Contents

Chapter 7

Infinite Sequence and Series

7.1 Sequences

Example 7.1.1. (1)

$$1, 3, 5, 7, \ldots$$

(2) n-th term is given by $(-1)^{n+1}1/n$:

$$1, -\frac{1}{2}, \frac{1}{3}, -\frac{1}{4}, \ldots, (-1)^{n+1}\frac{1}{n}, \ldots$$

(3) Certain rules

$$1, \frac{1}{2}, \frac{1}{2}, -\frac{1}{3}, -\frac{1}{3}, -\frac{1}{3}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4}, \ldots$$

(4) Constant sequence :

$$3, 3, 3, \ldots$$

(5) Digits after decimal point of $\sqrt{2}$

$$4, 1, 4, 1, 5, 9, \ldots$$

n-th term a_n

Definition 7.1.2. A **sequence** is a function with the set of natural numbers as domain.

Sequence as graph

Example 7.1.3. (1) $a_n = (n-1)/n$.

(2) $a_n = (-1)^n 1/n$.

(3) $a_n = \sqrt{n}$.

(4) $a_n = \sin(n\pi/6)$.

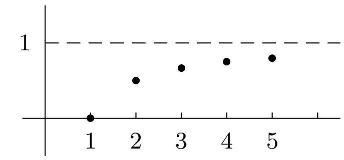


Figure 7.1: $a_n = (n-1)/n$

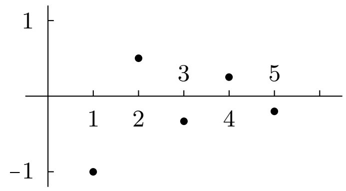


Figure 7.2: $a_n = (-1)^n 1/n$

(5) a_n is the n-th digit of π after decimal point.

Among these (1), (3), (4) are functions $(x-1)/x$, $\sqrt{x}$, $\ln x$ are restricted to N.

Subsequence

If all the terms of $\{a_n\}$ appears as some term in $\{b_n\}$ without changing orders we say $\{a_n\}$ is a **subsequence** of $\{b_n\}$.

Example 7.1.4. (1) $1, 1, 1, 1, \dots$ is a subsequence of $1, -1, 1, -1, \dots$.

(2) $\{9n\}$ $(n = 1, 2, 3, \dots)$ is a subsequence of $\{3n\}$ $(n = 1, 2, 3, \dots)$.

(3) $\{1+1/4^n\}$ $(n = 1, 2, 3, \dots)$ is a subsequence of $\{1+1/2^n\}$ $(n = 1, 2, 3, \dots)$.

Recursive relation

Some sequence are defined through recursive relation such as

$$\begin{aligned} a_1 &= 1, \\ a_{n+1} &= 2a_n + 1, \quad n = 1, 2, 3, \dots \end{aligned}$$

or

$$\begin{aligned} a_1 &= 1, \ a_2 = 2, \\ a_{n+2} &= a_{n+1} + a_n, \quad n = 1, 2, 3, \dots \end{aligned}$$

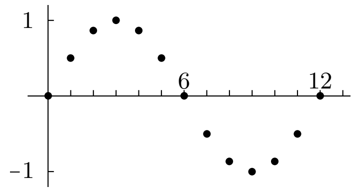


Figure 7.3: $a_n = \sin(n\pi/6)$

7.1.1 Convergence of a sequence

Definition 7.1.5. We say $\{a_n\}$ **converges** to L, if for any $\varepsilon > 0$ there exists some N s.t. for all $n > N$ it holds that

$$|a_n - L| < \varepsilon.$$

Otherwise, we say $\{a_n\}$ is said to **diverge**. If $\{a_n\}$ converges to L we write

$$\lim_{n\to\infty} a_n = L \quad \text{or} \quad \{a_n\} \to L.$$

We say L is the **limit** a_n.

Example 7.1.6. Show that $\{(n-1)/n\}$ converges to 1.

sol. We expect $L = 1$. For any ε, $|(n-1)/n - 1| < \varepsilon$ holds for n satisfying $|1/n| > \varepsilon$.

□

Example 7.1.7. Show that $\{\sqrt{n+2} - \sqrt{n}\}$ converges to 0.

sol. Let ε be given. We want to choose a number N so that

$$|\sqrt{n+2} - \sqrt{n} - 0| = \frac{2}{\sqrt{n+2} + \sqrt{n}}$$

is less than ε for all n greater than certain N. Since

$$\frac{2}{\sqrt{n+2} + \sqrt{n}} < \frac{1}{\sqrt{n}}$$

the conclusion will hold if n satisfies

$$\frac{1}{\sqrt{n}} < \varepsilon.$$

Thus we choose N any natural number greater than $1/\varepsilon^2$.

□

Theorem 7.1.8. *Suppose and subsequence b_n of a_n converges to L, then a_n also converges to L.*

Theorem 7.1.9 (Uniqueness)**.** *If $\{a_n\}$ converges, it has unique limit.*

Proof. Suppose $\{a_n\}$ has two limits L_1, L_2. Choose $\varepsilon = |L_1 - L_2|/2$ There exist N_1 s.t. for $n > N_1$ the following holds

$$|a_n - L_1| < \varepsilon.$$

Similarly, there exist N_2 s.t. for all $n > N_2$ it holds that

$$|a_n - L_2| < \varepsilon$$

Let N be the greater one of N_1, N_2. Then for all $n > N$

$$\begin{aligned}|L_1 - L_2| &= |L_1 - a_n + a_n - L_2| \leq |L_1 - a_n| + |a_n - L_2| \\ &< \varepsilon + \varepsilon = |L_1 - L_2|\end{aligned}$$

holds. A contradiction. So $L_1 = L_2$. □

Corollary 7.1.10. *If $\{a_n\}$ converges, we have $\lim_{n\to\infty}(a_n - a_{n+1}) = 0$.*

Remark 7.1.11. The above condition is not a sufficient for convergence. For example, the sequence $a_n = \ln(n+1)/n$ satisfies $a_{n+1} - a_n = \ln(n+1)/n \to 0$ but $\lim_{n\to\infty} a_n = \infty$.

Properties of limit

Theorem 7.1.12. *Suppose $\lim_{n\to\infty} a_n = A$, $\lim_{n\to\infty} b_n = B$. Then we have*

(1) $\lim_{n\to\infty}\{a_n + b_n\} = A + B$

(2) $\lim_{n\to\infty}\{a_n - b_n\} = A - B$

(3) $\lim_{n\to\infty}\{ka_n\} = kA$

(4) $\lim_{n\to\infty}\{a_n \cdot b_n\} = A \cdot B$

(5) $\lim_{n\to\infty}\left\{\frac{a_n}{b_n}\right\} = A/B,\ B \neq 0.$

$$\lim_{n\to\infty}\frac{n^2 - n}{n^2} = \lim_{n\to\infty} 1 - \frac{1}{n} = 1 - 0 = 1.$$

$$\lim_{n\to\infty}\frac{2 - 3n^5}{n^5 + 1} = \lim_{n\to\infty}\frac{2/n^5 - 3}{1 + 1/n^5} = -3.$$

Theorem 7.1.13 (Continuous function)**.** *Suppose the limit of a_n is L and a function f is defined on an interval containing all values of a_n and L, and continuous at L, then*

$$\lim_{n\to\infty} f(a_n) = f(L)$$

Proof. Since f is continuous at L, we have for any ε there is a δ such that for all a_n with $|a_n - L| < \delta$ it holds that $|f(a_n) - f(L)| < \varepsilon$. Since a_n converges to L, there is a natural number N s.t. for $n > N$ it holds that $|a_n - L| < \delta$. Hence $|f(a_n) - f(L)| < \varepsilon$ holds. ☐

Example 7.1.14. (1) $\lim_{n\to\infty} \sin\left(\frac{n\pi}{(2n+1)}\right) = 1$ (2) $\lim_{n\to\infty} 2^{\frac{1}{\sqrt{n}}} = 1$

sol. (1) Since the limit of $n\pi/(2n+1)$ is $\pi/2$ and the function $\sin x$ is continuous at $\pi/2$, we have $\lim_{n\to\infty} \sin(n\pi/(2n+1)) = 1$.

(2) Since $f(x) = 2^{\sqrt{x}}$ is continuous at $x = 0^+$ we have

$$\lim_{n\to\infty} 2^{1/\sqrt{n}} = 1$$

☐

Theorem 7.1.15. *Suppose $f(x)$ is defined for $x \geq 0$ and if $\{a_n\}$ is given by $a_n = f(n)$, $n = 1, 2, 3, \dots$ and if $\lim_{x\to\infty} f(x) = L$ then $\lim_{n\to\infty} a_n = L$.*

This theorem holds when $f(x) \to +\infty$ or $f(x) \to -\infty$.

Example 7.1.16. (1) $\lim_{n\to\infty} \frac{\ln n}{n} = 0$,

(2) $\lim_{n\to\infty} n(e^{\frac{1}{n}} - 1) = 1$

(3) Find $\lim_{n\to\infty} \left(\frac{n+1}{n-1}\right)^n$

sol. (1) Let $f(x) = \ln x/x$. Then

$$\lim_{n\to\infty} f(n) = \lim_{x\to\infty} f(x) = \lim_{x\to\infty} \frac{(\ln x)'}{x'} = \lim_{x\to\infty} \frac{1}{x} = 0$$

$$\lim_{n\to\infty} \ln n/n = 0$$

(2) Set $x = 1/n$. Then it corresponds to the limit of $f(x) = (e^x - 1)/x$ as $x \to 0$. By L'Hopital's rule

$$\lim_{x\to 0} f(x) = \lim_{x\to 0} e^x = 1$$

$$\lim_{n\to\infty} n(e^{1/n} - 1) = 1$$

□

Theorem 7.1.17 (Sanwich theorem)**.** *Suppose* a_n, b_n, c_n *satisfy* $a_n \leq b_n \leq c_n$ *and* $\lim_{n\to\infty} a_n = \lim_{n\to\infty} c_n = L$. *Then* $\lim_{n\to\infty} b_n = L$.

Useful Limits

Proposition 7.1.18.

(1) $\lim_{n\to\infty} \frac{\ln n}{n} = 0$

(2) $\lim_{n\to\infty} \sqrt[n]{n} = 1$

(3) $\lim_{n\to\infty} x^{1/n} = 1, \ x > 0$

(4) $\lim_{n\to\infty} x^n = 0, \ |x| < 1$

(5) $\lim_{n\to\infty} \left(1 + \frac{x}{n}\right)^n = e^x, \ x \in \mathbb{R}$

(6) $\lim_{n\to\infty} \frac{x^n}{n!} = 0, \ x \in \mathbb{R}$

Proof. (1) See Example 7.1.16.

(2) Let $a_n = n^{1/n}$ and take ln $\ln a_n = \ln n^{1/n} = \frac{\ln n}{n}$. Since this approaches 0 and e^x is continuous at 0 $a_n = e^{\ln a_n} \to e^0 = 1$ by theorem 7.1.15.

(3) Set $a_n = x^{1/n}$. Since the limit of $\ln a_n = \ln x^{1/n} = \frac{\ln x}{n}$ is 0, we see $x^{1/n} = a_n = e^{\ln a_n}$ converges to $e^0 = 1$.

(4) Use the definition. given $\varepsilon > 0$, we must find n, s.t. for $|x| < \varepsilon^{1/n}$ $|x^n - 0| < \varepsilon$ holds. Since $\lim_{n\to\infty} \varepsilon^{1/n} = 1$ there is an N s.t $|x| < \varepsilon^{1/N}$ holds. Now if $n > N$ we have $|x|^n < |x^N| < \varepsilon$.

(5) Let $a_n = (1 + x/n)^n$. Then $\lim_{n\to\infty} \ln a_n = \lim_{n\to\infty} \ln (1 + x/n)^n = n \ln (1 + x/n)$ and by L'Hopital's rule we see

$$\lim_{n\to\infty} \frac{\ln(1 + x/n)}{1/n} = \lim_{n\to\infty} \frac{x}{1 + x/n} = x$$

Hence $a_n = (1 + x/n)^n = e^{\ln a_n}$ converges to e^x.

(6) First we will show that

$$-\frac{|x|^n}{n!} \le \frac{x^n}{n!} \le \frac{|x|^n}{n!}$$

and $|x|^n/n! \to 0$. Then use Sandwich theorem. If $|x|$ is greater than M, then $|x|/M < 1$ and hence $(|x|/M)^n \to 0$. If $n > M$

$$\frac{|x|^n}{n!} = \frac{|x|^n}{1 \cdot 2 \cdots M(M+1) \cdots n} \le \frac{|x|^n}{M! M^{n-M}} = \frac{M^M}{M!}\left(\frac{|x|}{M}\right)^n$$

holds. But $M^M/M!$ is fixed number. As $n\infty$ $(|x|/M)^n$ approaches 0. So $|x|^n/n!$ approaches 0. Finally by Sandwich theorem 7.1.17 we get the result. $x^n/n! \to 0$.

□

Example 7.1.19. (1) $\lim_{n\to\infty}\left(\frac{1}{1000}\right)^{1/n} = 1$.

(2) $\lim_{n\to\infty}\left(10^{1000}n^2\right)^{1/n} = \lim_{n\to\infty}(10^{1/n})^{1000} \lim_{n\to\infty} n^{2/n} = 1 \cdot \lim_{n\to\infty}\left(n^{1/n}\right)^2 = 1.$

(3) $\lim_{n\to\infty}\left(1 - \frac{2}{n}\right)^n = e^{-2}$.

(4) $\lim_{h\to 0^+}(1+h)^{1/h} = \lim_{n\to\infty}\left(1 + \frac{1}{n}\right)^n = e$.

(5) $\lim_{n\to\infty}\frac{10^n}{n!} = 0$.

(6) The set of all x satisfying $\lim_{n\to\infty}\frac{|x|^n}{5^n} = 0$ is, $\{x : |x| < 5\}$.

Example 7.1.20. $\lim_{n\to\infty}\sqrt[n]{5n+1} = 1$.

sol. We see

$$\sqrt[n]{5n+1} = \exp(\ln(5n+1)^{1/n})$$

and

$$\lim_{n\to\infty}\frac{\ln(5n+1)}{n} = 0.$$

Hence by theorem 7.1.15, the above limit is

$$\lim_{n\to\infty}\sqrt[n]{5n+1} = \lim_{n\to\infty}\exp(\ln(5n+1)^{1/n}) = \exp(\lim_{n\to\infty}\frac{\ln(5n+1)}{n}) = e^0 = 1.$$

□

Example 7.1.21. Show that $\lim_{n\to\infty} \ln n/n^{\varepsilon} = 0$ for any $\varepsilon > 0$.

sol. By L'Hopital rule 3.6.5

$$\lim_{n\to\infty} \frac{\ln n}{n^{\varepsilon}} = \lim_{n\to\infty} \frac{1/n}{\varepsilon n^{\varepsilon-1}} = \lim_{n\to\infty} \frac{1}{\varepsilon n^{\varepsilon}} = 0.$$

□

Monotone Sequence

Definition 7.1.22. If a_n satisfies

$$a_1 \le a_2 \le \cdots \le a_n \le \cdots$$

then a_n is called an **nondecreasing sequence**(increasing sequence).

Definition 7.1.23. If there is a number M such that $a_n \le M$ for all n, then this sequence is called **bounded from above**. Any such M is called **upper bound**.

Example 7.1.24. For the sequence $a_n = 1 - 1/2^n$, $M = 1$ is an upper bound and any number bigger than 1 is an upper bound. The smallest such number(if exists) is **the least upper bound**.

Theorem 7.1.25. *If a nondecreasing sequence has an upper bound, it converges. Furthermore, it converges to the least upper bound.*

Suppose L is a least upper bound, we observe two things:

(1) $a_n \le L$ for all n, and

(2) for any $\varepsilon > 0$ there is a term a_N greater than $L - \varepsilon$.

Suppose there does not exist such a_N, it holds that $a_n \le L - \varepsilon$ for all n, which is a contradiction. Thus for $n \ge N$

$$L - \varepsilon < a_n \le L$$

Thus $|L - a_n| < \varepsilon$ and we have proved $a_n \to L$.

For a decreasing sequence, we have a similar definition and theorem.

Definition 7.1.26. If a_n satisfies

$$a_1 \ge a_2 \ge \cdots \ge a_n \cdots$$

a_n is called a **decreasing sequence**. If $s_n \ge N$, then N is called a **lower bound**(lower bound). The largest such number is called **the greatest lower bound**.

Theorem 7.1.27. *If a nonincreasing sequence has a lower bound, it converges. Furthermore, it converges to the greatest lower bound.*

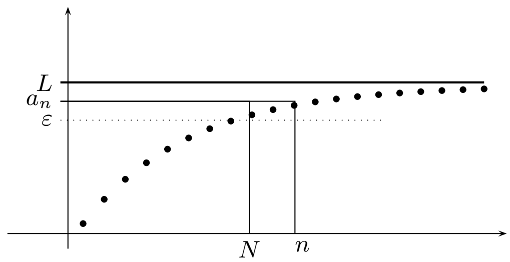


Figure 7.4: Nondecreasing(increasing) sequence and least upper bound L

7.2 Infinite Series

An infinite series is the sum of an infinite sequence of numbers.

Example 7.2.1. If we denote the sum of first n- term of $a_n = 1/2^n$ by s_n then

$$\begin{aligned} s_1 &= a_1 = \frac{1}{2} \\ s_2 &= a_1 + a_2 = \frac{1}{2} + \frac{1}{4} = \frac{3}{4} \\ s_3 &= a_1 + a_2 + a_3 = \frac{1}{2} + \frac{1}{4} + \frac{1}{8} = \frac{7}{8} \\ &\vdots \end{aligned}$$

The general term $\{s_n\}$ satisfies

$$s_n = a_1 + a_2 + a_3 + \cdots + a_n = \sum_{k=1}^{n} a_k$$

infinite series Write it as $\sum_{n=1}^{\infty} a_n$ or $\sum a_n$.

