Contents

Chapter 1

Functions and Limits

1.4 Limit of Functions

Limit of function values

Example 1.4.1. ff

Theorem 1.4.2. *If $f(x)$, $g(x)$ has limit at $x = a$ and values are L, M Then the following hold.*

(1) $\lim_{x\to a}(f(x)+g(x)) = L + M$ *(sum rule)*

(2) $\lim_{x\to a}(f(x)-g(x)) = L - M$ *(Difference rule)*

(3) $\lim_{x\to a}(kf(x)) = kL$ *(Constant multiple)*

(4) $\lim_{x\to a} f(x)\cdot g(x) = LM$ *(Product rule)*

(5) $\lim_{x\to a}\dfrac{f(x)}{g(x)} = \dfrac{L}{M},\ M \neq 0$ *(Quotient rule).*

Poly and rational function

Theorem 1.4.3. *For any polynomial $f(x) = a_0x^n + a_1x^{n-1} + \cdots + a_n$,*

$$\lim_{x\to a} f(x) = a_0a^n + a_1a^{n-1} + \cdots + a_n = f(a).$$

Example 1.4.4. Use Theorem 1.4.2 (3), (4) we see

$$\lim_{x\to a} 2x^2 = 2\lim_{x\to a} x^2 = 2\lim_{x\to a} x \lim_{x\to a} x = 2a\cdot a = 2a^2$$

Theorem 1.4.5. *If $f(x)$, $g(x)$ are polynomials and $g(a) \neq 0$ then*

$$\lim_{x\to a}\frac{f(x)}{g(x)} = \frac{f(a)}{g(a)}.$$

Example 1.4.6. Find

$$\lim_{x\to 2} \frac{x^3+x+2}{x+1}.$$

sol.

$$\lim_{x\to 2} \frac{x^3+x+2}{x+1} = \frac{2^3+2+2}{2+1} = 4$$

□

Eliminating zero denominator

Example 1.4.7.

$$\lim_{x\to 1} \frac{x^3-1}{x^2-1}.$$

sol. Now denominator is zero. But as long as $x \neq 1$ it holds that

$$\frac{x^3-1}{x^2-1} = \frac{(x-1)(x^2+x+1)}{(x-1)(x+1)} = \frac{x^2+x+1}{x+1}$$

$$\lim_{x\to 1} \frac{x^3-1}{x^2-1} = \lim_{x\to 1} \frac{x^2+x+1}{x+1} = \frac{3}{2}. \qquad \square$$

□

Example 1.4.8. $\lim_{x\to 0} \frac{\sqrt{x^2+1}-1}{x^2}$

sol.

$$\lim_{x\to 0} \frac{\sqrt{x^2+1}-1}{x^2} = \frac{x^2+1-1}{x^2(\sqrt{x^2+1}+1)} = \frac{1}{\sqrt{x^2+1}+1}$$

□

Example 1.4.9.

$$\lim_{x\to\infty} \frac{x^3+x}{2x^3+3} = \frac{1}{2}.$$

sol. Factor out x^3

$$\frac{x^3+x}{2x^3+3} = \frac{1+1/x^2}{2+3/x^3}$$

As $x \to \infty$ Theorem 1.4.2 (5) limit is 1/2.

□

Sandwich Theorem

Theorem 1.4.10 (Sandwich Theorem)**.** *If, for an interval* (α, β) *containing* a*, it holds* $g(x) \leq f(x) \leq h(x)$

$$\lim_{x \to a} g(x) = \lim_{x \to a} h(x) = L$$

then

$$\lim_{x \to a} f(x) = L$$

This hold for left or right limit also.

Example 1.4.11. Suppose that

$$v(x) - x^2 \leq u(x) \leq v(x) + 3x^2$$

holds for all $x \neq 0$. Then

$$\lim_{x \to 0} u(x) = \lim_{x \to 0} v(x)$$

provided the limit of the r.h.s term exists.

Example 1.4.12. We will later see that

$$-|\theta| \leq \sin \theta \leq |\theta|$$

for all θ. Hence $\lim_{x \to 0} \sin \theta = 0$ by Sandwich theorem.

Example 1.4.13. Similarly, from the inequality

$$0 \leq 1 - \cos \theta \leq |\theta|$$

for all θ. Hence $\lim_{x \to 0} \cos \theta = 1$.

Theorem 1.4.14 (Sandwich Theorem, inequality)**.** *If* $f(x) \leq g(x)$ *holds for all* x *in an interval* (α, β) *containing* c *except possibly* c*, then*

$$\lim_{x \to c} f(x) \leq \lim_{x \to c} g(x)$$

This hold for left- right limit also.

When limit do not exist ?

