $c\mathbf{i} + d\mathbf{j}$ *is* $|ad - bc|$. *This is the absolute value of the determinant of the matrix determined by two two vectors:*

$$det \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

*Proof.* Let $\mathbf{u} = a\mathbf{i} + b\mathbf{j}$, $\mathbf{v} = c\mathbf{i} + d\mathbf{j}$ and $\theta$ be the angle between them. Then the area of the parallelogram is

$$\begin{aligned} \|\mathbf{u}\| \, \|\mathbf{v}\| \, \sin\theta &= \|\mathbf{u}\| \, \|\mathbf{v}\| \sqrt{1 - \cos^2\theta} \\ &= \sqrt{\|\mathbf{u}\|^2 \, \|\mathbf{v}\|^2 - (\mathbf{u} \cdot \mathbf{v})^2} \\ &= \sqrt{(a^2 + b^2)(c^2 + d^2) - (ac + bd)^2} \\ &= \sqrt{a^2 d^2 + b^2 c^2 - 2abcd} \\ &= |ad - bc| \end{aligned}$$

□

## $3 \times 3$ matrix

A typical $3 \times 3$ matrix is given by

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}.$$

Here

$$[a_{31} \; a_{32} \; a_{33}], \qquad \begin{bmatrix} a_{12} \\ a_{22} \\ a_{32} \end{bmatrix}$$

are third row and second column. The **determinant** is defined as

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = a_{11}\begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - a_{12}\begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + a_{13}\begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix} \tag{1.2}$$

**Example 1.4.4.**

$$\begin{vmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{vmatrix} = 1\begin{vmatrix} 5 & 6 \\ 8 & 9 \end{vmatrix} - 2\begin{vmatrix} 4 & 6 \\ 7 & 9 \end{vmatrix} + 3\begin{vmatrix} 4 & 5 \\ 7 & 8 \end{vmatrix} = -3 + 12 - 9 = 0$$

$$\begin{vmatrix} 1 & 1 & 1 \\ 2 & 4 & 8 \\ 3 & 9 & 27 \end{vmatrix} = 1\begin{vmatrix} 4 & 8 \\ 9 & 27 \end{vmatrix} - 1\begin{vmatrix} 2 & 8 \\ 3 & 27 \end{vmatrix} + 1\begin{vmatrix} 2 & 4 \\ 3 & 9 \end{vmatrix} = 36 - 30 + 6 = 12$$

**Definition 1.4.5.** If we exchange rows and columns of the following matrices

$$\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}, \qquad \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

to get

$$\begin{bmatrix} a_{11} & a_{21} \\ a_{12} & a_{22} \end{bmatrix}, \qquad \begin{bmatrix} a_{11} & a_{21} & a_{31} \\ a_{12} & a_{22} & a_{32} \\ a_{13} & a_{23} & a_{33} \end{bmatrix}$$

then resulting matrices are called the **transpose**.

## Properties of determinant-skip for the time being

**Theorem 1.4.6.** *(1) Determinant of transposed matrix is the same the Determinant of original matrix.*

*(2) If we exchange any two rows(columns), then determinant changes signs.*

*(3)* $|det(\alpha A)| = \alpha^n |det(A)|$

*(4) Adding a scalar multiple of row (column) to another row (column) does not change determinant.*

*Proof.* (1) For $2\times 2$

$$\begin{vmatrix} a_{11} & a_{21} \\ a_{12} & a_{22} \end{vmatrix} = a_{11}a_{22} - a_{21}a_{12} = a_{11}a_{22} - a_{12}a_{21} = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}$$

For $3\times 3$

$$\begin{aligned}
\begin{vmatrix} a_{11} & a_{21} & a_{31} \\ a_{12} & a_{22} & a_{32} \\ a_{13} & a_{23} & a_{33} \end{vmatrix} &= a_{11}\begin{vmatrix} a_{22} & a_{32} \\ a_{23} & a_{33} \end{vmatrix} - a_{21}\begin{vmatrix} a_{12} & a_{32} \\ a_{13} & a_{33} \end{vmatrix} + a_{31}\begin{vmatrix} a_{12} & a_{22} \\ a_{13} & a_{23} \end{vmatrix} \\
&= a_{11}(a_{22}a_{33} - a_{32}a_{23}) - a_{21}(a_{12}a_{33} - a_{32}a_{13}) \\
&\qquad + a_{31}(a_{12}a_{23} - a_{22}a_{13}) \\
&= a_{11}(a_{22}a_{33} - a_{32}a_{23}) - a_{21}a_{12}a_{33} + a_{21}a_{32}a_{13} \\
&\qquad + a_{31}a_{12}a_{23} - a_{31}a_{22}a_{13} \\
&= a_{11}(a_{22}a_{33} - a_{23}a_{32}) - a_{12}a_{21}a_{33} + a_{12}a_{23}a_{31} \\
&\qquad + a_{13}a_{21}a_{32} - a_{13}a_{22}a_{31} \\
&= a_{11}(a_{22}a_{33} - a_{23}a_{32}) - a_{12}(a_{21}a_{33} - a_{23}a_{31}) \\
&\qquad + a_{13}(a_{21}a_{32} - a_{22}a_{31}) \\
&= a_{11}\begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - a_{12}\begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + a_{13}\begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix} \\
&= \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}
\end{aligned}$$

(4) $2\times 2$ case is easy.

For $3\times 3$, we see by expanding w.r.t. first row

$$\begin{vmatrix} a_{11}+ta_{21} & a_{12}+ta_{22} & a_{13}+ta_{23} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$$

$$\begin{aligned}
&= (a_{11}+ta_{21})\begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - (a_{12}+ta_{22})\begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} \\
&\qquad + (a_{13}+ta_{23})\begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}
\end{aligned}$$

$$= a_{11}\begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - a_{12}\begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + a_{13}\begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}$$
$$+ t\left( a_{21}\begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - a_{22}\begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + a_{23}\begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}\right)$$
$$= \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} + t\begin{vmatrix} a_{21} & a_{22} & a_{23} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$$

Exchange second and third rows, do not change the value. By (2) there must be a sign change. Hence it is 0.

$$\begin{vmatrix} a_{11}+ta_{21} & a_{12}+ta_{22} & a_{13}+ta_{23} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$$

Hence (4) holds. □

The RHS of 1.2 is expansion w.r.t **first row.** By theorem 1.4.6, (1), (2), we can expand w.r.t. any row or column, except we multiply $(-1)^{i+j}$. So if we expand w.r.t 2nd row

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = -a_{21}\begin{vmatrix} a_{12} & a_{13} \\ a_{32} & a_{33} \end{vmatrix} + a_{22}\begin{vmatrix} a_{11} & a_{13} \\ a_{31} & a_{33} \end{vmatrix} - a_{23}\begin{vmatrix} a_{11} & a_{12} \\ a_{31} & a_{32} \end{vmatrix}$$

If we expand w.r.t 3rd column

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = a_{13}\begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix} - a_{23}\begin{vmatrix} a_{11} & a_{12} \\ a_{31} & a_{32} \end{vmatrix} + a_{33}\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}$$

**Corollary 1.4.7.** *(1) Determinant of a matrix one of whose row is zero is zero.*
*(2) If any two rows (columns) are equal, the determinant is zero.*

**Example 1.4.8.** The followings are expanded w.r.t 2nd, 3rd row.

$$\begin{vmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{vmatrix} = -2\begin{vmatrix} 4 & 6 \\ 7 & 9 \end{vmatrix} + 5\begin{vmatrix} 1 & 3 \\ 7 & 9 \end{vmatrix} - 8\begin{vmatrix} 1 & 3 \\ 4 & 6 \end{vmatrix} = 12 - 60 + 48 = 0$$

$$\begin{vmatrix} 1 & 2 & 3 \\ 4 & 0 & 6 \\ 0 & 8 & 0 \end{vmatrix} = 0\begin{vmatrix} 2 & 3 \\ 0 & 6 \end{vmatrix} - 8\begin{vmatrix} 1 & 3 \\ 4 & 6 \end{vmatrix} + 0\begin{vmatrix} 1 & 2 \\ 4 & 0 \end{vmatrix} = 0 + 48 + 0 = 48$$

### Cross product-using determinant

In the previous section, we have defined the cross product using geometry, but did not show how to compute it. Now we can give a formula for the cross product using the determinant:

**Definition 1.4.9** (Alternative definition)**.** For $\mathbf{a} = a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k}$ and $\mathbf{b} = b_1\mathbf{i} + b_2\mathbf{j} + b_3\mathbf{k}$, the **cross product** $\mathbf{a} \times \mathbf{b}$ is defined by

$$\mathbf{a} \times \mathbf{b} = \begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix}\mathbf{i} - \begin{vmatrix} a_1 & a_3 \\ b_1 & b_3 \end{vmatrix}\mathbf{j} + \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix}\mathbf{k}. \tag{1.3}$$

Using the definition of determinant (1.2) symbolically, we have

$$\mathbf{a} \times \mathbf{b} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$$

**Example 1.4.10.** $\mathbf{i} \times \mathbf{i} = \mathbf{0}, \quad \mathbf{j} \times \mathbf{j} = \mathbf{0}, \quad \mathbf{k} \times \mathbf{k} = \mathbf{0}$.

