Contents

Chapter 5

Techniques of Integration

Basic integration formulas

(1) $\displaystyle\int x^n\,dx = \frac{x^{n+1}}{n+1} + C \quad (n \neq -1)$

(2) $\displaystyle\int \frac{1}{x}\,dx = \ln x + C$

(3) $\displaystyle\int \cos x\,dx = \sin x + C$

(4) $\displaystyle\int \sin x\,dx = -\cos x + C$

(5) $\displaystyle\int \tan x\,dx = -\ln|\cos x| + C$

(6) $\displaystyle\int \sec^2 x\,dx = \tan x + C$

(7) $\displaystyle\int \csc^2 x\,dx = -\cot x + C$

(8) $\displaystyle\int \sec x \tan x\,dx = \sec x + C$

(9) $\displaystyle\int \csc x \cot x\,dx = -\csc x + C$

(10) $\displaystyle\int \frac{1}{\sqrt{1-x^2}}\,dx = \sin^{-1} x + C$

(11) $\displaystyle\int \frac{1}{1+x^2}\,dx = \tan^{-1} x + C$

(12) $\displaystyle\int e^x\,dx = e^x + C$

(13) $\int a^x\,dx = \frac{a^x}{\ln a} + C \quad (a > 0, a \neq 1)$

(14) $\int \cosh x\,dx = \sinh x + C$

(15) $\int \sinh x\,dx = \cosh x + C$

(16) $\int \tanh x\,dx = \ln\cosh x + C$

(17) $\int \coth x\,dx = \ln|\sinh x| + C$

(18) $\int \operatorname{sech} x\,dx = \tan^{-1}\sinh x + C = \sin^{-1}\tanh x + C$

(19) $\int \operatorname{csch} x\,dx = \ln\left|\tanh\frac{x}{2}\right| + C$

For (18), use the identity $\operatorname{sech}^2 x = 1 - \tanh^2$ or

$$\operatorname{sech} x = \frac{1}{\cosh x} = \frac{1}{\sqrt{1+\sinh^2 x}}$$

Example 5.0.1. (1) $\int \frac{2x-9}{\sqrt{x^2-9x+1}}\,dx$

(2) $\int \frac{dx}{\sqrt{8x-x^2}}$

sol. 1. Substitute

$$\int \frac{2x-9}{\sqrt{x^2-9x+1}}\,dx = \int \frac{du}{\sqrt{u}}$$

2. Complete square

$$\begin{aligned}\int \frac{dx}{\sqrt{8x-x^2}} &= \int \frac{dx}{\sqrt{16-(x-4)^2}}\\ &= \int \frac{du}{\sqrt{a^2-u^2}} \quad \text{where } a = 4 \text{ and } u = x-4\\ &= \sin^{-1}(\frac{x-4}{4}) + C\end{aligned}$$

□

5.1 Integration by Parts

From Leibniz rule, we see

$$\frac{d}{dx}(uv) = u\frac{dv}{dx} + v\frac{du}{dx}$$

Integrating, we get

$$\begin{aligned} uv &= \int u\frac{dv}{dx}\,dx + \int v\frac{du}{dx}\,dx \\ &= \int u\,dv + \int v\,du \end{aligned}$$

Thus

Proposition 5.1.1 (Integration by Parts I)**.**

$$\int u\,dv = uv - \int v\,du \tag{5.1}$$

Proposition 5.1.2 (Integration by Parts II)**.**

$$\int f(x)g'(x)\,dx = f(x)g(x) - \int f'(x)g(x)\,dx \tag{5.2}$$

Proposition 5.1.3 (Definite integral)**.**

$$\int_a^b f(x)g'(x)\,dx = [f(x)g(x)]_a^b - \int_a^b f'(x)g(x)\,dx$$

Example 5.1.4. Find the following

(1) $\displaystyle\int_0^\pi x\sin x\,dx$

(2) $\displaystyle\int \ln x\,dx$

sol. (1) Let $u = x$, $dv = \sin x\,dx$. Then $du = dx$, $v = -\cos x$. (Fig 5.1)

$$\begin{aligned} \int_0^\pi x\sin x\,dx &= [x(-\cos x)]_0^\pi - \int_0^\pi (-\cos x)\,dx \\ &= \pi + [\sin x]_0^\pi \\ &= \pi. \end{aligned}$$

(2) Let $u = \ln x$, $dv = dx$. Then we have $du = (1/x)dx$, $v = x$.

$$\begin{aligned} \int \ln x\,dx &= (\ln x)x - \int x\cdot\frac{1}{x}\,dx \\ &= x\ln x - x + C. \end{aligned}$$

□

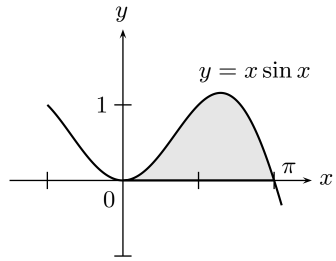


Figure 5.1:

Repeated integration by parts

		g and its integral
f and its derivative	$(+)$	$\int g$
f'	$(-)$	$\int\int g$
$(f')'$	$(+)$	$\int\int\int g$
$f^{(3)}$		

Example 5.1.5. Find $\displaystyle\int x^2 e^x\,dx$.

sol. $f(x) = x^2$, $g(x) = e^x$

□

		$g(x) = e^x$ and its integral
$f(x) = x^2$	$(+)$	e^x
$f'(x) = 2x$	$(-)$	e^x
$f^{(2)}(x) = 2$	$(+)$	e^x
0		e^x

Example 5.1.6. Find $\displaystyle\int x^2 \sin x\,dx$..

sol. Let $u = x^2$, $dv = \sin x\,dx$. Then $du = 2xdx$, $v = -\cos x$ and hence

$$\begin{aligned}\int x^2 \sin x\,dx &= x^2(-\cos x) - \int(-\cos x)2x\,dx\\ &= -x^2\cos x + \int 2x\cos x\,dx\end{aligned}$$

Again, set $u = 2x$, $dv = \cos x\,dx$. Then $du = 2\,dx$, $v = \sin x$.

$$\int x^2 \sin x\,dx = -x^2 \cos x + 2x \sin x - \int 2 \sin x\,dx$$
$$= -x^2 \cos x + 2x \sin x + 2 \cos x + C. \qquad \square$$

□

Example 5.1.7. Find $\int x^3 \sin x\,dx$.

Hence

$$\int x^3 \sin x\,dx = -x^3 \cos x + 3x^2 \sin x + 6x \cos x - 6 \sin x + C$$

f and its derivative		$g = \sin x$ and its integrals
x^3	$(+)$	$-\cos x$
$3x^2$	$(-)$	$-\sin x$
$6x$	$(+)$	$\cos x$
6	$(-)$	$\sin x$
0		

Example 5.1.8. Find $\int e^x \sin x\,dx$.

sol. If $u = e^x$, $dv = \sin x\,dx$, then $du = e^x\,dx$, $v = -\cos x$.

$$\int e^x \sin x\,dx = e^x(-\cos x) - \int e^x(-\cos x)\,dx$$
$$= -e^x \cos x + \int e^x \cos x\,dx$$

Again let $u = e^x$, $dv = \cos x\,dx$ so that $du = e^x\,dx$, $v = \sin x$.

$$\int e^x \sin x\,dx = -e^x \cos x + \int e^x \cos x\,dx$$
$$= -e^x \cos x + e^x \sin x - \int e^x \sin x\,dx.$$

Solving this for $\int e^x \sin x\,dx$ we obtain

$$\int e^x \sin x\,dx = \frac{1}{2} e^x(\sin x - \cos x) + C$$

□

Reduction formula

Example 5.1.9. Express $\int \cos^n x\,dx$ in terms of lower powers of $\cos x$.

sol.

$$\begin{aligned}\int \cos^{n-1} x \cos x\,dx &= \cos^{n-1}\sin x + (n-1)\int \sin^2 x \cos^{n-2} x\,dx \\ &= \cos^{n-1}\sin x + (n-1)\int (1-\cos^2 x)\cos^{n-2} x\,dx \\ &= \cos^{n-1}\sin x + (n-1)\int \cos^{n-2} x\,dx - (n-1)\int \cos^n x\,dx\end{aligned}$$

So

$$\int \cos^{n-1} x \cos x\,dx = \frac{\cos^{n-1}\sin x}{n} + \frac{n-1}{n}\int \cos^{n-2} x\,dx$$

□

Example 5.1.10. Prove

$$\int (a^2 \pm x^2)^n\,dx = \frac{x(a^2 \pm x^2)}{2n+1} + \frac{2na^2}{2n+1}\int (a^2 \pm x^2)^{n-1}\,dx \quad (n \neq -\frac{1}{2})$$

sol. Integration by parts

$$\begin{aligned}\int (a^2 \pm x^2)^n\,dx &= x(a^2 \pm x^2)^n - \int x \cdot n(a^2 \pm x^2)^{n-1}(\pm 2x)\,dx \\ &= x(a^2 \pm x^2)^n - \int 2n(a^2 \pm x^2)^{n-1}(a^2 \pm x^2 - a^2)\,dx \\ &= x(a^2 \pm x^2)^n - 2n\int (a^2 \pm x^2)^n\,dx \\ &\quad + 2na^2\int (a^2 \pm x^2)^{n-1}\,dx.\end{aligned}$$

If $n \neq -1/2$,

$$\int (a^2 \pm x^2)^n\,dx = \frac{x(a^2 \pm x^2)^n}{2n+1} + \frac{2na^2}{2n+1}\int (a^2 \pm x^2)^{n-1}\,dx$$

□

5.2 Integration of Trigonometric function

Products of powers of Sines and Cosines

Integral of $\sin^m x \cos^n x$

(1) If m is odd, then set $m = 2k+1$ and use $\sin^2 x = 1 - \cos^2 x$ $\sin x\, dx = -d(\cos x)$ to transform it to

$$\int \sin^{2k+1} x \cos^n x\, dx = -\int (1-\cos^2 x)^k \cos^n x\, d(\cos x)$$

(2) If n is odd $n = 2k+1$, use $\cos^2 x = 1 - \sin^2 x$ $\cos x\, dx = d(\sin x)$ to obtain

$$\int \sin^m x \cos^{2k+1} x\, dx = \int \sin^m x (1-\sin^2 x)^k\, d(\sin x)$$

(3) If both m, n are even, use $\sin^2 x = (1-\cos 2x)/2$, $\cos^2 x = (1+\cos 2x)/2$ lower the degree and repeat the previous technique

Example 5.2.1. Find $\displaystyle\int \sin^5 x\, dx$.

sol.
$$\begin{aligned}\int \sin^5 x\, dx &= -\int (1-\cos^2 x)^2\, d(\cos x)\\ &= -\int (1 - 2\cos^2 x + \cos^4 x)\, d(\cos x)\\ &= -\frac{1}{5}\cos^5 x + \frac{2}{3}\cos^3 x - \cos x + C.\end{aligned}$$

□

Example 5.2.2. Find $\displaystyle\int \sin^2 x \cos^3 x\, dx$.

sol.
$$\begin{aligned}\int \sin^2 x \cos^3 x\, dx &= \int \sin^2 x (1-\sin^2 x)\, d(\sin x)\\ &= -\frac{1}{5}\sin^5 x + \frac{1}{3}\sin^3 x + C.\end{aligned}$$

□

Example 5.2.3. Find $\displaystyle\int \sin^4 x \cos^2 x\, dx$.

sol.
$$\begin{aligned}\int \sin^4 x \cos^2 x\,dx &= \int \left(\frac{1-\cos 2x}{2}\right)^2 \left(\frac{1+\cos 2x}{2}\right) dx \\ &= \frac{1}{8}\int \left(1 - 2\cos 2x + \cos^2 2x\right)(1+\cos 2x)\; dx \\ &= \frac{1}{8}\int \left(1 - \cos 2x - \cos^2 2x + \cos^3 2x\right) dx \\ &= \frac{1}{8}\int \left(1 - \cos 2x - \frac{1+\cos 4x}{2} + (1-\sin^2 2x)\cos 2x\right) dx \\ &= \frac{1}{16}\int \left(1 - \cos 4x - \sin^2 2x \cdot 2\cos 2x\right) dx \\ &= \frac{1}{16}\left(x - \frac{1}{4}\sin 4x - \frac{1}{3}\sin^3 2x\right) + C.\end{aligned}$$

□

Integral of $\sqrt{1 \pm \sin ax}$, $\sqrt{1 \pm \cos ax}$

Use the double angle formula.

$$\begin{aligned}\sin 2A &= 2\sin A \cos A \\ \cos 2A &= 2\cos^2 A - 1 \\ &= 1 - 2\sin^2 A\end{aligned}$$

Change the form $1 \pm \sin ax$, $1 \pm \cos ax$ to complete square.