Three cases: Jump, infinity, oscillation:

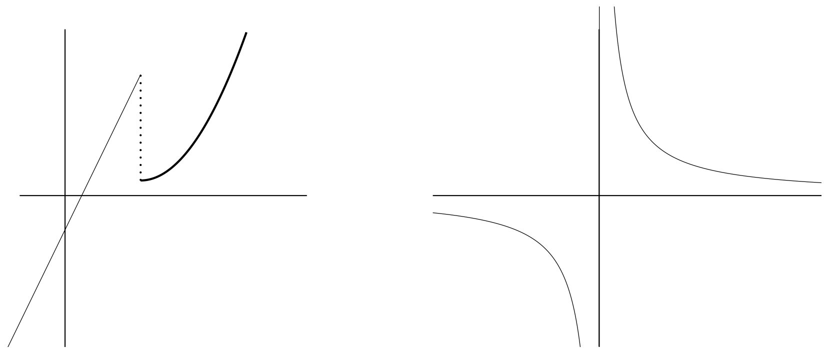


Figure 1.1: Broken graph or undefined(infinity)

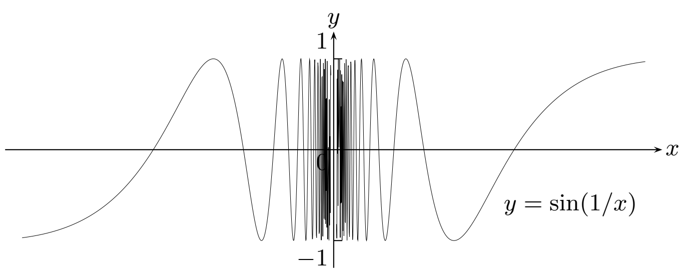


Figure 1.2: $y = \sin \dfrac{1}{x}$

1.5 Precise definition

Definition 1.5.1. Let $f(x)$ be defined on an open interval containing a except possibly a. If there is a number L such that for any positive ε, there exists $\delta > 0$ such that

$$0 < |x - a| < \delta \Rightarrow |f(x) - L| < \varepsilon$$

we say $f(x)$ has **limit** L at $x = a$ and write

$$\lim_{x \to a} f(x) = L$$

Remark 1.5.2. In general, δ depends on x, f and ε. Also, note that we do not care what happens at $x = a$. (This why we have strict inequality in $0 < |x - a|$.)

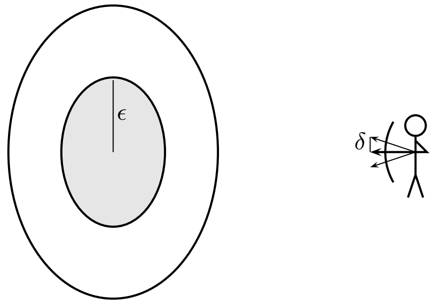


Figure 1.3: How to control bow to hit bull's eye within ϵ tolerance?

Remark 1.5.3. L is unique

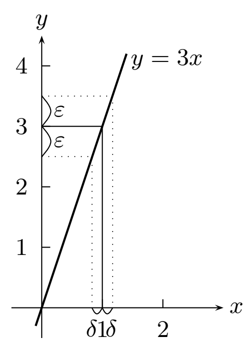


Figure 1.4: Linear case, $y = 3x$

Example 1.5.4 (Linear case)**.** Show $y = 3x + 1$ has limit 7 at $x = 2$.

sol. Suppose the following holds.

$$|y - 7| = |3x + 1 - 7| < \varepsilon$$

Then we take

$$3|x - 2| < \varepsilon, \ |x - 2| < \varepsilon/3$$

So $\delta = \dfrac{\varepsilon}{3}$.

□

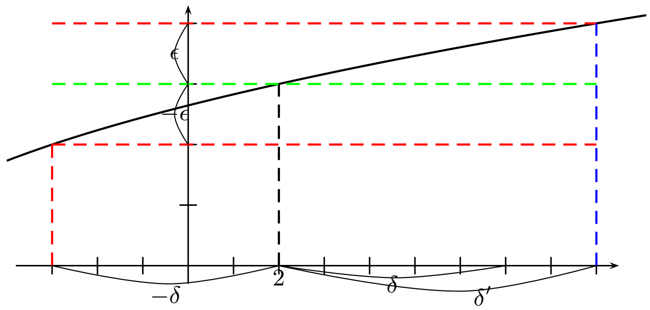


Figure 1.5: Choosing δ

Finding δ algebraically when ϵ is given

Example 1.5.5. For the limit $\lim_{x\to 2}\sqrt{x+7} = 3$, find a $\delta > 0$ that works for $\epsilon = 1$. Repeat with $\epsilon = 0.1,\ 0.001$, etc.

sol. Step 1) Solve the inequality $|\sqrt{x+7} - 3| < 2$:

$$\begin{aligned} -2 < \sqrt{x+7} - 3 &< 2 \\ 2 < \sqrt{x+7} &< 4 \\ 4 < x+7 &< 16 \\ -3 < x &< 9 \end{aligned}$$

Step 2). Find a value $\delta > 0$ to place the centered interval $(2-\delta, 2+\delta)$ inside the interval $-3 < x < 9$. A choice of delta is $\delta = 5$. Any value smaller than that works.

□

Example 1.5.6. Limit of $f(x) = (x^2-1)/(x-1)$ is 2 at $x = 1$.

sol. From

$$|x+1-2| < \varepsilon$$

we have $0 < |x-1| < \varepsilon$. Hence we may choose $\delta = \varepsilon$.