Definition 7.2.2. a_n is called n**-th term** $s_n = \sum_{k=1}^{n} a_k$ is n-th **partial sum** If the limit of $\{s_n\}$ is L then we say $\sum a_n$ converges to L and write $\sum_{n=1}^{\infty} a_n = L$ or $a_1 + a_2 + a_3 + \cdots = L$. If s series does not converges, we say it **diverges**.

Example 7.2.3 (Repeating decimals)**.** Write $0.1111\cdots$ as series.

sol. Writing $0.111\cdots = 0.1 + 0.01 + 0.001 + \cdots$ we see

$$\begin{aligned} a_1 &= 0.1, \\ a_2 &= 0.01, \\ &\vdots \\ a_n &= (0.1)^n \end{aligned}$$

Hence $0.111 = \sum_{k=1}^{\infty} 10^{-k}$.

□

Definition 7.2.4.

$$a + ar + ar^2 + \cdots$$

is called a **geometric series** and r is called a **ratio**.

We can compute the sum of a geometric series as follows: Note that

$$\begin{aligned} s_n &= a + ar + \cdots + ar^{n-1} \\ rs_n &= ar + ar^2 + \cdots + ar^n \\ s_n - rs_n &= a - ar^n \end{aligned}$$

Hence

$$s_n = a(1 - r^n)/(1 - r).$$

Example 7.2.5 (Telescoping Series)**.** $\sum_{n=1}^{\infty} \frac{1}{n(n+1)}$.

sol. Note that $\frac{1}{n(n+1)} = \frac{1}{n} - \frac{1}{n+1}$. Hence

$$s_n = \left(\frac{1}{1} - \frac{1}{2}\right) + \left(\frac{1}{2} - \frac{1}{3}\right) + \cdots + \left(\frac{1}{n} - \frac{1}{n+1}\right) = 1 - \frac{1}{n+1}.$$

Hence we see $s_n \to 1$.

□

Divergent Series

Example 7.2.6. $\sum_{n=1}^{\infty} \frac{(n+1)}{n}$ diverges since n-th term is greater than 1.

Example 7.2.7. $\sum_{n=1}^{\infty} \sin(\pi n/2)$ diverges.

sol.

$$1, 0, -1, 0, 1, \ldots$$

$$s_4 = s_8 = \cdots = s_{4n} = 0$$

but

$$s_2 = s_6 = \cdots = s_{4n+2} = 1$$

So s_n oscillates between 0 and 1.

□

Theorem 7.2.8 (n-th term test)**.** *If $\sum a_n$ converges then $a_n \to 0$.*

Proof. Suppose $\sum_{n=1}^{\infty} a_n$ converges then s_n and s_{n-1} must have the same limit. Since $a_n = s_n - s_{n-1}$ we see $\lim a_n = \lim s_n - \lim s_{n-1} = 0$. □

The converse is not true

Example 7.2.9. (1) Although $\sqrt{n+1} - \sqrt{n}$ converges to 0, the series $\sum \sqrt{n}$ diverges.

(2) Moreover, one can show the series $\sum \frac{1}{\sqrt{n}}$ diverges.

(3) The following sequence diverges even if each individual term approaches zero.

$$1 + \underbrace{\frac{1}{2} + \frac{1}{2}}_{\text{2 term}} + \underbrace{\frac{1}{3} + \frac{1}{3} + \frac{1}{3}}_{\text{3term}} + \cdots + \underbrace{\frac{1}{n} + \cdots + \frac{1}{n}}_{n\text{term}} + \cdots$$

Theorem 7.2.10 (nth term test for divergence)**.** *If $\lim a_n \not\to 0$ or $\lim a_n$ does not exists, then $\sum a_n$ diverges.*

Example 7.2.11. $\sum \frac{(n-1)}{n}$ diverges since $a_n = \frac{(n-1)}{n} \to 1$.

Example 7.2.12. $\sum (-1)^n \ln(\ln n)$ diverges since $\ln(\ln n) \to \infty$.

Theorem 7.2.13. *Suppose $\sum a_n, \sum b_n$ converges. Then*

(1) $\sum (a_n + b_n) = \sum a_n + \sum b_n$,

(2) $\sum (a_n - b_n) = \sum a_n - \sum b_n$,

(3) $\sum k a_n = k \sum a_n$.

Example 7.2.14.

(1) $$\sum_{n=1}^{\infty} \frac{2^n - 1}{3^n} = \sum_{n=1}^{\infty} \frac{2^n}{3^n} - \sum_{n=1}^{\infty} \frac{1}{3^n} = \frac{2}{3}\frac{1}{1-2/3} - \frac{1}{3}\frac{1}{1-1/3} = \frac{3}{2}.$$

(2) $$\sum_{n=1}^{\infty} \frac{3^n - 2^n}{6^n} = \sum_{n=1}^{\infty} \frac{3^n}{6^n} - \sum_{n=1}^{\infty} \frac{2^n}{6^n} = \sum_{n=1}^{\infty} \frac{1}{2^n} - \sum_{n=1}^{\infty} \frac{1}{3^n} = \frac{1}{2}.$$

Question: What's wrong with the following ?

$$1 = \left(1 - \frac{1}{2}\right) + \left(\frac{1}{2} - \frac{1}{3}\right) \cdots + \left(\frac{1}{n} - \frac{1}{n+1}\right) + \cdots = \sum_{n=1}^{\infty} \left(\frac{1}{n} - \frac{1}{n+1}\right) = \sum_{n=1}^{\infty} \frac{1}{n} - \sum_{n=1}^{\infty} \frac{1}{n+1}.$$

7.3 Series with nonnegative terms

We study the convergence or divergence of an infinite series whose n-th term is nonnegative.

Corollary 7.3.1. *A series $\sum a_n$ of nonnegative terms converges iff the partial sums are bounded from above.*

Integral Test

Example 7.3.2. Determine whether the following series converges or not.

$$\sum \frac{1}{n^2} = 1 + \frac{1}{4} + \frac{1}{9} + \cdots + \frac{1}{n^2} + \cdots$$

sol. We can compare the partial sum with the integral of a function. Set $f(x) = 1/x^2$. Then the partial sum is

$$s_n = 1 + \frac{1}{4} + \frac{1}{9} + \cdots + \frac{1}{n^2} = f(1) + f(2) + f(3) + \cdots + f(n)$$

and

$$\begin{aligned} f(2) &= \frac{1}{2^2} < \int_1^2 \frac{1}{x^2}\,dx \\ f(3) &= \frac{1}{3^2} < \int_2^3 \frac{1}{x^2}\,dx \\ &\vdots \\ f(n) &= \frac{1}{n^2} < \int_{n-1}^n \frac{1}{x^2}\,dx \end{aligned}$$

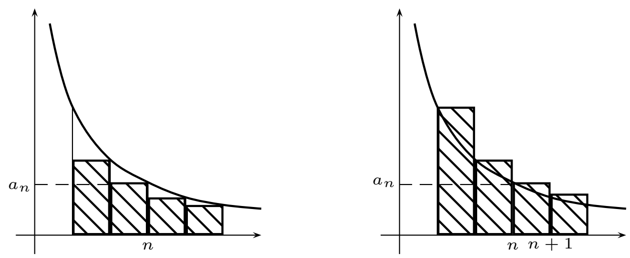
Hence

$$s_n = f(1) + f(2) + f(3) + \cdots + f(n) < 1 + \int_1^n \frac{1}{x^2}\,dx = 2 - \frac{1}{n}.$$

Thus s_n is bounded, increasing, and hence converges.

□

Theorem 7.3.3 (Integral Test)**.** *Suppose $f(x)$ is nonnegative, non-increasing for $x \geq 1$ and $a_n = f(n)$. Then the series $\sum_{n=1}^{\infty} a_n$ converges if the integral $\int_1^{\infty} f(x)\,dx$ converges. Conversely, if the integral $\int_1^{\infty} f(x)\,dx$ converges, then the series $\sum_{n=1}^{\infty} a_n$ also converges.*



(a) $a_n \le \int_{n-1}^{n} f(x)\,dx$ (b) $\int_{n}^{n+1} f(x)\,dx \le a_n$

Figure 7.5: Integral Test

Proof. Since f is decreasing and $f(n) = a_n$, we see from figure 7.5 (a), $a_n \le \int_{n-1}^{n} f(x)\,dx$, $(n = 2, 3, 4, \dots)$. Hence we have

$$a_2 + a_3 + \cdots + a_n \le \int_1^n f(x)\,dx$$

Conversely, we see from figure 7.5(b), $\int_n^{n+1} f(x)\,dx \le a_n$. So

$$\int_1^{n+1} f(x)\,dx \le a_1 + a_2 + \cdots + a_n$$

and from these two cases, we see

$$\int_1^{n+1} f(x)\,dx \le a_1 + a_2 + \cdots + a_n \le a_1 + \int_1^n f(x)dx.$$

Hence the conclusion follows. □

Example 7.3.4 (p-series)**.** Let p be a fixed number. Then

$$\sum_1^\infty \frac{1}{n^p} = \frac{1}{1^p} + \frac{1}{2^p} + \cdots + \frac{1}{n^p} + \cdots$$

converges when $p > 1$ and diverges when $p \le 1$. For $p = 1$, we see

$$\int_1^\infty \frac{1}{x}\,dx = \lim_{b\to\infty} [\ln b]_1^b = \infty$$

So the harmonic series

$$1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n} + \cdots$$

diverges.

Example 7.3.5. Test the convergence of

$$\sum_1^\infty \frac{1}{1+n^2}.$$

We see

$$\int_1^\infty \frac{dx}{1+x^2} = \lim_{b\to\infty}[\tan^{-1}x]_1^b = \lim_{b\to\infty}[\tan^{-1}b - \tan^{-1}1] = \frac{\pi}{4}.$$

7.3.1 Error estimation of integral test

Let $S = \sum_{n=1}^\infty a_n$ and $R_n = S - s_n = a_{n+1} + a_{n+2} + \cdots$ be the remainder of the partial sum. Then we see

$$\int_{n+1}^{n+2} f(x)\,dx < a_{n+1} \le \int_n^{n+1} f(x)\,dx.$$

Hence we have the estimate for the remainder:

$$\int_{n+1}^\infty f(x)\,dx < R_n < \int_n^\infty f(x)\,dx$$

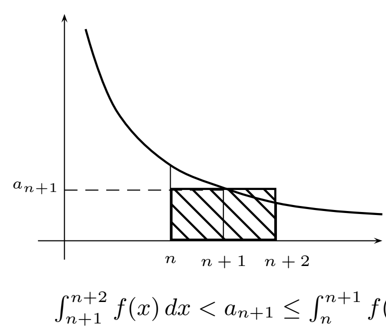


Figure 7.6: Error estimation

Example 7.3.6. Estimate the error when $\sum_{n=1}^\infty \frac{1}{n^2}$ is replaced by s_{10}.

sol. We see

$$\int_n^\infty \frac{1}{x^2}\,dx = \frac{1}{n}.$$

Thus

$$s_{10} + \frac{1}{11} < S < s_{10} + \frac{1}{10}.$$

Since $s_{10} \approx 1.54977$ and

$$\sum_{n=1}^{\infty} \frac{1}{n^2} \approx 1.65453$$

the actual error is 0.09553 which is between $\frac{1}{11} \approx 0.090909$ and $\frac{1}{10} = 0.1$.

□

7.3.2 Series with nonnegative terms-Comparison

$$\sum \frac{1}{n^3}, \quad \sum \frac{1}{3^n + 1}$$

Example 7.3.7. Investigate the convergence of $\sum_{n=1}^{\infty} \frac{1}{n^2}$.

sol. Useful inequality: $\frac{1}{n^2} < \frac{1}{n(n-1)}$.

$$\begin{aligned} s_n &= \frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \cdots + \frac{1}{n^2} \\ &< \frac{1}{1 \cdot 1} + \frac{1}{1 \cdot 2} + \frac{1}{2 \cdot 3} + \cdots + \frac{1}{n(n-1)} \\ &= 1 + \left(1 - \frac{1}{2}\right) + \left(\frac{1}{2} - \frac{1}{3}\right) + \cdots + \left(\frac{1}{n-1} - \frac{1}{n}\right) \\ &= 2 - \frac{1}{n} < 2. \end{aligned}$$

Hence s_n is bounded above and as a monotonic increasing sequence it converges.

□

Example 7.3.8 (Harmonic series)**.** The series

$$\sum \frac{1}{n} = 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n} + \cdots$$

diverges since

$$1 + \frac{1}{2} + \underbrace{\frac{1}{3} + \frac{1}{4}}_{>2/4} + \underbrace{\frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8}}_{>4/8} + \underbrace{\frac{1}{9} + \frac{1}{10} + \cdots + \frac{1}{16}}_{>8/16} + \cdots$$

is greater than

$$1 + \frac{1}{2} + \frac{1}{2} + \frac{1}{2} + \cdots$$

7.4 Comparison Test

Theorem 7.4.1 (The Comparison Test)**.** *Let $a_n \geq 0$.*

(a) The series $\sum a_n$ converges if $a_n \leq c_n$ for all $n > N$ and $\sum c_n$ converges

(b) The series $\sum a_n$ diverges if $a_n \geq d_n$ for all $n > N$ and $\sum d_n$ diverge.

Proof. In (a), the partial sum is bounded by

$$M = a_1 + a_2 + \cdots a_n + \sum_{n=N+1}^{\infty} c_n$$

In (b), the partial sum is greater than

$$M^* = a_1 + a_2 + \cdots a_n + \sum_{n=N+1}^{\infty} d_n$$

But the series $\sum_{n=N+1}^{\infty} d_n$ diverges. Hence so does $\sum a_n$.

□

Example 7.4.2. Look at the tail part of

$$3 + 600 + 5000 + \frac{1}{3!} + \frac{1}{4!} + \frac{1}{5!} + \cdots + \frac{1}{n!} + \cdots$$

Since $1/n! < 1/2^n$ for $n = 4, 5, 6, \ldots$ we compare it with a geometric series.

What about

$$\sum \frac{1}{n^{2.5} + 100n^4 + 3} \text{ or } \sum \frac{\ln n + 5}{n(\ln n)^2 + 3}$$

Limit Comparison Test

Example 7.4.3. Investigate the convergence of

$$\sum_{1}^{\infty} \frac{n}{2n^3 - n + 3}$$

sol. Since

$$a_n = \frac{n}{2n^3 - n + 3} = \frac{1}{2n^2 - 1 + 3/n}$$

we see the series $\sum a_n$ behaves similar to $\sum 1/2n^2$. If we let $c_n = 1/2n^2$, then $\lim_{n\to\infty} a_n/c_n = 1$. Hence for any ε there is N such that if $n > N$ for some N then the following holds:

$$1 - \varepsilon \leq \frac{a_n}{c_n} \leq 1 + \varepsilon.$$

In other words,

$$(1-\varepsilon)c_n \leq a_n \leq (1+\varepsilon)c_n, \quad n \geq N.$$

Since $\sum_{n\geq N} c_n$ converges, $\sum_{n\geq N} a_n$ converges by comparison.

□

Theorem 7.4.4 (Limit Comparison Test). *(1) Suppose $a_n > 0$ and there is a series $\sum c_n$ ($c_n > 0$) which converges and if*

$$\lim_{n\to\infty} \frac{a_n}{c_n} = c > 0$$

then $\sum a_n$ converges.

(2) Suppose $a_n > 0$ and there is a series $\sum d_n$ ($d_n > 0$) which diverges and if

$$\lim_{n\to\infty} \frac{a_n}{d_n} = c > 0$$

then $\sum a_n$ diverges.

Proof. We prove part (1). Since $c/2 > 0$ there is an N such that for all $n > N$ we have

$$\left|\frac{a_n}{b_n} - c\right| < \frac{c}{2}$$

Then

$$\begin{aligned} -\frac{c}{2} < \quad & \tfrac{a_n}{b_n} - c \quad < \frac{c}{2} \\ \frac{c}{2} < \quad & \tfrac{a_n}{b_n} \quad < \frac{3c}{2} \\ (\frac{c}{2})b_n < \quad & a_n \quad < \frac{3c}{2}b_n. \end{aligned}$$

Hence

$$(\frac{c}{2})\sum_{n\geq N}^{L} b_n < \sum_{n\geq N}^{L} a_n < \frac{3c}{2}\sum_{n\geq N}^{L} b_n$$

and the convergence of $\sum a_n$ follows that of $\sum b_n$. □

Example 7.4.5. (1) $\sum_1^\infty \frac{n+1}{100n^3+n+1}$ converges since $\sum_1^\infty \frac{1}{n^2}$ converges

(2) $\sum_{20}^\infty \frac{1}{3^n-1000n}$ converges since $\sum_1^\infty \frac{1}{3^n}$converge

(3) $\sum_1^\infty \frac{2n+1}{n^2+4n+1}$

(4) Does $\sum_2^\infty \frac{\ln n}{n^{3/2}}$ converge ? (compare $\ln < n^{0.1}$)

(5) Compare $\sum_1^\infty \frac{(\ln n)^{1/2}}{(n\ln n+1)}$ with $\sum_2^\infty \frac{1}{n(\ln n)^{1/2}}$. Use integral test.

$$\int_2^\infty \frac{dx}{x(\ln x)^{1/2}} = \int_{\ln 2}^\infty \frac{du}{u^{1/2}} = \infty$$

7.5 Ratio test and Root Tests

Example 7.5.1. It is not easy to find general term of $a_1 = 1,\ a_{n+1} = \frac{na_n}{3n+2}$. But its ratio is clearly seen.

Ratio Test

Theorem 7.5.2 (Ratio Test)**.** *Suppose $a_n > 0$ and if the limit exists.*

$$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \rho.$$

Then the following holds.