Example 5.2.4. Find $\int_0^\pi \sqrt{1-\sin x}\,dx$.

sol. Since $1 - \sin x = 1 - 2\sin(x/2)\cos(x/2) = (\sin(x/2) - \cos(x/2))^2$

$$\begin{aligned}\int_0^\pi \sqrt{1-\sin x}\,dx &= \int_0^\pi \left|\sin\frac{x}{2} - \cos\frac{x}{2}\right| dx \\ &= \int_0^{\pi/2} \left(\cos\frac{x}{2} - \sin\frac{x}{2}\right) + \int_{\pi/2}^{\pi} \left(\sin\frac{x}{2} - \cos\frac{x}{2}\right) dx \\ &= \left[2\sin\frac{x}{2} + 2\cos\frac{x}{2}\right]_0^{\pi/2} + \left[-2\cos\frac{x}{2} - 2\sin\frac{x}{2}\right]_{\pi/2}^{\pi} \\ &= (\sqrt{2} + \sqrt{2} - 2) + (-2 + \sqrt{2} + \sqrt{2}) \\ &= 4(\sqrt{2} - 1).\end{aligned}$$

□

Example 5.2.5. Find $\int_0^{\pi/2} \sqrt{1+\cos 2x}\, dx$.

sol. $1+\cos 2x = 2\cos^2 x$,

$$\begin{aligned}\int_0^{\pi/2} \sqrt{1+\cos 2x} &= \sqrt{2}\int_0^{\pi/2} |\cos x|\ dx \\ &= \sqrt{2}\,[\sin x]_0^{\pi/2} \\ &= \sqrt{2}.\end{aligned}$$

□

Tangent and secant

Recall $1+\tan^2 x = \sec^2 x$, $(\tan x)' = \sec^2 x$, $(\sec x)' = \sec x \tan x$.

Example 5.2.6. $\int \sec x\, dx$.

sol. Multiply $\sec x + \tan x$ both the numerator and denominator

$$\begin{aligned}\int \sec x\, dx &= \int \frac{\sec x(\sec x+\tan x)}{\sec x+\tan x}\, dx \\ &= \int \frac{(\sec x+\tan x)'}{\sec x+\tan x}\, dx \\ &= \ln|\sec x+\tan x| + C.\end{aligned}$$

□

Example 5.2.7. $\int \tan^2 x \sec x\, dx$.

sol. Since $\tan^2 x\sec x = (\sec^2 x - 1)\sec x = \sec^3 x - \sec x$, it reduces to find

$$\int \sec^3 x\, dx.$$

Let $u = \sec x$, $dv = \sec^2 x\, dx$ then $v = \tan x$, $du = \sec x \tan x\, dx$, we have

$$\begin{aligned}\int \sec^3 x\, dx = \int (\sec x)(\sec^2 x\, dx) &= \int u\, dv \\ &= uv - \int v\, du \\ &= \sec x\tan x - \int (\tan x)\sec x \tan x\, dx \\ &= \sec x\tan x - \int (\sec^2 x - 1)\sec x\, dx \\ &= \sec x\tan x - \int \sec^3 x\, dx + \int \sec x\, dx\end{aligned}$$

we obtain

$$\int \sec^3 x\, dx = \frac{1}{2}\sec x \tan x + \frac{1}{2}\int \sec x\, dx$$

Hence

$$\begin{aligned}
\int \tan^2 x \sec x\, dx &= \int \sec^3 x\, dx - \int \sec x\, dx \\
&= \frac{1}{2}\sec x \tan x - \frac{1}{2}\int \sec x\, dx \\
&= \frac{1}{2}\sec x \tan x - \frac{1}{2}\ln|\sec x + \tan x| + C.
\end{aligned}$$

□

Example 5.2.8. $\int \tan^6 x\, dx$.

sol. Since $\tan^2 x = \sec^2 x - 1$

$$\begin{aligned}
\int \tan^6 x\, dx &= \int \tan^4 x(\sec^2 x - 1)\, dx \\
&= \int \tan^4 x \sec^2 x\, dx - \int \tan^4 x\, dx \\
&= \int \tan^4 x \sec^2 x\, dx - \int \tan^2 x(\sec^2 x - 1)\, dx \\
&= \int \tan^4 x \sec^2 x\, dx - \int \tan^2 x \sec^2 x\, dx + \int \tan^2 x\, dx \\
&= \int \tan^4 x \sec^2 x\, dx - \int \tan^2 x \sec^2 x\, dx + \int (\sec^2 x - 1)\, dx \\
&= \frac{1}{5}\tan^5 x - \frac{1}{3}\tan^3 x + \tan x - x + C.
\end{aligned}$$

□

Example 5.2.9. Show

$$\int \sec^n x\, dx = \frac{\tan x \sec^{n-2} x}{n-1} + \frac{n-2}{n-1}\int \sec^{n-2} x\, dx.$$

sol. Since $\tan^2 x = \sec^2 x - 1$

$$\begin{aligned}
\int \sec^2 x \sec^{n-2} x\, dx &= \int \sec^2 x \sec^{n-2} x\, dx \\
&= \tan x \sec^{n-2} x - (n-2)\int \tan^2 x \sec^{n-2} x\, dx \\
&= \tan x \sec^{n-2} x - (n-2)\int (\sec^2 x - 1)\sec^{n-2} x\, dx \\
&= \frac{\tan x \sec^{n-2} x}{n-1} + \frac{n-2}{n-1}\int \sec^{n-2} x\, dx
\end{aligned}$$

□

Example 5.2.10. Find

$$\int (\sec x + \tan x)^2 \, dx$$

sol. Observe

$$(\sec x + \tan x)^2 = \sec^2 x + 2\sec x \tan x + \tan^2 x = 2\sec^2 x - 1 + 2\sec x \tan x$$

Hence

$$\begin{aligned} \int (\sec x + \tan x)^2 \, dx &= \int (2\sec^2 x - 1 + 2\sec x \tan x) \, dx \\ &= 2\tan x + 2\sec x - x + C. \end{aligned}$$

□

Remark 5.2.11. For integrals involving $\cot x$ or $\csc x$, use $1+\cot^2 x = \csc^2 x$, $(\cot x)' = -\csc^2 x$, $(\csc x)' = -\csc x \cot x$.

Example 5.2.12. Find

$$\int (\csc x + \cot x)^2 \, dx$$

sol. Observe

$$(\csc x + \cot x)^2 = \csc^2 x + 2\csc x \cot x + \cot^2 x = 2\csc^2 x - 1 + 2\csc x \cot x$$

Hence

$$\begin{aligned} \int (\csc x + \cot x)^2 \, dx &= \int (2\csc^2 x - 1 + 2\csc x \cot x) \, dx \\ &= -2\cot x - 2\csc x - x + C. \end{aligned}$$

□

Example 5.2.13.

$$\int_0^{\pi/4} \sqrt{1 + \cos 4x} \, dx$$

Using

$$\cos^2 \theta = \frac{1 + \cos 2\theta}{2}$$

Integrals of $\sin mx \sin nx$, $\sin mx \cos nx$, $\cos mx \cos nx$

Addition formula:

$$\begin{aligned}
\sin(A+B) &= \sin A \cos B + \cos A \sin B \\
\sin(A-B) &= \sin A \cos B - \cos A \sin B \\
\cos(A+B) &= \cos A \cos B - \sin A \sin B \\
\cos(A-B) &= \cos A \cos B + \sin A \sin B
\end{aligned}$$

From these we get(with $A = mx$, $B = nx$)

$$\begin{aligned}
\sin mx \sin nx &= \frac{1}{2}[\cos(m-n)x - \cos(m+n)x] \\
\sin mx \cos nx &= \frac{1}{2}[\sin(m-n)x + \sin(m+n)x] \\
\cos mx \cos nx &= \frac{1}{2}[\cos(m-n)x + \cos(m+n)x]
\end{aligned}$$

Example 5.2.14. $\displaystyle\int_0^{\pi/6} \sin 4x \sin 3x \, dx$.

sol. Figure **??**

$$\begin{aligned}
\int_0^{\pi/6} \sin 4x \sin 3x \, dx &= \frac{1}{2}\int_0^{\pi/6} (\cos x - \cos 7x) \, dx \\
&= \frac{1}{2}\left[\sin x - \frac{1}{7}\sin 7x\right]_0^{\pi/6} = \frac{2}{7}.
\end{aligned}$$

□

5.3 Trig Substitution

Quadratic term

For the integrals involving terms like $\sqrt{a^2-u^2}$, $\sqrt{a^2+u^2}$ and $\sqrt{u^2-a^2}$, trigonometric substitution is convenient.

Looking at the forms, we substitute by $u = a\sin\theta$, $u = a\tan\theta$, $u = a\sec\theta$ for each of them,

$$a^2 - u^2 = a^2 - a^2\sin^2\theta = a^2(1-\sin^2\theta) = a^2\cos^2\theta \tag{5.3}$$

$$a^2 + u^2 = a^2 + a^2\tan^2\theta = a^2(1+\tan^2\theta) = a^2\sec^2\theta \tag{5.4}$$

$$u^2 - a^2 = a^2\sec^2\theta - a^2 = a^2(\sec^2\theta - 1) = a^2\tan^2\theta \tag{5.5}$$

Thus we use the following substitution:

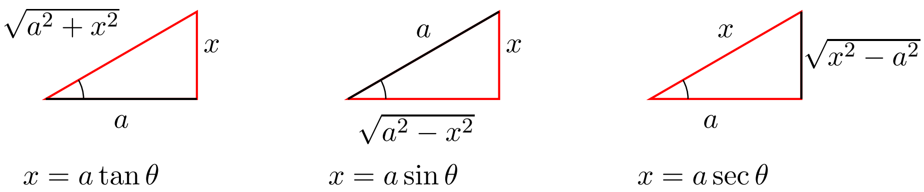


Figure 5.2: trig substitution

(1) $u = a\sin\theta$ defined on $-\pi/2 \le \theta \le \pi/2$.

(2) $u = a\tan\theta$ defined on $-\pi/2 < \theta < \pi/2$.

(3) $u = a\sec\theta$ defined on $|u| \ge a$. We choose the branch as(Do positive case only)

$$\begin{cases} 0 \le \theta < \pi/2 & \text{if } u \ge a \\ \pi/2 < \theta \le \pi & \text{if } u \le -a \end{cases}$$

Example 5.3.1. $\displaystyle\int \frac{du}{a^2+u^2}$.

sol. Use substitution $u = a\tan\theta$, $du = a\sec^2\theta\, d\theta$

$$\begin{aligned} \int \frac{du}{a^2+u^2} &= \int \frac{a\sec^2\theta\, d\theta}{a^2\sec^2\theta} \\ &= \int \frac{d\theta}{a} \\ &= \frac{1}{a}\cdot\theta + C \\ &= \frac{1}{a}\tan^{-1}\frac{u}{a} + C. \end{aligned}$$

□

Example 5.3.2. Find $\displaystyle\int \sqrt{a^2-u^2}\, du$. $(a > 0)$

sol. Use $u = a\sin\theta$, $du = a\cos\theta\, d\theta$

$$\begin{aligned}
\int \sqrt{a^2-u^2}\,du &= \int a\cos\theta \cdot a\cos\theta\,d\theta \\
&= \frac{a^2}{2}\int (1+\cos 2\theta)\,d\theta \\
&= \frac{a^2}{2}\left(\theta + \frac{\sin 2\theta}{2}\right) + C \\
&= \frac{a^2}{2}(\theta + \sin\theta\cos\theta) + C \\
&= \frac{a^2}{2}\left(\sin^{-1}\frac{u}{a} + \frac{u}{a}\sqrt{1-\frac{u^2}{a^2}}\right) + C \\
&= \frac{a^2}{2}\sin^{-1}\frac{u}{a} + \frac{1}{2}u\sqrt{a^2-u^2} + C.
\end{aligned}$$

□

Example 5.3.3. Find $\int \frac{du}{\sqrt{u^2-a^2}}$. $(|u| > a > 0)$

sol. We show positive case only. For negative, use substitution $u \to -u$. Let $u = a\sec\theta$

$$\begin{aligned}
u^2 - a^2 &= a^2(\sec^2\theta - 1) \\
&= a^2\tan^2\theta, \\
du &= a\sec\theta\tan\theta\,d\theta
\end{aligned}$$

Then

$$\begin{aligned}
\int \frac{du}{\sqrt{u^2-a^2}} &= \int \frac{a\sec\theta\tan\theta\,d\theta}{a|\tan\theta|} \\
&= \begin{cases} \int \sec\theta\,d\theta & (0<\theta<\pi/2) \\ -\int \sec\theta\,d\theta & (\pi/2<\theta<\pi) \end{cases} \\
&= \begin{cases} \ln|\sec\theta+\tan\theta| + C & (0<\theta<\pi/2) \\ -\ln|\sec\theta+\tan\theta| + C & (\pi/2<\theta<\pi) \end{cases} \\
&= \begin{cases} \ln\left|\frac{u}{a} + \frac{\sqrt{u^2-a^2}}{a}\right| + C & (u>a) \\ -\ln\left|\frac{u}{a} - \frac{\sqrt{u^2-a^2}}{a}\right| + C & (u<-a). \end{cases}
\end{aligned}$$

On the other hand,

$$\ln\left|\frac{u}{a}+\frac{\sqrt{u^2-a^2}}{a}\right| = \ln\left|u+\sqrt{u^2-a^2}\right| - \ln a$$

$$\begin{aligned}
-\ln\left|\frac{u}{a}-\frac{\sqrt{u^2-a^2}}{a}\right| &= \ln\left|\frac{a}{u-\sqrt{u^2-a^2}}\right| \\
&= \ln\left|\frac{a(u+\sqrt{u^2-a^2})}{(u-\sqrt{u^2-a^2})(u+\sqrt{u^2-a^2})}\right| \\
&= \ln\left|\frac{a(u+\sqrt{u^2-a^2})}{a^2}\right| \\
&= \ln\left|\frac{u+\sqrt{u^2-a^2}}{a}\right| \\
&= \ln\left|u+\sqrt{u^2-a^2}\right| - \ln a
\end{aligned}$$