**Example 1.4.11.** Compute $(2\mathbf{i} - \mathbf{j} + 3\mathbf{k}) \times (\mathbf{i} + \mathbf{j} + 2\mathbf{k})$.

**sol.** By the definition of cross product, we see

$$(2\mathbf{i} - \mathbf{j} + 3\mathbf{k}) \times (\mathbf{i} + \mathbf{j} + 2\mathbf{k}) = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & -1 & 3 \\ 1 & 1 & 2 \end{vmatrix} = -5\mathbf{i} - \mathbf{j} + 3\mathbf{k}$$

□

### Triple product

Let

$$\mathbf{a} = a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k}, \mathbf{b} = b_1\mathbf{i} + b_2\mathbf{j} + b_3\mathbf{k}, \mathbf{c} = c_1\mathbf{i} + c_2\mathbf{j} + c_3\mathbf{k}.$$

To see the relation with the geometric definition of the cross product, we consider the dot product between $(\mathbf{a}\times\mathbf{b})$ and $\mathbf{c}$. It is dot product $(\mathbf{a}\times\mathbf{b})\cdot\mathbf{c}$, called the triple product $(\mathbf{a}\times\mathbf{b})\cdot\mathbf{c}$ of three vectors, $\mathbf{a},\mathbf{b}$ and $\mathbf{c}$. We see

$$\begin{aligned}(\mathbf{a}\times\mathbf{b})\cdot\mathbf{c} &= \left(\begin{vmatrix} a_2 & a_3 \\ b_2 & b_3\end{vmatrix}\mathbf{i} - \begin{vmatrix} a_1 & a_3 \\ b_1 & b_3\end{vmatrix}\mathbf{j} + \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2\end{vmatrix}\mathbf{k}\right)\cdot(c_1\mathbf{i}+c_2\mathbf{j}+c_3\mathbf{k}) \\ &= \begin{vmatrix} a_2 & a_3 \\ b_2 & b_3\end{vmatrix}c_1 - \begin{vmatrix} a_1 & a_3 \\ b_1 & b_3\end{vmatrix}c_2 + \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2\end{vmatrix}c_3 \\ &= \begin{vmatrix} c_1 & c_2 & c_3 \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3\end{vmatrix} = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3\end{vmatrix}\end{aligned}$$

1) A **geometric meaning** follows: If $\mathbf{c}$ is a vector in the plane spanned by $\mathbf{a},\mathbf{b}$, then the third row in the determinant is a linear combination of the first and second row, and hence $(\mathbf{a}\times\mathbf{b})\cdot\mathbf{c}=0$. In other words, *the vector* $\mathbf{a}\times\mathbf{b}$ *is orthogonal to any vector in the plane spanned by* $\mathbf{a}$ *and* $\mathbf{b}$.

2) We compute length of $\mathbf{a}\times\mathbf{b}$.

$$\begin{aligned}\|\mathbf{a}\times\mathbf{b}\|^2 &= \begin{vmatrix} a_2 & a_3 \\ b_2 & b_3\end{vmatrix}^2 + \begin{vmatrix} a_1 & a_3 \\ b_1 & b_3\end{vmatrix}^2 + \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2\end{vmatrix}^2 \\ &= (a_2b_3-a_2b_2)^2 + (a_1b_3-b_1a_3)^2 + (a_1b_2-b_1a_2)^2 \\ &= (a_1^2+a_2^2+a_3^2)(b_1^2+b_2^2+b_3^2) - (a_1b_1+a_2b_2+a_3b_3)^2\end{aligned}$$

Hence

$$\|\mathbf{a}\times\mathbf{b}\|^2 = \|\mathbf{a}\|^2\|\mathbf{b}\|^2 - (\mathbf{a}\cdot\mathbf{b})^2 = \|\mathbf{a}\|^2\|\mathbf{b}\|^2(1-\cos^2\theta) = \|\mathbf{a}\|^2\|\mathbf{b}\|^2\sin^2\theta.$$

So we conclude that $\mathbf{a}\times\mathbf{b}$ is a vector perpendicular to the plane $\mathcal{P}$ spanned by $\mathbf{a}$ and $\mathbf{b}$ with length $\|\mathbf{a}\|\|\mathbf{b}\||\sin\theta|$. Hence we obtain

**Theorem 1.4.12** (Alternative cross Product)**.** *For* $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$, *it holds that*

*(1)* $\|\mathbf{a}\times\mathbf{b}\| = \|\mathbf{a}\|\|\mathbf{b}\|\sin\theta$, *the area of the parallelogram spanned by* $\mathbf{a}$ *and* $\mathbf{b}$.

*(2)* $\mathbf{a}\times\mathbf{b}$ *is perpendicular to* $\mathbf{a}$ *and* $\mathbf{b}$, *and the triple* $(\mathbf{a},\mathbf{b},\mathbf{a}\times\mathbf{b})$ *form a right-handed rule.*

*(3)* $(\mathbf{a}\times\mathbf{b})\cdot\mathbf{c}$ *is the volume of parallelepiped formed by three vectors* $\mathbf{a},\mathbf{b}$

*and* $\mathbf{c}$.*(See below)*

*Hence this alternative definition is the same as geometric definition of cross product given earlier.*

### Component formula using determinant

$$(a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k}) \times (b_1\mathbf{i} + b_2\mathbf{j} + b_3\mathbf{k}) = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$$
$$= (a_2b_3 - a_3b_2)\mathbf{i} - (a_1b_3 - a_3b_1)\mathbf{j} + (a_1b_2 - a_2b_1)\mathbf{k}$$

*Algebraic rules:*

*(1)* $\mathbf{a} \times \mathbf{b} = 0$, *iff* $\mathbf{a}$, $\mathbf{b}$ *are parallel or zero.*

*(2)* $\mathbf{a} \times \mathbf{b} = -(\mathbf{b} \times \mathbf{a})$

*(3)* $\mathbf{a} \times (\mathbf{b} + \mathbf{c}) = \mathbf{a} \times \mathbf{b} + \mathbf{a} \times \mathbf{c}$

*(4)* $(\mathbf{a} + \mathbf{b}) \times \mathbf{c} = \mathbf{a} \times \mathbf{c} + \mathbf{b} \times \mathbf{c}$

*(5)* $(\alpha\mathbf{a}) \times \mathbf{b} = \alpha(\mathbf{a} \times \mathbf{b})$.

*Multiplication rules:*

*(1)* $\mathbf{i} \times \mathbf{j} = \mathbf{k}, \quad \mathbf{j} \times \mathbf{k} = \mathbf{i}, \quad \mathbf{k} \times \mathbf{i} = \mathbf{j}$.

*(2)* $\mathbf{u} \times \mathbf{v} = -(\mathbf{v} \times \mathbf{u})$

*(3)* $\mathbf{i} \times \mathbf{i} = \mathbf{j} \times \mathbf{j} = \mathbf{k} \times \mathbf{k} = 0$

**Example 1.4.13.** Find $(\mathbf{i} + \mathbf{j}) \times (\mathbf{j} - 2\mathbf{k})$.

**sol.** $(\mathbf{i} + \mathbf{j}) \times (\mathbf{j} - 2\mathbf{k}) = \mathbf{i} \times j - 2\mathbf{i} \times k + \mathbf{j} \times j - 2\mathbf{j} \times k = -2\mathbf{i} + 2\mathbf{j} + \mathbf{k}$.

□

**Theorem 1.4.14** (Cross product II)**.**

*(1)* $\|\mathbf{u} \times \mathbf{v}\|^2 = \|\mathbf{u}\|^2\|\mathbf{v}\|^2 - (\mathbf{u} \cdot \mathbf{v})^2$. *In particular,* $\mathbf{u} \times \mathbf{u} = \mathbf{0}$.

*(2) If is $\theta$ the angle between $\mathbf{u}$ and $\mathbf{v}$, $\|\mathbf{u}\times\mathbf{v}\| = \|\mathbf{u}\|\,\|\mathbf{v}\|\,\sin\theta$. Hence nec. and suff. condition for $\mathbf{u}$ and $\mathbf{v}$ are parallel is $\mathbf{u}\times\mathbf{v} = \mathbf{0}$.*

*(3)* $(\mathbf{u}\times\mathbf{v})\cdot\mathbf{w} = \mathbf{u}\cdot(\mathbf{v}\times\mathbf{w})$

*(4)* $(\mathbf{u}\times\mathbf{v})\cdot\mathbf{u} = (\mathbf{u}\times\mathbf{v})\cdot\mathbf{v} = 0$, *i.e, $\mathbf{u}\times\mathbf{v}$ is orthogonal to $\mathbf{u}$ and $\mathbf{v}$.*

*Proof.* Let $\mathbf{u} = a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k}$, $\mathbf{v} = b_1\mathbf{i} + b_2\mathbf{j} + b_3\mathbf{k}$, $\mathbf{w} = c_1\mathbf{i} + c_2\mathbf{j} + c_3\mathbf{k}$.

(1) $\|\mathbf{u}\times\mathbf{v}\|^2 = \|\mathbf{u}\|^2\|\mathbf{v}\|^2 - (\mathbf{u}\cdot\mathbf{v})^2$ was shown before.

So $\mathbf{u}\times\mathbf{u} = \mathbf{0}$.

(2) Since $\mathbf{u}\cdot\mathbf{v} = \|\mathbf{u}\|\,\|\mathbf{v}\|\,\cos\theta$, we have by (1)

$$\begin{aligned}\|\mathbf{u}\times\mathbf{v}\| &= \sqrt{\|\mathbf{u}\|^2\|\mathbf{v}\|^2 - (\mathbf{u}\cdot\mathbf{v})^2} \\ &= \|\mathbf{u}\|\,\|\mathbf{v}\|\sqrt{1-\cos^2\theta} \\ &= \|\mathbf{u}\|\,\|\mathbf{v}\|\,\sin\theta\end{aligned}$$

(3)

$$\begin{aligned}(\mathbf{u}\times\mathbf{v})\cdot\mathbf{w} &= \left(\begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix}\mathbf{i} - \begin{vmatrix} a_1 & a_3 \\ b_1 & b_3 \end{vmatrix}\mathbf{j} + \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix}\mathbf{k}\right)\cdot(c_1\mathbf{i} + c_2\mathbf{j} + c_3\mathbf{k}) \\ &= \begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix} c_1 - \begin{vmatrix} a_1 & a_3 \\ b_1 & b_3 \end{vmatrix} c_2 + \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix} c_3 \\ &= \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}\end{aligned}$$

Now exapnding w.r.t first row

$$\begin{aligned}&= a_1\begin{vmatrix} b_2 & b_3 \\ c_2 & c_3 \end{vmatrix} - a_2\begin{vmatrix} b_1 & b_3 \\ c_1 & c_3 \end{vmatrix} + a_3\begin{vmatrix} b_1 & b_2 \\ c_1 & c_2 \end{vmatrix} \\ &= (a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k})\cdot\left(\begin{vmatrix} b_2 & b_3 \\ c_2 & c_3 \end{vmatrix}\mathbf{i} - \begin{vmatrix} b_1 & b_3 \\ c_1 & c_3 \end{vmatrix}\mathbf{j} + \begin{vmatrix} b_1 & b_2 \\ c_1 & c_2 \end{vmatrix}\mathbf{k}\right) \\ &= \mathbf{u}\cdot(\mathbf{v}\times\mathbf{w})\end{aligned}$$