□

Example 1.5.7. Show that $y = x^2$ has limit 4 at $x = 2$.

sol. (Method 1) We try to solve the exact values of x where

$$|x^2 - 4| = |(x-2)(x+2)| < \varepsilon \tag{1.1}$$

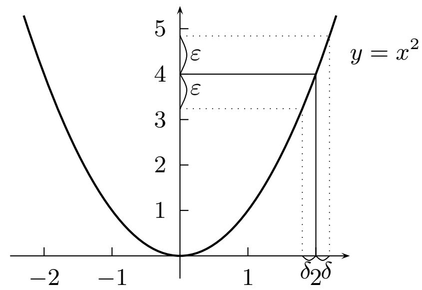


Figure 1.6: The graph of $y = x^2$

holds. Thus we see

$$-\varepsilon < x^2 - 4 < \varepsilon$$

from which we get

$$\sqrt{4-\varepsilon} < x < \sqrt{4+\varepsilon}$$

Now for any x in the interval $(2-\delta_1, 2+\delta_2)$, (1.1) holds. But the interval is not an symmetric interval.Thus the idea is to choose $\delta := \min\{\delta_1, \delta_2\}$ so that

$$|x-2| < \delta \Rightarrow x \in (2-\delta_1, 2+\delta_2) \Rightarrow |x^2 - 4| < \varepsilon$$

(Method 2) Choose certain interval near $x = 2$ where the following holds:

$$|x^2 - 4| = |(x-2)(x+2)| < \varepsilon \tag{1.2}$$

Suppose $\delta < 1$. So $0 < |x-2| < 1$ and Since $|x+2|$ between 3 and 5 we have

$$|x^2 - 4| = |(x-2)(x+2)| \le |x-2| \cdot \max_{1\le x\le 3} |x+2| \le 5|x-2|$$

To satisfy (1.2) we take $|x-2| < \varepsilon/5$. Since δ was chosen to satisfy $\delta < 1$, we may choose $\delta = \min\{1, \varepsilon/5\}$.

□

Example 1.5.8. Show $y = \sqrt{x-1}$ has limit 2 at $x = 5$.

sol. We need to find an interval near $x = 5$ where the inequality is true:

$$|\sqrt{x-1} - 2| < \varepsilon$$

(Method 1) Just solve the inequality exactly.

$$\begin{aligned} -\varepsilon < \sqrt{x-1} - 2 < \varepsilon \\ 2-\varepsilon < \sqrt{x-1} < 2+\varepsilon \\ (2-\varepsilon)^2 < x-1 < (2+\varepsilon)^2 \\ -4\varepsilon + \varepsilon^2 < x-5 < 4\varepsilon + \varepsilon^2 \end{aligned}$$

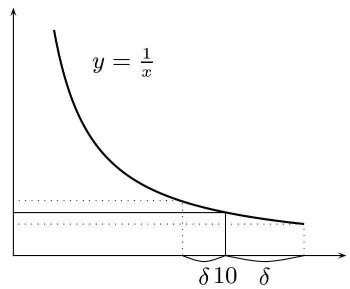


Figure 1.7: Choose smaller δ

We may assume $0 < \varepsilon < 1$. Hence we can choose

$$\delta = \min\{4\varepsilon - \varepsilon^2, 4\varepsilon + \varepsilon^2\} = 4\varepsilon - \varepsilon^2.$$

(Method 2)

$$\begin{aligned}\sqrt{x-1} - 2 &= \frac{(\sqrt{x-1}-2)(\sqrt{x-1}+2)}{\sqrt{x-1}+2} \\ &= \frac{x-5}{\sqrt{x-1}+2}\end{aligned}$$

Taking absolute values

$$\begin{aligned}\left|\sqrt{x-1} - 2\right| &= \left|\frac{(\sqrt{x-1}-2)(\sqrt{x-1}+2)}{\sqrt{x-1}+2}\right| \\ &= \left|\frac{x-5}{\sqrt{x-1}+2}\right| \\ &\leq \left|\frac{x-5}{2}\right|\end{aligned}$$

This will be less than ϵ if $|x-5| < 2\epsilon$. Hence $\delta = 2\varepsilon$.

□

Proving theorems with definitions

Example 1.5.9. Assume $\lim_{x\to a} f(x) = L, \quad \lim_{x\to a} g(x) = M$. Prove $\lim\limits_{x\to a} (f(x) + g(x)) = L + M$.

sol. Let ϵ be given. Note that

$$|f(x)+g(x)-(L+M)| = |f(x)-L+g(x)-M| \le |f(x)-L|+|g(x)-M|.$$

We want to show this term is less than ϵ when x lies within certain interval. From the definitions, we see there exists two deltas $\delta_1, \delta_2 > 0$ such that

$$0 < |x-a| < \delta_1 \Rightarrow |f(x)-L| < \epsilon/2$$
$$0 < |x-a| < \delta_2 \Rightarrow |g(x)-L| < \epsilon/2.$$

Thus for those x with $0 < |x-a| < \min(\delta_1, \delta_2)$, we see the desired inequality holds.