Hence

$$\int \frac{du}{\sqrt{u^2-a^2}} = \ln\left|u+\sqrt{u^2-a^2}\right| + C'$$

□

Example 5.3.4. $\displaystyle\int \frac{dx}{\sqrt{x^2+9}}$.

sol. Let $x = 3\tan\theta$ $(-\pi/2 < \theta < \pi/2)$. Then $dx = 3\sec^2\theta\, d\theta$ and

$$\begin{aligned}
\int \frac{dx}{\sqrt{x^2+9}} &= \int \sec\theta\, d\theta \\
&= \ln|\sec\theta+\tan\theta| + C \\
&= \ln\left|\sqrt{\left(\frac{x}{3}\right)^2+1}+\frac{x}{3}\right| + C \\
&= \ln\left|x+\sqrt{x^2+9}\right| + C.
\end{aligned}$$

□

Involving $ax^2 + bx + c-$ Completing square

For $ax^2 + bx + c$, $(a, b \neq 0)$ use $u = x + b/(2a)$ to get

$$a(x + b/(2a))^2 - \frac{b^2 - 4ac}{4a} = a(u^2 \pm p^2).$$

Example 5.3.5. Find $\int \sqrt{2x - x^2}\, dx$.

sol. Since $2x - x^2 = 1 - (x-1)^2$ $u = x - 1$ we have by example 5.3.2 with $a = 1$

$$\begin{aligned}\int \sqrt{2x - x^2}\, dx &= \int \sqrt{1 - u^2}\, du \\ &= \frac{1}{2}\sin^{-1} u + \frac{1}{2}u\sqrt{1 - u^2} + C \\ &= \frac{1}{2}\sin^{-1}(x-1) + \frac{1}{2}(x-1)\sqrt{2x - x^2} + C.\end{aligned}$$

□

Example 5.3.6. $\int \frac{dx}{x^2 + x + 1}$.

sol. Since $x^2 + x + 1 = (x + 1/2)^2 + 3/4$, we let $u = x + 1/2$ and $a = \sqrt{3}/2$. Thus

$$\begin{aligned}\int \frac{dx}{x^2 + x + 1} &= \int \frac{du}{u^2 + 3/4} \\ &= \frac{2}{\sqrt{3}} \tan^{-1} \frac{2u}{\sqrt{3}} + C \\ &= \frac{2}{\sqrt{3}} \tan^{-1} \frac{2x + 1}{\sqrt{3}} + C.\end{aligned}$$

□

Integral tables

(1) $\displaystyle\int \frac{1}{a^2+u^2}\,du = \frac{1}{a}\tan^{-1}\frac{u}{a} \quad (a>0)$

(2) $\displaystyle\int \frac{1}{\sqrt{a^2-u^2}}\,du = \sin^{-1}\frac{u}{a} \quad (a>0)$

Example 5.3.7. For $\int 1/(4+9x^2)\,dx$, use substitution. Let $3x = u$ then $3dx = du$, and

$$\begin{aligned}\int \frac{1}{4+9x^2}\,dx &= \frac{1}{3}\int \frac{1}{2^2+u^2}\,du \\ &= \frac{1}{3}\left(\frac{1}{2}\tan^{-1}\frac{u}{2}\right)+C \\ &= \frac{1}{6}\tan^{-1}\frac{3}{2}x + C\end{aligned}$$

5.4 Integration of Rational functions by partial fraction

When $p(x)$, $q(x)$ are rational functions we can always write $p(x)$ in the form

$$\frac{p(x)}{q(x)} = Q(x) + \frac{r(x)}{q(x)}$$

where deg of $r(x)$ less than deg of $q(x)$. Thus, it suffices to consider the case when the fraction is proper;

Distinct linear factors

$\alpha_1,\dots,\alpha_r$ are distinct and $p(x)$ is polynomial of degree of is less than r. Then

$$\frac{p(x)}{(x-\alpha_1)\cdots(x-\alpha_r)} = \frac{A_1}{x-\alpha_1}+\cdots+\frac{A_r}{x-\alpha_r} \tag{5.6}$$

Here A_i are obtained by method of undetermined coefficients(or by evaluating at certain points).

$$\int \frac{dx}{(x-\alpha_1)\cdots(x-\alpha_r)} = \sum_{i=1}^{r} A_i \ln|x-\alpha_i| + C$$

Example 5.4.1. Find $\displaystyle\int \frac{x+1}{x(x+2)}\,dx$.

sol.

$$\frac{x+1}{x(x+2)} = \frac{1}{2x} + \frac{1}{2(x+2)}$$

$$\int \frac{x+1}{x(x+2)}\,dx = \frac{1}{2}\int\left(\frac{1}{x}+\frac{1}{x+2}\right)dx$$
$$= \frac{1}{2}\ln|x(x+2)|+C. \qquad \square$$

$\square$

Example 5.4.2. Find $\int \frac{2x+1}{x^3-x}\,dx$.

sol. Since $x^3-x = x(x-1)(x+1)$

$$\frac{2x+1}{x^3-x} = \frac{A}{x}+\frac{B}{x-1}+\frac{C}{x+1}$$

Solving for A, B, C we get $A=-1,\ B=3/2,\ C=-1/2$.((In this case, evaluating at $x=0,1,-1$ would be convenient!) Hence

$$\int \frac{2x+1}{x^3-x}\,dx = \int\left(\frac{-1}{x}+\frac{3/2}{x-1}+\frac{-1/2}{x+1}\right)dx$$
$$= -\ln|x|+\frac{3}{2}\ln|x-1|-\frac{1}{2}\ln|x+1|+C.$$

$\square$

Repeated linear factor

When deg of $p(x)$ is less than r, we let

$$\frac{p(x)}{(x-\alpha)^r} = \frac{A_1}{x-\alpha}+\frac{A_2}{(x-\alpha)^2}+\cdots+\frac{A_r}{(x-\alpha)^r}$$

Find $A_1, A_2, \dots, A_r$. Then

$$\int \frac{p(x)}{(x-\alpha)^r}\,dx = \int\left(\frac{A_1}{x-\alpha}+\frac{A_2}{(x-\alpha)^2}+\cdots+\frac{A_r}{(x-\alpha)^r}\right)dx$$
$$= A_1\ln|x-\alpha| - \frac{A_2}{x-\alpha} - \cdots - \frac{(r-1)A_r}{(x-\alpha)^{r-1}} + C$$

Example 5.4.3. Find $\int \frac{x^2}{(x-2)^3}\,dx$ we see

sol. Since $x^2 = (x-2)^2 + 4(x-2) + 4$,

$$\frac{x^2}{(x-2)^3} = \frac{1}{x-2} + \frac{4}{(x-2)^2} + \frac{4}{(x-2)^3}$$

Hence

$$\begin{aligned}\int \frac{x^2}{(x-2)^3}\,dx &= \int \left(\frac{1}{x-2} + \frac{4}{(x-2)^2} + \frac{4}{(x-2)^3}\right) dx \\ &= \ln|x-2| - \frac{4}{x-2} - \frac{8}{(x-2)^2} + C.\end{aligned}$$

□

Irreducible quadratic factor

Suppose $x^2 + \beta_1 x + \gamma_1, \ldots, x^2 + \beta_r x + \gamma_r$ are distinct quadratic factor without having real roots, and suppose $p(x)$ is polynomial of degree less than $2r$. Then we can set

$$\frac{p(x)}{(x^2+\beta_1 x+\gamma_1)\cdots(x^2+\beta_r x+\gamma_r)} = \sum_{i=1}^{r} \frac{B_i x + C_i}{x^2+\beta_i x+\gamma_i}$$

for some constants $B_1, \ldots, B_r$, $C_1, \ldots, C_r$.We have

$$\begin{aligned}B_i x + C_i &= \frac{B_i}{2}(2x+\beta_i) + D_i \\ &= \frac{B_i}{2}(x^2+\beta_i x+\gamma_i)' + D_i\end{aligned}$$

where $D_i = C_i - B_i\beta_i/2$. So

$$\begin{aligned}\int \frac{B_i x + C_i}{x^2+\beta_i x+\gamma_i}\,dx &= \int \left(\frac{B_i}{2}\frac{(x^2+\beta_i x+\gamma_i)'}{x^2+\beta_i x+\gamma_i} + \frac{D_i}{x^2+\beta_i x+\gamma_i}\right) dx \\ &= \frac{B_i}{2}\ln(x^2+\beta_i x+\gamma_i) + \int \frac{D_i}{x^2+\beta_i x+\gamma_i}\,dx\end{aligned}$$

Use the following

$$\int \frac{du}{u^2+a^2} = \frac{1}{a}\tan^{-1}\frac{u}{a} + C$$

Example 5.4.4. Find $\displaystyle\int \frac{2x}{x^4+x^2+1}\,dx$

sol. Since $x^4 + x^2 + 1 = (x^2 - x + 1)(x^2 + x + 1)$, we can set

$$\frac{2x}{x^4 + x^2 + 1} = \frac{B_1 x + C_1}{x^2 - x + 1} + \frac{B_2 x + C_2}{x^2 + x + 1}$$

We find $B_1 = B_2 = 0$, $C_1 = 1$, $C_2 = -1$. Since $x^2 \pm x + 1 = (x \pm 1/2)^2 + (\sqrt{3}/2)^2$ we see

$$\begin{aligned}
&\int \frac{2x}{x^4 + x^2 + 1}\, dx \\
&\quad = \int \left(\frac{1}{(x - 1/2)^2 + (\sqrt{3}/2)^2} - \frac{1}{(x + 1/2)^2 + (\sqrt{3}/2)^2} \right) dx \\
&\quad = \frac{2}{\sqrt{3}} \left(\tan^{-1} \frac{2x - 1}{\sqrt{3}} - \tan^{-1} \frac{2x + 1}{\sqrt{3}} \right) + C.
\end{aligned}$$

□

Repeated irreducible quadratic factor

Suppose $p(x)$ is polynomial of degree less than $2r$, and $x^2 + \beta x + \gamma$ does not have real roots. Then we set

$$\frac{p(x)}{(x^2 + \beta x + \gamma)^r} = \frac{B_1 x + C_1}{x^2 + \beta x + \gamma} + \frac{B_2 x + C_2}{(x^2 + \beta x + \gamma)^2} + \cdots + \frac{B_r x + C_r}{(x^2 + \beta x + \gamma)^r}$$

for some $B_1, B_2, \ldots, B_r, C_1, C_2, \ldots, C_r$. with $D_i = C_i - B_i \beta/2$ we have

$$\begin{aligned}
B_i x + C_i &= \frac{B_i}{2}(2x + \beta) + D_i \\
&= \frac{B_i}{2}(x^2 + \beta x + \gamma)' + D_i
\end{aligned}$$

we see

$$\begin{aligned}
\int \frac{B_i x + C_i}{(x^2 + \beta x + \gamma)^i}\, dx &= \int \left(\frac{B_i}{2} \frac{(x^2 + \beta x + \gamma)'}{(x^2 + \beta x + \gamma)^i} + \frac{D_i}{(x^2 + \beta x + \gamma)^i} \right) dx \\
&= -\frac{B_i}{2(i-1)(x^2 + \beta x + \gamma)^{i-1}} + \int \frac{D_i}{(x^2 + \beta x + \gamma)^i}\, dx
\end{aligned}$$

For the integral of $D_i/(x^2 + \beta x + \gamma)^i$ $(i \geq 2)$, use the recurrence relation

Proposition 5.4.5.

$$\int \frac{du}{(u^2 + a^2)^i} = \frac{u}{a^2(2i - 2)(u^2 + a^2)^{i-1}} + \frac{2i - 3}{a^2(2i - 2)} \int \frac{du}{(u^2 + a^2)^{i-1}}$$

Proof.

$$\begin{aligned}
\int \frac{a^2}{(u^2+a^2)^i}du &= \int \frac{u^2+a^2-u^2}{(u^2+a^2)^i}du \\
&= \int \frac{du}{(u^2+a^2)^{i-1}} - \frac{1}{2}\int \frac{u\cdot 2udu}{(u^2+a^2)^i} \\
&= \int \frac{du}{(u^2+a^2)^{i-1}} + \frac{1}{2(i-1)}\left[\frac{u}{(u^2+a^2)^{i-1}} - \int \frac{du}{(u^2+a^2)^{i-1}}\right] \\
&= \frac{1}{2(i-1)}\frac{u}{(u^2+a^2)^{i-1}} + \frac{2i-3}{2(i-1)}\int \frac{du}{(u^2+a^2)^{i-1}}
\end{aligned}$$

□

Example 5.4.6. Find $\displaystyle\int \frac{x^4+2x^3+5x^2+6}{(x^2+2)^3}\,dx.$

sol.

$$\frac{x^4+2x^3+5x^2+6}{(x^2+2)^3} = \frac{A_1x+B_1}{x^2+2} + \frac{A_2x+B_2}{(x^2+2)^2} + \frac{A_3x+B_3}{(x^2+2)^3}$$

Multiply $(x^2+2)^3$ to see

$$\begin{aligned}
x^4+2x^3+5x^2+6 = A_1x^5 &+ B_1x^4 + (4A_1+A_2)x^3 + (4B_1+B_2)x^2 \\
&+ (4A_1+2A_2+A_3)x + 4B_1+2B_2+B_3
\end{aligned}$$