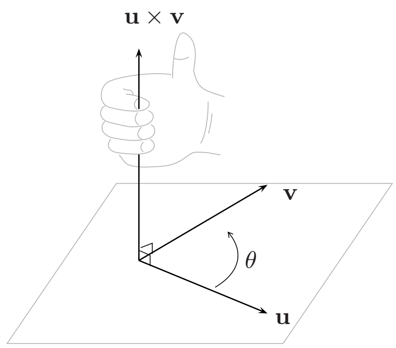



Figure 1.16: right handed rule

(4) Using (3) and corollary 1.4.7, we see

$$(\mathbf{u}\times\mathbf{v})\cdot\mathbf{u} = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ a_1 & a_2 & a_3 \end{vmatrix} = 0$$

$$(\mathbf{u}\times\mathbf{v})\cdot\mathbf{v} = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ b_1 & b_2 & b_3 \end{vmatrix} = 0$$

□

### Geometric Meaning of Cross Product

### Geometry of Determinant

$2\times 2$ matrix: If $\mathbf{a} = (a_1, a_2)$ and $\mathbf{b} = (b_1, b_2)$ then

$$\mathbf{a}\times\mathbf{b} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a_1 & a_2 & 0 \\ b_1 & b_2 & 0 \end{vmatrix} = \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix}\mathbf{k}$$

$\|\mathbf{a}\times\mathbf{b}\|$ is the absolute value of the determinant and is the area of parallelogram determined by two vectors.

$$\begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix} = a_1b_2 - a_2b_1.$$

**Example 1.4.15.** Find the area of triangle with vertices at $(1,1),(0,2)$ and $(3,2)$. Sol. Two sides are $(0,2)-(1,1)=(-1,1)$ and $(3,2)-(1,1)=(2,1)$. Thus $\frac{1}{2}\begin{vmatrix} -1 & 1 \\ 2 & 1 \end{vmatrix} = -\frac{3}{2}$.

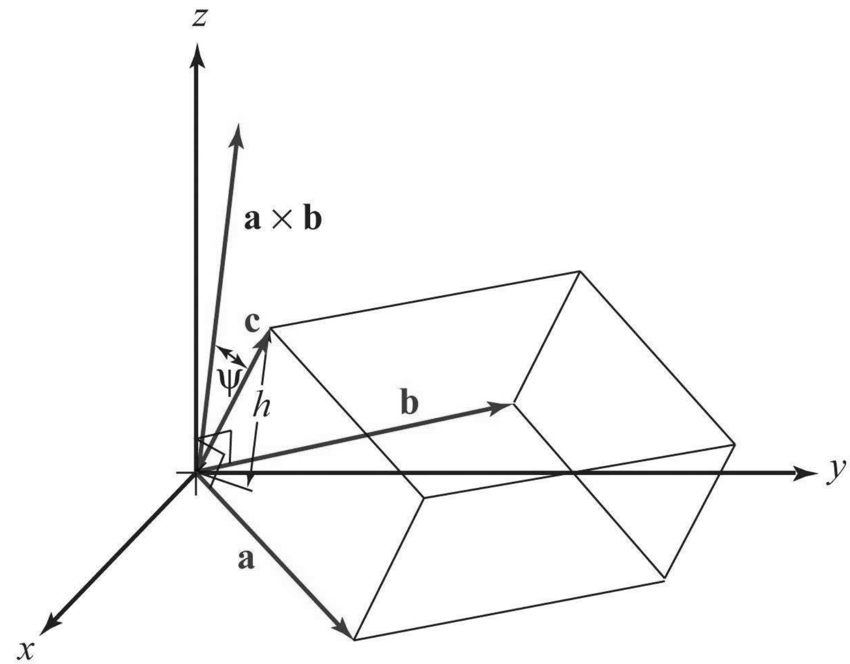
**Proposition 1.4.16.** *The volume of parallelepiped with sides* $\mathbf{a}$*,* $\mathbf{b}$*,* $\mathbf{c}$ *is give by the absolute value of triple product* $(\mathbf{a}\times\mathbf{b})\cdot\mathbf{c}$ *which is the determinant*

$$|(\mathbf{a}\times\mathbf{b})\cdot\mathbf{c}| = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}.$$

*Proof.* Consider a parallelogram with two sides $\mathbf{a}$, $\mathbf{b}$ as bottom of the parallelepiped. Then the height is length of the orthogonal projection of $\mathbf{c}$ onto $\mathbf{a}\times\mathbf{b}$ which is $\left\|\frac{(\mathbf{a}\times\mathbf{b})\cdot\mathbf{c}}{\|\mathbf{a}\times\mathbf{b}\|^2}\mathbf{a}\times\mathbf{b}\right\|$. Hence the volume is

$$Area(bottom)\times height = \|\mathbf{a}\times\mathbf{b}\|\left\|\frac{(\mathbf{a}\times\mathbf{b})\cdot\mathbf{c}}{\|\mathbf{a}\times\mathbf{b}\|^2}\mathbf{a}\times\mathbf{b}\right\| = |(\mathbf{a}\times\mathbf{b})\cdot\mathbf{c}|$$

□




**Example 1.4.17.** Three points $A(1,2,3)$, $B(0,1,2)$, $C(0,3,2)$ are given. Find the volume of hexahedron having three vectors $OA$, $OB$, $OC$ as sides.

sol. By proposition 1.4.16, we have

$$\begin{vmatrix} 1 & 2 & 3 \\ 0 & 1 & 2 \\ 0 & 3 & 2 \end{vmatrix} = 1\begin{vmatrix} 1 & 2 \\ 3 & 2 \end{vmatrix} - 0\begin{vmatrix} 2 & 3 \\ 3 & 2 \end{vmatrix} + 0\begin{vmatrix} 2 & 3 \\ 1 & 2 \end{vmatrix} = -4$$

□

## Torque

Imagine we are trying to fasten a bolt with a wrench. If one apply the force **F** as the figure, we see the amount force acting to the action of bolt is $\|\mathbf{r}\|\|\mathbf{F}\|\sin\theta$.

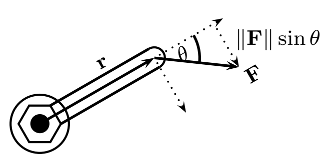



Figure 1.17: Turning a hexagonal bolt with a wrench with force **F**. Torque vector is $\mathbf{r} \times \mathbf{F}$.

Then

$$\text{Amount of Torque } = \|\mathbf{r}\|\|\mathbf{F}\|\sin\theta = \|\mathbf{r}\times\mathbf{F}\|.$$

Also, the direction of the vector $\mathbf{r}\times\mathbf{F}$ is the same direction as the bolt moves. Hence it is natural to define $\mathbf{r}\times\mathbf{F}$ to be the torque vector.

### Rotation of a rigid body

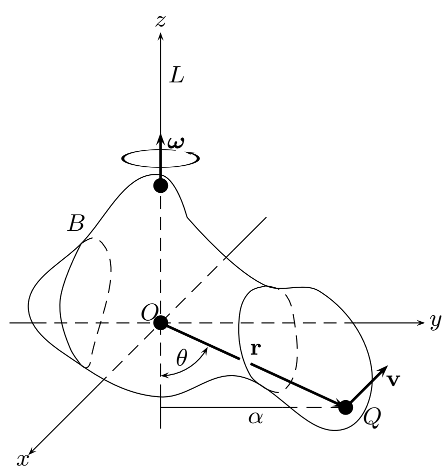



Figure 1.18: velocity $\mathbf{v}$ and angular velocity $\boldsymbol{\omega}$ has relation $\mathbf{v}=\boldsymbol{\omega}\times\mathbf{r}$.

Consider a rigid body $B$ rotating about an axis $L$. (See fig 1.18 ). **The rotational motion of $B$ can be described by a vector along axis of rotation $\boldsymbol{\omega}$.** Let $\boldsymbol{\omega}$ the vector along $z$-axis s.t. $\omega = ||\boldsymbol{\omega}||$. The vector $\boldsymbol{\omega}$ is called the **angular velocity vector** and $\omega$ is angular speed. $\omega = ||\boldsymbol{\omega}||$

Assume $L$ is $z$-axis $Q$ is any point on the body $B$, $\alpha$ is distance from $Q$ to $L$. Then $\alpha = ||\mathbf{r}||\sin\theta$ ($\mathbf{r}$ points to $Q$). Consider the tangent vector $\mathbf{v}$ at $Q$. Since $Q$ moves around a circle of radius $\alpha$ perpendicular to $\boldsymbol{\omega}$ (parallel to $xy$-plane, counterclockwise), we see,

$$\begin{aligned}\Delta\mathbf{r} &\approx \text{(radius of circle )( angle swept by } Q)\\ &= ||\mathbf{r}||\sin\theta(\Delta\phi)\end{aligned}$$

Thus

$$\left\|\frac{\Delta\mathbf{r}}{\Delta t}\right\| \approx ||\mathbf{r}||\sin\theta\frac{\Delta\phi}{\Delta t}.$$

As $\Delta t \to 0$, we obtain the (line) velocity and angular velocity by

$$\mathbf{v} = \lim_{\Delta t \to 0} \frac{\Delta \mathbf{r}}{\Delta t}, \quad \omega = \lim_{\Delta t \to 0} \frac{\Delta \phi}{\Delta t}$$

Hence

$$||\mathbf{v}|| = \omega \alpha = \omega ||\mathbf{r}|| \sin\theta = ||\boldsymbol{\omega}|| ||\mathbf{r}|| \sin\theta. \tag{1.4}$$

Then by definition of cross product,

$$\mathbf{v} = \boldsymbol{\omega} \times \mathbf{r}. \tag{1.5}$$

## 1.5 Equations of Planes

Let $\mathcal{P}$ be a plane and $P_0 = (x_0, y_0, z_0)$ a point on that plane, and suppose that $\mathbf{n} = A\mathbf{i} + B\mathbf{j} + C\mathbf{k}$ is a normal vector. Let $P = (x, y, z)$ be any point in $\mathbb{R}^3$. Then $P$ lies in the plane iff the vector $\overrightarrow{P_0P} = (x - x_0, y - y_0, z - z_0)$ is perpendicular to $\mathbf{n}$, that is, $\overrightarrow{P_0P} \cdot \mathbf{n} = 0$. In other words,

$$(A\mathbf{i} + B\mathbf{j} + C\mathbf{k}) \cdot [(x - x_0)\mathbf{i} + (y - y_0)\mathbf{j} + (z - z_0)\mathbf{k}] = 0.$$

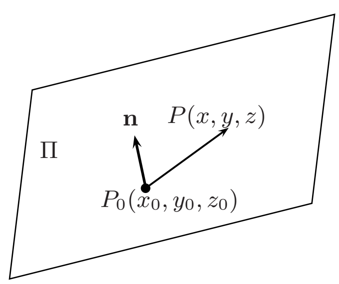



Figure 1.19: A plane is det'd by a point and normal vector

**Proposition 1.5.1.** *Equation of plane through* $(x_0, y_0, z_0)$ *that has normal vector* $\mathbf{n}$ *is*

$$A(x - x_0) + B(y - y_0) + C(z - z_0) = 0$$

*or*

$$Ax + By + Cz - D = 0$$

*where* $D = -(Ax_0 + By_0 + Cz_0)$.