(1) The sum $\sum a_n$ converges if $\rho < 1$

(2) The sum $\sum a_n$ diverges if $\rho > 1$

(3) The test is inconclusive if $\rho = 1$.

Proof. The motive is to compare with a geometric series. (1) Let $\rho < 1$. Then choose any r between ρ and 1 and set $\varepsilon = r - \rho$. Then since

$$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \rho$$

there exists a natural number N such that for all $n > N$,

$$\left| \frac{a_{n+1}}{a_n} - \rho \right| < \varepsilon$$

holds. Solving

$$\rho - \varepsilon < \frac{a_{n+1}}{a_n} < \varepsilon + \rho$$

for all $n > N$. Hence we see

$$(\rho - \varepsilon)a_n < a_{n+1} < (\varepsilon + \rho)a_n = ra_n, \quad n > N \tag{7.1}$$

and

$$\begin{aligned} a_{N+1} &< ra_N \\ a_{N+2} &< ra_{N+1} < r^2 a_N \\ &\vdots \\ a_{N+m} &< ra_{N+m-1} < r^m a_N \end{aligned}$$

We compare a_n with a series general term is $r^m a_N$. Since $\sum_{m=1}^{\infty} r^m a_N$ converges, $\sum_{n=N+1}^{\infty} a_n$ converges. (2) Suppose $\rho > 1$. Then exist an M such that for $n > M$ and by (7.1) it holds that

$$\frac{a_{n+1}}{a_n} > r$$

And note that

$$a_M < a_{M+1} < a_{M+2} < \cdots$$

so the series diverges.

(3) The case: $\rho = 1$. Both the series $\sum 1/n^2$ and $\sum 1/n$. But the former converges and the latter diverges. □

Example 7.5.3.

(1) $\sum \frac{n!n!}{(2n)!}$

(2) $\sum \frac{(2^n+5)}{3^n}$

(3) $\sum \frac{2^n}{n!}$

sol. Ratio Test

(1)

$$\begin{aligned}\frac{a_{n+1}}{a_n} &= \frac{(n+1)!(n+1)!(2n)!}{n!n!(2n+2)(2n+1)(2n)!} \\ &= \frac{(n+1)(n+1)}{(2n+2)(2n+1)} = \frac{n+1}{4n+2} \to \frac{1}{4}\end{aligned}$$

(2) $\frac{a_{n+1}}{a_n} = \frac{(2^{n+1}+5)3^n}{3^{n+1}(2^n+5)} = \frac{2^{n+1}+5}{3(2^n+5)} \to \frac{2}{3}$

(3) $\frac{a_{n+1}}{a_n} = \frac{2^{n+1}n!}{(n+1)!2^n} = \frac{2}{n+1} \to 0$

□

Example 7.5.4. Find the range of x which makes the following converge.

$$1 + \frac{x^2}{2} + \frac{x^4}{4} + \frac{x^6}{6} + \cdots$$

sol. For $n > 1$, $a_n = \frac{x^{2n-2}}{(2n-2)}$.

$$\frac{a_{n+1}}{a_n} = \frac{x^{2n}(2n-2)}{2nx^{2n-2}} = \frac{(2n-2)x^2}{2n} \to x^2$$

So it converges if $|x| < 1$ and diverges if $|x| > 1$. When $|x| = 1$ the series behaves like

$$1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{6} \cdots = 1 + \frac{1 + 1/2 + 1/3 + \cdots}{2}$$

□

Estimate error

For $\rho < 1$ If the series is approximated by its N- partial sum, then the error is

$$a_{N+1} + a_{N+2} + \cdots$$

So if N is large, for some r with $\rho < r < 1$ we have

$$\frac{a_{n+1}}{a_n} < r, \quad n \geq N$$

$$a_{N+1} + a_{N+2} + \cdots \leq r a_N + r^2 a_N + \cdots = a_N \cdot \frac{r}{1-r}$$

is the estimate of errors.

Example 7.5.5 (Ratio test does not work)**.** Investigate

$$\frac{1}{3} + \frac{2}{9} + \frac{1}{27} + \frac{4}{81} + \cdots + \frac{f(n)}{3^n} + \cdots$$

$$\text{where } f(n) = \begin{cases} n, & n \text{ even} \\ 1, & n \text{ odd} \end{cases}$$

sol. Since $a_n = \frac{f(n)}{3^n}$ we have

$$\frac{a_{n+1}}{a_n} = \frac{f(n+1)}{3f(n)} = \begin{cases} \frac{1}{3n}, & n \text{ even} \\ \frac{n+1}{3}, & n \text{ odd} \end{cases}$$

So we cannot use ratio test. However if we take n-th root,

$$\sqrt[n]{a_n} = \frac{\sqrt[n]{f(n)}}{3} = \begin{cases} \sqrt[n]{\frac{n}{3}}, & n \text{ even} \\ \frac{1}{3}, & n \text{ odd} \end{cases}$$

and $\sqrt[n]{n}$ converges to 1. Hence we see

$$\lim_{n\to\infty} \sqrt[n]{a_n} = \frac{1}{3}.$$

Now we can compare this series with $\sum(\frac{1}{3})^n$.

□

n-th Root Test

Theorem 7.5.6 (n-th Root Test)**.** *Suppose* $\sqrt[n]{a_n} \to \rho$. *Then*

(1) $\sum a_n$ *converges if* $\rho < 1$.

(2) $\sum a_n$ *diverges if* $\rho > 1$.

(3) Inconclusive if $\rho = 1$.

Proof. The motive is again to compare with a geometric series:

$$a + a\rho + a\rho^2 + \cdots + a\rho^n + \cdots (a > 0)$$

i.e, if $a_n \sim a\rho^n (n \geq N))$ for some $0 < \rho < 1$, then we would have

$$\sqrt[n]{\frac{a_n}{a}} \doteq \rho, \quad n \geq N$$

which is eventually equivalent to

$$\sqrt[n]{a_n} \doteq \rho, \quad n \geq N.$$

Now the remaining task it to prove it rigorously.

(1) Suppose $\rho < 1$. Choose r between ρ and 1 and set $\varepsilon = \rho - r > 0$. Since $\sqrt[n]{a_n}$ converges to ρ there is some integer N such that when n is greater than N, then it holds that

$$|\sqrt[n]{a_n} - \rho| < \varepsilon,$$

i.e,

$$\sqrt[n]{a_n} < \rho + \varepsilon = r < 1.$$

Hence

$$a_n < (\rho + \varepsilon)^n$$

holds. Since $\sum(\rho+\varepsilon)^n$ converges the series $\sum_{n=N}^{\infty} a_n$ converges by comparison.

(2) Suppose $\rho > 1$. Then $\sqrt[n]{a_n} > 1$ for suff. large n and hence $a_n > 1$. So the series diverges.

(3) The case $\rho = 1$: the test is inclusive: It may converge or may diverge. See $\sum \frac{1}{n}, \sum \frac{1}{n^2}$. Both series has $\rho = 1$ but one diverges while the other converges. □

Example 7.5.7. $\sum_{n=1}^{\infty} \frac{n}{2^n}$ converges since $\sqrt[n]{\frac{n}{2^n}} = \sqrt[n]{\frac{n}{2}} \to \frac{1}{2}$.

Example 7.5.8. $\sum_{n=1}^{\infty} \frac{3^n}{n^n}$ converges since $\sqrt[n]{\frac{3^n}{n^n}} = \frac{3}{n} \to 0$.

7.6 Alternating Series, absolute and conditional convergence

Alternating Series

Definition 7.6.1. Suppose $a_n > 0$ for all n. A series of the form

$$a_1 - a_2 + a_3 - a_4 + \cdots$$

is called an **alternating series**.

The followings are alternating series

$$1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \cdots$$
$$1 - 2 + 3 - 4 + 5 - 6 + \cdots$$

But following is not an alternating series.

$$1 - \frac{1}{2} - \frac{1}{3} + \frac{1}{4} + \frac{1}{5} - \frac{1}{6} - \frac{1}{7} + \cdots$$

Theorem 7.6.2 (Alternating Series Test, Leibniz theorem)**.** *Suppose the following three conditions hold.*

(1) $a_n > 0$.

(2) $a_n \geq a_{n+1}$ *for all* $n \geq N$ *for some integer* N.

(3) $a_n \to 0$.

Then the series

$$\sum_{n=1}^{\infty} (-1)^{n+1} a_n = a_1 - a_2 + a_3 - a_4 + \cdots$$

converges.

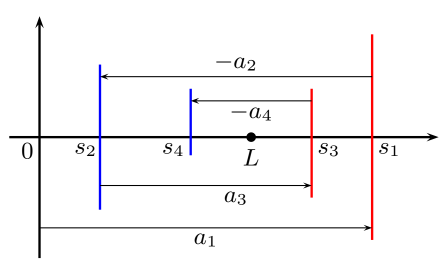


Figure 7.7: Partial sum of alternating series

Proof. Suppose n is even ($n = 2m$) then the partial sum

$$s_{2m} = (a_1 - a_2) + (a_3 - a_4) + \cdots + (a_{2m-1} - a_{2m})$$

is increasing. But we also see

$$s_{2m} = a_1 - (a_2 - a_3) - (a_4 - a_5) - \cdots - (a_{2m-2} - a_{2m-1}) - a_{2m}.$$

Hence s_{2m} is less than a_1. In other words, s_{2m} is bounded above, hence as an increasing sequence, it converges. Let L be its limit.

$$\lim s_{2m} = L.$$

Now suppose n is odd ($n = 2m + 1$). Then

$$s_{2m+1} = s_{2m} + a_{2m+1}$$

Then since $a_{2m+1} \to 0$, we see $\lim s_{2m+1} = \lim(s_{2m} + a_{2m+1}) = L$. □

Remark 7.6.3. By graphical interpretation, we can see that the limit L lies between any tow consecutive sum s_n and s_{n+1}. Hence we can also show $|s_n - L| < a_{n+1}$. This gives some estimation theorem(later).

Example 7.6.4.

$$\sum (-1)^{n+1}\frac{1}{n} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots$$

converges.

Example 7.6.5.

$$\sum (-1)^{n+1}\frac{1}{\sqrt{n}} = 1 - \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} - \frac{1}{\sqrt{4}} + \cdots$$

converges.

Example 7.6.6.

$$\sum (-1)^{n+1}\frac{\sqrt{n}}{\sqrt{n+1}} = \frac{1}{\sqrt{2}} - \frac{\sqrt{2}}{\sqrt{3}} + \frac{\sqrt{3}}{\sqrt{4}} - \frac{\sqrt{4}}{\sqrt{5}} + \cdots$$

diverges by n-th term test.

Example 7.6.7.

$$\frac{2}{1} - \frac{1}{1} + \frac{2}{3} - \frac{1}{3} + \frac{2}{4} - \frac{1}{4} + \frac{2}{5} - \frac{1}{5} + \cdots + \frac{2}{2n-1} - \frac{1}{2n-1} + \cdots$$

is alternating. But

$$\left(\frac{2}{1} - \frac{1}{1}\right) + \left(\frac{2}{3} - \frac{1}{3}\right) + \left(\frac{2}{4} - \frac{1}{4}\right) + \left(\frac{2}{5} - \frac{1}{5}\right) + \cdots$$
$$+ \left(\frac{2}{2n-1} - \frac{1}{2n-1}\right) + \cdots = 1 + \frac{1}{3} + \frac{1}{5} + \cdots + \frac{1}{2n-1} + \cdots$$

So it diverges.

Example 7.6.8. Investigate the convergence of the series $\sum_{n=2}^{\infty}(-1)^n \frac{\ln n}{n+1}$.

sol. The conditions (1) and (3) are easy to check. To see if (2) is satisfied, we have to check if

$$\frac{\ln n}{n+1} \geq \frac{\ln(n+1)}{n+n}$$

which is not an easy task. The idea is to consider a function $f(x)$ such that $f(n) = \ln n/(n+1)$ and use derivative test. We let

$$f(x) = \frac{\ln x}{x+1}.$$

Then $f(n) = \ln n/(n+1)$ and take derivative:

$$f'(x) = \frac{(x+1)/x - \ln x}{(x+1)^2} = \frac{(x+1) - x\ln x}{x(x+1)^2}.$$

We can show $(x+1) - x\ln x < 0$, for sufficiently large x. Hence $f(x)$ is decreasing function for sufficiently large x. For example, for $x \geq 8$, $f(x)$ is decreasing. So $a_n = f(n)$ is decreasing for $n \geq 8$. By Leibniz theorem the series converges.

□

Partial Sum of Alternating Series

We look at the partial sums of an alternating series:

$$\begin{aligned}
s_1 &= a_1, & \\
s_2 &= a_1 - a_2, & \text{So } s_2 < s_1. \\
s_3 &= a_1 - a_2 + a_3 = a_1 - (a_2 - a_3), & \text{So } s_2 < s_3 < s_1. \\
s_4 &= a_1 - a_2 + a_3 - a_4 = a_1 - a_2 + (a_3 - a_4), & \text{So } s_2 < s_4 < s_3 < s_1.
\end{aligned}$$

Thus s_{2m+1} is decreasing and s_{2m} is increasing. Let L be its sum. Then

$$\underbrace{\underbrace{s_{2m} < s_{2m+2} < \cdots < L}_{|s_{2m}-L|} < \cdots < s_{2m+1}}_{|s_{2m}-s_{2m+1}|} < s_{2m-1}$$

But since

$$\begin{aligned}
|s_{2m} - L| &< |s_{2m} - s_{2m+1}| = a_{2m+1}, \\
|s_{2m+1} - L| &< |s_{2m+2} - s_{2m+1}| = a_{2m+2}
\end{aligned}$$

we see that for all n,

$$|s_n - L| < a_{n+1}.$$

In other words, partial sum is an approximation to the true sum with error bound a_{n+1}. Since a_n is decreasing s_{n+1} is better approximation than s_n.

Theorem 7.6.9 (Alternating Series Estimation Theorem)**.** *Suppose* $\sum(-1)^{n+1}a_n$ *is an alternating series satisfying the conditions of Leibniz theorem. Then the partial sum*

$$s_n = a_1 - a_2 + a_3 - \cdots + (-1)^{n+1}a_n$$

is a good approximation with error bound less than a_{n+1}.

Example 7.6.10. Estimate

$$\sum_{n=0}^{\infty} \frac{(-1)^n}{2^n} = 1 - \frac{1}{2} + \frac{1}{4} + \cdots = \frac{1}{1-(-\frac{1}{2})} = \frac{2}{3}$$

with first six term.

sol. Let $s_n = \sum_{k=0}^{n} \frac{(-1)^n}{2^n}$. Error bound for $|s_5 - L|$ is $a_6 = 1/64$. The actual value up to six term(a_5) is

$$s_5 = 1 - \frac{1}{2} + \frac{1}{4} - \frac{1}{8} + \frac{1}{16} - \frac{1}{32} = \frac{21}{32}.$$

So true error is $|2/3 - 21/32| = 1/96$ which is less than $a_6 = 1/64$, the estimate of the theorem .

□

Example 7.6.11. Use s_{10} or s_{100} to estimate

$$\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} = 1 - \frac{1}{2} + \frac{1}{3} - \cdots = \ln 2 = 0.69314\cdots$$

sol. We have

$$s_{10} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots - \frac{1}{10} = 0.64563\cdots$$

and the error of s_{10} is $|0.64563 - \ln 2| = 0.0475\cdots < a_{11} = 1/11$. Also,

$$s_{100} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots - \frac{1}{100} = 0.68881\cdots$$

and the error of s_{100} is $|0.68881 - \ln 2| = 0.00433\cdots < a_{111} = 1/111$. In either case, the actual error is smaller than the error predicted by the theory.

□

Absolute convergence and Conditional Convergence

Example 7.6.12. Suppose we want to compute the series:

$$1-\frac{1}{5}+\frac{1}{3}-\frac{1}{5^2}+\frac{1}{3^2}-\frac{1}{5^3}+\cdots$$

It would be good if we compute positive term first and then negative terms: Hence

$$1+\frac{1}{3}+\frac{1}{3^2}+\frac{1}{3^3}+\cdots-\left(\frac{1}{5}+\frac{1}{5^2}+\frac{1}{5^3}+\cdots\right)=\frac{1}{1-1/3}-\frac{1}{1-1/5}=\frac{3}{2}-\frac{5}{4}=\frac{1}{4}.$$

Fortunately, this is correct. Next example is the following series which is convergent:

$$1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\frac{1}{5}-\frac{1}{6}+\frac{1}{7}-\frac{1}{8}+\frac{1}{9}+\cdots$$

Consider a rearrangement:

$$\left(1+\frac{1}{3}+\frac{1}{5}+\frac{1}{7}+\frac{1}{9}\cdots-\right)-\left(\frac{1}{2}+\frac{1}{4}+\frac{1}{6}+\frac{1}{8}\cdots\right)$$

Then the sum is not defined! Thus we have to be careful when we add infinite series.

Definition 7.6.13. If $\sum |a_n|$ converges then $\sum a_n$ is said to **converge absolutely**. A series which converges but does not converge absolutely is said to **converges conditionally**.

Example 7.6.14. (1) $\sum_{n=1}^{\infty}(-1)^{n+1}\frac{1}{n^2}=1-\frac{1}{4}+\frac{1}{9}+\cdots$ converges absolutely since $\sum\frac{1}{n^2}$ converges.

(2) $\sum\frac{\cos n}{n^2}$ satisfies $|a_n|=\frac{|\cos n|}{n^2}\le\frac{1}{n^2}$. Since $\sum\frac{1}{n^2}$ converges, $\sum\frac{\cos n}{n^2}$ converges.(absolutely)

(3) The series

$$\sum(-1)^{n+1}\frac{1}{n}=1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\cdots$$

converges. But $\sum|a_n|=\sum\frac{1}{n}$ diverges. Hence $\sum(-1)^{n+1}\frac{1}{n}$ converges conditionally.