Solving we get $A_1 = 0,\ A_2 = 2,\ A_3 = -2,\ B_1 = 1,\ B_2 = 1,\ B_3 = 0$. Hence the integral is

$$\frac{x^4+2x^3+5x^2+6}{(x^2+2)^3} = \frac{1}{x^2+2} + \frac{2x+1}{(x^2+2)^2} + \frac{-4x}{(x^2+2)^3}$$

$$\begin{aligned}
&\int \frac{x^4+2x^3+5x^2+6}{(x^2+2)^3}\,dx \\
&= \int \frac{dx}{x^2+2} + \int \frac{2x}{(x^2+2)^2}\,dx + \int \frac{1}{(x^2+2)^2}\,dx + \int \frac{-4x}{(x^2+2)^3}\,dx \\
&= \frac{1}{\sqrt{2}}\tan^{-1}\frac{x}{\sqrt{2}} - \frac{1}{x^2+2} + \frac{x}{4(x^2+2)} + \frac{1}{4}\int \frac{1}{x^2+2}\,dx + \frac{1}{(x^2+2)^2} \\
&= \frac{5}{4\sqrt{2}}\tan^{-1}\frac{x}{\sqrt{2}} + \frac{x-4}{4(x^2+2)} + \frac{1}{(x^2+2)^2} + C.
\end{aligned}$$

□

Heaviside cover up method for linear factors

Example 5.4.7.

$$\frac{x^2+1}{(x-1)(x-2)(x-3)} = \frac{A}{x-1} + \frac{B}{x-2} + \frac{C}{x-3}$$

Here the coefficients A can be obtained by multiplying the factor $(x-1)$, and substituting $x=1$. This can be viewed as follows:

$$A = \frac{(1)^2+1}{\underbrace{(x-1)}_{cover}(1-2)(1-3)}$$

$$B = \frac{(2)^2+1}{(2-1)\boxed{(x-2)}_{cover}(2-3)} = \frac{5}{(1)(-1)} = -5$$

$$B = \frac{(3)^2+1}{(3-1)(3-2)\boxed{(x-3)}_{cover}} = \frac{10}{(2)(1)} = 5$$

Example 5.4.8. Do the same with

$$\int \frac{x+4}{x(x-2)(x+5)}$$

sol. Note

$$\frac{x+4}{x(x-2)(x+5)} = \frac{A}{x} + \frac{B}{x-2} + \frac{C}{x+5}$$

$$\begin{aligned} A &= \frac{0+4}{\boxed{x}(0-2)(0+5)} \\ B &= \frac{2+4}{2\boxed{(x-2)}(2+5)} \\ C &= \frac{-5+4}{(-5)(-5-2)\boxed{(x+5)}} = -\frac{1}{35} \end{aligned}$$

□

Using differentiation-repeated factors

Example 5.4.9.

$$\frac{x-1}{(x+1)^3} = \frac{A}{x+1} + \frac{B}{(x+1)^2} + \frac{C}{(x+1)^3}$$

Write

$$x - 1 = A(x+1)^2 + B(x+1) + C$$

Substitute $x = -1$. Then take derivative

$$1 = 2A(x+1) + B$$

and Substitute $x = -1$. to get B etc.

Example 5.4.10. Do the same with

$$\int \frac{x+4}{x(x-2)(x+5)}$$

CAS

Find

$$\int x^2\sqrt{x^2+a^2}$$

by Maple. Define the function;
Then

$$> int(f,x);$$

Maple will return answer

5.5 Tables

5.6 Numerical Integration

One can use an interpolating polynomial $F(x)$ to evaluate the integral

$$\int_a^b f(x)\,dx \approx \int_a^b F(x)\,dx$$

5.6.1 Approximation based on polynomial interpolation-equal intervals

We would like to design a quadrature for the following integral:

$$\int_a^b f(x)\,dx.$$

One point formula-rectangle rule

If $F(x)$ is a constant interpolation, at x_0 then we have a rectangle rule:

$$\int_a^b f(x)\,dx \approx (b-a)f(x_0) \tag{5.7}$$

The error of the rectangle rule is $\frac{(b-a)^2}{2}\max|f'(\xi)|$. Thus one point formula is exact for constant polynomial, so it is called a **0-th order method**.
Two point formula-trapezoidal rule

$$\int_{x_0}^{x_1} f(x)\,dx = \frac{h}{2}(f(x_0)+f(x_1)) - \frac{h^3}{12}f''(\xi). \tag{5.8}$$

$h = x_1 - x_0$. This is exact for linear polynomial, hence it is a **first order method**.

Repeated use of trapezoidal rule

Let $[a,b]$ is divided into n-intervals $a = x_0 < x_1 < \cdots < x_n = b$. Using the rule repeatedly over n-intervals, we see

$$\int_{x_0}^{x_n} f(x)\,dx \approx \frac{h}{2}[y_0 + 2y_1 + 2y_2 + \cdots + 2y_{n-1} + y_n]$$

Error bound is

$$|E_T| = \frac{1}{12}\sum\left(\frac{b-a}{n}\right)^3 |f''(\xi_i)| = \frac{(b-a)^3}{12n^2}M$$

Three point formula-Simpson rule. Use quadratic interpolation at three equally spaced points x_0, x_1, x_2. For simplicity, we can assume $x_0 = -h, x_1 = 0$ and $x_2 = h$. Let $Ax^2 + Bx + C$ be the interpolating polynomial. Then the integral is approximated by the following formula:

$$\int_{x_0}^{x_2} f(x)\,dx \approx \int_{-h}^{h}(Ax^2+Bx+C)\,dx = \frac{h}{3}(2Ah^2+6C). \tag{5.9}$$

Now we need to find coefficients:

$$\begin{aligned} y_0 &= Ah^2 - Bh + C \\ y_1 &= C \\ y_2 &= Ah^2 + Bh + C. \end{aligned}$$

Solving, we obtain $A = \frac{y_0-2y_1+y_2}{2h^2}, B = \frac{y_2-y_0}{2h}, C = y_1$. Hence

$$\int_{-h}^{h}(Ax^2+Bx+C)\,dx = \frac{h}{3}(2Ah^2+6C) = \frac{h}{3}(y_0+4y_1+y_2). \tag{5.10}$$

So Simpson's is exact for quadratic. In fact we have:

Proposition 5.6.1. *The Simpson's rule is exact for cubic polynomials and the error satisfies*

$$|E_S| \le \frac{(b-a)^5}{180n^4}M$$

Proof. The proof is based on the fact

$$\int_{x_0}^{x_2} f(x)\,dx = \frac{h}{3}(f(x_0)+4f(x_1)+f(x_2)) - \frac{h^5}{90}f^{(4)}(\xi_1), \quad \xi_1 \in [x_0, x_2]. \quad (5.11)$$

If f is a cubic polynomial, the 4-th derivative vanishes and hence the result. Also the error can be estimated as

$$|E_S| \le \frac{1}{90}\sum_{i=1}^{n/2}\left(\frac{b-a}{n}\right)^5 |f^{(4)}(\xi_i)| \le \frac{(b-a)^5}{180n^4}M$$

□

We use it repeatedly: When n is even,

$$\begin{aligned}\int_{x_0}^{x_n} f(x)\,dx &\approx \frac{h}{3}[y_0+4y_1+y_2]+\frac{h}{3}[y_2+4y_3+y_4]+\cdots+\frac{h}{3}[y_{n-2}+4y_{n-1}+y_n]\\ &= \frac{h}{3}(y_0+4y_1+2y_2+4y_3+\cdots+2y_{n-2}+4y_{n-1}+y_n) \quad (5.12)\end{aligned}$$

Example 5.6.2. Estimate the error when $\int_0^2 5x^4\,dx$ is approx. by Simpson's rule.

$f^{(4)} = 120$. Hence $E_S = \frac{120\cdot 2^5\cdot 1}{1804^4} = \frac{1}{12}$.

5.7 Improper Integral

Improper Integral

Example 5.7.1. Find the area surrounded by $y = 1/\sqrt{x}$, x-axis, y-axis, $x = 1$(fig 5.3).

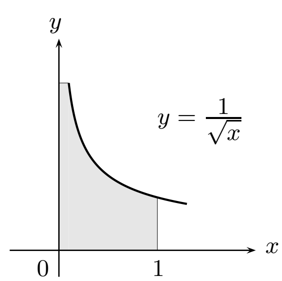


Figure 5.3:

sol. Function $1/\sqrt{x}$ is not defined at $x = 0$. So one cannot write the region as

$$\int_0^1 \frac{dx}{\sqrt{x}}$$

But we can use limit to interpret

$$\begin{aligned}(\text{Area}) &= \lim_{\varepsilon\to 0^+}\int_\varepsilon^1 \frac{dx}{\sqrt{x}}\\ &= \lim_{\varepsilon\to 0^+}\left[2x^{1/2}\right]_\varepsilon^1\\ &= \lim_{\varepsilon\to 0^+}\left(2-2\varepsilon^{1/2}\right)\\ &= 2.\end{aligned}$$

□

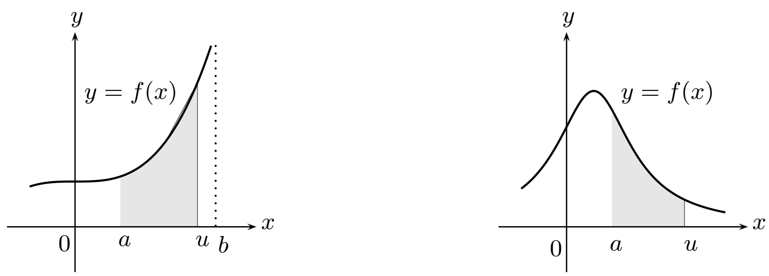


Figure 5.4: Improper integral on $[a, b)$

Definition 5.7.2 (Convergence of Improper integral)**.**

(1) Suppose $f(x)$ is integrable on all closed subinterval of $[a, b)$ and we have $\lim_{x\to b^-} f(x) = \pm\infty$. If the limit

$$\lim_{u\to b^-}\int_a^u f(x)\,dx \tag{5.13}$$

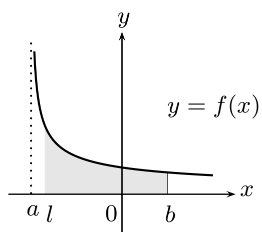
exists then we call the limit the **improper integral**, or we say the improper integral **converges**.

(2) Similarly, suppose $f(x)$ is integrable on all closed subinterval of $(a, b]$ and if $\lim_{x\to a^+} f(x) = \pm\infty$ and the limit

$$\lim_{\ell\to a^+}\int_\ell^b f(x)\,dx \tag{5.14}$$

exists, we say the improper integral **converges**. In either cases, we write

$$\int_a^b f(x)\,dx.$$



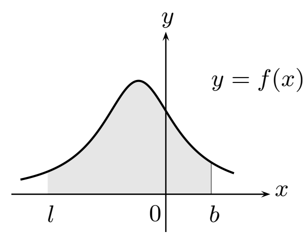


Figure 5.5: Improper integral on $(a, b]$

(3) If the limit does not converge, we say it **diverges**.

The same definition holds if $a = -\infty$ or $b = \infty$.

Example 5.7.3. $\displaystyle\int_{-1}^{1} \frac{1}{\sqrt{1-x^2}}\, dx.$

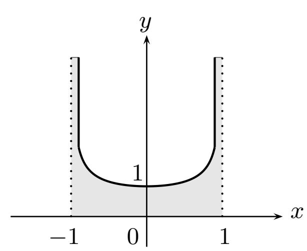


Figure 5.6:

sol. We distinguish two case: $(-1, 0]$ and $[0, 1)$.

$$\begin{aligned}
\int_{-1}^{0} \frac{1}{\sqrt{1-x^2}}\, dx &= \lim_{\ell \to -1^+} \int_{\ell}^{0} \frac{1}{\sqrt{1-x^2}}\, dx \\
&= \lim_{\ell \to -1^+} \left[\sin^{-1} x\right]_{\ell}^{0} \\
&= -\sin^{-1}(-1) \\
&= \frac{\pi}{2} \\
\int_{0}^{1} \frac{1}{\sqrt{1-x^2}}\, dx &= \lim_{u \to 1^-} \int_{0}^{u} \frac{1}{\sqrt{1-x^2}}\, dx \\
&= \lim_{u \to 1^-} \left[\sin^{-1} x\right]_{0}^{u} \\
&= \sin^{-1} 1 \\
&= \frac{\pi}{2}
\end{aligned}$$

Hence

$$\int_{-1}^{1} \frac{1}{\sqrt{1-x^2}}\,dx = \int_{-1}^{0} \frac{1}{\sqrt{1-x^2}}\,dx + \int_{0}^{1} \frac{1}{\sqrt{1-x^2}}\,dx = \pi.$$

□

Example 5.7.4. $\displaystyle\int_0^2 \frac{dx}{(x-1)^{2/3}}$.

sol. The function $1/(x-1)^{2/3}$ is not defined at $x = 1$. Hence we separate

$$\int_0^2 \frac{dx}{(x-1)^{2/3}} = \int_0^1 \frac{dx}{(x-1)^{2/3}} + \int_1^2 \frac{dx}{(x-1)^{2/3}}$$

$$\begin{aligned}\int_0^1 \frac{dx}{(x-1)^{2/3}} &= \lim_{u\to 1^-} \int_0^u \frac{dx}{(x-1)^{2/3}} \\ &= \lim_{u\to 1^-} \Big[3(x-1)^{1/3}\Big]_0^u \\ &= \lim_{u\to 1^-} (u-1)^{1/3} + 3 \\ &= 3\end{aligned}$$

$$\begin{aligned}\int_1^2 \frac{dx}{(x-1)^{2/3}} &= \lim_{u\to 1^-} \Big[3(x-1)^{1/3}\Big]_1^2 \\ &= \lim_{u\to 1^-} 3\end{aligned}$$

□

The function $1/x^p$(Typical example)

The integral of $1/x^p$ on $(0,1]$ depends on the value of p.