□

1.6 One sided limit

Definition 1.6.1. We say $f(x)$ has right-handed limit L at x_0 and write

$$\lim_{x \to x_0} f(x) = L$$

if for every positive ε, there is a number $\delta > 0$ such that for all

$$x_0 < x < x_0 + \delta \Rightarrow |f(x) - L| < \varepsilon.$$

Similar definition can be given for left-handed limit.(Write it down as an exercise)

Limit involving $\sin\theta$ etc.

Useful limits:

Theorem 1.6.2. *(1)* $\lim_{\theta \to 0} \sin\theta = 0$

(2) $\lim_{\theta \to 0} \cos\theta = 1$

(3) $\lim_{\theta \to 0} \dfrac{\sin\theta}{\theta} = 1$

Proof. (1) Referring to Fig 1.8, let P be the point on the unit circle so that θ is the angle between the x-axis and line OP. Then the length of segment $\overline{PQ}$ is $\sin\theta$. The arc from $(1, 0)$ to P has length $s = \theta$. Hence

$$0 < \overline{PQ} < \overline{PA} < \theta$$

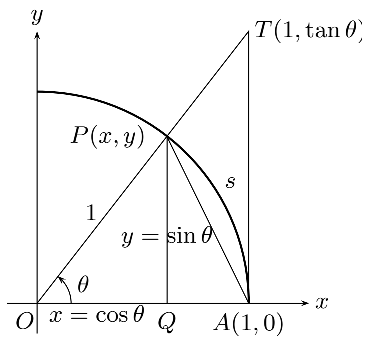


Figure 1.8:

and it holds that

$$0 < \sin\theta < \theta.$$

As θ approaches 0, we see

$$\lim_{\theta\to 0^+} \sin\theta = 0$$

Since $\sin(-\theta) = -\sin\theta$

$$-\theta < \sin(-\theta) < 0$$

Hence

$$\lim_{\theta\to 0^-} \sin\theta = 0$$

(2) Comparing $\overline{QA}$ and $\overline{PA}$, we see

$$0 < \overline{QA} < \overline{PA} < \theta$$

$$0 < 1 - \cos\theta < \theta$$

Also, as θ approaches to 0 $1 - \cos\theta$ approaches to 0.

(3) We note that

$$\begin{array}{ccccc} \overline{PQ} & \leq & \theta & \leq & \overline{TA} \\ \sin\theta & \leq & \theta & \leq & \tan\theta \\ 1 & \leq & \theta/\sin\theta & \leq & 1/\cos\theta \end{array}$$

As θ approaches 0, $1/\cos\theta$ approaches 1 by (2). Hence $\lim_{\theta\to 0} \frac{\sin\theta}{\theta} = 1$. □

Example 1.6.3. $\displaystyle\lim_{\theta\to 0} \frac{\sin 2\theta}{\theta} = \lim_{\theta\to 0} 2\frac{\sin 2\theta}{2\theta} = 2\lim_{x\to 0} \frac{\sin x}{x} = 2.$

Example 1.6.4.

$$\lim_{\theta\to 0} \frac{\tan\theta\sin\theta}{\theta^2} = 1.$$

sol. Thm 1.4.2 (4)

$$\begin{aligned}\lim_{\theta\to 0}\frac{\tan\theta\sin\theta}{\theta^2} &= \lim_{\theta\to 0}\frac{\sin^2\theta}{\theta^2}\cdot\frac{1}{\cos\theta}\\ &= \lim_{\theta\to 0}\left(\frac{\sin\theta}{\theta}\right)^2\cdot\lim_{\theta\to 0}\frac{1}{\cos\theta}\\ &= \lim_{\theta\to 0}\left(\frac{\sin\theta}{\theta}\right)\cdot\lim_{\theta\to 0}\left(\frac{\sin\theta}{\theta}\right)\\ &= 1.\end{aligned}$$

□

1.7 Continuity

Assume f is defined on the interval $[a, b]$.

Definition 1.7.1. If a function $f(x)$ is **continuous** at an interior point $x = c$ if

$$\lim_{x\to c} f(x) = f(c)$$

A function $f(x)$ is **continuous at a left end point** $x = a$ (resp. right end point $x = b$) if

$$\lim_{x\to a^+} f(x) = f(a), \quad (\text{resp. } \lim_{x\to b^-} f(x) = f(b))$$

If f is continuous at all points of its domain, we say f is a **continuous function**.

Equivalently, we have

Definition 1.7.2. For a point c in (a, b), the function $f(x)$ is continuous at $x = c$ iff

(1) $f(c)$ exists

(2) $\lim_{x\to c} f(x)$ exists.

(3) $\lim_{x\to c} f(x) = f(c)$ holds.

Example 1.7.3. Consider the function $y = [x]$ defined by the largest integer not exceeding x. For example, $[5] = 5$, $[2.4] = 2$ and $[-1.3] = -2$, etc. This function is (greatest integer) is right-continuous at every integer, but not left-continuous.

Theorem 1.7.4. *If f, g are continuous at $x = c$ then*

(1) $f \pm g$ *is continuous at* $x = c$.