(4) $\sum\frac{(-1)^n}{n^p}$ converges for any $p>0$. But $\sum\frac{1}{n^p}$ converges for $p>1$ only. Hence $\sum\frac{(-1)^n}{n^p}$ converges conditionally for all $p>0$, but converges absolutely for $p>1$.

Theorem 7.6.15. *If $\sum|a_n|$ converges then so does $\sum a_n$.*

Proof.

$$-|a_n| \le a_n \le |a_n|$$

holds for all n. Hence

$$0 \le a_n + |a_n| \le 2|a_n|.$$

Since $\sum |a_n|$ converges and $a_n + |a_n| \ge 0$, the series

$$\sum (a_n + |a_n|)$$

converges by comparison test. Subtracting converging series, we have

$$\sum a_n = \sum (a_n + |a_n|) - \sum |a_n|$$

and so $\sum a_n$ converges. □

Corollary 7.6.16. *If $\sum a_n$ diverges so does $\sum |a_n|$.*

Rearrangement of Series for Absolutely Convergent Series

Theorem 7.6.17 (Rearrangement of Series)**.** *Suppose b_n is a rearrangement of a_n(i.e, $b_k = a_{n(k)}$ for some 1-1 function $n(k)$). If $\sum a_n$ converges then the series $\sum b_n$ converges to the same sum.*

Proof. First assume $a_n \ge 0$ for all n. Suppose

$$|\sum_{n=1}^{k} a_n - L| < \epsilon, \quad \text{for all } k \ge N$$

Choose N_1 so large that $\{b_1, b_2, \cdots, b_{N_1}\}$ contains all of the terms in $\{a_1, a_2, \cdots, a_N\}$. Then

$$\sum_{n=1}^{N} a_n \le \sum_{n=1}^{N_1} b_n \le L.$$

Hence

$$\epsilon > L - \sum_{n=1}^{N} a_n \ge L - \sum_{n=1}^{N_1} b_n > 0.$$

This is true if N_1 is replaced by any larger index. Now we allow a_n negative. As in the proof of the previous theorem, we have

$$\sum b_n = \sum (b_n + |b_n|) - \sum |b_n|.$$

Now the result for positive terms shows that $\sum (b_n + |b_n|) = \sum (a_n + |a_n|)$ and $\sum |b_n| = \sum |a_n|$. Hence

$$\begin{aligned} \sum b_n &= \sum (b_n + |b_n|) - \sum |b_n| \\ &= \sum (a_n + |a_n|) - \sum |a_n| \\ &= \sum a_n. \end{aligned}$$

□

Example 7.6.18. We know the following converges absolutely:

$$1 - \frac{1}{2} + \frac{1}{4} - \frac{1}{8} + \frac{1}{16} - \frac{1}{32} + \cdots$$

Hence a rearrangement

$$1 + \frac{1}{4} - \frac{1}{2} + \frac{1}{16} + \frac{1}{64} - \frac{1}{8} + \cdots$$

converges to the same limit.

Now the series

$$1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \cdots$$

converges to $\ln 2$ but not absolutely. Hence its rearrangement may not converge or it may converge to a different value.

Consider one rearrangement:

$$\begin{aligned}
&\left(1 - \frac{1}{2}\right) + \left(\frac{1}{3} + \frac{1}{5} - \frac{1}{4}\right) + \left(\frac{1}{7} + \frac{1}{9} - \frac{1}{6}\right) + \left(\frac{1}{11} + \frac{1}{13} - \frac{1}{8}\right) + \cdots \\
\doteq\ & 0.5 + (0.53333 - 0.25) + (0.365079 - 0.166666) + (0.16783 - 0.125) + \cdots \\
=\ & 0.5 + 0.2833333 + 0.198413 + \text{ positive terms} \\
=\ & 0.7833333 + 0.198413 + \text{ positive terms}
\end{aligned}$$

Then sum is bigger than $\ln 2 = 0.69314\cdots$.

Product of two series

Suppose $\sum_{n=0}^{\infty} a_n, \sum_{n=0}^{\infty} b_n$ converge absolutely. Then

$$\left(\sum_{n=0}^{\infty} a_n\right) \times \left(\sum_{n=0}^{\infty} b_n\right) = (a_0 + a_1 + \cdots + a_n + \cdots) \times (b_0 + b_1 + \cdots + b_n + \cdots).$$

The product of finite partial sum is

$$(a_0 + a_1 + \cdots + a_n) \times (b_0 + b_1 + \cdots + b_n).$$

We multiply it out and write it as

$$\begin{aligned}
&a_0b_0 + (a_0b_1 + a_1b_0) + (a_0b_2 + a_1b_1 + a_2b_0) + \cdots \\
&+ \cdots + (a_0b_n + a_1b_{n-1} + \cdots + a_{n-1}b_1 + a_nb_0) + \cdots
\end{aligned}$$

In other words,

$$\left(\sum_{k=0}^{n} a_k\right) \times \left(\sum_{k=0}^{n} b_k\right) = \sum_{k=0}^{n} c_k + \text{ extra terms,}$$

where $c_0 = a_0b_0,\ c_1 = a_0b_1 + a_1b_0, \cdots, c_n = (a_0b_n + a_1b_{n-1} + \cdots + a_{n-1}b_1 + a_nb_0)$.

In the limit, (use the fact $\lim A_n \cdot \lim B_n = \lim(A_n B_n)$ when both sequence converge) we have

$$\left(\sum_{n=0}^{\infty} a_n\right) \times \left(\sum_{n=0}^{\infty} b_n\right) = \sum_{n=0}^{\infty} c_n.$$

Since it converges absolutely, its value does not change.

Theorem 7.6.19. *Suppose both $\sum_{n=0}^{\infty} a_n$ and $\sum_{n=0}^{\infty} b_n$ converge absolutely. If we set $c_n = \sum_{n=0}^{k} a_k b_{n-k}$ then $\sum c_n$ converge absolutely and*

$$\sum_{n=0}^{\infty} c_n = \left(\sum_{n=0}^{\infty} a_n\right) \times \left(\sum_{n=0}^{\infty} b_n\right).$$

7.7 Power Series

Definition 7.7.1. A **power series** about $x = 0$ is a series of the form

$$\sum_{n=0}^{\infty} a_n x^n = a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n + \cdots$$

A power series about $x = a$ is a series of the form

$$\sum_{n=0}^{\infty} a_n (x-a)^n$$

a_n are coefficients and a is the center.

Example 7.7.2. (1) $\sum_{n=1}^{\infty} \frac{(x-1)^n}{2^n} = \frac{1}{2^1} + \frac{(x-1)^2}{2^2} + \frac{(x-1)^3}{2^3} + \cdots$

(2) $\sum_{n=1}^{\infty} (-1)^{n-1} \frac{x^n}{n} = x - \frac{x^2}{2} + \frac{x^3}{3} - \cdots$

(3) $\sum_{n=1}^{\infty} (-1)^{n-1} \frac{x^{2n-1}}{2n-1} = x - \frac{x^3}{3} + \frac{x^5}{5} - \cdots$

(4) $\sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$

(5) $\sum_{n=0}^{\infty} n! x^n = 1 + x + 2! x^2 + 3! x^3 + \cdots$

Theorem 7.7.3 (Convergenec of Power Series)**.** *(1) Suppose the power series $\sum_{n=0}^{\infty} a_n (x-a)^n$ converges at a point $x_1\,(\neq a)$, then it converges absolutely for all points with $|x-a| < |x_1 - a|$.*

(2) Suppose the power series $\sum_{n=0}^{\infty} a_n (x-a)^n$ diverges at x_2, then it diverges for all x with $|x-a| > |x_2 - a|$.

Proof. (1) Suppose $\sum_{n=0}^{\infty} a_n(x_1 - a)^n$ converges, and let x be any number satisfying $|x - a| < |x_1 - a|$. Then $\lim_{n\to\infty} a_n(x_1 - a)^n = 0$. Hence for suff. large n, it holds that $|a_n(x_1 - a)^n| \le 1$ and

$$|a_n(x-a)^n| = |a_n(x_1-a)^n| \left|\frac{x-a}{x_1-a}\right|^n \le \left|\frac{x-a}{x_1-a}\right|^n.$$

Hence by comparison, the series $\sum_{n=0}^{\infty} a_n(x-a)^n$ converges. (2) Now suppose the series $\sum_{n=0}^{\infty} a_n(x_2 - a)^n$ diverges. If there is an x with $|x - a| > |x_2 - a|$ for which the series $\sum_{n=0}^{\infty} a_n(x - a)^n$ converge. Then by (1) the series $\sum_{n=0}^{\infty} a_n(x_2 - a)^n$ should converge. Hence a contradiction. Thus for any x with $|x - a| > |x_2 - a|$, the series diverges. □

By Theorem 7.7.3, there are three possibilities:

(1) There exists a positive number $R(0 < R < \infty)$ such that the series converges absolutely for all x with $|x - a| < R$, and the series diverges for all x with $|x - a| > R$.

(2) It converges for a only; In this case we can put $R = 0$.

(3) It converges absolutely for all x; In this case we can put $R = \infty$.

The value R is called **the radius of convergence** of $\sum_{n=0}^{\infty} a_n(x - a)^n$.

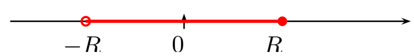


Figure 7.8: Interval of convergence

Theorem 7.7.4. *For $\sum_{n=0}^{\infty} a_n(x-a)^n$, R is given by any one of the following formula(provided the limits exist):*

$$R = \lim_{n\to\infty} \left|\frac{a_n}{a_{n+1}}\right| \tag{7.2}$$

$$R = \lim_{n\to\infty} \frac{1}{\sqrt[n]{|a_n|}} \tag{7.3}$$

Proof. Suppose the limit in (7.2) exists. Then

$$\lim_{n\to\infty} \left|\frac{a_{n+1}(x-a)^{n+1}}{a_n(x-a)^n}\right| = \lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| |x-a| = \frac{|x-a|}{R}$$

and by the ratio test (Thm 7.5.2), the power series converges absolutely for all x with $|x - a|/R < 1$, and diverges for all x with $|x - a|/R > 1$. Hence R is given by (7.2). Next (7.3) is obtained from n-th root test (Thm 7.5.6). Fill-in some gaps. □

Definition 7.7.5. From the discussions above, we see the set of all points for which the series converges will be an interval(open, half open or closed) I, where

$$(a-R, a+R) \subset I \subset [a-R, a+R]$$

I is called **interval of convergence**.

Example 7.7.6. Find the interval of convergence of the following power series.

(1) $\displaystyle\sum_{n=0}^{\infty} n^n x^n$

(2) $\displaystyle\sum_{n=1}^{\infty} \frac{x^n}{n^2}$

(3) $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n-1}x^n}{n}$

(4) $\displaystyle\sum_{n=0}^{\infty} \frac{x^n}{n!}$

sol.

(2)

$$R = \lim_{n\to\infty} \frac{(n+1)^2}{n^2} = 1$$

When $x = \pm 1$, the series $\sum_{n=1}^{\infty}((\pm 1)^n/n^2)$ converges absolutely.

(3)

$$R = \lim_{n\to\infty} \frac{n+1}{n} = 1$$

For $x = 1$, the series $\sum_{n=1}^{\infty}((-1)^{n-1}/n)$ satisfies alternating series test, so conditionally converges. While for $x = -1$ the sequence is $\sum_{n=1}^{\infty}(-1/n)$ which diverges. Hence $I = (-1, 1]$.

(4)

$$R = \lim_{n\to\infty} \frac{(n+1)!}{n!} = \infty$$

□

Theorem 7.7.7 (Term by term differentiation)**.** *Suppose* $\sum_{n=0}^{\infty} a_n(x-a)^n$ *converges for* $R > 0$*. If we define a function by*

$$f(x) = \sum_{n=0}^{\infty} a_n(x-a)^n, \quad |x-a| < R, \tag{7.4}$$

then we have

(i) $f(x)$ is differentiable on $(a-R, a+R)$ and its derivative can be computed term by term

$$f'(x) = \sum_{n=1}^{\infty} n a_n (x-a)^{n-1}, \quad |x-a| < R \tag{7.5}$$

(ii) $f(x)$ is integrable on $(a-R, a+R)$ and its integral can be computed term by term

$$\int f(x)\,dx = \sum_{n=0}^{\infty} a_n \frac{(x-a)^{n+1}}{n+1} + C, \quad |x-a| < R \tag{7.6}$$

The radius convergence of (7.5) *and* (7.6) *are also R.*

Proof. The proof of term by term computation is out of the scope of this book, hence skipped. Instead, we verify the radius of convergence. Suppose the following limit exists:

$$R = \lim_{n\to\infty} \left| \frac{a_n}{a_{n+1}} \right|.$$

Then the radius of convergence of (7.5) is by Thm 7.7.4

$$\lim_{n\to\infty} \left| \frac{(n+1)a_{n+1}}{(n+2)a_{n+2}} \right| = \lim_{n\to\infty} \left| \frac{a_{n+1}}{a_{n+2}} \right| = R$$

The case for (7.6) is the same. □

Corollary 7.7.8. *In fact, the function $f(x)$ in Thm 7.7.7 is differentiable infinitely many times on $(a-R, a+R)$ and the derivatives are given by*

$$f^{(k)}(x) = \sum_{n=k}^{\infty} n(n-1)\cdots(n-k+1) a_n (x-a)^{n-k}, \quad |x-a| < R,\ k = 0, 1, 2, \dots . \tag{7.7}$$

Product of two Power series

Theorem 7.7.9. *Suppose both $A(x) = \sum_{n=0}^{\infty} a_n x^n$, $B(x) = \sum_{n=0}^{\infty} b_n x^n$ converge absolutely for $|x| < R$ and let*

$$c_n = a_0 b_n + a_1 b_{n-1} + \cdots + a_n b_0 = \sum_{n=0}^{k} a_k b_{n-k}.$$

Then the series $\sum_{n=0}^{\infty} c_n x^n$ converge absolutely for $|x| < R$, and

$$\left(\sum_{n=0}^{\infty} a_n x^n \right) \times \left(\sum_{n=0}^{\infty} b_n x^n \right) = \sum_{n=0}^{\infty} c_n x^n.$$

Example 7.7.10. Use

$$\sum_{n=0}^{\infty} x^n = 1 + x + x^2 + \cdots = \frac{1}{1-x}, \text{ for } |x| < 1$$

to obtain the power series of $1/(1-x)^2$ about $x = 0$.

sol. Formally we have

$$\frac{1}{(1-x)^2} = \frac{1}{(1-x)} \cdot \frac{1}{(1-x)} = (\sum_{n=0}^{\infty} x^n) \cdot (\sum_{n=0}^{\infty} x^n).$$

We let $A(x) = B(x) = \sum_{n=0}^{\infty} x^n$. Then we see

$$c_n = a_0 b_n + a_1 b_{n-1} + \cdots + a_n b_0 = \sum_{n=0}^{k} a_k b_{n-k} = n + 1$$

Hence by the above theorem

$$A(x)B(x) = \sum_{n=0}^{\infty} c_n x^n = \sum_{n=0}^{\infty} (n+1) x^n.$$

Alternatively this series could be obtained by differentiation.

□

Example 7.7.11.

$$\begin{aligned}
\cos x \cdot \sin x &= \left(1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots\right) \cdot \left(x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots\right) \\
\frac{1}{2} \sin 2x &= \frac{1}{2}\left(2x - \frac{2^3 x^3}{3!} + \frac{2^5 x^5}{5!} - \frac{2^7 x^7}{7!} \cdots\right) \\
&= \left(x - \frac{2^2 x^3}{3!} + \frac{2^4 x^5}{5!} - \frac{2^6 x^7}{7!} \cdots\right)
\end{aligned}$$

On the other hand, by multiplying out

$$\cos x \cdot \sin x = x - \left(\frac{1}{1!2!} + \frac{1}{3!}\right) x^3 + \left(\frac{1}{1!4!} + \frac{1}{2!3!}\right) x^5 - \left(\frac{1}{1!6!} + \frac{1}{2!5!} + \frac{1}{3!4!}\right) x^7 + \cdots$$

Comparing the coefficients, we see

$$\begin{aligned}
\frac{2^2}{3!} &= \frac{1}{0!3!} + \frac{1}{1!2!} \\
\frac{2^4}{5!} &= \frac{1}{0!5!} + \frac{1}{1!4!} + \frac{1}{2!3!} \\
\frac{2^6}{7!} &= \frac{1}{0!7!} + \frac{1}{1!6!} + \frac{1}{2!5!} + \frac{1}{3!4!} \\
&= \cdots \\
\frac{2^{2n}}{(2n+1)!} &= \frac{1}{0!(2n+1)!} + \frac{1}{1!(2n)!} + \frac{1}{2!(2n-1)!} + \cdots + \frac{1}{n!(n+1)!}
\end{aligned}$$

Example 7.7.12.

$$\begin{aligned}\frac{\ln(1-x)}{1-x} &= -(1+x+x^2+x^3+\cdots)\left(x+\frac{x^2}{2}+\frac{x^3}{3}+\cdots+\frac{x^n}{n}+\cdots\right)\\ &= -\left(x+(1+\frac{1}{2})x^2+(1+\frac{1}{2}+\frac{1}{3})x^3+\cdots+(1+\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{n})x^n+\cdots\right)\end{aligned}$$

Now integrating the lhs,

$$\int_0^x \frac{\ln(1-t)}{1-t}dt = -\int u\,du = -\frac{(\ln(1-x))^2}{2}$$

while the integral of the right hand side is

$$= -\left(\frac{x^2}{2}+(1+\frac{1}{2})\frac{x^3}{3}+(1+\frac{1}{2}+\frac{1}{3})\frac{x^4}{4}+\cdots+(1+\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{n})\frac{x^{n+1}}{n+1}+\cdots\right)$$

The direct power series expansion of $\frac{(\ln(1-x))^2}{2}$ is

$$\frac{(\ln(1-x))^2}{2} = \frac{1}{2}\left(x+\frac{x^2}{2}+\frac{x^3}{3}+\cdots+\frac{x^n}{n}+\cdots\right)^2$$
$$= \frac{1}{2}\left(x^2+(\frac{1}{2}+\frac{1}{2})x^3+(\frac{1}{3}+\frac{1}{2\cdot 2}+\frac{1}{3})x^4+\cdots+(\frac{1}{n}+\frac{1}{2\cdot n-1}+\cdots+\frac{1}{n})x^{n+1}+\cdots\right)$$

Comparing the coefficients of x^{n+1}, we obtain

$$\left(1+\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{n}\right)\frac{1}{n+1} = \frac{1}{2}\left(\frac{1}{n}+\frac{1}{2\cdot n-1}+\cdots+\frac{1}{n}\right)$$

Or by multiplying $(n+1)/2$, we get

$$\left(1+\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{n}\right) = \left(\frac{1}{n}+\frac{1}{2\cdot n-1}+\cdots+\frac{1}{n}\right)\frac{n+1}{2}$$

Use trapezoidal rule to estimate $\int_1^n \frac{1}{x}\,dx$. Get approximation formula for $\ln n$.