**Example 1.5.2.** Find the equation of plane through the points $A(-3,0,-1)$, $B(-2,3,2)$, $C(1,1,3)$.

sol. Draw some graph describing the normal vector.

Find a vector $\mathbf{n}$ orthogonal to plane.

$$\begin{aligned}\mathbf{n} &= \overrightarrow{AB} \times \overrightarrow{AC}\\ &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -2-(-3) & 3-0 & 2-(-1) \\ 1-(-3) & 1-0 & 3-(-1) \end{vmatrix}\\ &= \begin{vmatrix} 3 & 3 \\ 1 & 4 \end{vmatrix}\mathbf{i} - \begin{vmatrix} 1 & 3 \\ 4 & 4 \end{vmatrix}\mathbf{j} + \begin{vmatrix} 1 & 3 \\ 4 & 1 \end{vmatrix}\mathbf{k}\\ &= 9\mathbf{i} + 8\mathbf{j} - 11\mathbf{k}\end{aligned}$$

By proposition 1.5.1, the equation is

$$9(x+3) + 8(y-0) - 11(z+1) = 0$$

or $9x + 8y - 11z + 16 = 0$.

□

## Distance from a point to plane

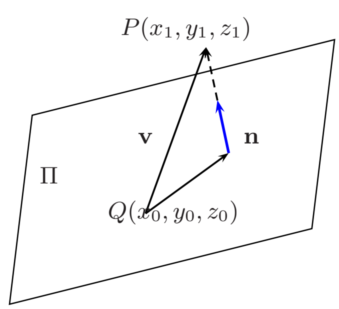



Figure 1.20: Distance from a point to plane

**Proposition 1.5.3.** *The distance from $P(x_1, y_1, z_1)$ to the plane $Ax + By + Cz + D = 0$ is*

$$\frac{|Ax_1 + By_1 + Cz_1 + D|}{\sqrt{A^2 + B^2 + C^2}}$$

*Proof.* Let $\mathbf{n}$ be a normal vector to the plane. If $Q(x_0, y_0, z_0)$ lies in the plane, the distance from $P$ to the plane is the orthogonal projection of $\overrightarrow{PQ}$ along $\mathbf{n}$. Note that from $A(x-x_0)+B(y-y_0)+C(z-z_0)=0$, we see $\mathbf{n}//A\mathbf{i}+B\mathbf{j}+C\mathbf{k}$. Hence length of the orthogonal projection of $\overrightarrow{PQ}$ along $\mathbf{n}$ is

$$\begin{aligned}
\left\| \frac{\mathbf{n}\cdot\overrightarrow{PQ}}{\|\mathbf{n}\|^2}\mathbf{n} \right\| &= \frac{|\mathbf{n}\cdot\overrightarrow{PQ}|}{\|\mathbf{n}\|} \\
&= \frac{|A(x_0-x_1)+B(y_0-y_1)+C(z_0-z_1)|}{\sqrt{A^2+B^2+C^2}} \\
&= \frac{|Ax_0+By_0+Cz_0-Ax_1-By_1-Cz_1|}{\sqrt{A^2+B^2+C^2}} \\
&= \frac{|-D-Ax_1-By_1-Cz_1|}{\sqrt{A^2+B^2+C^2}} = \frac{|Ax_1+By_1+Cz_1+D|}{\sqrt{A^2+B^2+C^2}}
\end{aligned}$$

□

**Example 1.5.4.** Find the distance from $(3, 4, -2)$ to the plane $2x-y+z-4=0$.

**sol.** Using above proposition, distance is

$$\frac{|2\cdot 3 - 1\cdot 4 + 1\cdot(-2) - 4|}{\sqrt{4+1+1}} = \frac{|-4|}{\sqrt{6}} = \frac{2\sqrt{6}}{3}$$

□

**Example 1.5.5.** Find a unit vector perpendicular to the plane $4x-3y+z-4=0$ and express it as a cross product of two unit orthogonal vectors lying in the plane.

**sol.** Let $\mathcal{S}$ the given plane. By proposition 1.5.1 we see $4\mathbf{i}-3\mathbf{j}+\mathbf{k}$ is orthogonal to $\mathcal{S}$. Hence

$$\mathbf{n} = \pm\frac{4\mathbf{i}-3\mathbf{j}+\mathbf{k}}{\sqrt{4^2+(-3)^2+1^2}} = \pm\frac{1}{\sqrt{26}}(4\mathbf{i}-3\mathbf{j}+\mathbf{k}).$$

Now in order to express this as a cross product of two vectors lying in the plane, we need to choose three points $(1,0,0), (0,0,4), (2,1,-1)$ in $\mathcal{S}$. Then we obtain two vectors

$$\mathbf{u} = (1,0,0) - (2,1,-1) = -\mathbf{i} - \mathbf{j} + \mathbf{k}$$
$$\mathbf{v} = (0,0,4) - (2,1,-1) = -2\mathbf{i} - \mathbf{j} + 5\mathbf{k}$$

which are parallel to the plane $\mathcal{S}$. Let us orthogonalize them.Let $\mathbf{a}$ be the orthogonal projection of $\mathbf{v}$ onto $\mathbf{u}$. Then let $\mathbf{b} = \mathbf{v} - \mathbf{a}$.

$$\begin{aligned}
\mathbf{a} &= \frac{\mathbf{u}\cdot\mathbf{v}}{\|\mathbf{u}\|^2}\mathbf{u} = \frac{8}{3}(-\mathbf{i} - \mathbf{j} + \mathbf{k}) \\
\mathbf{b} &= \mathbf{v} - \frac{\mathbf{u}\cdot\mathbf{v}}{\|\mathbf{u}\|^2}\mathbf{u} = (-2\mathbf{i} - \mathbf{j} + 5\mathbf{k}) - \frac{8}{3}(-\mathbf{i} - \mathbf{j} + \mathbf{k}) \\
&= \frac{1}{3}(2\mathbf{i} + 5\mathbf{j} + 7\mathbf{k}).
\end{aligned}$$

Now normalize them.

$$\mathbf{a}_1 = \frac{\mathbf{a}}{\|\mathbf{a}\|} = \frac{1}{\sqrt{3}}(-\mathbf{i} - \mathbf{j} + \mathbf{k}), \quad \mathbf{b}_1 = \frac{\mathbf{b}}{\|\mathbf{b}\|} = \frac{1}{\sqrt{78}}(2\mathbf{i} + 5\mathbf{j} + 7\mathbf{k}).$$

We can check that

$$\begin{aligned}
\mathbf{a}_1 \cdot \mathbf{b}_1 &= \frac{(-1)\cdot 2 + (-1)\cdot 5 + 1\cdot 7}{\sqrt{3}\cdot\sqrt{78}} = 0 \\
\mathbf{a}_1 \times \mathbf{b}_1 &= \frac{1}{3\sqrt{26}} \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -1 & -1 & 1 \\ 2 & 5 & 7 \end{vmatrix} \\
&= \frac{1}{3\sqrt{26}} \left( \begin{vmatrix} -1 & 1 \\ 5 & 7 \end{vmatrix} \mathbf{i} - \begin{vmatrix} -1 & 1 \\ 2 & 7 \end{vmatrix} \mathbf{j} + \begin{vmatrix} -1 & -1 \\ 2 & 5 \end{vmatrix} \mathbf{k} \right) \\
&= -\frac{1}{\sqrt{26}}(4\mathbf{i} - 3\mathbf{j} + \mathbf{k})
\end{aligned}$$

□

## Parametric equation of a plane

**Proposition 1.5.6.** *A parametric equation for a plane the through the point* $P_0 = (c_1, c_2, c_3)$ *and parallel to* $\mathbf{a}$ *and* $\mathbf{b}$ *is given by*

$$\mathbf{x}(s,t) = s\mathbf{a} + t\mathbf{b} + \mathbf{c}.$$

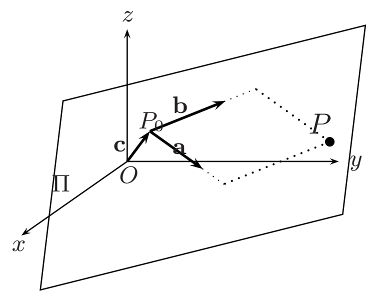



Figure 1.21: A plane is det'd by a point and two vectors

## Distance between a point and a line

**Example 1.5.7.** Find the distance from the point $P_0(2,1,3)$ to the line $\ell(t) = t(-1,1,-2) + (2,3,-2)$.

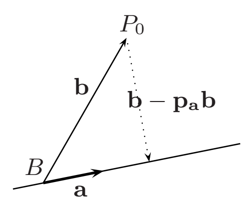



Figure 1.22: Distance from a point to a line

**sol.** Choose any point $B$ on the line and find an orthogonal decomposition of $\overrightarrow{BP_0}$ onto the direction vector $\mathbf{a} = (-1,1,-2)$ of the line. Then the length of the orthogonal complement is the distance. Choose $B = (2,3,-2)$. Then

$$\begin{aligned} \overrightarrow{BP_0} := \mathbf{b} &= (2,1,3) - (2,3,-2) \\ &= (0,-2,5). \end{aligned}$$

Hence the orthogonal projection onto $\mathbf{a}$ is

$$\begin{aligned} \mathbf{p_a b} &= \frac{\mathbf{a} \cdot \mathbf{b}}{\|\mathbf{a}\|^2}\mathbf{a} \\ &= (2,-2,4). \end{aligned}$$

Thus the distance is

$$\|\mathbf{b} - \mathbf{p_a b}\| = \|(0,-2,5) - (2,-2,4)\| = \sqrt{5}.$$

□

### Distance between two parallel planes

To find the distance between two parallel planes, we need to compute the length of the common normal vector. Choose one point from each planes, say $P_i$ from the plane $\Pi_i (i = 1, 2)$. Then find the projection of $\overrightarrow{P_1 P_1}$ onto the common normal vector.