(2) For any constant k, *the function* kf *is continuous at* $x = c$

(3) fg *is continuous at* $x = c$

(4) f/g *is continuous at* $x = c$, *provided* $g(c) \neq 0$.

Corollary 1.7.5. *Polynomials* $f(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$ *are continuous at all points.*

Corollary 1.7.6. *For two polynomials* $f(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$ *and* $g(x) = b_m x^m + b_{m-1}x^{m-1} + \cdots + b_1 x + b_0$, *the rational function* $f(x)/g(x)$ *is continuous where* $g(x)$ *does not vanish.*

Composite function

Theorem 1.7.7. *Suppose* f *is continuous at* c *and* g *continuous at* $f(c)$ *then* $g \circ f$ *is continuous at* c.

Proof. Since g is continuous at $f(c)$ and $\lim_{x\to c} f(x) = f(c)$

$$\lim_{x\to c} g(f(x)) = g(\lim_{x\to c} f(x)) = g(f(c))$$

□

Does it hold if any one of the functions f or g is not continuous?

Theorem 1.7.8 (Intermediate Value theorem)**.** *Suppose* f *is continuous on* $[a, b]$. *Then for any value* y_0 *between* $f(a)$ *and* $f(b)$ *there is a point* $c \in [a, b]$ *such that* $f(c) = y_0$.

When does this breaks down?

1.8 Limit Involving Infinite and vertical asymptote

We know $f(x) = 1/x \to 0$ as $x \to \infty$. We use ε-δ to define it.

Definition 1.8.1. We say $f(x)$ has limit L as x approaches ∞, if for every positive ε, there is M such that for all

$$x > M \Rightarrow |f(x) - L| < \varepsilon.$$

We write

$$\lim_{x\to\infty} f(x) = L$$

Example 1.8.2. Explain

$$\lim_{x\to\infty} \frac{x+2}{x+1} = 1$$

sol. $f(x) = 1 + 1/(x+1)$ We want to know when the following holds:

$$|f(x) - 1| = \frac{1}{|x+1|}$$

Thus we solve $|x+1| > 1/\varepsilon$ for x. It will holds when $x > M$ where

$$M = \frac{1}{\varepsilon} - 1$$

□

Example 1.8.3.

$$\lim_{x\to 0} \frac{1}{x}$$

$$\lim_{x\to\infty} \left(x + \frac{1}{x}\right)$$

Horizontal Asymptote

Definition 1.8.4. A line $y = b$ is a **horizontal asymptote** of the graph of a function $y = f(x)$ if either

$$\lim_{x\to\infty} f(x) = b \text{ or } \lim_{x\to-\infty} f(x) = b$$

Example 1.8.5. Find the horizontal asymptote of the graph of

$$y = \frac{x^3+2}{|x|^3+1}$$

sol.

$$\lim_{x\to\infty} \frac{x^3+2}{|x|^3+1} = \lim_{x\to\infty} \frac{1+2/x^3}{1+1/x^3} = 1$$

For $x < 0$, we see

$$\lim_{x\to-\infty} \frac{x^3+2}{|x|^3+1} = \lim_{x\to\infty} \frac{1+2/x^3}{1-1/x^3} = -1$$

Hence we have two horizontal asymptotes $y = \pm 1$.

□

Sandwich theorem revisited

Sandwich theorem holds when $x \to \infty$.

The concept of ∞ as a limit

We give a precise meaning of the symbol ∞ using similar idea used for limit.

Definition 1.8.6. We say f **approaches infinity as** x **approaches** x_0 and write

$$\lim_{x \to x_0} f(x) = \infty$$

if for every real number B there exists a corresponding $\delta > 0$ such that for all x with

$$0 < |x - x_0| < \delta \Rightarrow f(x) > B.$$

Give a similar definition for $-\infty$ as an exercise.

Vertical Asymptotes

Definition 1.8.7. The line $x = a$ is called a **vertical asymptote** of a graph of a function $y = f(x)$ if either

$$\lim_{x \to a^+} f(x) = \pm\infty \text{ or } \lim_{x \to a^-} f(x) = \pm\infty.$$

Example 1.8.8. (1) Find a vertical asymptote of $f(x) = \frac{x+3}{x+1}$.

(2) Find a horizontal and vertical asymptote of $f(x) = -\frac{8}{x^2-4}$.

Oblique asymptote?

Chapter 2

Differentiation

2.1 Tangents and Derivatives at a point

Finding tangent to the graph of a function

Definition 2.1.1. The slope of the curve $y = f(x)$ at a point $P = (x_0, f(x_0))$ is the number

$$\lim_{h \to 0} \frac{f(x_0 + h) - f(x_0)}{h}$$

provided it exists. The **tangent line** to the curve at the point P is the line through P with this slope.

The rate of change of $y = f(x)$ between $P(x_0, y_0)$ and $Q(x_1, y_1)$ is

$$m_{sec} = \frac{f(x_1) - f(x_0)}{x_1 - x_0}$$

In the limit, it is the slope of tangent line.