7.8 Taylor and Maclaurin Series

In the previous discussions we have seen that a power series defines a continuous function on some interval I. How about its converse? Suppose f is differentiable n-times. Is it possible to represents it with a power series ? To answer to this question, let us assume that a power series $\sum_{n=0}^{\infty} a_n(x-a)^n$ represents a function $f(x)$ on its interval of convergence I. Then we have

$$f(x) = \sum_{n=0}^{\infty} a_n(x-a)^n, \quad x \in I$$

If we can find the coefficients a_n, then we would have obtained a power series representation of $f(x)$.

We shall later show that if f has derivative of any order

$$\begin{aligned}&\sum_{n=0}^{\infty}\frac{f^{(n)}(a)}{n!}(x-a)^n\\ =&f(a)+f'(a)(x-a)+\cdots+\frac{f^{(n)}(a)}{n!}(x-a)^n+\cdots\end{aligned}$$

This is called the **Taylor series** of $f(x)$ at a.(If $a=0$, it is also called **Maclaurin series**).

Example 7.8.1. Find Taylor series of $f(x)=1/x$ at $a=2$.

sol.

$$f(x)=\frac{1}{x},\quad f'(x)=-x^{-2},\quad f''(x)=2!x^{-3},\cdots,\quad f^{(n)}(x)=(-1)^n n!x^{-(n+1)},$$

$$f(2)=\frac{1}{2},\quad f'(2)=-\frac{1}{2^2},\quad \frac{f''(2)}{2!}=\frac{1}{2^{-3}},\cdots,\quad \frac{f^{(n)}(2)}{n!}=\frac{(-1)^n}{2^{n+1}}$$

Thus

$$f(x)=\frac{1}{2}-\frac{(x-2)}{2^2}+\frac{(x-2)^2}{2^3}+\cdots+(-1)^n\frac{(x-2)^n}{2^{n+1}}+\cdots$$

We can check this series converges for $0<x<4$.

□

Taylor Polynomial

Consider

$$y=P_1(x):=f(a)+f'(a)(x-a)$$

This is linear approximation to $f(x)$ Similarly we can consider

$$y=P_2(x):=f(a)+f'(a)(x-a)+\frac{1}{2}f''(a)(x-a)^2$$

which has same derivative up to second order. By the same way one can find a polynomial $P_n(x)$ of degree n. It is called **a Taylor polynomial of degree** n Then we see

$$P_n^{(k)}(a)=f^{(k)}(a),\quad k=0,1,\cdots,n$$

$$P_n(x)=f(a)+f'(a)(x-a)+\cdots+\frac{f^{(n)}(a)}{n!}(x-a)^n \tag{7.8}$$

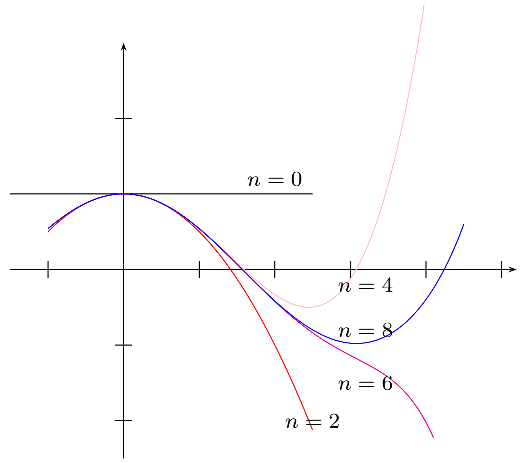


Figure 7.9: Taylor approx. of $\cos x$, p_8 is blue colored

The difference(error) is defined as

$$R_n(x) = f(x) - P_n(x)$$

and called the **remainder**

$$f(x) = P_n(x) + R_n(x)$$

is called n-th **Taylor formula** of $f(x)$ at a.

Example 7.8.2. Find Taylor polynomial for $\cos x$.

Example 7.8.3.

$$f(x) = \begin{cases} \exp(-1/x^2), & x \neq 0 \\ 0, & x = 0 \end{cases}$$

is infinitely differentiable at 0, but the Taylor series converges only at $x = 0$. In fact, we can show that $f^{(n)}(0) = 0$, $n = 0, 1, \ldots$. So the Taylor polynomial $P_n(x) = 0$ and $R_{n+1}(x) = f(x)$. Hence $P_n(x) \not\to f(x)$.

7.9 Convergence of Taylor Series, Error estimates

If $R_n(x) \to$ on I, then Taylor polynomial becomes Taylor series.

Theorem 7.9.1 (Taylor's Theorem with Remainder)**.** *Suppose $f(x)$ is differentiable $n+1$ times on I containing a and $P_n(x)$ is the Taylor polynomial given by* (7.8)*. Then*

$$R_n(x) = \frac{f^{(n+1)}(c)}{(n+1)!}(x-a)^{n+1}. \tag{7.9}$$

Corollary 7.9.2. *Suppose there is some number M such that $f(x)$ satisfies $|f^{(n+1)}(x)| \le M$ for all $x \in I$. Then*

$$|R_n(x)| \le M \frac{|x-a|^{n+1}}{(n+1)!}, \quad x \in I \tag{7.10}$$

Example 7.9.3. At $a = 0$, we have

$$e^x = 1 + x + \cdots + \frac{x^n}{n!} + R_n(x)$$

Here

$$|R_n(x)| \le e^c \frac{x^{n+1}}{(n+1)!}.$$

Definition 7.9.4. Suppose $x \in I$ and $f(x)$ is infinitely differentiable on $I = (a, b)$

$$\lim_{n\to\infty} R_n(x) = 0, \quad x \in I$$

then we say $f(x)$ is **analytic** at a. Here $R_n(x) = f(x) - P_n(x)$ is the remainder.

In this case, we write

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(x)}{n!}(x-a)^n, \quad x \in I$$

Example 7.9.5. (1) Maclaurin series of $\sin x$, $\cos x$ and e^x are:

$$\sin x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!}, \quad -\infty < x < \infty$$

$$\cos x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!}, \quad -\infty < x < \infty$$

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}, \quad -\infty < x < \infty$$

(2) Maclaurin series of $\ln(1+x)$ on $(0, \infty)$ is

$$\ln(1+x) = \sum_{n=1}^{\infty} \frac{(-1)^{n-1} x^n}{n}, \quad -1 < x \le 1$$

(3) Maclaurin series of $1/(1-x)$

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n, \quad -1 < x < 1$$

(4) $\sqrt{x}$ is analytic on $(0, \infty)$.

Example 7.9.6 (Substitution)**.** Find series for $\cos x^2$ near $x = 0$.

Example 7.9.7 (Multiplication)**.** Find series for $x \sin x^2$ near $x = 0$.

Example 7.9.8 (Truncation Error)**.** For what values of x can we replace $\sin x$ with error less than $x \times 10^{-4}$?

$$\sin x \approx x - \frac{x^3}{3!}$$

Here error term is

$$\frac{|x|^5}{5!}.$$

Euler's identity

$$\begin{aligned} e^{i\theta} &= 1 + \frac{i\theta}{1!} + \frac{i^2\theta^2}{2!} + \frac{i^3\theta^3}{3!} + \frac{i^4\theta^4}{4!} + \cdots \\ &= \left(1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} - \frac{\theta^6}{6!} + \cdots\right) + i\left(\theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - \cdots\right) \\ &= \cos\theta + i \sin\theta \end{aligned}$$

Proof of Taylor's Formula with Remainder

We shall show that for a function f analytic near $x = a$, we have

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n = f(a) + f'(a)(x-a) + \cdots + \frac{f^{(n)}(a)}{n!}(x-a)^n + \cdots$$

We set

$$\phi_n(x) = P_n(x) + K(x-a)^{n+1}.$$

This function has same first n-derivative as f at a. We can choose K so that $\phi_n(x)$ agrees with $f(x)$. We shall show that K is indeed given by the form $\frac{f^{(n+1)}(c)}{(n+1)!}$. The idea is to fix $x = b$ and choose K so that $\phi_n(b)$ agrees with $f(b)$. So

$$f(b) = P_n(b) + K(b-a)^{n+1}, \text{ or } K = \frac{f(b) - P_n(b)}{(b-a)^{n+1}} \tag{7.11}$$

and

$$F(x) = f(x) - \phi_n(x)$$

is the error. We use Rolle's theorem. First since $F(b) = F(a) = 0$

$$F'(c_1) = 0, \quad \text{for some } c_1 \in (a, b).$$

Next, because $F'(a) = F'(c_1) = 0$ we have

$$F''(c_2) = 0, \quad \text{for some } c_2 \in (a, c_1).$$

Now repeated application of Rolle's theorem to F'', *etc* show that there exist

$$\begin{array}{rll} c_3 & \text{in } (a, c_2) & \text{such that } F'''(c_3) = 0, \\ c_4 & \text{in } (a, c_3) & \text{such that } F^{(4)}(c_4) = 0, \\ & \vdots & \\ c_n & \text{in } (a, c_{n-1}) & \text{such that } F^{(n)}(c_n) = 0 \\ c_{n+1} & \text{in } (a, c_n) & \text{such that } F^{(n+1)}(c_{n+1}) = 0. \end{array}$$

But since $F(x) = f(x) - \phi_n(x) = f(x) - P_n(x) - K(x-a)^{n+1}$, we see

$$F^{(n+1)}(c) = f^{(n+1)}(c) - 0 - (n+1)!K.$$

Hence

$$K = \frac{f^{(n+1)}(c)}{(n+1)!}, \quad c = c_{n+1}.$$

Thus we have

$$f(b) = P_n(b) + \frac{f^{(n+1)}(c)}{(n+1)!}(b-a)^{n+1}. \tag{7.12}$$

Now since b is arbitrary, we can set $b = x$. Furthermore, if $R_n \to 0$ as $n \to \infty$, we obtain Taylor's theorem.

7.10 Application

Binomial Series

Consider for any real m

$$(1+x)^m = 1 + mx + \frac{m(m+1)}{2!}x^2 + \cdots + \binom{m}{n}x^n + R_n(x). \tag{7.13}$$

It can be shown that this series converges for $-1 < x < 1$. This is true .

$$\lim_{n\to\infty} R_n(x) = 0, \quad -1 < x < 1$$

Here

$$\binom{m}{n} = \frac{m(m-1)\cdots(m-n+1)}{n!}, \quad n = 0, 1, 2, \ldots$$

We can show $R = 1$.

Proof.

$$\begin{aligned} f'(x) &= m(1+x)^{m-1} \\ f''(x) &= m(m-1)(1+x)^{m-2} \\ &\cdots \\ f^{(n)}(x) &= m(m-1)\cdots(m-n+1)(1+x)^{m-n} \end{aligned}$$

We see

$$f^{(n)}(0) = \binom{m}{n} n!, \quad n = 0, 1, 2, \cdots$$

Hence equation (7.13) is the Taylor formula of $f(x)$ at 0 and its remainder. □

Example 7.10.1.

$$(1+x)^{1/2} = 1 + \frac{x}{2} - \frac{x^2}{8} + \frac{x^3}{16} - \cdots$$

Example 7.10.2. Find $\int \sin^2 x\, dx$ as power series.

Estimate $\int_0^1 \sin^2 x\, dx$ within error less than 0.001.

Example 7.10.3. Find Maclaurin series of $\arctan x$.

sol. Note that for $|x| < 1$ the $\arctan x$ has convergent power series:

$$(\arctan x)' = \frac{1}{1+x^2} = \sum_{n=0}^{\infty} (-1)^n x^{2n}.$$

Integrate it from 0 to x

$$\begin{aligned} \arctan x &= \int_0^x \sum_{n=0}^{\infty} (-1)^n t^{2n}\, dt \\ &= \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{2n+1}, \quad |x| < 1. \end{aligned}$$

Thus

$$\arctan x = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \cdots$$

For example,

$$\frac{\pi}{4} = \arctan 1 = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots$$

Remark 7.10.4. We can actually use the given formula to estimate π. As it turns out it, however, is not an effective method. Let us estimate the error when we use this formula to approximate

$$\pi \approx 4(1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots)$$

The error using n-term is about $4/(2n+1)$. So to get the error less than 10^{-4}, we need $2n+1 \approx 10000/4$, $n = 1200$ terms! Too many! Fortunately there are more effective ways.

□

Example 7.10.5. Suppose $1 > a_n \geq 0$ and $\sum a_n$ converges. Determine whether the following series converges or not.

(1) $\sum \frac{a_n}{1+a_n}$

(2) $\sum \frac{a_n}{1-a_n}$

(3) $\sum a_n^2$

Sol. 1)

$$\sum \frac{a_n}{1+a_n} \leq \sum a_n$$

2)

$$\sum_{n=k} \frac{a_n}{1-a_n} \leq 2\sum_{n=k} a_n (\text{ for sufficiently large } k \text{ so } a_n < \frac{1}{2})$$

3) Suppose $\sum_{n=1}^{\infty} a_n^2$ diverges. Then for any $L > 0$ there is N such that $\sum_1^n a_n^2 > L$ for $n \geq N$. So $(\sum_1^N a_n)^2 \geq \sum_1^N a_n^2 \geq L$. Hence

$$\sum_1^n a_n \geq \sqrt{L}$$

for all $n \geq N$. Thus $\sum_1^{\infty} a_n$ diverges.

7.10.1 Term by term differentiation and integration

Theorem 7.10.6. *Suppose the radius of convergence R of $\sum_{n=0}^{\infty} a_n(x-a)^n$ is lager than 0.*

$$f(x) = \sum_{n=0}^{\infty} a_n(x-a)^n, \quad |x-a| < R \tag{7.14}$$

Then

(i) $f(x)$ is differentiable on $(a-R, a+R)$ and the derivative is given by term by term differentiation. Hence

$$f'(x) = \sum_{n=1}^{\infty} na_n(x-a)^{n-1}, \quad |x-a| < R \tag{7.15}$$

(ii) $f(x)$ has an anti-derivative on $(a-R, a+R)$ and it is given by

$$\int f(x)\,dx = \sum_{n=0}^{\infty} a_n \frac{(x-a)^{n+1}}{n+1} + C, \quad |x-a| < R \tag{7.16}$$

The radius of convergence of (7.15) *and* (7.16) *do not change.* .

We repeat theorem 7.7.4. Then

Corollary 7.10.7. *By theorem* 7.7.4*, the function* $f(x)$ *is differentiable in* $(a-R, A+R)$ *and*

$$f^{(k)}(x) = \sum_{n=k}^{\infty} n(n-1)\cdots(n-k+1)a_n(x-a)^{n-k}, \qquad |x-a| < R, \tag{7.17}$$

$k = 0, 1, \dots$ *The radius of convergence is again* R.

Theorem 7.10.8 (Uniqueness)**.** *Suppose* $f(x)$ *has continuous derivative up to order* $(n+1)$ *in a nhd* $I = (a, b)$ *of* a*. Suppose*

$$f(x) = a_0 + a_1(x-a) + \cdots + a_n(x-a)^n + r(x), \quad x \in I$$

for some $r(x)$ *and* M *s.t.*

$$|r(x)| \le M|x-a|^{n+1}, \quad x \in I.$$

Then a_k *is the Taylor coefficients. i.e,*

$$a_k = \frac{1}{k!} f^{(k)}(a), \quad k = 0, 1, \dots, n.$$

Proof. Taylor coefficient $C_k = (1/k!)f^{(k)}(a)$. Then by theorem 7.9.1

$$\begin{aligned} f(x) &= C_0 + C_1(x-a) + \cdots + C_n(x-a)^n + R_{n+1}(x) \\ &= a_0 + a_1(x-a) + \cdots + a_n(x-a)^n + r(x) \end{aligned}$$

Hence with $b_k = C_k - a_k$ we have

$$b_0 + b_1(x-a) + \cdots + b_n(x-a)^n = r(x) - R_{n+1}(x)$$

Set $x = a$, then we have $b_0 = 0$, i.e, $a_0 = C_0$.