### Distance between two skewed lines

Two lines are said to be **skewed** if they are neither intersecting nor parallel. It follows they must *lie in parallel planes* and the distance between the lines is equal to the distance between the planes. Let us describe how to find the distance between them.

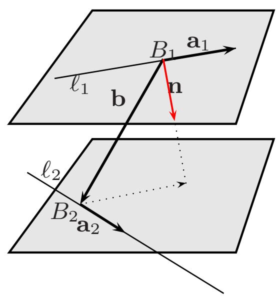



Figure 1.23: Distance between two lines

Assume we have two parallel planes $\Pi_1$ and $\Pi_2$ containing each lines. They share a common normal vector $\mathbf{n}$. This vector is obtained by the cross product of two vectors lying in the planes. Let $\mathbf{a}_1$ and $\mathbf{a}_2$ are two direction vectors and let $B_1$, $B_2$ are any two points on the lines. Then we compute the projection of $\overrightarrow{B_1 B_2}$ onto $\mathbf{n}$. Its length is the desired distance.

**Example 1.5.8.** Find the distance between the two lines

$$\ell_1(t) = (0, 5, -1) + t(2, 1, 3), \text{ and } \ell_2(t) = (-1, 2, 0) + t(1, -1, 0).$$

Choose $B_1 = (2, 6, 2)$ and $B_2 = (0, 1, 0)$. Let $\mathbf{b} = (2, 6, 2) - (0, 1, 0) =$

(2, 5, 2). While

$$\mathbf{n} = (2, 1, 3) \times (1, -1, 0) = (3, 3, -3).$$

Normalizing, we let $\mathbf{n} = (1, 1, -1)/\sqrt{3}$. Now the projection of $\mathbf{b}$ onto $\mathbf{n}$ is

$$proj_{\mathbf{n}}\mathbf{b} = (\mathbf{b} \cdot \mathbf{n})\mathbf{n} = \frac{(2+5-2)}{\sqrt{3}} \frac{(1, 1, -1)}{\sqrt{3}} = \frac{5}{3}(1, 1, -1)$$

Hence the distance is

$$\left\| \frac{5}{3}(1, 1, -1) \right\| = \frac{5}{\sqrt{3}}.$$

## 1.6 $n$-dim Euclidean space

### Vectors in $n$ -dim space

The set of all points with $n$-coordinates

$$\mathbb{R}^n = \{(a_1, a_2, \dots, a_n) \mid a_1, a_2, \dots, a_n \text{ real}\}$$

is called $n$**-dimensional Euclidean space**. Addition and scalar multiplication can be defined as

$$\begin{aligned} (a_1, a_2, \dots, a_n) + (b_1, b_2, \dots, b_n) &= (a_1 + b_1, a_2 + b_2, \dots, a_n + b_n) \\ s(a_1, a_2, \dots, a_n) &= (sa_1, sa_2, \dots, sa_n) \end{aligned}$$

The identity $(0, 0, \dots, 0)$ in $\mathbb{R}^n$ is the **zero element**. The inverse of $(a_1, a_2, \dots, a_n)$ is $(-a_1, -a_2, \dots, -a_n)$, or $-(a_1, a_2, \dots, a_n)$. For two points $P(a_1, a_2, \dots, a_n)$ and $Q(b_1, b_2, \dots, b_n)$, the set

$$\overline{PQ} = \{(1-t)(a_1, a_2, \dots, a_n) + t(b_1, b_2, \dots, b_n) \mid 0 \le t \le 1\}$$

is called the **line segment** $PQ$ and

$$\sqrt{(a_1 - b_1)^2 + (a_2 - b_2)^2 + \cdots + (a_n - b_n)^2}$$

is **length** $PQ$. Also the set

$$\overleftrightarrow{PQ} = \{(1-t)(a_1, a_2, \dots, a_n) + t(b_1, b_2, \dots, b_n) \mid -\infty < t < \infty\}$$

is line $PQ$.

For three points $P(a_1, \ldots, a_n)$ $Q(b_1, \ldots, b_n)$, $R(c_1, \ldots, c_n)$ not on the same line, the set

$$\{r(a_1, \ldots, a_n) + s(b_1, \ldots, b_n) + t(c_1, \ldots, c_n) \mid -\infty < r, s, t < \infty, r + s + t = 1\}$$

is called the plane **determined by** $P,\ Q,\ R$.

## Standard basis vector

We let

$$\begin{aligned}
\mathbf{e}_1 &= (1, 0, 0, \ldots, 0) \\
\mathbf{e}_2 &= (0, 1, 0, \ldots, 0) \\
\mathbf{e}_3 &= (0, 0, 1, \ldots, 0) \\
&\vdots \\
\mathbf{e}_n &= (0, 0, 0, \ldots, 1)
\end{aligned}$$

Then any vector in $\mathbb{R}^n$ can be written as a scalar combination of $\mathbf{e}_1,\ \mathbf{e}_2, \ldots,\ \mathbf{e}_n$:

$$(a_1, a_2, \ldots, a_n) = a_1\mathbf{e}_1 + a_2\mathbf{e}_2 + \cdots + a_n\mathbf{e}_n.$$

The vectors $\mathbf{e}_1,\ \mathbf{e}_2, \ldots,\ \mathbf{e}_n$ are the **standard basis vectors** of $\mathbb{R}^n$. Clearly, we have

$$\mathbf{e}_i \cdot \mathbf{e}_j = \begin{cases} 0 & (i \neq j) \\ 1 & (i = j) \end{cases}$$

[1]

**Theorem 1.6.1.** *We have the following:*

**(i)** $(\alpha\mathbf{x} + \beta\mathbf{y}) \cdot \mathbf{z} = \alpha\mathbf{x} \cdot \mathbf{z} + \beta\mathbf{y} \cdot \mathbf{z}$ *(associate law)*

**(ii)** $\mathbf{x} \cdot \mathbf{y} = \mathbf{y} \cdot \mathbf{x}$ *(commutative law)*

**(iii)** $\mathbf{x} \cdot \mathbf{x} \geq 0$

**(iv)** $\mathbf{x} \cdot \mathbf{x} = 0$ *iff* $\mathbf{x} = 0$

**Example 1.6.2.** Let $\mathbf{u} = 3\mathbf{e}_1 - 4\mathbf{e}_2 + 2\mathbf{e}_4$, $\mathbf{v} = \mathbf{e}_1 + 2\mathbf{e}_2 + 2\mathbf{e}_3 - 3\mathbf{e}_4$ be in $\mathbb{R}^4$. Express $2\mathbf{u} - 7\mathbf{v}$ using standard basis vector.

[1] By definition 1.1.8, the vectors $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$ are denoted by $\mathbf{i}, \mathbf{j}, \mathbf{k}$ in $\mathbb{R}^3$

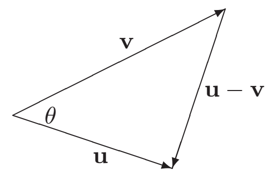



Figure 1.24: angle between two vectors

**sol.** Using standard basis vector, $2\mathbf{u} - 7\mathbf{v}$ is

$$\begin{aligned} 2\mathbf{u} - 7\mathbf{v} &= 2(3\mathbf{e}_1 - 4\mathbf{e}_2 + 2\mathbf{e}_4) - 7(\mathbf{e}_1 + 2\mathbf{e}_2 + 2\mathbf{e}_3 - 3\mathbf{e}_4) \\ &= (6\mathbf{e}_1 - 8\mathbf{e}_2 + 4\mathbf{e}_4) + (-7\mathbf{e}_1 - 14\mathbf{e}_2 - 14\mathbf{e}_3 + 21\mathbf{e}_4) \\ &= (6-7)\mathbf{e}_1 + (-8-14)\mathbf{e}_2 + (0-14)\mathbf{e}_3 + (4+21)\mathbf{e}_4 \\ &= -\mathbf{e}_1 - 22\mathbf{e}_2 - 14\mathbf{e}_3 + 25\mathbf{e}_4 \end{aligned}$$

□

For two vector $\mathbf{u} = a_1\mathbf{e}_1 + a_2\mathbf{e}_2 + \cdots + a_n\mathbf{e}_n$, $\mathbf{v} = b_1\mathbf{e}_1 + b_2\mathbf{e}_2 + \cdots + b_n\mathbf{e}_n$, their **inner product** is defined as

$$\mathbf{u} \cdot \mathbf{v} = a_1b_1 + a_2b_2 + \cdots + a_nb_n = \sum_{i=1}^{n} a_ib_i$$

This satisfies proposition 1.3.3. The length of a vector $\mathbf{u}$ is defined as

$$\|\mathbf{u}\| = (a_1^2 + \cdots + a_n^2)^{1/2} = \sqrt{\mathbf{u} \cdot \mathbf{u}}$$

and the distance between two vectors $\mathbf{u}$ and $\mathbf{v}$ is defined as $\|\mathbf{u} - \mathbf{v}\|$.