On $(0,1]$

Example 5.7.5. Find $\displaystyle\int_0^1 \frac{dx}{x^p}$ ($p > 0$.)

sol.

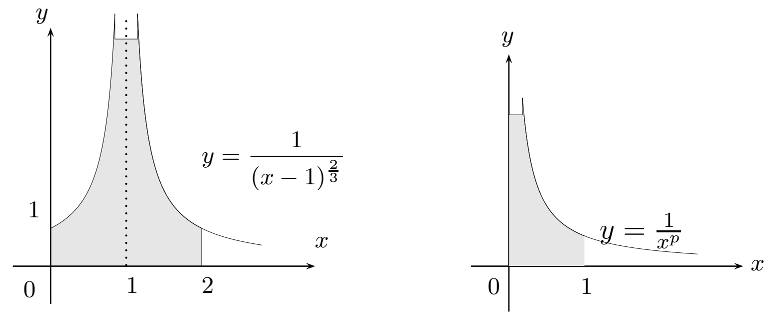


Figure 5.7: $y = \frac{1}{(x-1)^{\frac{2}{3}}}$ on $[0,2]\backslash\{1\}$ and $y = \frac{1}{x^p}$ on $(0,1]$

(1) For $0 < p < 1$

$$\int_0^1 \frac{dx}{x^p} = \lim_{\ell\to 0^+} \int_\ell^1 \frac{dx}{x^p} = \lim_{\ell\to 0^+} \left[\frac{x^{1-p}}{1-p}\right]_\ell^1 = \lim_{\ell\to 0^+} \frac{1-\ell^{1-p}}{1-p} = \frac{1}{1-p}.$$

(2) For $p = 1$

$$\int_0^1 \frac{dx}{x^p} = \lim_{\ell\to 0^+} \int_1^u \frac{dx}{x} = \lim_{\ell\to 0^+} [\ln x]_\ell^1 = \lim_{\ell\to 0^+} (-\ln \ell) = \infty.$$

(3) For $p > 1$

$$\int_0^1 \frac{dx}{x^p} = \lim_{\ell\to 0^+} \int_\ell^1 \frac{dx}{x^p} = \lim_{\ell\to 0^+} \left[\frac{x^{1-p}}{1-p}\right]_\ell^1 = \lim_{\ell\to 0^+} \frac{1-\ell^{1-p}}{1-p} = \infty.$$

□

The same definition holds if $a = -\infty$ or $b = \infty$.

Definition 5.7.6 (Convergence of Improper integral)**.**

(1) Suppose $f(x)$ is integrable on any closed subinterval of $[a,\infty)$. If the limit

$$\lim_{u\to\infty} \int_a^u f(x)\,dx \tag{5.15}$$

exists then we say the improper integral **converges** and write $\int_a^\infty f(x)\,dx$.

(2) Similarly, suppose $f(x)$ is integrable on all closed subinterval of $(-\infty, b]$ and if the limit

$$\lim_{\ell\to-\infty}\int_{\ell}^{b} f(x)\,dx \tag{5.16}$$

exists, we say the improper integral **converges** and write $\int_{-\infty}^{b} f(x)\,dx$.

(3) If the limit does not converge, we say it **diverges**.

On $[1, \infty)$

Example 5.7.7. Find $\displaystyle\int_1^\infty \frac{dx}{x^p}$. ($p > 0$.)

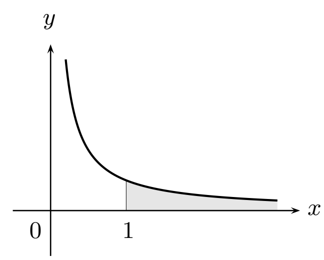


Figure 5.8: Improper integral on $[1, \infty)$

sol.

(1) For $0 < p < 1$,

$$\int_1^\infty \frac{dx}{x^p} = \lim_{u\to\infty}\int_1^u \frac{dx}{x^p} = \lim_{u\to\infty}\left[\frac{x^{1-p}}{1-p}\right]_1^u = \lim_{u\to\infty}\frac{u^{1-p}-1}{1-p} = \infty,$$

(2) For $p = 1$

$$\int_1^\infty \frac{dx}{x^p} = \lim_{u\to\infty}\int_1^u \frac{dx}{x} = \lim_{u\to\infty}[\ln x]_1^u = \lim_{u\to\infty}\ln u = \infty,$$

(3) For $p > 1$

$$\int_1^\infty \frac{dx}{x^p} = \lim_{u\to\infty}\int_1^u \frac{dx}{x^p} = \lim_{u\to\infty}\left[\frac{x^{1-p}}{1-p}\right]_1^u = \lim_{u\to\infty}\frac{u^{1-p}-1}{1-p} = \frac{1}{p-1}.$$

□

Test for Convergence

Sometimes, it is not easy to find exact values for an improper integral. So it is necessary to know whether the integral converges or not.

Theorem 5.7.8 (Direct Comparison test)**.** *Let $0 \le f(x) \le g(x)$ for all $x > a$. Then*

(1) If $\displaystyle\int_a^\infty g(x)\,dx$ converges, then $\displaystyle\int_a^\infty f(x)\,dx$ also converges.

(2) If $\displaystyle\int_a^\infty f(x)\,dx$ diverges, then $\displaystyle\int_a^\infty g(x)\,dx$ also diverges.

Example 5.7.9. Test whether $\displaystyle\int_0^\infty \frac{dx}{1+x^3}$ converges or not.

sol. We see

$$1/(1+x^3) \le 1/x^3, \quad x \ge 1$$

holds. By example 5.7.7 we see $\int_1^\infty 1/x^3\,dx = 1/2$. Hence by Comparison test, $\int_1^\infty 1/(1+x^3)\,dx$ converges. On the other hand, the integral $\int_0^1 1/(1+x^3)\,dx$ is well defined on $[0,1]$. Hence $\int_0^\infty 1/(1+x^3)\,dx$ converges and the value is $\int_0^1 1/(1+x^3)\,dx + \int_1^\infty 1/(1+x^3)\,dx$. See Fig 5.9

□

Example 5.7.10. Test whether $\displaystyle\int_0^\infty \frac{|\sin x| dx}{1+\cos^2 x + x^2}$ converges or not.

Theorem 5.7.11 (Limit Comparison Test)**.** *Assume $f(x)$, $g(x)$ are positive on $[a,\infty)$ and suppose*

$$\lim_{x\to\infty} \frac{f(x)}{g(x)} = L\,(0 < L < \infty).$$

Then the two integra $\displaystyle\int_a^\infty f(x)\,dx$ and $\displaystyle\int_a^\infty g(x)\,dx$ both converge or both diverge.

Proof. (1) Suppose $\int_a^\infty g(x)\,dx$ converges: We know that there is a number N such that $f(x)/g(x) \le L+1$ holds for all $x \ge N$. So we have $0 \le f(x) \le (L+1)g(x)$ for such x and by Limit Comparison Test, $\int_N^\infty f(x)\,dx$ converge. Hence $\int_a^\infty f(x)\,dx$ converges to $\int_a^N f(x)\,dx + \int_N^\infty f(x)\,dx$.

(2) Suppose $\int_a^\infty g(x)\,dx$ diverges: Again there exists a number N s.t. for all $x \ge N$, $f(x)/g(x) \ge L - L/2 = L/2$ holds. Hence $f(x) \ge (L/2)g(x) \ge 0$ and by Limit Comparison Test, $\int_N^\infty f(x)\,dx$ diverges. So does $\int_a^\infty f(x)\,dx$. □

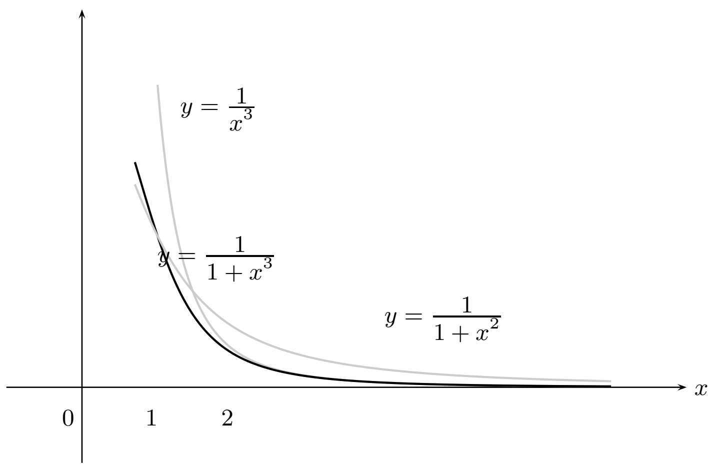


Figure 5.9:

Example 5.7.12. Test whether the following integral converges or not.

(1) $\displaystyle\int_0^\infty \frac{dx}{1+e^x}$

(2) $\displaystyle\int_0^\infty \frac{dx}{1+x+3x^2}$

sol. Let $f(x) = 1/(1+e^x)$, $g(x) = 1/e^x$. Then we see

$$\lim_{x\to\infty} \frac{f(x)}{g(x)} = \lim_{x\to\infty} \frac{e^x}{1+e^x} = 1$$

and

$$\int_0^\infty \frac{dx}{e^x} = \lim_{u\to\infty} \int_0^u \frac{dx}{e^x} = \lim_{u\to\infty} \left[-e^{-x}\right]_0^u = \lim_{u\to\infty} \left(-e^{-u}+1\right) = 1.$$

Hence by Limit Comparison Test, $\int_0^\infty 1/(1+e^x)\,dx$ converges.

□

Example 5.7.13. Test for convergence $\displaystyle\int_2^\infty \sqrt{\frac{x}{x^2-1}}\,dx$.

sol. Set $f(x) = \sqrt{\dfrac{x}{x^2-1}}$, $g(x) = \dfrac{1}{\sqrt{x}}$

Then

$$\lim_{x\to\infty} \frac{f(x)}{g(x)} = \lim_{x\to\infty} \sqrt{\frac{x^2}{x^2-1}} = 1$$

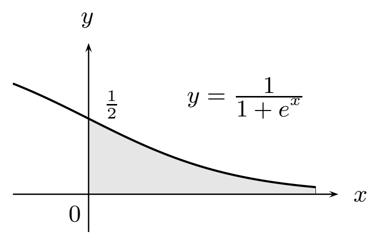


Figure 5.10:

$$\int_2^\infty \frac{dx}{\sqrt{x}} = \lim_{u\to\infty} \left[2\sqrt{x}\right]_2^u = \lim_{u\to\infty} \left(2\sqrt{u} - 2\sqrt{2}\right) = \infty.$$

By Limit Comparison Test, $\displaystyle\int_2^\infty \sqrt{\frac{x}{x^2-1}}\,dx$ diverges.

□

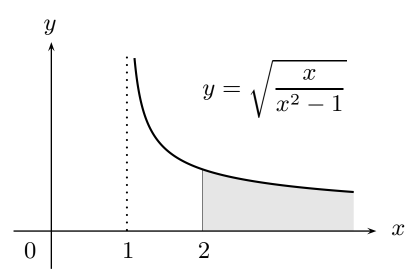


Figure 5.11:

Chapter 6

Application of integrals

6.1 Volume

Volume: Slice method

Definition 6.1.1. The volume of a solid with known cross-sectional area $A(x)$ from a to b is given by

$$V = \int_a^b A(x)\,dx$$

(1) Sketch solid and its cross section.

(2) Find a formula for $A(x)$.

(3) Find limits of integration.

Example 6.1.2 (Vol of Pyramid)**.** Suppose cross section $A(x) = x^2$ $(0 \le x \le 3)$, then

$$V = \int_0^3 x^2 dx = \left.\frac{x^3}{3}\right|_0^3 = 9$$

Cavalieri principle

Solid of revolution: Disk method

The solid generated by rotating a plane region about an axis is called **Solid of revolution**. The cross sectional area is $A(x) = \pi R(x)^2$. When the region under the graph of $y = f(x)(\ x_i \le x \le x_{i+1})$ is rotated about x-axis, the volume is approximately $\pi(f(x_i))^2 \Delta x_i$. Hence the total volume is approximated by

$$\sum_{i=0}^{n-1} \pi(f(x_i))^2(x_{i+1} - x_i)$$

and the volume is

$$\int_a^b \pi f(x)^2\,dx.$$

Note the cross-section is obtained by cutting with a plane perpendicular to the axis of rotation.