Induction : Assume $b_0 = b_1 = \cdots = b_{m-1} = 0$ for all m with $1 \le m \le n$. Then

$$b_m(x-a)^m + \cdots + b_n(x-a)^n = r(x) - R_{n+1}(x)$$

Divide by $(x-a)^m$ and let $x \to a$. Then we see $b_m = 0$. Hence by induction,

$$b_0 = b_1 = \cdots = b_n = 0$$

or

$$a_0 = C_0, \quad a_1 = C_1, \quad \dots \quad , a_n = C_n.$$

□

Example 7.10.9. (1)

$$\frac{1}{1-2x} = 1 + 2x + (2x)^2 + (2x)^3 + (2x)^4 + \cdots$$

(2)

$$\frac{1}{x} = \frac{1}{1+x-1} = 1 - (x-1) + (x-1)^2 - (x-1)^3 + \cdots$$

(3)

$$-\frac{1}{x^2} = -1 + 2(x-1) - 3(x-1)^2 + 4(x-1)^3 - \cdots$$

(4) Application

$$\frac{2}{(1-2x)^2} = 2 + 2\cdot 2(2x) + 3\cdot 2(2x)^2 + 4\cdot 2(2x)^3 + \cdots + n\cdot 2(2x)^{n-1} + \cdots$$

$$\begin{aligned} f(x) &= \frac{1}{(1-2x)^2} \\ f'(x) &= \frac{2^2}{(1-2x)^3} \\ f''(x) &= \frac{2^3\cdot 3}{(1-2x)^4} \\ &= \cdots \\ f^{(n)}(x) &= \frac{2^{n+1}\cdot (n+1)!}{(1-2x)^{n+2}} \end{aligned}$$

For constant, check!

Example 7.10.10. Find Taylor polynomial of degree 3 of x^3+3x^2+2x+1 at $a=1$.

sol. Set $x=t+1$, $t=x-1$ and then f is

$$t^3 + 6t^2 + 11t + 7$$

$$x^3 + 3x^2 + 2x + 1 = (x-1)^3 + 6(x-1)^2 + 11(x-1) + 7.$$

By theorem 7.10.8 Taylor polynomial is

$$(x-1)^3 + 6(x-1)^2 + 11(x-1) + 7.$$

□

Example 7.10.11. Estimate sin(0.1) up to third digit 3.

sol. Taylor polynomial of $\sin x$ at $a = 0$

$$\sin x = \sum_{k=0}^{n} \frac{1}{k!} \left(\frac{d}{dx}\right)^k \sin x \bigg|_{x=0} x^k + R_n(x)$$

Since $|\sin x| \le 1$, for $|\cos x| \le 1$

$$|R_n(x)| \le \frac{|x|^{n+1}}{(n+1)!}$$

If $n = 3$

$$|R_3(0.1)| \le \frac{(0.1)^3}{3!} < 10^{-3}$$

we have $\sin(0.1) \approx 0.1$ and the error is less than $\pm(1/6) \times 10^{-3}$.

□

Example 7.10.12. Find

$$\lim_{x\to 0} \frac{\sin x - x + (x^3/6)}{x^4}$$

sol. $a = 0$ Taylor polynomial of $\sin x$ at $a = 0$ is

$$\sin x = x - \frac{x^3}{6} + R(x) \qquad |R(x)| \le \frac{|x|^5}{5!}$$

Hence

$$\left|\frac{\sin x - x + (x^3/6)}{x^4}\right| = \left|\frac{R(x)}{x^4}\right| \le \frac{|x|}{5!}$$

and limit is 0.

□

Example 7.10.13. Estimate

$$\ln 2 = \ln(1+1) = 1 - \frac{1}{2} + \cdots + \frac{(-1)^{n-1}}{n} + R_n(1)$$

Since

$$|R_n(1)| \le \frac{1}{n+1}$$

we need to take large n. However, we can do the following:

$$\ln 2 = \ln \frac{4}{3} \cdot \ln \frac{3}{2} 2 = \ln(1 + \frac{1}{3}) + \ln(1 + \frac{1}{2})$$

and use Taylor series.

Theorem 7.10.14 (Binomial series)**.** *For any real s*

$$(1+x)^s = 1 + sx + \frac{s(s+1)}{2!}x^2 + \cdots + \binom{s}{n}x^n + R_{n+1}(x), \qquad (7.18)$$
$$-1 < x < 1$$

and

$$\lim_{n\to\infty} R_{n+1}(x) = 0, \quad -1 < x < 1$$

Here

$$\binom{s}{n} = \frac{s(s-1)\cdots(s-n+1)}{n!}, \quad n = 0, 1, 2, \ldots$$

Example 7.10.15. Find $\sqrt{1.2}$ up to two decimal point.

sol. Let $f(x) = \sqrt{1+x}$. Then $\sqrt{1.2} = f(0.2)$. Hence from equation (7.18) We see Taylor series at $a = 0$ is

$$f(x) = 1 + \frac{1}{2}x + \cdots + \binom{1/2}{n}x^n + R_n(x),$$
$$R_n(x) = \frac{f^{(n+1)}(\bar{x})}{(n+1)!}x^{n+1} \quad (0 \le \bar{x} \le 0.2)$$

For $n = 1$,

$$R_1(0.2) = (\frac{1}{2})f''(\bar{x})(0.2)^2 = -0.005(1+\bar{x})^{-3/2} \quad (0 \le \bar{x} \le 0.2)$$

Hence $\sqrt{1.2} \approx 1 + (\frac{1}{2})(0.2) = 1.1$ and the error satisfies $|R_2(0.2)| < 0.005$.

□

Chapter 8

Conic Sections and Polar Coordinates

8.1 Polar coordinate

In **polar coordinate system** the origin O is called a **pole**, and the half line from O in the positive direction x is **polar axis**

Given P let the distance from O to P be r the angle $\overrightarrow{OP}$ is θ in **radian**. Then P is denoted by (r, θ). (figure 8.1)

We allow r and θ to have negative value, i.e, if $r < 0$, (r, θ) represent the opposite point $(|r|, \theta)$. While if $\theta < 0$ (r, θ) represents $(r, |\theta|)$ (figure 8.1)

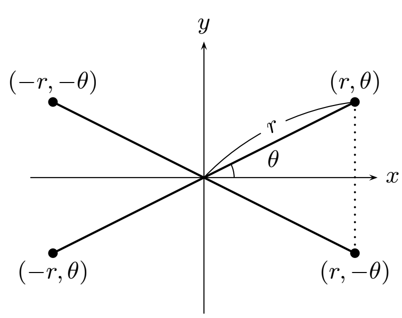


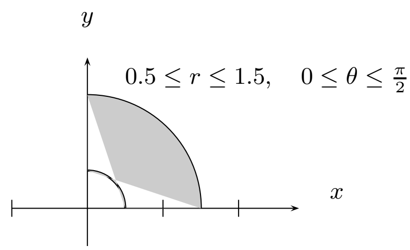
Figure 8.1:

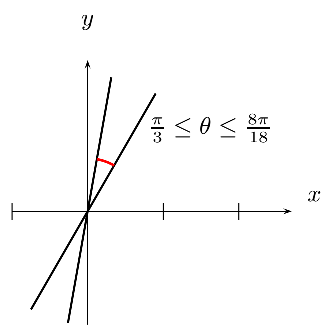
Nonuniqueness of polar coordinate

Polar equations and graphs

Example 8.1.1. (1) $r = a$

(2) $1 \leq r \leq 2, 0 \leq \theta \leq \frac{\pi}{2}$





(3) $\frac{\pi}{3} \le \theta \le \frac{8\pi}{18}$

Relation with Cartesian coordinate

If $(r, \theta) = (x, y)$

Proposition 8.1.2. *(1)* $x^2 + y^2 = r^2$

(2) $x = r\cos\theta$

(3) $y = r\sin\theta$

Example 8.1.3. Draw

(1) Line through the origin: $\theta = c$

(2) Line through the origin: $r\cos(\alpha - \theta) = d$ where d is the distance from the origin to the line.

8.2 Drawing in Polar Coordinate

Example 8.2.1. Draw the graph of

$$r = 2\cos\theta$$

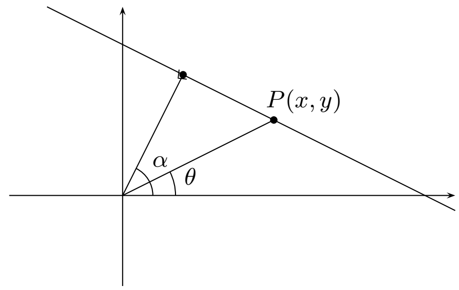


Figure 8.2: Equation of line in polar coord.

sol. Since $r = 2\cos\theta$, we have $r^2 = 2r\cos\theta$. Then we obtain $x^2 + y^2 = 2x$, or $(x-1)^2 + y^2 = 1$.

□

θ	r	θ	r
0	3	$\pm 2\pi/3$	0
$\pm\pi/6$	$1+\sqrt{3}$	$\pm 3\pi/4$	$1-\sqrt{2}$
$\pm\pi/4$	$1+\sqrt{2}$	$\pm 5\pi/6$	$1-\sqrt{3}$
$\pm\pi/3$	2	$\pm\pi$	-1
$\pm\pi/2$	1		

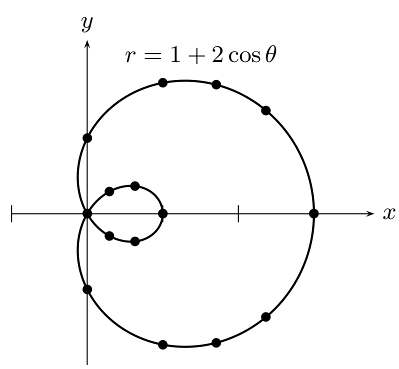


Figure 8.3: $y = 1 + 2\cos\theta$

Equation of circles

Circles of radius a centered at (r_0, θ_0) is described by

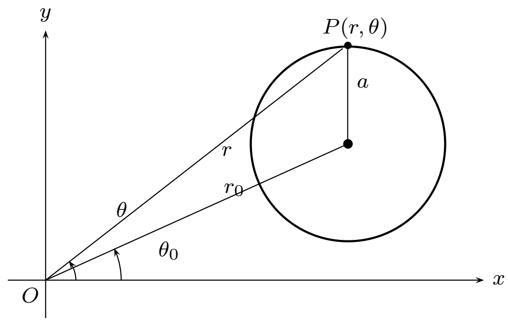
$$a^2 = r^2 + r_0^2 - 2rr_0\cos(\theta - \theta_0)$$

If the circle pass the origin, $a = r_0$ and the equation is $r = a\cos(\theta - \theta_0)$

Example 8.2.2. Draw $r = 1 + 2\cos\theta$

sol. Multiply r to have $r^2 = r + 2r\cos\theta$.

$$x^2 + y^2 = \sqrt{x^2+y^2} + 2x \qquad (r \geq 0)$$
$$x^2 + y^2 = -\sqrt{x^2+y^2} + 2x \quad (r < 0)$$



□

Example 8.2.3. Draw the graph of $r = 1 - \sin\theta$.

sol.
Figure 8.5

□

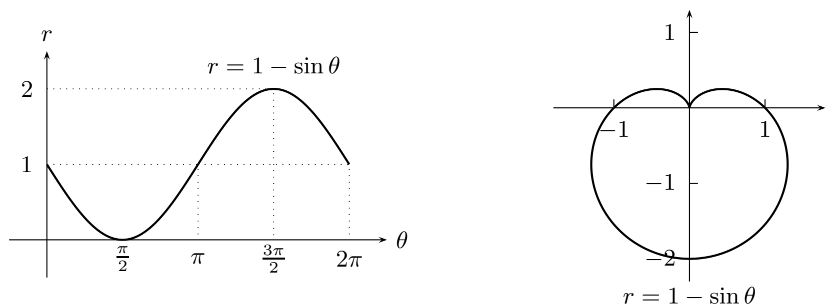


Figure 8.4: $r = 1 - \sin\theta$

Example 8.2.4. Find cartesian equation of

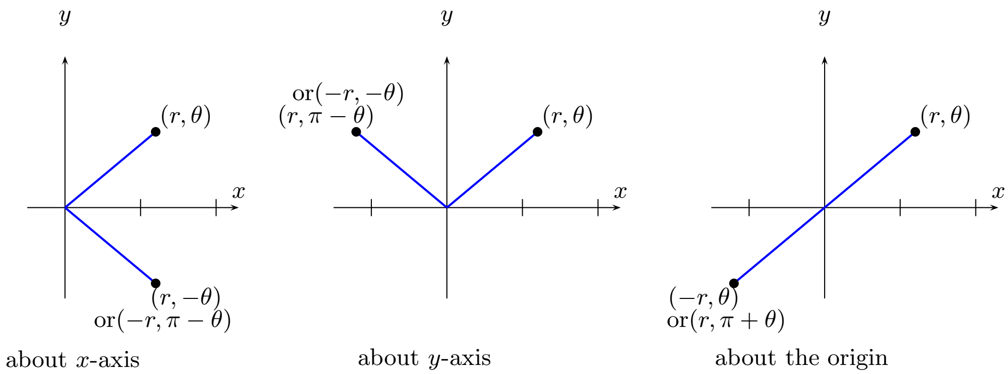
(1) $r\cos\theta = -4$

(2) $r^2 = 4r\cos\theta$

(3) $r = \frac{4}{2\cos\theta - \sin\theta}$ (line)

sol.
Check

□



Symmetry

A point symmetric to x axis of (r, θ) is $(r, -\theta)$ or $(-r, \pi - \theta)$. a point symmetric to y-axis is $(r, \pi - \theta)$ or $(-r, -\theta)$.

$(-r, \theta)$ or $(r, \pi + \theta)$ is symmetric about the origin.

Proposition 8.2.5. *The graph of $f(r, \theta) = 0$ is symmetric w.r.t*

(1) x-axis if $f(r, -\theta) = f(r, \theta)$ $f(-r, \pi - \theta) = f(r, \theta)$

(2) y-axis if $f(r, \pi - \theta) = f(r, \theta)$ or $f(-r, -\theta) = f(r, \theta)$,

(3) the origin if $f(-r, \theta) = f(r, \theta)$ or $f(r, \pi + \theta) = f(r, \theta)$.

Example 8.2.6. Find the symmetry of $r^2 = \sin 2\theta$.

sol. Set $f(r, \theta) = r^2 - \sin 2\theta$. Then

$$f(-r, \theta) = (-r)^2 - \sin 2\theta = f(r, \theta)$$

is **symmetric about the origin**. On the other hand,

$$f(r, -\theta) = r^2 - \sin(-2\theta) \neq f(r, \theta)$$

and

$$f(-r, \pi - \theta) = r^2 - \sin(2\pi - 2\theta) \neq f(r, \theta)$$

Hence it is not symmetric about the x-axis. Also because

$$f(r, \pi - \theta) = r^2 - \sin(2\pi - 2\theta) = r^2 + \sin 2\theta \neq f(r, \theta)$$

$$f(-r, -\theta) = r^2 - \sin(-2\theta) = r^2 + \sin 2\theta \neq f(r, \theta)$$

it is not symmetric about y-axis either.

□

Example 8.2.7. For the graph $r = 2\cos 2\theta$, we let $f(r,\theta) = r - \cos 2\theta$ and we replace the x-axis symmetric point $(-r, \pi - \theta)$ for (r,θ) then

$$f(-r, \pi - \theta) = -r - \cos 2(\pi - \theta) = -r - \cos 2\theta \neq f(r,\theta)$$

This looks different from the given relation. However, if we replace another expression of the same x-axis symmetric point $(r, -\theta)$ for (r,θ), then

$$f(r, -\theta) = r - \cos(-2\theta) = r - \cos 2\theta = f(r,\theta)$$

Hence it is symmetric about x-axis.

Slope of tangent

Caution: The slope of a polar curve $r = f(\theta)$ is given by dy/dx, not given by $r' = df/d\theta$, because the slope is measured as the ratio between the increase in y and increase in x(i.e, $\Delta y/\Delta x$). Let us use the parametric expression

$$x = r\cos\theta = f(\theta)\cos\theta, \quad y = f(\theta)\sin\theta$$

Using the parametric derivative, we have

$$\begin{aligned} \frac{dy}{dx} &= \frac{dy/d\theta}{dx/d\theta} = \frac{\frac{d}{d\theta}[f(\theta)\sin\theta]}{\frac{d}{d\theta}[f(\theta)\cos\theta]} \\ &= \frac{\frac{df}{d\theta}\sin\theta + f(\theta)\cos\theta}{\frac{df}{d\theta}\cos\theta - f(\theta)\sin\theta} \end{aligned}$$

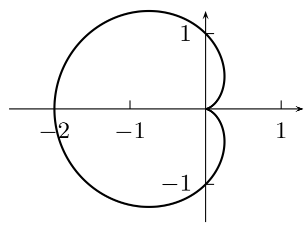
Hence

$$\frac{dy}{dx} = \frac{f'(\theta)\sin\theta + f(\theta)\cos\theta}{f'(\theta)\cos\theta - f(\theta)\sin\theta}.$$

As a special case, when the curve pass the origin at $\theta_0 = 0$, then

$$\left.\frac{dy}{dx}\right|_{0,\theta_0} = \frac{f'(\theta_0)\sin\theta_0}{f'(\theta_0)\cos\theta_0} = \tan\theta_0.$$

Example 8.2.8. Draw the curve: $r = 1 - \cos\theta$(This is another Cardioid). Also, find the slope of tangent at the origin.



$r = 1 - \cos\theta$

Figure 8.5: $r = 1 - \cos\theta$

Problems Caused by Polar Coordinates

Example 8.2.9. Show the point $(2, \pi/2)$ lies on $r = 2\cos 2\theta$.

sol. Substitute $(r, \theta) = (2, \pi/2)$ into $r = 2\cos 2\theta$, we see

$$2 = 2\cos\pi = -2$$

does not holds. However, if we use alternative expression for the same point $(-2, -\pi/2)$, then

$$-2 = 2\cos 2(-\pi/2) = -2$$

So the point $(2, \pi/2) = (-2, -\pi/2)$ line on the curve.