One can even define the angle between $\mathbf{u}$ and $\mathbf{v}$ by

$$\theta = \cos^{-1} \frac{\mathbf{u} \cdot \mathbf{v}}{\|\mathbf{u}\| \, \|\mathbf{v}\|} = \cos^{-1} \frac{a_1b_1 + \cdots + a_nb_n}{(a_1^2 + \cdots + a_n^2)^{1/2}(b_1^2 + \cdots + b_n^2)^{1/2}}$$

**Example 1.6.3.** Find the inner product of $\mathbf{u} = \mathbf{e}_1 - 2\mathbf{e}_2 + 3\mathbf{e}_3 + 2\mathbf{e}_4$, $\mathbf{v} = 2\mathbf{e}_1 + \mathbf{e}_2 - 3\mathbf{e}_3 - \mathbf{e}_4$.

**sol.**

$$\mathbf{u} \cdot \mathbf{v} = 2 - 2 - 9 - 2 = -11$$

□

**Example 1.6.4.** Find the angle between $\mathbf{u} = \mathbf{e}_1 - \mathbf{e}_2 + \mathbf{e}_4$, $\mathbf{v} = -\mathbf{e}_2 - \mathbf{e}_3 + 2\mathbf{e}_4$.

**sol.** The angle between $\mathbf{u}$ and $\mathbf{v}$ is

$$\cos^{-1}\frac{0+1+0+2}{\sqrt{(1+1+0+1)(0+1+1+4)}} = \cos^{-1}\frac{1}{\sqrt{2}} = \frac{\pi}{4}$$

□

**Theorem 1.6.5** (Cauchy-Schwarz inequality)**.** *For any two vectors* $\mathbf{a}$, $\mathbf{b}$ *in* $n$*-dim space the following holds. Equality holds iff* $\mathbf{a}$ *and* $\mathbf{b}$ *are parallel.*

$$|\mathbf{a}\cdot\mathbf{b}| \le \|\mathbf{a}\|\,\|\mathbf{b}\|$$

*Proof.* For $n > 3$ our early proof is unclear. Thus we prove this again. We may assume none of the vectors are zero. Recall the orthogonal decomposition of $\mathbf{b}$ onto $\mathbf{a}$, i.e, we write

$$\mathbf{b} = k\mathbf{a} + \mathbf{c}$$

where $k\mathbf{a}$ is the projection of $\mathbf{b}$ and $\mathbf{c} = \mathbf{b} - k\mathbf{a}$ is the orthogonal complement. By orthogonality($\mathbf{a}\cdot\mathbf{c} = 0$),

$$\|\mathbf{a}\|^2\|\mathbf{b}\|^2 = \|\mathbf{a}\|^2(k^2\|\mathbf{a}\|^2 + \|\mathbf{c}\|^2) \ge k^2\|\mathbf{a}\|^2\|\mathbf{a}\|^2.$$

Thus

$$k^2\|\mathbf{a}\|^2 \le \|\mathbf{b}\|^2.$$

Since $k = \frac{\mathbf{b}\cdot\mathbf{a}}{\|\mathbf{a}\|^2}$, we see

$$\|\mathbf{a}\cdot\mathbf{b}\|^2 \le \|\mathbf{a}\|^2\|\mathbf{b}\|^2.$$

□

**Theorem 1.6.6** (Triangle inequality)**.** *For any two vectors* $\mathbf{u}$, $\mathbf{v}$ *in* $n$*-dim space the following holds. Equality holds iff* $\mathbf{u}$ *and* $\mathbf{v}$ *are parallel and same direction.*

$$\|\mathbf{u}+\mathbf{v}\| \le \|\mathbf{u}\| + \|\mathbf{v}\|$$

*Proof.*

$$\begin{aligned} \|\mathbf{u}+\mathbf{v}\|^2 &= (\mathbf{u}+\mathbf{v})\cdot(\mathbf{u}+\mathbf{v}) \\ &= \mathbf{u}\cdot\mathbf{u}+2\mathbf{u}\cdot\mathbf{v}+\mathbf{v}\cdot\mathbf{v} \\ &\leq \mathbf{u}\cdot\mathbf{u}+2\|\mathbf{u}\|\|\mathbf{v}\|+\mathbf{v}\cdot\mathbf{v} \\ &= (\|\mathbf{u}\|+\|\mathbf{v}\|)^2 \end{aligned}$$

□

## General matrix

Let $m, n$ be any natural numbers. The arrays $a_{ij}$ $(1 \leq i \leq m,\ 1 \leq j \leq n)$

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}$$

is called $m \times n$ **matrix** and denote by

$$\begin{bmatrix} a_{ij} \end{bmatrix}_{\substack{1 \leq i \leq m \\ 1 \leq j \leq n}}, \quad \begin{bmatrix} a_{ij} \end{bmatrix}_{m \times n} \quad \text{or} \quad [a_{ij}]$$

If $m = 1$, then $1 \times n$ matrix consists of one row and is called **row vector**, and if $n = 1$ then $m \times 1$ matrix is **column vector**. If $m = n$, it is called **square matrix**. $a_{ij}$ is called $ij$**-entry**. The $1 \times n$ matrix

$$\begin{bmatrix} a_{i1} & a_{i2} & \cdots & a_{in} \end{bmatrix}$$

is $i$**-th row vector**, $m \times 1$ matrix

$$\begin{bmatrix} a_{1j} \\ a_{2j} \\ \vdots \\ a_{mj} \end{bmatrix}$$

is $j$**-th column vector**.

**Example 1.6.7.** What is 4-th row and second column of the $4 \times 3$ matrix?

$$\begin{bmatrix} 0 & -2 & 12 \\ 3 & 1 & 4 \\ -1 & 0 & 5 \\ 1 & -3 & 7 \end{bmatrix}$$

**sol.** 4-th row and second column is

$$\begin{bmatrix} 1 & -3 & 7 \end{bmatrix}, \qquad \begin{bmatrix} -2 \\ 1 \\ 0 \\ -3 \end{bmatrix}$$

□

## Matrix addition, multiplication

Let $A$ and $B$ be two $m \times n$ matrices. Then the **matrix sum** $A + B$ is the $m \times n$ matrix whose $(i, j)$ entries are the sum of $a_{i,j}$ and $b_{i,j}$. If $k$ is any scalar, define the scalar multiplication $kA$ by

$$(kA)_{ij} = ka_{ij}.$$

i.e, each entry is multiplied by $k$.

**Definition 1.6.8** (Matrix multiplication)**.** If $A = [a_{ij}]$ is $m \times n$ matrix and $B = [b_{kl}]$ is $n \times p$ matrix, then the $m \times p$ matrix

$$\left[\sum_{k=1}^{n} a_{ik}b_{kj}\right]_{\substack{1 \le i \le m \\ 1 \le j \le p}}$$

is the **product** of $A$ and $B$ denoted by $AB$. In other words, the product of $A$ and $B$ is $AB$ and its $ij$-component is the inner product of $i - th$ row of $A$ and $j - th$ column of $B$.

**Example 1.6.9.** Product of $2 \times 3$ and $3 \times 4$ matrices

$$\begin{bmatrix} 2 & 3 & 1 \\ -1 & 1 & 4 \end{bmatrix} \begin{bmatrix} 0 & 2 & 3 & -2 \\ -2 & 1 & 5 & -3 \\ 2 & 0 & -2 & 1 \end{bmatrix} = \begin{bmatrix} -4 & 9 & 19 & -12 \\ 6 & -1 & -6 & 3 \end{bmatrix}$$

**Example 1.6.10.** Product of $1 \times 3$ and $3 \times 2$ matrices

$$\begin{bmatrix} 2 & 1 & -3 \end{bmatrix} \begin{bmatrix} -3 & 5 \\ 1 & -3 \\ 0 & -2 \end{bmatrix} = \begin{bmatrix} -5 & 13 \end{bmatrix}$$

**Example 1.6.11.** Product of $3 \times 4$ and $4 \times 1$ matrices

$$\begin{bmatrix} 0 & 2 & 3 & 1 \\ -1 & 2 & 0 & -3 \\ 2 & 0 & 1 & 4 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix} = \begin{bmatrix} 2y + 3z + w \\ -x + 2y - 3w \\ 2x + z + 4w \end{bmatrix}$$

**Definition 1.6.12.** The following $n \times n$ matrix is

$$\begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \end{bmatrix}$$

$n \times n$ **identity matrix** and denote it by $I_n$.

**Proposition 1.6.13** (Properties of matrix multiplication)**.** *Let $A, B$ and $C$ are matrices where the multiplication $AB$ and $BC$ etc, makes sense. Then*

*(1)* $A(BC) = (AB)C$.

*(2)* $k(AB) = (kA)B = A(kB)$

*(3)* $A(B + C) = AB + AC$

*(4)* $(A + B)C = AC + BC$

*whenever the multiplication makes sense.*

Transpose.

$$\begin{aligned} (A^T)^T &= A \\ (AB)^T &= B^T A^T \end{aligned}$$

$$\mathbf{a} \cdot \mathbf{b} = a^T \mathbf{b}$$

**Lemma 1.6.14.** *For nay $m \times n$ matrix $A$ and $n \times p$ matrix $B$, we have*

$$A\, I_n = A, \qquad I_n\, B = B$$

*Also, for any $n \times n$ matrix $A$, it holds that*

$$A\, I_n = I_n\, A = A$$

*$I_n$ is identity element in multiplication.*

**Example 1.6.15.**

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \qquad \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

**Definition 1.6.16.** If for any vector $\mathbf{u},\ \mathbf{v} \in \mathbb{R}^n$ and scalar $\alpha \in \mathbb{R}$, a function $T\colon \mathbb{R}^n \to \mathbb{R}^m$ is defined by

(1) $T(\mathbf{u}+\mathbf{v}) = T(\mathbf{u}) + T(\mathbf{v})$

(2) $T(\alpha\, \mathbf{u}) = \alpha\, T(\mathbf{u})$

we say $T$ **linear transformation(mapping, functions).**

**Example 1.6.17.** Express a given linear transformation $T\colon \mathbb{R}^n \to \mathbb{R}^m$ using the standard basis vector.