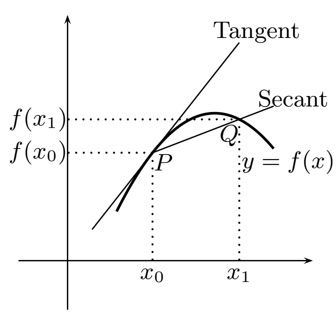


Figure 2.1: Tangent and secant

Hence the slope of tangent line at P is

$$m_{tan} = \lim_{x_1 \to x_0} \frac{f(x_1) - f(x_0)}{x_1 - x_0} = \lim_{h \to 0} \frac{f(x_0 + h) - f(x_0)}{h}$$

Rate of change:Derivative at a point

Definition 2.1.2. The derivative of a function $f(x)$ at a point is given by

$$f'(x_0) = \lim_{h\to 0} \frac{f(x_0+h) - f(x_0)}{h}$$

exists, it is called **derivative** at $x = x_0$.

Example 2.1.3. Interpretations:

(1) The slope of $y = f(x)$ at x_0

(2) The slope of tangent to the curve $y = f(x)$ at x_0

(3) The rate of change of f with respect to x at x_0

(4) The derivative $f'(x_0)$ at the point.

2.2 Derivative as a function

Definition 2.2.1. If the derivative of a function $f(x)$ at a

$$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h}$$

exists, it is called the **derivative** (function) of $f(x)$. (Treated as a function) f is said to be **differentiable** at x. If f is differentiable at all points of domain we say f is **differentiable.** We also use the notation df/dx, $(d/dx)f$ for f'.

Alternative formula for the derivative is

$$f'(x) = \lim_{z\to x} \frac{f(z) - f(x)}{z - x}$$

Try some examples in the text.

One sided derivative

Definition 2.2.2. Suppose f is defined on $[a, b]$. Then at each end point the one sided derivative is defined by

$$f'(a^+) = \lim_{h\to 0^+} \frac{f(a+h) - f(a)}{h},$$
$$f'(b^-) = \lim_{h\to 0^-} \frac{f(b+h) - f(b)}{h} = \lim_{h\to 0^+} \frac{f(b) - f(b-h)}{h}.$$

Example 2.2.3. Using definition, find $(d/dx)x^3$.

sol. Set $f(x) = x^3$

$$f(x+h) - f(x) = h\{(x+h)^2 + x(x+h) + x^2\}$$

$$\lim_{h\to 0} \frac{1}{h}\{f(x+h) - f(x)\} = \lim_{h\to 0}\{(x+h)^2 + x(x+h) + x^2\} = 3x^2$$

So $f'(x) = 3x^2$

□

Example 2.2.4. The one sided derivatives of $f(x) = |x|$ at $x = 0$ are $f'(0^-) = -1$ and $f'(0^+) = 1$. Hence f is not differentiable at $x = 0$. (Figure 2.2)

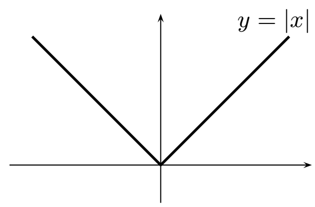


Figure 2.2: $y = |x|$

When does a function do not have derivative at a point?

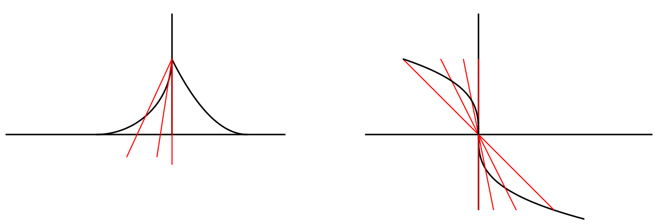


Figure 2.3: Cases with no derivative

Example 2.2.5. The function defined by

$$f(x) = \begin{cases} x\sin(1/x), & x \neq 0 \\ 0, & x = 0 \end{cases}$$

is conti at $x = 0$ but not differentiable.

sol. Since $-|x| \le f(x) \le$ and $\lim_{x\to 0}|x| = \lim_{x\to 0} = 0$ by theorem f is conti. at $x=0$. But the limit

$$\lim_{h\to 0}\frac{1}{h}\left\{h\sin\frac{1}{h} - 0\right\} = \lim_{h\to 0}\sin\frac{1}{h}$$

does not exist, f is not differentiable $x = 0$.

□

Differentiable functions are continuous

Theorem 2.2.6. *If f is differentiable at $x = a$, then f is conti. at $x = a$.*

Proof. By definition of derivative we have

$$\lim_{h\to 0}\{f(a+h) - f(a)\} = \left(\lim_{h\to 0}\frac{1}{h}\{f(a+h)-f(a)\}\right)\cdot\left(\lim_{h\to 0} h\right) = 0.$$

Hence $f(x)$ is conti. at $x = a$. □

Intermediate Value property of derivatives(Darboux's theorem)

Theorem 2.2.7. *If a, b are any two points in an interval where f is differentiable, then f' takes any value between $f'(a)$ and $f'(b)$.*

2.3 Differentiation Rules

Proposition 2.3.1. *Suppose f, g are differentiable functions. Then*

(1) For any constant C, $\frac{dC}{dx} = 0$ for any constant C.