□

Example 8.2.10 (Draw only $r^2 = 4\cos\theta$)**.** Find all the intersections of $r^2 = 4\cos\theta$ and $r = 1 - \cos\theta$.

sol. [Draw only $r^2 = 4\cos\theta$]. First solve

$$r^2 = 4\cos\theta$$
$$r = 1 - \cos\theta$$

Substitute $\cos\theta = r^2/4$ into $r = 1 - \cos\theta$ to see

$$r = 1 - \cos\theta = 1 - r^2/4$$

$r = -2 \pm 2\sqrt{2}$ among those $r = -2 - 2\sqrt{2}$ is too large, we only choose $r = -2 + 2\sqrt{2}$

$$\theta = \cos^{-1}(1 - r) = \cos^{-1}(3 - 2\sqrt{2}) \approx 80°.$$

But if we see the graph 8.6 there are four points A, B, C, D. These parameter θ in two equation is not necessarily the same(they run on different time.) That is

The curve $r = 1 - \cos\theta$ passes C when $\theta = \pi$, while the curve $r^2 = 4\cos\theta$ passed C when $\theta = 0$. Same phenomena happens with D.

□

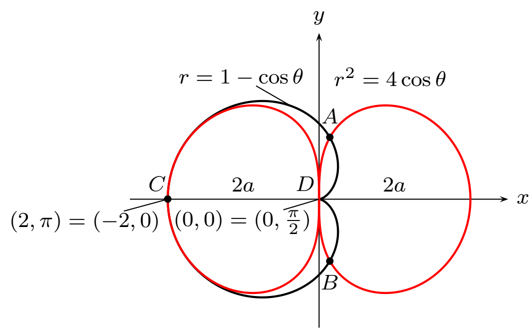


Figure 8.6: Intersection of two curves

8.3 Areas and Lengths in Polar Coordinates

Areas

The function represents certain region.

$$r = f(\theta), \quad \theta = a, \quad \theta = b$$

Let $P = \{\theta_0, \theta_1, \dots, \theta_n\}$ be the partition of $[a, b]$(angle) and $r_i = r(\theta_i)$. Each region is approx'd by n sectors given by the figure 8.7. Let $\Delta\theta_i = \theta_{i+1} - \theta_i$. Then the area of the sector determined by

$$r = f(\theta), \quad \theta_i \le \theta \le \theta_{i+1}$$

is approx'd by $\frac{r_i^2}{2}\Delta\theta_i$. Hence the total area is given by

$$\lim_{n\to\infty} \sum_{i=0}^{n-1} \frac{1}{2} {r_i}^2 \Delta\theta_i.$$

(See fig 8.8). In the limit, it is

$$\int_a^b \frac{1}{2} r^2 \, d\theta.$$

Example 8.3.1. Find the area enclosed by the cardioid: $r = 2(1 + \cos\theta)$.

sol. (fig 4.6) $\theta \in [0, 2\pi]$

$$\int_0^{2\pi} \frac{1}{2}(2 + 2\cos\theta)^2 \, d\theta = 6\pi$$

□

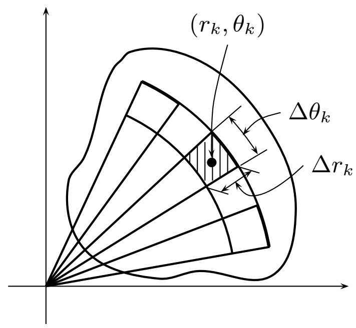


Figure 8.7: Area of region in polar coord.-partition along constant angle

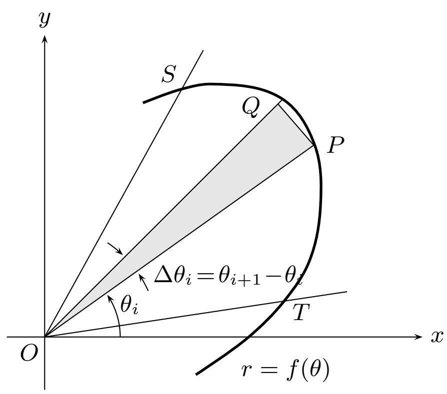


Figure 8.8: Area of sector OST is approx't by sum of triangles such as OPQ

Area between two curves $r = f_1(\theta)$ and $r = f_2(\theta)$

$$A = \int_a^b \frac{1}{2}(r_2^2 - r_1^2)d\theta$$

Example 8.3.2. Find the area of the region that lies inside the circle $r = 1$ and outside the cardioid $r = 1 - \cos\theta$. (Fig 8.5)

sol. Find points of intersection. $r = 1, \theta = \pm\pi/2$. Let $r_2 = 1$ and $r_1 =$

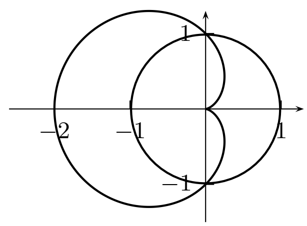


Figure 8.9: region between $r = 1 - \cos\theta$ and $r = 1$

$1 - \cos\theta$.

$$\begin{aligned} A &= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{1}{2}(r_2^2 - r_1^2)d\theta \\ &= \int_0^{\frac{\pi}{2}} (r_2^2 - r_1^2)d\theta \\ &= \int_0^{\frac{\pi}{2}} (1 - (1 - 2\cos\theta + \cos^2\theta))d\theta \\ &= 2 - \frac{\pi}{4}. \end{aligned}$$

□

Arc Length

Find the arc-length of the curve given by polar corrdinate

$$r = f(\theta), \quad \theta \in [a, b]$$

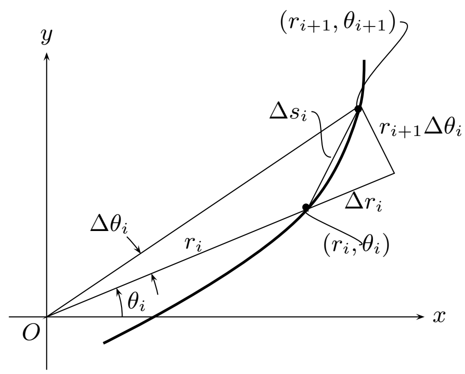


Figure 8.10: $r_i = r(\theta_i),\ \Delta r_i = r_{i+1} - r_i,\ \Delta\theta_i = \theta_{i+1} - \theta_i$

Let $P = \{\theta_0, \theta_1, \ldots, \theta_n\}$ be the partition of $[a, b]$ and $r_i = r(\theta_i)$. The line segment connecting (r_i, θ_i), (r_{i+1}, θ_{i+1}) has length

$$\sqrt{(r_{i+1}(\theta_{i+1} - \theta_i))^2 + (r_{i+1} - r_i)^2}$$

Thus total curve length is approx'ed by(see fig 8.10).

$$\sum_{i=0}^{n-1} \sqrt{(r_{i+1}\Delta\theta_i)^2 + (\Delta r_i)^2}$$

Dividing by $\Delta\theta_i$

$$\sum_{i=0}^{n-1} \sqrt{r_{i+1}^2 + \left(\frac{\Delta r_i}{\Delta\theta_i}\right)^2}\ \Delta\theta_i.$$

$$\int_a^b \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\, d\theta$$

Example 8.3.3. Find the length of closed curve $r = 1 - \cos\theta$.

sol.

$$r = 1 - \cos\theta, \quad \frac{dr}{d\theta} = \sin\theta$$

$$\begin{aligned} r^2 + (\frac{dr}{d\theta})^2 &= (1 - \cos\theta)^2 + \sin^2\theta \\ &= 2 - 2\cos\theta \end{aligned}$$

$$L = \int_0^{2\pi} \sqrt{2 - 2\cos\theta} d\theta = 8 \tag{8.1}$$

□

Area of a Surface of Revolution in Polar coordinate-Skip

Recall the formula

$$\text{about } x\text{-axis } S = \int_a^b 2\pi y \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} dt \tag{8.2}$$

$$\text{about } y\text{-axis } S = \int_a^b 2\pi x \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} dt \tag{8.3}$$

Since $x = r\cos\theta, \quad y = r\sin\theta$. Changing it to polar coordinates; we have

$$\left(\frac{dx}{d\theta}\right)^2 + \left(\frac{dy}{d\theta}\right)^2 = r^2 + \left(\frac{dr}{d\theta}\right)^2$$

If the graph is revolved

(1)

$$\text{about } x\text{-axis } S = \int_a^b 2\pi r \sin\theta \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2} d\theta$$

(2)

$$\text{about } y\text{-axis } S = \int_a^b r \cos\theta \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2} d\theta$$

Example 8.3.4. Revolve the right hand loop of lemniscate $r^2 = \cos 2\theta$ about y-axis

8.4 Polar Coordinates of Conic Sections

Classifying Conic sections by Eccentricity

Consider the **ellipse** with $a \geq b$

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

Let $c = \sqrt{a^2 - b^2}$. Then $(\pm c, 0)$ are foci and $(\pm a, 0)$ are vertices.

For the **hyperbola**

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$$

Let c be defined by $c = \sqrt{a^2 + b^2}$. Foci are $(\pm c, 0)$ and vertices are $(\pm a, 0)$.

Definition 8.4.1. (1) **eccentricity** of the ellipse $x^2/a^2 + y^2/b^2 = 1$ $(a > b)$ is defined by

$$e = \frac{c}{a} = \frac{\sqrt{a^2 - b^2}}{a} < 1$$

(2) **eccentricity** of the hyperbola $x^2/a^2 - y^2/b^2 = 1$ is defined by

$$e = \frac{c}{a} = \frac{\sqrt{a^2 + b^2}}{a} > 1$$

(3) **eccentricity** of the parabola is $e = 1$.

eccentricity and directrix

From definition of parabola we see that for any point P, PF the distance to focus F is the same as the distance to the directrix PD. i.e,

$$PF = PD$$

Or with $e = 1$

$$PF = e \cdot PD$$

This holds for other quadratic curves too!

Definition 8.4.2. The **Focus-directrix equation** is defined as follows:

$$PF = e \cdot PD \tag{8.4}$$

where the eccentricity $e = \frac{c}{a}$ and the directrix ℓ is the line $x = \pm\frac{a}{e}$.

Proposition 8.4.3. ***eccentricity****(eccentricity) e is defined by*

$$\begin{aligned} e &= \frac{\textit{Distance between two focus}}{\textit{Distance between two vertices}} \\ &= \frac{2c}{2a} \\ &= \frac{c}{a} \end{aligned}$$

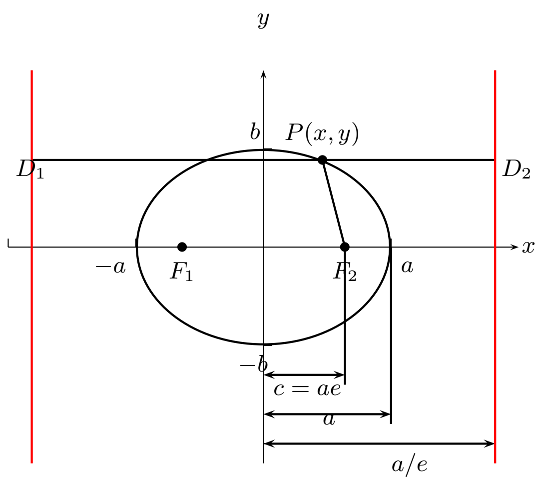


Figure 8.11: $x^2/a^2 + y^2/b^2 = 1$

We now define conic sections using eccentricity and directrix

Definition 8.4.4. Suppose a point F and a line ℓ. If P satisfies

$$PF = e \cdot PD$$

Then

(1) ellipse when $e < 1$

(2) parabola when $e = 1$

(3) hyperbola when $e > 1$

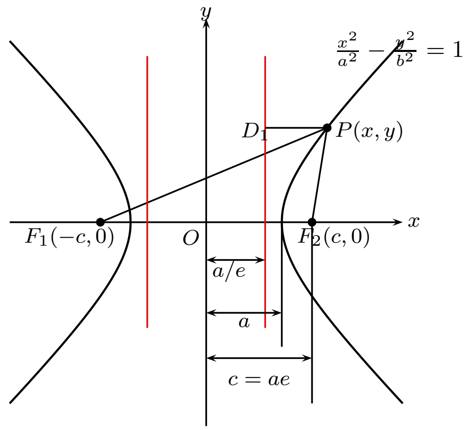


Figure 8.12: $x^2/a^2 - y^2/b^2 = 1$

Relation to Cartesian Coordinate-Skip

For ellipse $x^2/a^2 + y^2/b^2 = 1 (a > b)$, the line

$$x = \pm\frac{a}{e} = \pm\frac{a^2}{\sqrt{a^2 - b^2}}$$

is directrix. If $b > a$, the lines

$$y = \pm\frac{b}{e} = \pm\frac{b^2}{\sqrt{b^2 - a^2}}$$

are directrix.

For hyperbola $x^2/a^2 - y^2/b^2 = 1$, the directrix is

$$x = \pm\frac{a}{e} = \pm\frac{a^2}{\sqrt{a^2 + b^2}}$$

and for the hyperbola $-x^2/a^2 + y^2/b^2 = 1$, directrix are

$$y = \pm\frac{b}{e} = \pm\frac{b^2}{\sqrt{b^2 + a^2}}$$

Example 8.4.5. Find the equation of hyperbola with center at the origin and focus at $F = (\pm 3, 0)$ and directrix is the line $x = 1$.

sol. $F = (3, 0)$ $c = 3$. Since $x = a/e = 1$ is directrix. we see $a = e$. Since $e = c/a$

$$e = \frac{c}{a} = \frac{3}{e}$$

holds. So $e = \sqrt{3}$. From $PF = e \cdot PD$ we see

$$\sqrt{(x-3)^2 + y^2} = \sqrt{3}|x-1| \Rightarrow \quad \frac{x^2}{3} - \frac{y^2}{6} = 1$$

□

Polar equation of conic section

$$PF = e \cdot PD$$

Assume the focus F is at the origin and the directrix ℓ is the line $x = k$, $k > 0$.

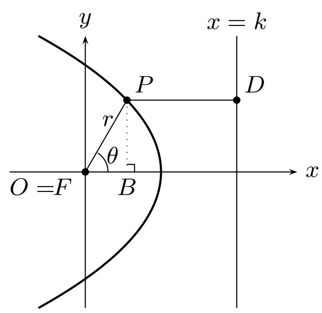


Figure 8.13:

Let D be the foot of P to directrix ℓ, while the foot on the x-axis is B. Then

$$PF = r, \qquad PD = k - FB = k - r\cos\theta$$

So by (8.4)

$$r = PF = e \cdot PD = e(k - r\cos\theta) \tag{8.5}$$

Proposition 8.4.6. *The polar equation of a conic section with eccentricity e, directrix $x = k$, $k > 0$ having focus at the origin is*

$$r = \frac{ke}{1 + e\cos\theta} \tag{8.6}$$

Remark 8.4.7. If $k < 0$, we see (Draw graph) $r = PF = e \cdot PD = e(r\cos\theta + k)$. Hence we have

$$r = \frac{ke}{1 - e\cos\theta}. \tag{8.7}$$

Example 8.4.8. Find the polar equation of a conic section with $e = 2$ directrix $x = -2$ and focus at origin

sol. Since $k = -2$ and $e = 2$ we have from equation (8.7)

$$r = \frac{2(-2)}{1 - 2\cos\theta} = \frac{4}{2\cos\theta - 1}$$

□

Example 8.4.9. Identify

$$r = \frac{-3}{1 - 3\cos\theta}$$

sol. Since $e = 3$ it is hyperbola and from $ke = -3$, we have $k = -1$. Hence directrix is $x = -1$.

□

Example 8.4.10. Identify

$$r = \frac{10}{2 + \cos\theta}$$

sol. From standard form $r = \frac{5}{1+\frac{1}{2}\cos\theta}$, we see $e = 1/2$. Thus ellipse and $ke = 5$. So $k = 10$.

□

Example 8.4.11. Find polar equation of conic section with Directrix $y = 2$, eccentricity $e = 3$ focus at origin.

sol. Fig 8.14

$$PF = r, \qquad PD = 2 - r\sin\theta.$$

So $r = 3(2 - r\sin\theta)$ and

$$r = \frac{6}{1 + 3\sin\theta}$$

Note that $\cos\theta$ has been replaced by $\sin\theta$.

□

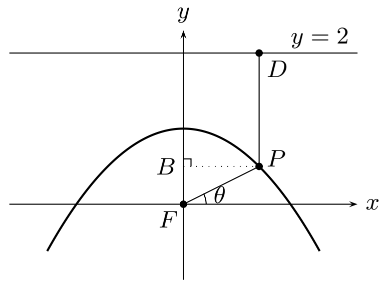


Figure 8.14:

If $x = a$ is the major semi-axis (ellipse or hyperbola as in Fig 8.11, 8.12, placing the focus at the origin) then the distance between the focus and directrix is $k = dist(F, D)$ and it is given by

$$k = \begin{cases} \frac{a}{e} - ae \text{ if } e < 1 \\ ae - \frac{a}{e} \text{ if } e > 1 \end{cases}$$

Thus the equation of a conic section(ellipse of hyperbola) with major axis at $x = a$ becomes

$$r = \frac{ke}{1 + e\cos\theta} = \begin{cases} \frac{a(1-e^2)}{1+e\cos\theta} \text{ if } e < 1 \\ \frac{a(e^2-1)}{1+e\cos\theta} \text{ if } e > 1 \end{cases} \tag{8.8}$$

8.5 Plane curves

Parameterized curve

Definition 8.5.1. If there is a continuous function γ defined on $I = [a, b]$ $\gamma \colon I \to \mathbb{R}^2$, then its image (or the function itself) $C = \gamma(I)$ is called a **parameterized curve**

The point $\gamma(a)$ is **initial point** of γ, $\gamma(b)$ is **end point** of γ.

sol. For the unit circle $x^2 + y^2 = 1$, we can represent it

$$x(t) = \cos(2\pi t), \qquad y(t) = \sin(2\pi t), \quad t \in [0, 1].$$

Another one is

$$\gamma_2 = (\cos(-4\pi t + \frac{\pi}{2}), \qquad \sin(-4\pi t + \frac{\pi}{2}))$$

□

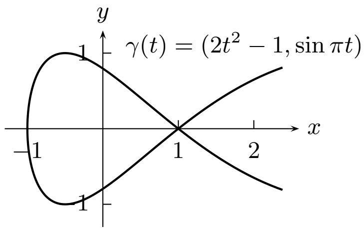


Figure 8.15: $\gamma(t) = (2t^2 - 1, \sin \pi t)$

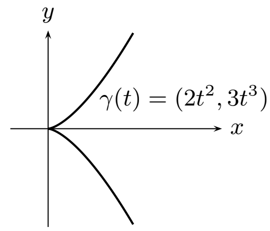


Figure 8.16: $\gamma(t) = (2t^2, 3t^3)$

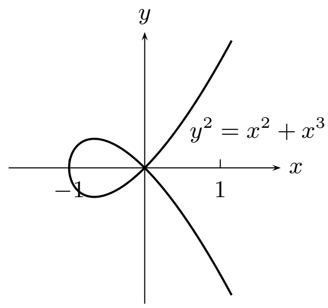


Figure 8.17: $y^2 = x^2 + x^3$

Drawing

Example 8.5.2. Draw the graph of $\gamma(t) = (2t^2 - 1, \sin \pi t)$ on $[0, 1]$.