Since any vector in $\mathbb{R}^n$ can be written as $a_1\mathbf{e}_1 + a_2\mathbf{e}_2 + \cdots + a_n\mathbf{e}_n$ and $T$ is determined by the values at these vectors.

$$\begin{aligned} T(a_1\mathbf{e}_1 + a_2\mathbf{e}_2 + \cdots + a_n\mathbf{e}_n) &= T(a_1\mathbf{e}_1) + T(a_2\mathbf{e}_2) + \cdots + T(a_n\mathbf{e}_n) \\ &= a_1 T(\mathbf{e}_1) + a_2 T(\mathbf{e}_2) + \cdots + a_n T(\mathbf{e}_n) \end{aligned}$$

Since $T(\mathbf{e}_1), T(\mathbf{e}_2), \ldots, T(\mathbf{e}_n)$ are in $\mathbb{R}^m$, we can write it as linear combinations of $\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_m$. Hence there are $t_{ij}$ $(1 \le i \le m, 1 \le j \le n)$ s.t.

$$T(\mathbf{e}_j) = \sum_{i=1}^{m} t_{ij}\mathbf{e}_i \quad (1 \le j \le n) \tag{1.6}$$

Hence

$$T(a_1\mathbf{e}_1 + a_2\mathbf{e}_2 + \cdots + a_n\mathbf{e}_n) = \sum_{j=1}^{n} a_j T(\mathbf{e}_j) = \sum_{i=1}^{m} \left( \sum_{j=1}^{n} t_{ij} a_j \right) \mathbf{e}_i \tag{1.7}$$

This procedure can be written in matrix form Eq. (1.6). The matrix having $t_{ij}$ as $ij$-th component

$$\mathrm{mat}(T) = \begin{bmatrix} t_{11} & t_{12} & \cdots & t_{1n} \\ t_{21} & t_{22} & \cdots & t_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ t_{m1} & t_{m2} & \cdots & t_{mn} \end{bmatrix}$$

is called **matrix of** $T$. Let us multiply the column vector $a_1\mathbf{e}_1 + a_2\mathbf{e}_2 + \cdots + a_n\mathbf{e}_n$ to the right of this matrix.

$$\begin{bmatrix} t_{11} & t_{12} & \cdots & t_{1n} \\ t_{21} & t_{22} & \cdots & t_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ t_{m1} & t_{m2} & \cdots & t_{mn} \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{bmatrix} = \begin{bmatrix} t_{11}a_1 + t_{12}a_2 + \cdots + t_{1n}a_n \\ t_{21}a_1 + t_{22}a_2 + \cdots + t_{2n}a_n \\ \vdots \\ t_{m1}a_1 + t_{m2}a_2 + \cdots + t_{mn}a_n \end{bmatrix}$$

Compare this with equation (1.7). Then rhs vector has $T(a_1\mathbf{e}_1 + a_2\mathbf{e}_2 + \cdots + a_n\mathbf{e}_n)$ as its component. Conversely, any $m \times n$ matrix $[t_{ij}]$ is given, then it determines linear transformation $T\colon \mathbb{R}^n \to \mathbb{R}^m$ as in equation (1.7). Hence linear transformation $T\colon \mathbb{R}^n \to \mathbb{R}^m$ has one-one correspondence with $m \times n$ matrix as follows:

$$\mathrm{mat}\colon T \mapsto \left[\ \mathbf{e}_i \cdot T(\mathbf{e}_j)\ \right]_{\substack{1 \le i \le m \\ 1 \le j \le n}}$$

**Proposition 1.6.18.** *For two linear transformations* $T\colon \mathbb{R}^n \to \mathbb{R}^m$, $U\colon \mathbb{R}^p \to \mathbb{R}^n$ *it holds that*

$$\mathrm{mat}(T \circ U) = \mathrm{mat}(T)\ \mathrm{mat}(U)$$

**Example 1.6.19.** For the given two linear transformations $T\colon \mathbb{R}^n \to \mathbb{R}^m$, $U\colon \mathbb{R}^p \to \mathbb{R}^n$ check Proposition 1.6.18 holds.

$$\begin{aligned} T(x, y, z) &= (3y - z, x + y) \\ U(s, t) &= (2s - t, s + 2t, -3s) \end{aligned}$$

**sol.** The matrices for $T$ and $U$ are

$$\text{mat}(T) = \begin{bmatrix} 0 & 3 & -1 \\ 1 & 1 & 0 \end{bmatrix}, \quad \text{mat}(U) = \begin{bmatrix} 2 & -1 \\ 1 & 2 \\ -3 & 0 \end{bmatrix}$$

Hence

$$\text{mat}(T)\ \text{mat}(U) = \begin{bmatrix} 0 & 3 & -1 \\ 1 & 1 & 0 \end{bmatrix} \begin{bmatrix} 2 & -1 \\ 1 & 2 \\ -3 & 0 \end{bmatrix} = \begin{bmatrix} 6 & 6 \\ 3 & 1 \end{bmatrix}$$

On the other hand

$$\begin{aligned}(T \circ U)(s,t) &= T(2s - t, s + 2t, -3s) \\ &= (3(s+2t) - (-3s), (2s - t) + (s + 2t)) \\ &= (6s + 6t, 3s + t)\end{aligned}$$

So

$$\text{mat}(T \circ U) = \begin{bmatrix} 6 & 6 \\ 3 & 1 \end{bmatrix}$$

Hence the following holds.

$$\text{mat}(T \circ U) = \text{mat}(T)\ \text{mat}(U)$$

□

## Determinant

We have seen $3 \times 3$ $2 \times 2$. Using these, we define determinant of $n \times n$ matrix by induction. We expand w.r.t 1st column.

$$\begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \cdots & \cdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{vmatrix} = a_{11} \begin{vmatrix} a_{22} & a_{23} & \cdots & a_{2n} \\ a_{32} & a_{33} & \cdots & a_{3n} \\ \cdots & \cdots & \ddots & \vdots \\ a_{n2} & a_{n3} & \cdots & a_{nn} \end{vmatrix} +$$

$$\cdots + (-1)^{1+i} a_{1i} \begin{vmatrix} a_{21} & \cdots & a_{2(i-1)} & a_{2(i+1)} & \cdots & a_{2n} \\ a_{31} & \cdots & a_{3(i-1)} & a_{3(i+1)} & \cdots & a_{3n} \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ a_{n1} & \cdots & a_{n(i-1)} & a_{n(i+1)} & \cdots & a_{nn} \end{vmatrix} + \cdots$$

$$+(-1)^{1+n} a_{1n} \begin{vmatrix} a_{21} & a_{22} & \cdots & a_{2(n-1)} \\ a_{31} & a_{32} & \cdots & a_{3(n-1)} \\ \cdots & \cdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{n(n-1)} \end{vmatrix}$$

The $i$-th term on the right is $(-1)^{1+i} a_{1i}$ times the determinant of $(n-1) \times (n-1)$ obtained by deleting first row and $i$-column.

Thm 1.4.6 and corollary 1.4.7 hold for any square matrices.

**Expansion with respect to any row**

Let$A_{ij}$ be the $(n-1) \times (n-1)$ matrix obtained by deleting $i$-row and $j$-th column. Expand w.r.t $i$-th row, we see

$$|A| = \sum_{j=1}^{n} (-1)^{i+j} a_{ij} |A_{ij}|$$

if we expand w.r.t $j$-th row, we see

$$|A| = \sum_{i=1}^{n} (-1)^{i+j} a_{ij} |A_{ij}|$$

**Example 1.6.20.** Expand w.r.t 2nd row

$$\begin{vmatrix} 2 & -1 & 3 & 0 \\ 0 & -1 & 0 & 4 \\ 3 & 1 & 0 & 2 \\ 2 & 0 & -3 & 0 \end{vmatrix} = -0\begin{vmatrix} -1 & 3 & 0 \\ 1 & 0 & 2 \\ 0 & -3 & 0 \end{vmatrix} + (-1)\begin{vmatrix} 2 & 3 & 0 \\ 3 & 0 & 2 \\ 2 & -3 & 0 \end{vmatrix}$$

$$-0\begin{vmatrix} 2 & -1 & 0 \\ 3 & 1 & 2 \\ 2 & 0 & 0 \end{vmatrix} + 4\begin{vmatrix} 2 & -1 & 3 \\ 3 & 1 & 0 \\ 2 & 0 & -3 \end{vmatrix}$$

$$= -2\begin{vmatrix} 0 & 2 \\ -3 & 0 \end{vmatrix} + 3\begin{vmatrix} 3 & 2 \\ 2 & 0 \end{vmatrix}$$

$$+ 4 \cdot 2\begin{vmatrix} -1 & 3 \\ 1 & 0 \end{vmatrix} + 4(-3)\begin{vmatrix} 2 & -1 \\ 3 & 1 \end{vmatrix}$$

$$= -2 \cdot 6 + 3 \cdot (-4) + 8 \cdot (-3) - 12 \cdot 5$$

$$= -108$$

**Example 1.6.21.** Solve

$$\begin{array}{ccccccc} 3x & + & 2y & + & z & = & 1 \\ & & y & + & z & = & 0 \\ x & + & y & & & = & 3 \end{array}$$

**sol.** Use Cramer's rule

$$\begin{vmatrix} 3 & 2 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 0 \end{vmatrix} = 3\begin{vmatrix} 1 & 1 \\ 1 & 0 \end{vmatrix} - 0\begin{vmatrix} 2 & 1 \\ 1 & 0 \end{vmatrix} + 1\begin{vmatrix} 2 & 1 \\ 1 & 1 \end{vmatrix} = -2$$

Then $x_1.x_2$ and $x_3$ are

$$x_1 = \frac{1}{-2}\begin{vmatrix} 1 & 2 & 1 \\ 3 & 1 & 1 \\ 0 & 1 & 0 \end{vmatrix} = -1 \quad x_2 = \frac{1}{-2}\begin{vmatrix} 3 & 1 & 1 \\ 0 & 0 & 1 \\ 1 & 3 & 0 \end{vmatrix} = 4 \quad x_3 = \frac{1}{-2}\begin{vmatrix} 3 & 2 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 3 \end{vmatrix} = -4$$

□

## 1.7 Cylindrical and spherical coordinate

### Cylindrical coordinate system

Given a point $P = (x, y, z)$, we can use polar coordinate for $(x, y)$-plane. Then it holds that

$$\text{Cylindrical to Cartesain} \begin{cases} x &= r\cos\theta, \\ y &= r\sin\theta, \\ z &= z \end{cases}$$

We say $(r, \theta, z)$ is **cylindrical coordinate** of $P$. Conversely, the cylindrical coordinate $(r, \theta, z)$ is given by

$$\text{Cartesain to Cylindrical} \begin{cases} r^2 &= x^2 + y^2, \\ \tan\theta &= \frac{y}{x}, \\ z &= z \end{cases}$$

The expression $(r, \theta, z)$ is not unique.