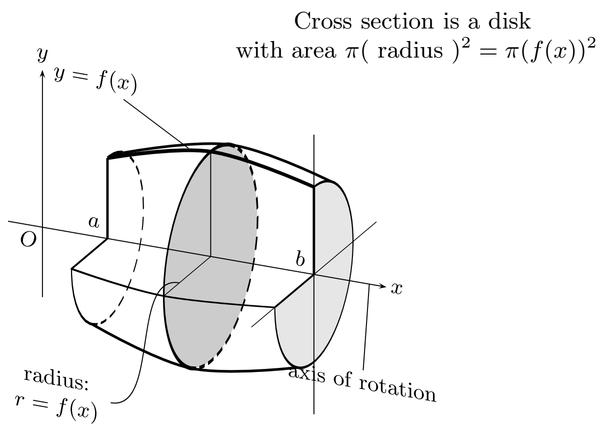


Figure 6.1: Slice method

Example 6.1.3. The region $y = \sqrt{x}$ $(0 \le x \le 4$) is rotated about x-axis.

sol.

$$V = \int_0^4 \pi(\sqrt{x})^2 dx = 8\pi$$

□

Example 6.1.4. The region bounded by $y = \sqrt{x}$, x-axis$(0 \le x \le 4)$ is rotated about $y = 1$.

sol.

$$V = \int_1^4 \pi(\sqrt{x} - 1)^2 dx = 7\pi/6$$

□

Example 6.1.5. Find the volume of solid obtained when the area $\{(x, y) \mid 0 \le x \le \pi, \quad 0 \le y \le \sin x\}$ is rotated about x-axis.

sol. $\pi \int_0^\pi \sin^2 x\,dx = \pi \int_0^\pi \frac{1 - \cos 2x}{2}\,dx = \frac{\pi^2}{2}.$

□

When the graph of $x = g(y)$ on $[c, d]$ is rotated about y-axis the volume is

$$\int_c^d \pi g(y)^2 \, dy$$

Example 6.1.6. The region bounded by $y = 2/x$, y-axis between $1 \le y \le 4$ is rotated about $y = 1$.

sol.

$$\pi \int_1^4 (\frac{2}{y})^2 dy = 3\pi.$$

□

Example 6.1.7. The region between $x = y^2 + 1$, $x = 3$ $1 \le x \le 3$ is revolved about $x = 3$.

sol. We have $r = 3 - (y^2 + 1) = 2 - y^2$. Hence

$$\begin{aligned} V &= \pi \int_{-\sqrt{2}}^{\sqrt{2}} (2 - y^2)^2 dy \\ &= \pi \int_{-\sqrt{2}}^{\sqrt{2}} [4 - 4y^2 + y^4] dy \\ &= \frac{64\pi\sqrt{2}}{15} \end{aligned}$$

□

6.1.1 Washer method

When a region is revolved, it sometimes happens that the cross section has a hole in it. In this case, we obtain a washer of outer radius R and inner radius$r$. In this case, the volume is

$$V = \int_a^b \pi(R(x)^2 - r(x)^2) dx$$

Example 6.1.8. The region between $y = x^2 + 1$ and $y = -x + 3$ is rotated about x-axis.

sol. Find points of intersection. $x^2 + 1 = -x + 3$, so $x = -2, x = 1$. Considering graphs,

$$\begin{aligned} V &= \pi \int_{-2}^{1} [(-x+3)^2 - (x^2+1)^2] dx \\ &= \frac{117\pi}{5} \end{aligned}$$

□

6.2 Volume by cylindrical shells (Shell method)

We introduce still another method of finding the volume of solid of revolution: Let $a \geq 0$ and $f(x) \geq 0$ on $[a, b]$. The volume of solid by rotating the region bounded by $y = f(x)$, $y = 0$, $x = a$, $x = b$ about y-axis can be obtained by .... If we cut the region with lines parallel to the axis of rotation, we get hollow shells whose volume we can approximate easily.

Rough approximation

Consider the part of the shell obtained by rotating the stritp between x_i and x_{i+1}. The side of the shell has area between $2\pi x_i$ and $2\pi x_{i+1}$. Now the height is $f(\bar{x}_i)$ for some $\bar{x}_i \in [x_i, x_{i+1}]$. Hence each shell has volume between $2\pi x_i f(\bar{x}_i)\Delta x_i$ and $2\pi x_{i+1} f(\bar{x}_i)\Delta x_i$. By adding this, we get $\sum 2\pi x_i f(\bar{x}_i)\Delta x_i$ and in the limit the volume becomes

$$\int_a^b 2\pi x f(x)\, dx = \int_a^b 2\pi \text{ (shell radius)} \times \text{ (shell height)}\, dx.$$

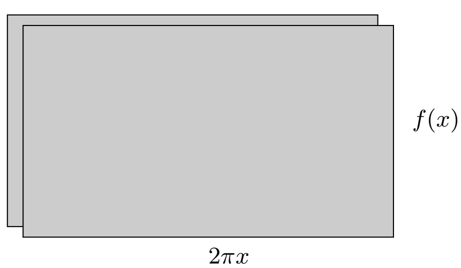


Figure 6.2: Shell method

Rigorous interpretation

Let $P = \{x_0, x_1, \dots, x_n\}$ be a partition of $[a, b]$. The volume of the shell between x_i and x_{i+1} is

$$\Delta V_i = \text{ Area of bottom washer } \times \text{height} = \pi(x_{i+1}^2 - x_i^2) f(x_i)$$

Hence total volume is

$$\sum_{i=0}^{n-1} \pi(x_{i+1} + x_i) f(x_i)(x_{i+1} - x_i) = \sum_{i=0}^{n-1} \pi(x_{i+1} + x_i) f(x_i) \Delta x_i$$

In the limit the volume is

$$\int_a^b 2\pi x f(x)\, dx = \int_a^b 2\pi \text{ (shell radius)} \times \text{ (shell height)}\, dx.$$

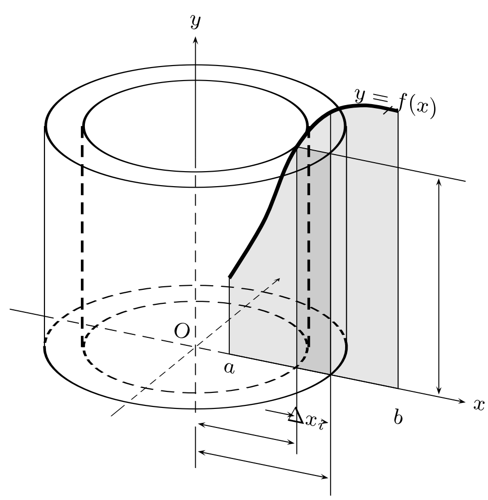


Figure 6.3: Shell method

Example 6.2.1. $\{(x, y) \mid 0 \le y \le \sqrt{x}, \quad 0 \le x \le 4\}$ about y-axis

sol. $$\int_0^4 2\pi x\sqrt{x}\, dx = 2\pi \int_0^4 x^{3/2} dx = 128\pi/5$$

□

Example 6.2.2. $\{(x, y) \mid 0 \le y \le \sqrt{x}, \quad 0 \le x \le 4\}$ is rotated about x-axis

sol. $\int_0^2 2\pi y(4-y^2)\,dy = 8\pi.$

□

Example 6.2.3. Find the volume of solid rotating the region $\{(x,y) \mid 0 \le x \le \pi, \quad 0 \le y \le \sin x\}$ about y-axis.

sol. $\int_0^\pi 2\pi x \sin x\,dx = 2\pi\left[\sin x - x\cos x\right]_0^\pi = 2\pi^2.$

□

Similarly if a region $x = g(y)$ $c \le y \le d$ is rotated about x-axis, then

$$\int_c^d 2\pi g(y)\,dy$$

6.3 Length of plane curves

Length of parametric curves

Find the length of a curve given by $(x(t), y(t))$, $(a \le t \le b)$.

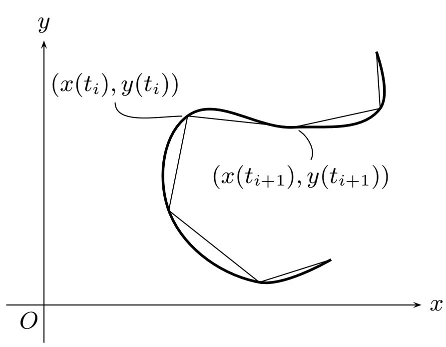


Figure 6.4: Sum of line segments

Let $P = \{t_0, t_1, \ldots, t_n\}$ be a partition of $[a,b]$. The sum of the length of each line segment is

$$\sum_{i=0}^{n-1} \sqrt{(x(t_{i+1}) - x(t_i))^2 + (y(t_{i+1}) - y(t_i))^2},$$

$$\sum_{i=0}^{n-1} \sqrt{\left(\frac{x(t_{i+1}) - x(t_i)}{t_{i+1} - t_i}\right)^2 + \left(\frac{y(t_{i+1}) - y(t_i)}{t_{i+1} - t_i}\right)^2}\,(t_{i+1} - t_i)$$

(fig 6.4). Thus as $\|P\| \to 0$

$$\frac{x(t_{i+1}) - x(t_i)}{t_{i+1} - t_i} \to x'(t_i), \quad \frac{y(t_{i+1}) - y(t_i)}{t_{i+1} - t_i} \to y'(t_i)$$

$$x'(t_i) \to x'(t), \quad y'(t_i) \to y'(t), \quad \Delta t_i = t_{i+1} - t_i \to dt$$

Hence

$$\int_a^b \sqrt{(x'(t))^2 + (y'(t))^2}\, dt$$

Using Leibniz notation

$$\begin{aligned} L &\approx \sum_{i=1}^{n} \sqrt{(\Delta x_i)^2 + (\Delta y_i)^2} \\ &= \sum_{i=1}^{n} \sqrt{\left(\frac{\Delta x_i}{\Delta t_i}\right)^2 + \left(\frac{\Delta y_i}{\Delta t_i}\right)^2}\, \Delta t_i \end{aligned}$$

In the limit,

$$\int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$$

Example 6.3.1. Find the length

$$(x(t), y(t)) = (\cos t, \sin t), \quad 0 \le t \le \pi/2$$

sol. $$\int_0^{\pi/2} \sqrt{(-\sin t)^2 + (\cos t)^2}\, dt = \int_0^{\pi/2} dt = \frac{\pi}{2}.$$

□

Example 6.3.2 (Astroid)**.**

$$x = \cos^3 t, y = \sin^3 t, (0 \le t \le 2\pi)$$

sol. We can easily see that

$$\sqrt{(x')^2 + (y')^2} = 3\cos t \sin t.$$

Hence the length(of the first quadrant) is

$$\int_0^{\pi/2} 3\cos t \sin t\, dt = \frac{3}{2}\int_0^{\pi/2} \sin 2t\, dt = \frac{3}{2}$$

□

Example 6.3.3. Find the length of the curve $y = \ln(\cos x), \quad 0 \le x \le \pi/3$.

sol. Since

$$1 + (\frac{dy}{dx})^2 = 1 + \tan^2 x$$

we have

$$\begin{aligned} L &= \int_0^{\pi/3} \sqrt{1 + \tan^2 x} dx \\ &= \int_0^{\pi/3} |\sec^2 x| dx = [\ln|\sec x + \tan x|]_0^{\pi/2} \\ &= \ln(2 + \sqrt{3}) - \ln(1 + 0). \end{aligned}$$

□

Length of a curve $y = f(x)$

The length of a curve $y = f(x)$ for $(a \le x \le b)$ is given by

$$\int_a^b \sqrt{1 + (f'(x))^2}\, dx$$

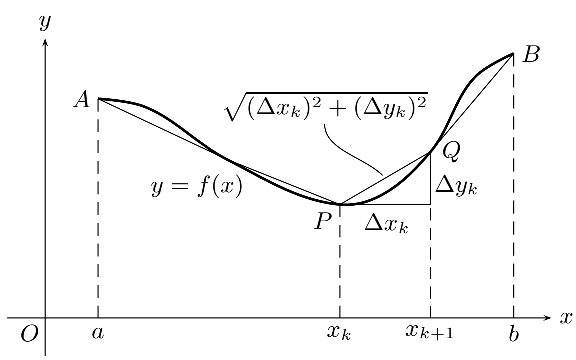


Figure 6.5: $PQ = \sqrt{(\Delta x_k)^2 + (\Delta y_k)^2}$, $\Delta y_k = f(x_{k+1}) - f(x_k)$

Example 6.3.4. Find the length of the curve

$$y = \frac{1}{2}(e^x + e^{-x}), \quad 0 \le x \le 2.$$

sol. Since $y' = \frac{1}{2}(e^x - e^{-x})$

$$\int_0^2 \sqrt{1 + \frac{1}{4}(e^x - e^{-x})^2}\, dx = \frac{1}{2}(e^2 + e^{-2}) - 1.$$

The case with $x = g(y)$

When the curve is given by $x = g(y)$, $c \le y \le d$, the length is

$$L = \int_c^d \sqrt{1 + \left(\frac{dx}{dy}\right)^2}\, dy$$

□

Differential formula

Since $\Delta L = \sqrt{(\Delta x_i)^2 + (\Delta y_i)^2}$ is approximate length, it is natural to introduce a parameter $ds = \sqrt{dx^2 + dy^2}$(called **arclength parameter**). Then we have

$$L(s) = \int_0^s ds = \int \sqrt{dx^2 + dy^2}.$$

This in turn ca be written as

$$= \int \sqrt{1 + \left(\frac{dy}{dx}\right)^2} dx \text{ or } \int \sqrt{1 + \left(\frac{dx}{dy}\right)^2} dy.$$

Arclength parameter has the property that

$$\frac{dL}{ds} = 1$$

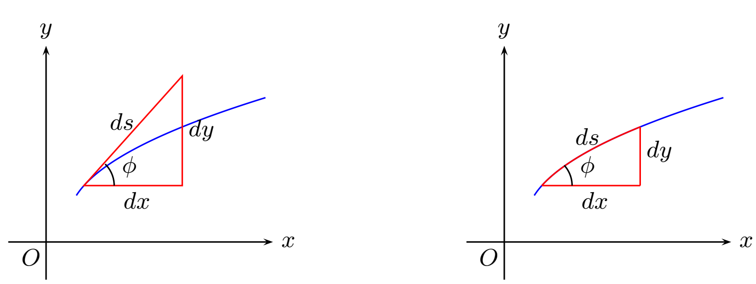


Figure 6.6: $ds = \sqrt{(dx)^2 + (dy)^2}$ Fig. on the right is rough

6.4 Exponential Change

Exponential function as inverse of log function

Separable equation

Suppose we have a differential equation of the form

$$\frac{dy}{dx} = \frac{g(x)}{h(y)}.$$

Then we see

$$h(y)\frac{dy}{dx} = g(x)$$

Hence

$$\int h(y)\frac{dy}{dx}\,dx = \int g(x)\,dx \Rightarrow \int h(y)dy = \int g(x)dx$$

These are called **separable equations**.