Example 8.5.3. Find a parameterized representation of $y^2 = x^2 + x^3$.

sol. First see the graph in fig 8.17. We introduce a small trick to find a parametrization. Noting the curve pass $(0, 0)$, let $y = tx$. Then substituting into $y^2 = x^2 + x^3$, we obtain

$$x^2(t^2 - 1 - x) = 0$$

Set $x = t^2 - 1$ then $y = t(t^2 - 1)$. Hence $(t^2 - 1, t(t^2 - 1))$ lie on the curve. Hence $\gamma(t) = (t^2 - 1, t(t^2 - 1))$ is a parametrization.

□

Find a parametrization of the equation $y^3 = x^2 + x^3$. Set $y = tx$. Then

$$(tx)^3 = x^2 + x^3, \quad x^2(t^3 - 1 - x) = 0$$

So

$$x = t^3 + 1, \quad y = t(t^3 + 1)$$

Example 8.5.4. A parametrization of $y^3 = x^2 + x^3 + 1$.

$$(y-1)(y^2+y+1) = x^2 + x^3$$

Let $y = 1 + tx$. Then

$$tx(1 + 2tx + t^2x^2 + tx + 2) = x^2 + x^3$$

$$t(1 + 2tx + t^2x^2 + tx + 2) = x + x^2 = x(1+x)$$

Cycloid

Assume circle of radius a rolling on x-axis. Let P be a point starting to move from the origin. Fig 8.18 If circle rotates by t radian then the point P is

$$x = at + a\cos\theta, \qquad y = a + a\sin\theta \tag{8.9}$$

Since $\theta = (3\pi)/2 - t$ we have

$$x = a(t - \sin t), \qquad y = a(1 - \cos t).$$

8.6 Conic Sections and Quadratic Equations

Remark 8.6.1. The upside down cycloid has two names **brachistochrones** and **tautochrones** each one of which has some physical meanings.

Parabola

Definition 8.6.2. The set of all points in a plane equidistant from a *fixed point* and a *fixed line* is a **parabola** The fixed point is called a **focus** and the line is called a **directrix**

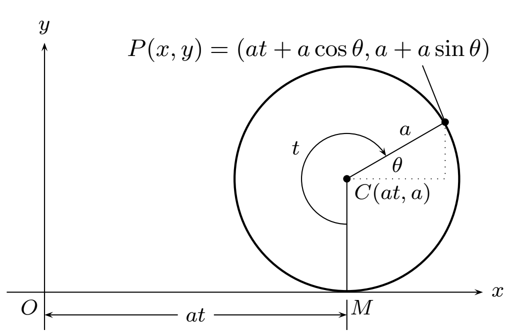


Figure 8.18: Cycloid

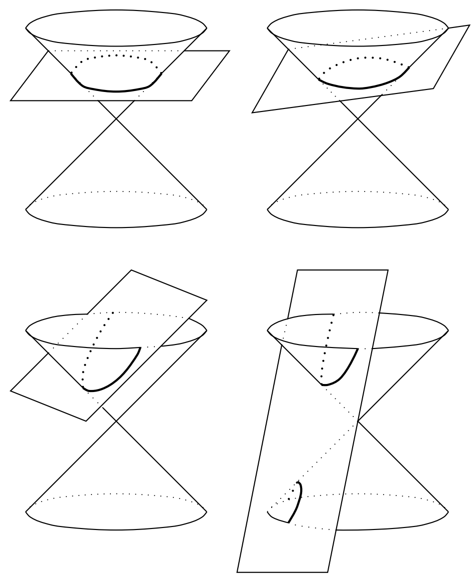


Figure 8.19: Conic sections

Find equ of parabola whose focus is at $F = (p, 0)$ and directrix ℓ is $x = -p$ Figure 8.20 Q P By definition it holds that $\overline{PQ} = \overline{PF}$. Thus

$$(x - p)^2 + y^2 = (x + p)^2$$

is the equation of parabola.

$$y^2 = 4px \tag{8.10}$$

The point closest to the curve is called

vertex the line connecting vertex and focus is **axis** $y^2 = 4px$ F is $(0, 0)$ and x-axis is the axis of parabola.

If $F = (0, p)$ directrix ℓ is $y = -p$ then

$$x^2 = py$$

Example 8.6.3. Find parabola whose directrix is $x = 1$, focus is at $(0, 3)$

sol.

$$x^2 + (y - 3)^2 = (x - 1)^2$$

So $y^2 - 6y + 2x + 8 = 0$.

□

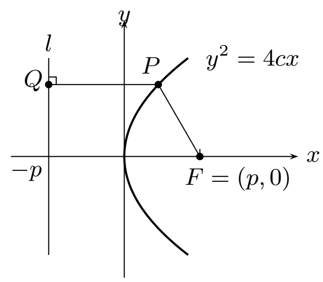


Figure 8.20: Parabola ($y^2 = 4cx$)

Ellipse

Definition 8.6.4. The set of all points in a plane whose sum of distances from two given focuses is a **ellipse** If two points are identical, it becomes a **circle**.

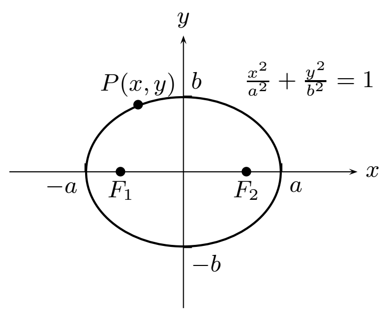


Figure 8.21: Ellipse ($x^2/a^2 + y^2/b^2 = 1$)

Now given two points $F_1 = (-c, 0)$ and $F_2 = (c, 0)$. Find the set of all points where the sum of distances from focuses are constant. Fig 8.21 $P = (x, y)$. This is an ellipse

$$PF_1 + PF_2 = 2a$$

$$\sqrt{(x+c)^2 + y^2} + \sqrt{(x-c)^2 + y^2} = 2a$$

$$\frac{x^2}{a^2} + \frac{y^2}{a^2 - c^2} = 1 \tag{8.11}$$

Let assume $b > 0$ satisfies

$$b^2 = a^2 - c^2$$

Then $b \leq a$ and hence from (8.11) we get

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \tag{8.12}$$

If $x = 0$ then $y = \pm b$ and if $y = 0$ we have $x = \pm a$. Two points $(\pm a, 0)$ are intersection of ellipse with x-axis $(0, \pm b)$ are intersection of ellipse with y-axis

major axis minor axis vertex $(\pm a, 0)$ are vertices.

Foci $F_1 = (0, -c)$ and $F_2 = (0, c)$ The set of all points whose sum of distance to these $2b$

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

$(0, \pm b)$ are vertices.

Example 8.6.5. Foci $(\pm 1, 0)$ sum of distance is 6

sol. $c = 1$ and $a = 3$. Thus $b^2 = a^2 - c^2 = 9 - 1 = 8$. Hence

$$\frac{x^2}{9} + \frac{y^2}{8} = 1$$

□

More generally, foci may not lie on the convenient axis.

Example 8.6.6. Find ellipse whose foci are $(1, 0)$ and $(1, 4)$ sum of distance is 8

sol. New coordinates $X = x - 1$, $Y = y - 2$ then on XY-plane the foci are $(0, \pm 2)$ Hence

$$\frac{X^2}{12} + \frac{Y^2}{16} = 1 \tag{8.13}$$

$$\frac{(x-1)^2}{12} + \frac{(y-2)^2}{16} = 1$$

□

Hyperbola

Definition 8.6.7. The difference of distances from given two foci are constant, we obtain **hyperbola**

Two foci are $F_1 = (-c, 0)$, $F_2 = (c, 0)$ The sum of distance is $2a$. Fig 8.22. $P = (x, y)$ satisfies $|PF_1 - PF_2| = 2a$

$$\sqrt{(x+c)^2 + y^2} - \sqrt{(x-c)^2 + y^2} = \pm 2a$$

Or

$$\frac{x^2}{a^2} + \frac{y^2}{a^2 - c^2} = 1 \tag{8.14}$$

We see $2a < 2c$. Thus

$$a^2 - c^2 < 0.$$

Let $b^2 = c^2 - a^2$. Then we obtain two **asymptotes**: (8.14)

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1 \tag{8.15}$$

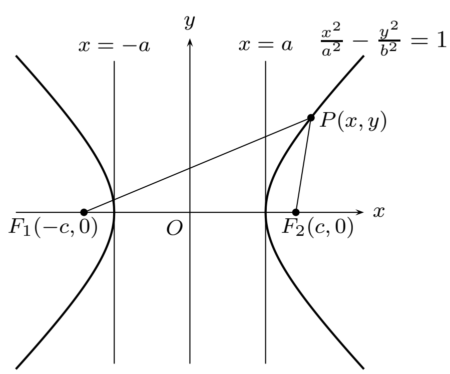


Figure 8.22: hyperbola $x^2/a^2 - y^2/b^2 = 1$

On the other hand if the distances from two foci $(0, \pm c)$ is $2b$, then the equation of hyperbola is

$$-\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

$x^2/a^2 - y^2/b^2 = 1$ has asymptotes

$$y = \pm \frac{b}{a} x$$

Example 8.6.8. Foci are $(\pm 2, 0)$ Find the locus whose difference is 2.

sol. Since $a = 1$, $c = 2$, $b = \sqrt{3}$

$$x^2 - \frac{y^2}{3} = 1$$

Asymptote are $y = \pm\sqrt{3}x$, vertices $(\pm 1, 0)$.

□

Classifying Conic Sections by Eccentricity

8.7 Quadratic Equations and Rotations

General quadratic curves are give by

$$Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0 \tag{8.16}$$

The case $B = 0$, i.e, no xy-term

In this case the equation (8.16) is

$$Ax^2 + Cy^2 + Dx + Ey + F = 0 \tag{8.17}$$

If $AC \neq 0$ then are again classified into three classes:

(1) If $AC = 0$, but $A^2 + C^2 \neq 0$, we have a parabola:

$$A(x - \alpha)^2 + Ey = \delta$$

(2) $AC > 0$: Ellipse(Assume $A > 0$)

$$\frac{(x-\alpha)^2}{C\gamma^2} + \frac{(y-\beta)^2}{A\gamma^2} = \frac{1}{AC\gamma}$$

$$A(x-\alpha)^2 + C(y-\beta)^2 = \gamma \tag{8.18}$$

(3) $AC < 0$: Hyperbola (Assume $A > 0$)

$$\frac{(x-\alpha)^2}{|C|\gamma^2} - \frac{(y-\beta)^2}{A\gamma^2} = \frac{\gamma}{|AC\gamma^2|}$$

Theorem 8.7.1. *For*

$$Ax^2 + Cy^2 + Dy^2 + Ey + F = 0$$

(1) $A = C = 0$ and one of D E is nonzero, then we have a line

(2) If one of A or C is zero, it is parabola

(3) $AC > 0$, ellipse

(4) $AC < 0$, hyperbola

The case $B \neq 0$, i.e presence of xy-term

Example 8.7.2. Find eq. of hyperbola Two foci are $F_1 = (-3,-3)$, $F_2 = (3,3)$ where difference of the distances are 6

sol. From $|PF_1 - PF_2| = 6$

$$\sqrt{(x+3)^2+(y+3)^2} - \sqrt{(x-3)^2+(y-3)^2} = \pm 6$$

$$2xy = 9$$

□

Rotation

Rotate xy-coordinate by α and call new coordinate $x'y'$- Then $P(x,y)$ is represented by (x',y') in $x'y'$-coordinate.

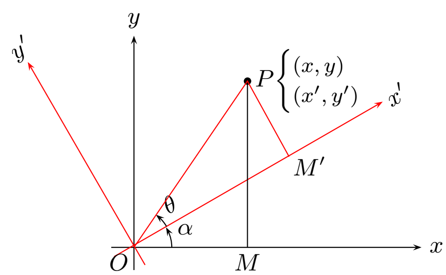


Figure 8.23: Rotation of axis

From fig 8.23 we see

$$\begin{aligned} x &= OM = OP\cos(\theta+\alpha) = OP\cos\theta\cos\alpha - OP\sin\theta\sin\alpha \\ y &= MP = OP\sin(\theta+\alpha) = OP\cos\theta\sin\alpha + OP\sin\theta\cos\alpha \end{aligned}$$

On the other hand,

$$OP\cos\theta = OM' = x', \qquad OP\sin\theta = M'P' = y'$$

Proposition 8.7.3. *Let $P = (x,y)$ be denoted by (x',y') in $x'y'$-coordinate. Then*

$$\begin{aligned} x &= x'\cos\alpha - y'\sin\alpha \\ y &= x'\sin\alpha + y'\cos\alpha \end{aligned}$$

We see from proposition 8.7.3

$$A'x'^2 + B'x'y' + C'y'^2 + D'x' + E'y' + F' = 0 \tag{8.19}$$

So

$$\begin{aligned}
A' &= A\cos^2\alpha + B\cos\alpha\sin\alpha + C\sin^2\alpha \\
B' &= B\cos 2\alpha + (C-A)\sin 2\alpha \\
C' &= A\sin^2\alpha - B\sin\alpha\cos\alpha + C\cos^2\alpha \\
D' &= D\cos\alpha + E\sin\alpha \\
E' &= -D\sin\alpha + E\cos\alpha \\
F' &= F
\end{aligned}$$

We set $B' = 0$. Then

$$B' = B\cos\alpha + (C-A)\sin\alpha = 0$$

Theorem 8.7.4. *For*

$$Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0$$

If we choose

$$\tan 2\alpha = \frac{B}{A-C}$$

then cross product term disappears.

Example 8.7.5.

$$x^2 + xy + y^2 - 6 = 0$$

sol. From $\tan 2\alpha = B/(A-C)$

$$2\alpha = \frac{\pi}{2}, \quad \text{i.e,} \quad \alpha = \frac{\pi}{4}$$

$$\begin{aligned}
x &= x'\cos\alpha - y'\sin\alpha = \frac{\sqrt{2}}{2}x' - \frac{\sqrt{2}}{2}y' \\
y &= x'\sin\alpha + y'\cos\alpha = \frac{\sqrt{2}}{2}x' + \frac{\sqrt{2}}{2}y'
\end{aligned}$$

Substitute into $x^2 + xy + y^2 - 6 = 0$ to get

$$\frac{x'^2}{4} + \frac{y'^2}{12} = 1$$

See Fig 8.24.

□

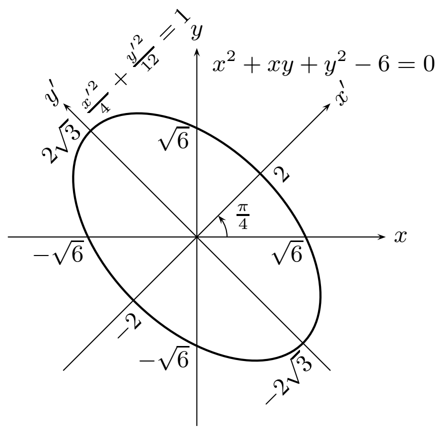


Figure 8.24: $x^2 + xy + y^2 - 6 = 0$

Invariance of Discriminant

Given a quadratic curve in xy-coordinate, we rotated the axis and obtain new equation in $x'y'$-coordinate. In this case, one can choose the angle so that no $x'y'$ term exists. However, if we are only interested in classification, there is a simple way.

$$Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0$$

$$A'x'^2 + B'x'y' + C'x'^2 + D'x' + E'y' + F' = 0$$

After some computation we can verify that

$$B^2 - 4AC = B'^2 - 4A'C' \tag{8.20}$$

Theorem 8.7.6. *For the quadratic curves given in x, y*

$$Ax^2 + Bxy + Cx^2 + Dx + Ey + F = 0$$

we have the following classification:

(1) $B^2 - 4AC = 0$ *parabola*

(2) $B^2 - 4AC < 0$ *ellipse*

(3) $B^2 - 4AC > 0$ *hyperbola*

Example 8.7.7. (1) $3x^2 - 5xy + y^2 - 2x + 3y - 5 = 0$ has $B^2 - 4AC = 25 - 12 > 0$. Thus a hyperbola.

(2) $x^2 + xy + y^2 - 5 = 0$ has $B^2 - 4AC = -3 < 0$. Thus ellipse.

(3) $x^2 - 2xy + y^2 - 5x - 3 = 0$ satisfies $B^2 - 4AC = 0$, a parabola.