**Example 1.7.1.** The set of all points $r = a$ in cylindrical coordinate is

$$\{(x, y, z) \mid x^2 + y^2 = a^2\}.$$

This is a cylinder (Figure 1.25).

**Example 1.7.2.** $r = 3\cos\theta$ gives

$$r^2 = 3r\cos\theta \Rightarrow x^2 + y^2 = 3x$$

This is again a cylinder.

**Example 1.7.3.** Change cylindrical coordinate $(6, \pi/3, 4)$ to Cartesian coordinate.

**sol.**

$$x = 6\cos(\pi/3) = 3, \quad y = 6\sin(\pi/3) = 3\sqrt{3}, \quad z = 4.$$

So $(x, y, z) = (3, 3\sqrt{3}, 4)$.

□

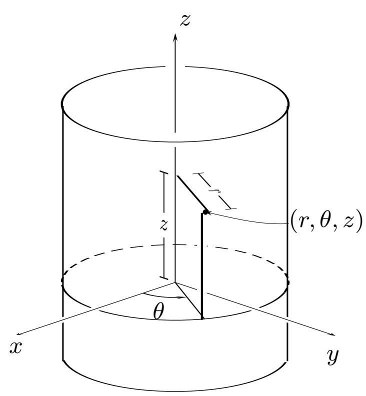



Figure 1.25: cylindrical coordinate

$$\theta = \begin{cases} \tan^{-1}(y/x) & (x > 0,\ y \geq 0) \\ 2\pi + \tan^{-1}(y/x) & (x > 0,\ y < 0) \\ \pi + \tan^{-1}(y/x) & (x < 0) \\ \pi/2 & (x = 0,\ y > 0) \\ 3\pi/2 & (x = 0,\ y < 0) \end{cases}$$

**Example 1.7.4.** Identify the surface given by the equation $z = 2r$ in cylindrical coordinate.

**sol.** $z^2 = 4r^2 = 4(x^2 + y^2)$. This is a cone.

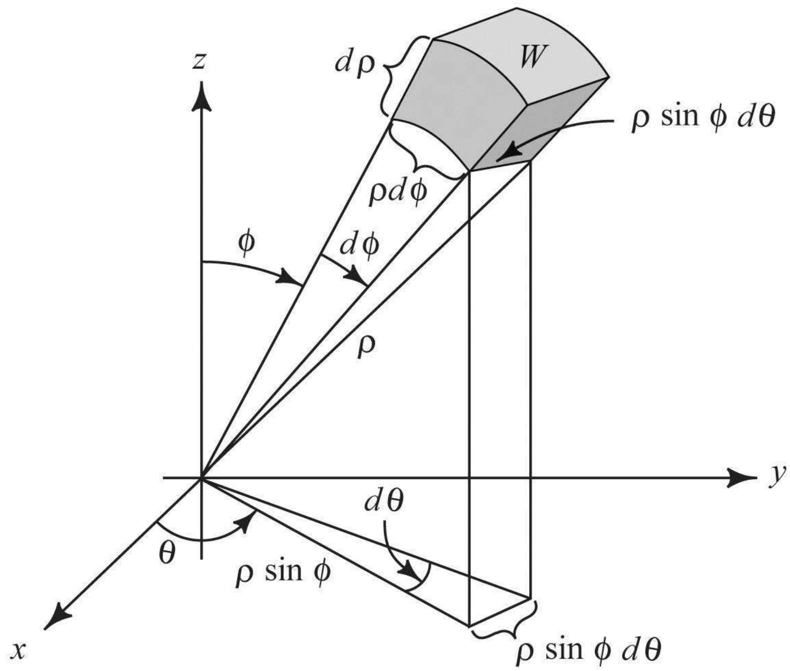
□

**Example 1.7.5.** Change the equation $x^2 + y^2 - z^2 = 1$ to cylindrical coordinate.

**sol.** $r^2 - z^2 = 1$.

□

**Spherical coordinate system**




We call $(\rho, \phi, \theta)$ to be the **spherical coordinate** of $P$.

$$\text{Shperical to cylindrical} \begin{cases} r = \rho \sin \phi \\ \theta = \theta \\ z = \rho \cos \phi \end{cases} \quad \begin{cases} \rho^2 = r^2 + z^2 \\ \tan \phi = \dfrac{r}{z} \\ \theta = \theta \end{cases}$$

For $P = (x, y, z)$ we have

$$\text{Shperical to Cartesian} \begin{cases} x = \rho \sin \phi \cos \theta \\ y = \rho \sin \phi \sin \theta \\ z = \rho \cos \phi \end{cases} \quad \begin{pmatrix} \rho \geq 0 \\ 0 \leq \theta < 2\pi \\ 0 \leq \phi \leq \pi \end{pmatrix}$$

Conversely, we can write $\rho$, $\phi$, $\theta$ in terms of $x$, $y$, $z$.

$$\text{Cartesian to shperical} \begin{cases} \rho = \sqrt{x^2 + y^2 + z^2} \\ \cos \phi = \dfrac{z}{\rho} \\ \tan \theta = \dfrac{y}{x} \end{cases}$$

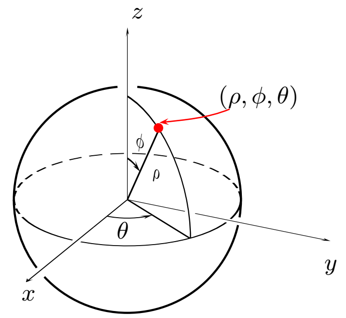



Figure 1.26: Spherical-coordinate

Now the second condition has an alternate expression:We see from the figure

$$r = \rho \sin\phi, \quad z = \rho\cos\phi.$$

Hence $\cos\phi = \frac{z}{\rho}$ can be replaced by

$$\tan\phi = \frac{r}{z}.$$

**Example 1.7.6.** (1) Find spherical coord. of $(1, -1, 1)$ and plot.

(2) Find cartesian coord. of $(3, \pi/6, \pi/4)$.

(3) Find spherical coord. of $(2, -3, 6)$.

(4) Find spherical coord. of $(-3, -3, \sqrt{6})$.

**sol.** (1) $\rho = \sqrt{3}$.

$$\phi = \cos^{-1}(\frac{z}{\rho}) = \cos^{-1}(\frac{1}{\sqrt{3}}) \approx 0.955 \approx 54.74^o$$

Since the point $(1, -1)$ lies in the 4-th quadrant, we see

$$\theta = \arctan(\frac{y}{x}) = \frac{7\pi}{4}.$$

(3) $\rho = \sqrt{2^2 + (-3)^2 + 6^2} = \sqrt{2^2 + (-3)^2 + 6^2} = 7.$

$$\phi = \cos^{-1}(\frac{z}{\rho}) = \cos^{-1}\frac{6}{7}$$

Also, the point lies in the fourth quadrant, we have

$$\theta = 2\pi + \tan^{-1}(-3/2).$$

(4)

$$\begin{aligned}
\rho &= \sqrt{9+9+6} = 2\sqrt{6} \\
\phi &= \cos^{-1}\frac{\sqrt{6}}{2\sqrt{6}} = \cos^{-1}\frac{1}{2} = \frac{\pi}{3} \\
\theta &= \pi + \tan^{-1}(\frac{-1}{-1}) = \pi + \frac{\pi}{4} = \frac{5\pi}{4}
\end{aligned}$$

Hence spherical coordinate is $(2\sqrt{6}, \pi/3, 5\pi/4)$.

□

**Example 1.7.7.** Express the surface (1) $xz = 1$ and (2) $x^2 + y^2 - z^2 = 1$ in spherical coordinate.

**sol.** (1) Since $xz = \rho^2 \sin\phi\cos\theta\cos\phi = 1$, we have the equation

$$\rho^2 \sin 2\phi \cos\phi = 2.$$

(2) Since $x^2+y^2-z^2 = x^2+y^2+z^2-2z^2 = \rho^2-2(\rho\cos\phi)^2 = \rho^2(1-2\cos^2\phi)$, the equation is $1 + \rho^2\cos 2\phi = 0$.

□

## Standard basis for cylindrical and spherical coordinates

For cylindrical coordinates, the following sets are standard basis vectors:

$$\mathbf{e_r} = \frac{x\mathbf{i} + y\mathbf{j}}{\sqrt{x^2+y^2}}, \quad \mathbf{e}_\theta, \quad \mathbf{e}_z = \mathbf{k}$$

These vary depending on the points and are defined so that only the coordinate indicated by the subscript increases. Now $\mathbf{e}_\theta$ is given by

$$\mathbf{e}_\theta = \mathbf{e_z} \times \mathbf{e_r} = \frac{-y\mathbf{i} + x\mathbf{j}}{\sqrt{x^2+y^2}}.$$

In this way $(\mathbf{e_r}, \mathbf{e}_\theta, \mathbf{e}_z)$ form a right handed coordinate system.

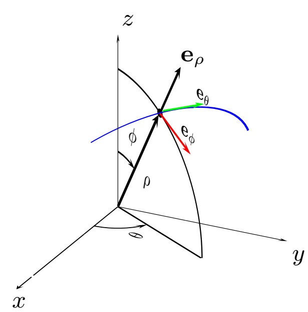



Figure 1.27: Standard basis for spherical coordinate

For spherical coordinates the followings are standard basis vectors.

$$\begin{aligned}
\mathbf{e}_\rho &= \frac{x\mathbf{i} + y\mathbf{j} + z\mathbf{j}}{\sqrt{x^2+y^2+z^2}} = \sin\phi\cos\theta\mathbf{i} + \sin\phi\sin\theta\mathbf{j} + \cos\phi\mathbf{k} \\
\mathbf{e}_\phi &= \cos\phi\cos\theta\mathbf{i} + \cos\phi\sin\theta\mathbf{j} - \sin\phi\mathbf{k} \\
\mathbf{e}_\theta &= \frac{-y\mathbf{i} + x\mathbf{j}}{\sqrt{x^2+y^2}} = -\sin\theta\mathbf{i} + \cos\theta\mathbf{j}
\end{aligned}$$