(2) When n is positive integer , $\frac{d}{dx}x^n = nx^{n-1}$.

(3) For any constant C, $\frac{d(Cu)}{dx} = C\frac{du}{dx}$.

(4) $\frac{d}{dx}(u \pm v) = \frac{du}{dx} \pm \frac{dv}{dx}$.

(5) $\frac{d(uv)}{dx} = v\frac{du}{dx} + u\frac{dv}{dx}$.

(6) $\frac{d}{dx}\left(\frac{u}{v}\right) = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$.

(7) For any real number n, $\frac{d}{dx}x^n = nx^{n-1}$.

Proof. (2) Use

$$z^n - x^n = (z-x)(z^{n-1} + z^{n-2}x + \cdots + x^{n-2}z + x^{n-1})$$

(4)

$$\begin{aligned}\frac{d}{dx}(uv) &= \lim_{h\to 0}\frac{u(x+h)v(x+h)-u(x)v(x)}{h}\\ &= \lim_{h\to 0}\left\{\frac{u(x+h)-u(x)}{h}v(x+h) + u(x)\frac{v(x+h)-v(x)}{h}\right\}\\ &= u'(x)v(x)+u(x)v'(x).\end{aligned}$$

Try to prove the product rule and draw Figure for product rule. □

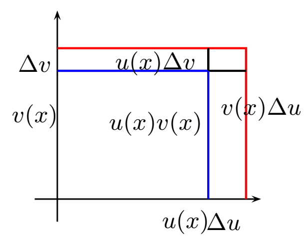


Figure 2.4: product rule

Higher order derivative

$$f''(x) = \frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right)$$

$$y^{(n)} = \frac{dy^{(n-1)}}{dx} = \frac{d^ny}{dx^n}$$

2.4 The Derivative as a Rate of change

Definition 2.4.1. The **instantaneous rate of change** of f at x_0 is

$$f'(x_0) = \lim_{h\to 0}\frac{f(x_0+h)-f(x_0)}{h}$$

provided the limit exists.

How to describe a moving object?

How fast ? etc. First we assume an object is moving along a line(coordinate line) and its position is given as a function of time:

$$s = f(t)$$

Then the **displacement** of the object over the time interval t to $t + \Delta t$ is

$$\Delta s = f(t + \Delta t) - f(t)$$

and define the **average velocity** of the object over that time interval is

$$v_{av} = \frac{\text{displacement}}{\text{elapsed time}} = \frac{\Delta s}{\Delta t} = \frac{f(t + \Delta t) - f(t)}{\Delta t}$$

Definition 2.4.2. Velocity(instantaneous velocity) is the derivative of a position function w.r.t time. If a moving object position is given by $s = f(t)$, then the **velocity** at t is

$$v(t) = \lim_{h \to 0} \frac{f(t + h) - f(t)}{h}$$

provided the limit exists.

Definition 2.4.3. (Speed) is the absolute value of the velocity, i.e, **Speed**

$$\text{Speed } = |v(t)| = \left| \frac{ds}{dt} \right|$$

Acceleration is

$$a(t) = \frac{dv}{dt} = \frac{d^2 s}{dt^2}$$

Third derivative is called a Jerk

2.5 Derivative of Trig functions

Use definition to find the derivative of $f(x) = \sin x$.

$$\begin{aligned}
&= \lim_{h \to 0} \frac{\sin(x + h) - \sin x}{h} \\
&= \lim_{h \to 0} \frac{\sin x \cos h + \cos x \sin h - \sin x}{h} \\
&= \sin x \lim_{h \to 0} \frac{\cos h - 1}{h} + \cos x \lim_{h \to 0} \frac{\sin h}{h} \\
&= \sin x \cdot 0 + \cos x \cdot 1 = \cos x.
\end{aligned}$$

Hence $\frac{d}{dx}\sin x = \cos x$. Similarly, we have

$$\frac{d}{dx}\cos x = -\sin x$$

Other trigonometric functions are defined by the following relation and their derivatives can be found using differentiation rules:

$$\tan x = \frac{\sin x}{\cos x}, \quad \sec x = \frac{1}{\cos x}, \quad \csc x = \frac{1}{\sin x}, \quad \cot x = \frac{\cos x}{\sin x}$$

The derivative of $\tan x$ is

$$\frac{d}{dx}\tan x = \frac{(\sin x)'\cos x - (\cos x)'\sin x}{\cos^2 x} = \frac{1}{\cos^2 x} = \sec^2 x$$