Exponential Growth and decay

The decay of radioactive material or money earning interests in bank account, temperature between a cup of hot water and room air it sits, etc follows the law of *exponential change*.

Suppose $y(t)$ denotes some quantity which changes according to the exponential law: *The rate of change of* y *is proportional to* y.

$$\frac{dy}{dt} = ky$$

with the I.C. $y(0) = A$. Then the solution is $y = Ae^{kt}$.

Example 6.4.1. Solve the D.E.

$$\frac{dy}{dx} = \frac{(1+y^2)}{\sqrt{1-x^2}}.$$

sol.

$$\begin{aligned}
dy &= \frac{(1+y^2)}{\sqrt{1-x^2}}dx \\
\frac{dy}{1+y^2} &= \frac{dx}{\sqrt{1-x^2}} \\
\int \frac{dy}{1+y^2} &= \int \frac{dx}{\sqrt{1-x^2}} \\
\tan^{-1} y &= \sin^{-1} x + C \\
y &= \tan(\sin^{-1} x + C)
\end{aligned}$$

Check with I.C.

□

Population Growth

Example 6.4.2. Assume a disease is spreading "Entero virus", "A.I" Let y be the number of people infected by some epidemic. Assume we start to cure people as soon as we find out what it is. Assume dy/dt is proportional to y.(The more people infected, the more cured) Suppose for each year the number is reduced by 20% and 10,000 people infected today, how many years will it take to reduce the number to $1,000$?

sol. We use $y = Ae^{kt}$, $A = 10,000$. Since it is reduced by 20% each year, we see

$$0.8A = Ae^{k\cdot 1} \Rightarrow k = \ln 0.8 < 0$$

So we have $y = 10,000e^{(\ln 0.8)t}$. Since we want $10,000e^{(\ln 0.8)t} = 1,000$ we have $e^{(\ln 0.8)t} = \frac{1}{10}$. $\ln(0.8)t = \ln(0.1)$. $t = \frac{\ln(0.1)}{\ln(0.8)} \approx 10.32$ yrs.

□

Radioactive Decay-Half Life

Example 6.4.3 (**Half life** of a radioactive material)**.** The decay of radioactive material follows exponential law. Hence $y = y_0 e^{-kt}$. To see half life, we set $y_0 e^{-kt_H} = \frac{1}{2} y_0$. so

$$t_H = \frac{\ln 2}{k}.$$

Example 6.4.4 (Carbon 14)**.** It is estimated the half life of Carbon 14 is 5,700 yrs. A wooden artifact was found from an ancient site. This contains carbon 14 about 10% less than the living tree. How old is the site?

$$k = \frac{\ln 2}{\text{Half life}} = \frac{\ln 2}{5700}.$$

From $y = y_0 e^{-kt} = 0.9y_0$, we see $e^{-kt} = 0.9$. Hence

$$t = -5700\frac{\ln 0.9}{\ln 2} = 866 \text{ yrs.}$$

Heat Transfer-Newton law of Cooling

If H is the temperature of an object and H_s the surrounding temperature. Then the rate of change(cooling) is proportional to the temperature difference. Thus the relation is

$$\frac{dH(t)}{dt} = -k(H(t) - H_s).$$

Let $y = H(t) - H_s$ and solving we get

$$H(t) - H_s = (H_0 - H_s)e^{-kt}.$$

Example 6.4.5 (Law of Cooling)**.** A boiled egg at 98^o is put in the sink of 18^o to cool down. In 5 min, the egg was 38^o. How much longer will it take to reach 20^o?

sol. The environment $H_s = 18$ and initial temperature is $H_0 = 98$. Hence

$$H(t) - 18 = (98 - 18)e^{-kt}.$$

Set $H(5) = 38$. Then $20 = 80e^{-5k}$,

$$k = -\frac{\ln 1/4}{5} = 0.2 \ln 4 \approx 0.28.$$

Hence

$$H = 18 + 80e^{-(0.2 \ln 4)t}.$$

If $H(t) = 20$, we get $t \approx 13$ min.

□

6.5 Work and Fluid forces

Work done by a constant force

When a body moves a distance d along a straight line by a constant force F in the direction of the motion, we define the **work** W done by the force on the body with the formula

$$W = Fd \tag{6.1}$$

Work done by a variable force

$$W \approx \sum_{k=1}^{n} F(c_k)\Delta x_k$$

In the limit

$$W = \int_a^b F(x)\, dx \tag{6.2}$$

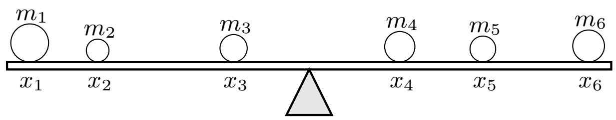


Figure 6.7: Balanced Seasaw

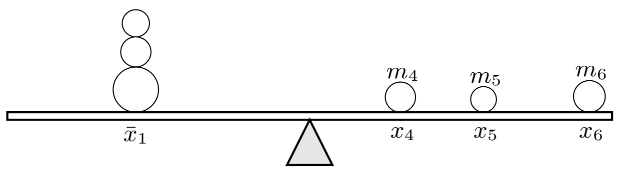


Figure 6.8: Mass at a single point has same effect of turning-Torque

6.6 Moment and Center of mass

Moment

Suppose masses m_1, m_2, m_3 are placed on a fulcrum(Jiret-Tae) at the origin. Each mass exerts a downward force $m_k g$. This force has *tendency of turning the axis* about the origin. This effect is **torque**. It is measured by multiplying the force $m_k g$ by the signed distance x_k from the point of application to the origin. The sum is called a **system torque.**

$$\text{System torque } = m_1 g x_1 + m_2 g x_2 + m_3 g x_3 = \underbrace{g}_{\text{feature of env}} \underbrace{(m_1 x_1 + m_2 x_2 + m_3 x_3)}_{\text{feature of the system}}$$

The number $(m_1 x_1 + m_2 x_2 + m_3 x_3)$ is called the **moment of the system about the origin.**

$$\text{Moment of the system about the origin } = \sum m_k x_k$$

We want to know where to place the place the fulcrum to balance the system. That is, at what point $\bar{x}$ does the system torque add to zero.

The torque of each mass about the fulcrum in this special position is

$$\begin{aligned} \text{Torque of } m_k \text{ about } \bar{x} &= \begin{pmatrix} \text{signed dist} \\ \text{of } m_k \text{ from } \bar{x} \end{pmatrix} \begin{pmatrix} \text{downward} \\ \text{force} \end{pmatrix} \\ &= (x_k - \bar{x}) m_k g \end{aligned}$$

We want the system torque about $\bar{x}$ is zero to balance the fulcrum:

$$\text{The system torque about } \bar{x} = g \sum (x_k - \bar{x}) m_k = 0$$

Hence

$$\bar{x} = \frac{\sum m_k x_k}{\sum m_k} = \frac{\text{system moment}}{\text{system mass}}$$

This point $\bar{x}$ is called the **center of mass**.

Masses over a plane region

Suppose a finite collection of masses m_k are located at (x_k, y_k). Then system mass M, moments and the center of mass $(\bar{x}, \bar{y})$ are given by

$$\begin{aligned}
\text{System Mass: } & M = \sum m_k \\
\text{Moments about } x\text{-axis: } & M_x = \sum m_k y_k \\
\text{Moments about } y\text{-axis: } & M_y = \sum m_k x_k \\
\text{Center of Mass: } & \bar{x} = \frac{M_y}{M} = \frac{\sum m_k x_k}{\sum m_k} \\
& \bar{y} = \frac{M_x}{M} = \frac{\sum m_k y_k}{\sum m_k}
\end{aligned}$$

Thin flat plates

Suppose we need to find the center of mass of a thin plate. In this case, we assume a continuous distribution of mass. Imagine a plate is occupying a region which we slice by strips parallel to one of the coordinate axis, say y-axis. Assume D is described as follows:

$$D = \{(x, y) : a \le x \le b,\ g(x) \le y \le h(x)\}.$$

Let $f(x) = h(x) - g(x)$. We approximate the mass and moment with the finite collection of masses distributed over the region and then take the limit.

Let $(\tilde{x}, \tilde{y})$ be the center of mass of the *vertical strip* of mass Δm.

Then M, M_x and M_y are approximated by

$$M = \sum \Delta m_k,\ M_x = \sum \Delta m_k \tilde{y}_k,\ M_y = \sum \Delta m_k \tilde{x}_k$$

and taking the limit, we see

$$\begin{aligned}
\text{System Mass: } & M = \int_a^b dm \\
\text{Moments about } x\text{-axis: } & M_x = \int \tilde{y}\, dm \\
\text{Moments about } y\text{-axis: } & M_y = \int \tilde{x}\, dm
\end{aligned}$$

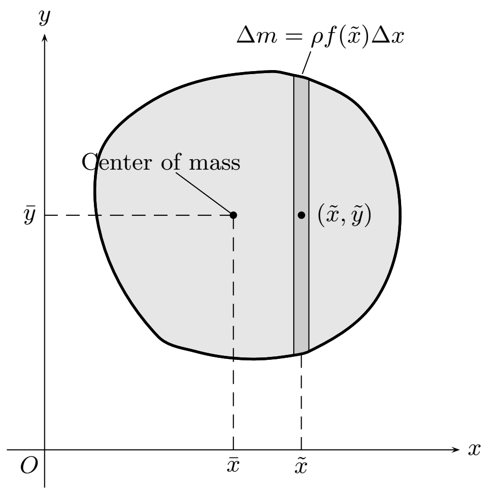


Figure 6.9: Vertical strip of mass Δm

Finally the center of Mass is given by

$$\bar{x} = \frac{M_y}{M}, \quad \bar{y} = \frac{M_x}{M}.$$

In particular, if the strip has uniform mass distribution, then $(\tilde{x}, \tilde{y})$ is the mid point of each strip. We can also use the horizontal strip to obtain the same result. The center of mass when $\delta = 1$ is called **Centroid**.

Example 6.6.1. Find the centroid of the region D surrounded by $y = 0$, $x = 1$ and $y = x^2$.

sol. [Method 1. along vertical strip] For each strip, we have

$$\begin{aligned}
\text{center of mass } &: (\tilde{x}, \tilde{y}) = (x, \frac{x^2}{2}) \\
\text{length } &: x^2 \\
\text{width } &: dx \\
\text{Area } &: x^2\, dx \\
\text{mass } &: dm = \delta dA = x^2\, dx \\
\text{dist. from } y\text{-axis } &: \tilde{x} = x
\end{aligned}$$

The moment of the strip about the y-axis is

$$\tilde{x}\, dm = x^3 dx$$

$$M_y = \int \tilde{x}\, dm = \int_0^1 x^3\, dx = \frac{x^4}{4}\Big|_0^1 = \frac{1}{4}$$

Since the mass is

$$M = \int_0^1 x^2\, dx = \frac{1}{3}$$

we see

$$\bar{x} = \frac{M_y}{M} = \frac{\int_0^1 x^3\, dx}{M} = \frac{3}{4}.$$

□

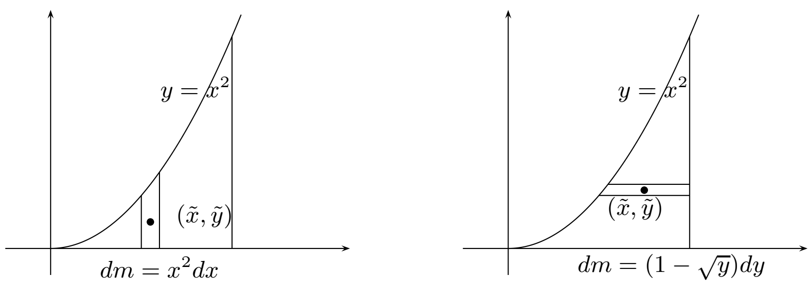


Figure 6.10: Vertical and horizontal strip of mass Δm

To find $\bar{y}$, we see the center of mass, length, width, area and mass are the same as before. The only difference is the distance from the axis(from the x-axis).

$$\text{dist. from } x\text{-axis } : \tilde{y} = \frac{x^2}{2}$$

The moment of the strip about the y-axis is

$$\tilde{y}\, dm = \frac{x^4}{2} dx$$

$$M_x = \int_0^1 \frac{x^4}{2}\, dx = \left.\frac{x^5}{10}\right|_0^1 = \frac{1}{10}$$

Hence

$$\bar{y} = \frac{M_x}{M} = \frac{3}{10}.$$

[Method 2. along horizontal strip]

$$\begin{aligned}
&\text{center of mass } : (\tilde{x}, \tilde{y}) = (\frac{\sqrt{y}+1}{2}, y) \\
&\text{length } : 1 - \sqrt{y} \\
&\text{width } : dy \\
&\text{Area } : (1 - \sqrt{y})\, dy \\
&\text{mass } : dm = \delta dA = (1 - \sqrt{y})\, dy \\
&\text{ave. dist. from } y\text{-axis } : \tilde{x} = \frac{\sqrt{y}+1}{2} \\
&\text{dist. from } x\text{-axis } : \tilde{y} = y
\end{aligned}$$

$$\begin{aligned}
M_x &= \int \tilde{y}\, dm = \int_0^1 y(1 - \sqrt{y})\, dy = \frac{1}{10}, \quad \bar{y} = \frac{M_x}{M} = \frac{3}{10} \\
M_y &= \int \tilde{x}\, dm = \int_0^1 \frac{1-y}{2} dy = \frac{1}{4}, \quad \bar{x} = \frac{M_y}{M} = \frac{3}{4}.
\end{aligned}$$

Continuous Mass-Rods or Wires

A thin rod is placed along x-axis on $[a, b]$. Suppose the density is given by $\delta(x)$. We consider the following:

$$\text{System torque } = \sum x_k \Delta m_k.$$

Here $\Delta m_k \approx \delta(x_k)\Delta x_k$. Hence

$$\bar{x} \approx \frac{\sum x_k \Delta m_k}{\sum \Delta m_k} = \frac{\sum x_k \delta(x_k)\Delta x_k}{\sum \delta(x_k)\Delta x_k}.$$

In the limit, the total mass and the center of mass are

$$M = \int_a^b \delta(x)\, dx, \quad Mom. = \int_a^b x\, \delta(x)\, dx, \quad \bar{x} = \frac{\int_a^b x\, \delta(x)\, dx}{\int_a^b \delta(x)\, dx}.$$

Similarly, we can compute mass, moments and the center of mass a wire lying over a curve:

Example 6.6.2 (Constant density wire)**.** Suppose a wire with constant density δ is lying over the following quarter circle.

$$(x(t), y(t)) = (a\cos t, a\sin t), \quad 0 \le t \le \pi/2.$$

Find the center of mass.

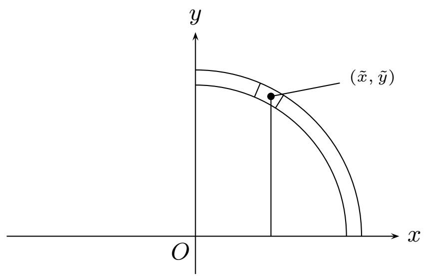


Figure 6.11: Center of mass of a wire

sol. If density is 1, the length is same as the mass. Hence the typical segment has

$$\begin{aligned} \text{length } ds &= a d\theta \\ \text{mass } dm &= ds = a d\theta \\ \text{dist. to } x\text{-axis } \tilde{y} &= a \sin\theta \end{aligned}$$

The total mass is

$$M = a \int_0^{\pi/2} dt = \frac{\pi a}{2}$$

The center of mass of a fraction Δm is $(\tilde{x}, \tilde{y}) = (a \cos t, a \sin t)$. Hence

$$M_y = \int_0^{\pi/2} \tilde{x} dm = \int_0^{\pi/2} a^2 \sin t\, dt, \quad M_x = \int_0^{\pi/2} \tilde{y} dm = \int_0^{\pi/2} a^2 \sin t\, dt$$

Thus

$$\begin{aligned} \bar{x} &= \frac{M_y}{M} = \frac{a^2 \int_0^{\pi/2} \cos t\, dt}{a\pi/2} = \frac{2a}{\pi}, \\ \bar{y} &= \frac{M_x}{M} = \frac{a^2 \int_0^{\pi/2} \sin t\, dt}{a\pi/2} = \frac{2a}{\pi}. \end{aligned}$$

□

6.7 Surface area of revolution and Pappus theorem

The area of surface obtained when we rotate the graph of $y = f(x)$ about x-axis. Let $P = \{x_0, x_1, \dots, x_n\}$ be a partition of $[a, b]$. The average height is $\frac{f(x_i)+f(x_{i-1})}{2}$, slant length is

$$\sqrt{\Delta x_i^2 + \Delta y_i^2}$$

So

$$2\pi\left(\frac{f(x_i)+f(x_{i-1})}{2}\right)\sqrt{\Delta x_i^2+\Delta y_i^2},$$

$$\pi\left(f(x_i)+f(x_{i-1})\right)\sqrt{1+\left(\frac{f(x_i)-f(x_{i-1})}{x_i-x_{i-1}}\right)^2}\,\Delta x_i$$

Hence

$$\int_a^b 2\pi f(x)\sqrt{1+(f'(x))^2}\,dx \tag{6.3}$$

On the other hand, if the $x=g(y)$ is rotated about y-axis, the surface area is

$$\int_a^b 2\pi g(y)\sqrt{1+(g'(y))^2}\,dy \tag{6.4}$$

Example 6.7.1. The line segment $x=1,\ 0\le y\le 1$ is revolved about y-axis. Find the area.

sol. $x=1-y$ So $dx/dy=-1$. $\sqrt{1+(g'(y))^2}=\sqrt{2}$ and the area is

$$S=2\pi\int_0^1\sqrt{2}(1-y)\,dy=\pi\sqrt{2}$$

□

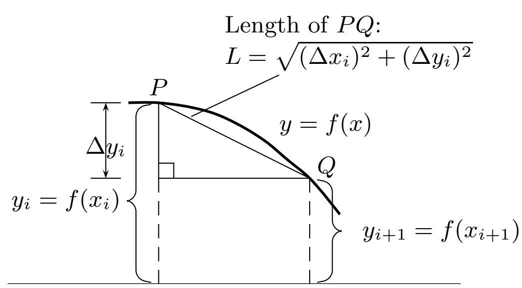


Figure 6.12: Surface area of revolution

Example 6.7.2. Surface area of revolution

$$y=\cos x,\quad 0\le x\le\frac{\pi}{2}$$

is revolved about x-axis

sol. By (6.3),

$$\int_0^{\pi/2}2\pi\cos x\sqrt{1+(-\sin x)^2}\,dx=2\pi\frac{2}{3}\left[(1+\sin^2 x)^{3/2}\right]_0^{\pi/2}=\frac{4\pi}{3}(2^{3/2}-1).$$

□

6.7.1 Parameterized curve

The length differential of a parameterized curve $(f(t), g(t))$ is

$$\sqrt{[f'(x)]^2 + [g'(x)]^2}\,dx = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} \tag{6.5}$$

Hence surface area of revolution is

$$\text{about } x\text{-axis, } S = \int_a^b 2\pi y \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\,dt \tag{6.6}$$

$$\text{about } y\text{-axis, } S = \int_a^b 2\pi x \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\,dt \tag{6.7}$$

Example 6.7.3. The circle of radius 1 at $(0, 1)$ is

$$x = \cos t, \quad y = 1 + \sin t, \quad (0 \le t \le 2\pi)$$

Find the area of doughnut shaped surface obtained when it is rotated it about x-axis.

$$\begin{aligned} S &= \int_0^{2\pi} 2\pi y \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\,dt \\ &= \int_0^{2\pi} 2\pi(1 + \sin t)dt = 4\pi^2 \end{aligned}$$

Differential form

$$S = \int_a^b 2\pi y \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\,dx \text{ and } \int_a^b 2\pi x \sqrt{1 + \left(\frac{dx}{dy}\right)^2}\,dy$$

are often written as

$$S = \int_a^b 2\pi y\,ds \text{ and } \int_a^b 2\pi x\,ds$$

or

$$S = \int_a^b 2\pi \text{ (radius) } \times \text{ (band width) } = \int_a^b 2\pi\rho\,ds$$

Example 6.7.4. Find area of surface revolving $y = x^2$, $0 \le x \le 1/2$.

sol.

$$y' = 3x^2, \quad ds = \sqrt{1+9x^4}dx$$

Hence

$$\begin{aligned} S &= \int_0^{1/2} 2\pi y\, ds = \int_0^{1/2} 2\pi y\sqrt{1+9x^4}\, dx \\ &= \int_0^{1/2} 2\pi x^2\sqrt{1+9x^4}\, dx \\ &= \frac{61\pi}{1728}. \end{aligned}$$

□

6.8 Relative Rate of Growth

Definition 6.8.1. Suppose $f(x), g(x)$ are positive for sufficiently large x.

(1) f grows faster than g as $x \to \infty$ if

$$\lim_{x\to\infty} \frac{f(x)}{g(x)} = \infty$$

(2) f grows at the same rate as g as $x \to \infty$ if

$$\lim_{x\to\infty} \frac{f(x)}{g(x)} = L, \text{ for some postive finite number } L$$

Example 6.8.2. (1) e^x grows faster than x^3 as $x \to \infty$

(2) 3^x grows faster than 2^x as $x \to \infty$

(3) x grows faster than $\ln x$ as $x \to \infty$

Definition 6.8.3. (1) $f(x)$ grows at a smaller order than $g(x)$ as $x \to \infty$ if

$$\lim_{x\to\infty} \frac{f(x)}{g(x)} = 0.$$

In this case we write $f = o(g)$.

(2) f grows at the same order as g as $x \to \infty$ if

$$\lim_{x\to\infty} \frac{f(x)}{g(x)} = L, \text{ for some postive finite number } L.$$

In this case we write $f = O(g)$.

Example 6.8.4. (1) $\ln x = o(x)$ as $x \to \infty$

(2) $x^2 = 0(x^3)$ as $x \to \infty$

(3) $x + \sin x = O(x)$

(4) $x = O(e^x)$

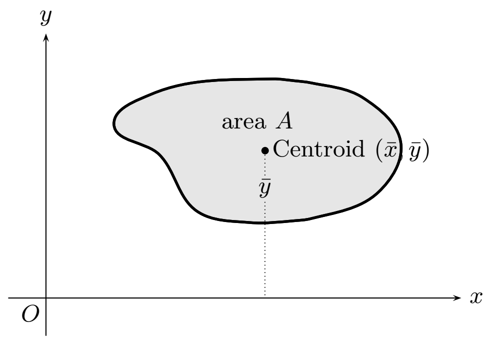


Figure 6.13: Volume of solid of revolution about x-axis is $2\pi\bar{y}A$.

Pappus's theorem

Theorem 6.8.5. *If a plane region is revolved about a line that does not cut through the region's interior, then the volume of the solid is*

$$2\pi\rho, A$$

where A is area of the region and ρ is the distance from the centroid to the line L.

Proof. The idea is to express the volume by **shell method**. Assume the region D is between $y = c$ and $y = d(0 \leq c \leq d)$. For simplicity, assume x-axis is the axis of rotation and we slice ti by lines parallel to x-axis. Then the y-coord of the centroid is $\bar{y} = \rho$ and $f(y)$ is the length of the line segment intersecting with D. Then

$$\rho = \frac{\int_c^d y f(y)\, dy}{A}.$$

While the volume using the *shell method* is

$$V = \int_c^d 2\pi y f(y)\, dy = 2\pi \int_c^d y f(y)\, dy = 2\pi\rho A$$

□

Example 6.8.6. The area of volume obtained rotating the circle $\{(x, y) \mid (x - b)^2 + y^2 = a^2\}$ about y-axis $(0 < a < b)$

sol. $A = \pi a^2$, $\rho = b$. So volume is $(2\pi b) \times (\pi a^2) = 2\pi^2 ba^2$.

□

Example 6.8.7 (Centroid of circular region)**.** Find y-coord of centroid of the region of semi-circle $y = \sqrt{a^2 - x^2}$.

sol. $\bar{y} = \frac{V}{2\pi A} = \frac{4}{3\pi}a$.

□

Theorem 6.8.8 (Pappus' theorem for surface area)**.** *If an arc of a smooth curve is revolved about a line not cutting through arc, then area of revolution generated by the arc is*

$$\textit{arc is length } \times \textit{ the distance traveled by the centroid} = 2\pi\rho L.$$

Here ρ is the distance from centroid to the axis.

Proof. Assume the x-axis is the axis of rotation. Then surface area is

$$S = 2\int 2\pi y\, ds$$

But the y-coord of centroid of arc is

$$\bar{y} = \frac{\int y\, ds}{L}$$

Hence

$$\int y\, ds = \bar{y}L$$

So

$$S = 2\pi\bar{y}L = 2\pi\rho L$$

□

Example 6.8.9. The area of surface obtained rotating the circle $\{(x, y) \mid (x-b)^2 + y^2 = a^2\}$ about y-axis $(0 < a < b)$

sol. Length is $2\pi a$ centroid is $(b, 0)$. Distance from centroid to axis is b. So the surface area is $(2\pi b) \times (2\pi a) = 4ab\pi^2$.

□