Summarizing, we have

Proposition 2.5.1. *(1)* $\frac{d}{dx}\sin x = \cos x$

(2) $\frac{d}{dx}\cos x = -\sin x$

(3) $\frac{d}{dx}\tan x = \sec^2 x$

(4) $\frac{d}{dx}\sec x = \sec x \tan x$

(5) $\frac{d}{dx}\csc x = -\csc x \cot x$

(6) $\frac{d}{dx}\cot x = -\csc^2 x$

2.6 Exponential functions

Let $a > 0$. For any rational number $x = n/m$, we let $a^x = (a^n)^{1/m}$, while any real number x we define

$$a^x = \lim_{r\to x} a^r, \quad \text{rational}$$

Rules for exponentiation. For $a, b > 0$ we have

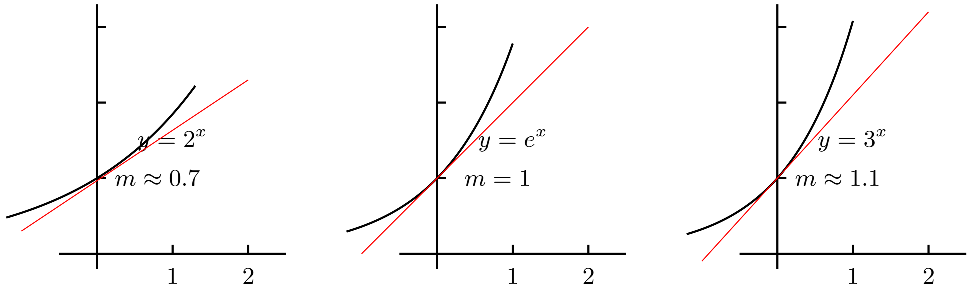
(1) $a^x \cdot a^y = a^{x+y}$

(2) $\frac{a^x}{a^y} = a^{x-y}$

(3) $(a^x)^y = (a^y)^x = a^{xy}$

(4) $a^x b^x = (ab)^x$

(5) $\frac{a^x}{b^x} = \left(\frac{a}{b}\right)^x$



The natural exponential function e^x

We define e to be the number such that the slope of tangent line to $f(x) = e^x$ at 0 is 1. e is a irrational number $e = 2.78182\cdots$. Why do we use this strange number? This simplifies computation in later sections(diff. integration)

Differentiation of exponential function

Let us compute the derivative of $f(x) = a^x$.

$$\begin{aligned}\lim_{h\to 0}\frac{a^{x+h}-a^x}{h} &= \lim_{h\to 0}\frac{a^x a^h - a^x}{h}\\ &= \lim_{h\to 0} a^x\left(\frac{a^h-1}{h}\right)\\ &= a^x\lim_{h\to 0}\frac{a^h-1}{h}\end{aligned}$$

Note that when $x = 0$ this limit $\lim_{h\to 0}\frac{1}{h}(a^h - 1)$ is the slope of tangent line to a^x at $x = 0$. As a special case, if $a = e$, then we know

$$f'(0) = \lim_{h\to 0}\frac{e^h-1}{h} = 1$$

by definition of the number e! So we have

$$\frac{d}{dx}e^x = e^x.$$

In Chapter 3 we will see this number is obtained from $(1 + \frac{1}{x})^x$ as x approaches unboundedly.

Exponential growth

Example 2.6.1 (exponential growth)**.** It is used to model the amount of money to grow when the interest is compounded. Say P dollar is initially invested in the bank with interest rate of r per year, t is the time in years, then the total amount of money after t years is

$$y = Pe^{rt}$$

Example 2.6.2 (exponential decay)**.** It is used to model the amount of radioactive material. e.g, Carbon 14 case, If A is initial original amount of carbon 14.

$$y = Ae^{-1.2\times 10^{-4}t}$$

Carbon 14 is used to detect how old is the dead organisms such as shells, wooden artifacts.

2.7 Chain rule

Chain Rule

Theorem 2.7.1. *(1) If $f(u)$ is differentiable at $u = g(x)$ and g is differentiable at x, then the composite function $f \circ g$ is differentiable at x and*

$$(f \circ g)'(x) = f'(g(x))g'(x)$$

If $y = f(u)$, $u = g(x)$ then

$$\left.\frac{dy}{dx}\right|_x = \left.\frac{dy}{du}\right|_{u=g(x)} \cdot \left.\frac{du}{dx}\right|_x$$

Proof. (Intuitive) Let $\Delta u = g(x+\Delta x)-g(x)$ be the change of u corresponding to the change of Δx. (We assume $g(x + \Delta x) \neq g(x)$) We might consider

$$\frac{\Delta y}{\Delta x} = \frac{\Delta y}{\Delta u}\frac{\Delta u}{\Delta x}, (\Delta u \neq 0)$$

$$\begin{aligned}
\lim_{\Delta x\to 0}\frac{\Delta y}{\Delta x} &= \lim_{\Delta x\to 0}\frac{\Delta y}{\Delta u}\frac{\Delta u}{\Delta x} \\
&= \lim_{\Delta u\to 0}\frac{\Delta y}{\Delta u}\lim_{\Delta x\to 0}\frac{\Delta u}{\Delta x} \\
&= \frac{dy}{du}\frac{du}{dx}
\end{aligned}$$

□

Repeated Use

Example 2.7.2. $y = \cos(1 + x^4)^5$

sol. Given function is the composite of $y = \cos u$ and $u = (1 + x^4)^5$. Hence

$$\frac{dy}{du} = -\sin u\frac{du}{dx}.$$

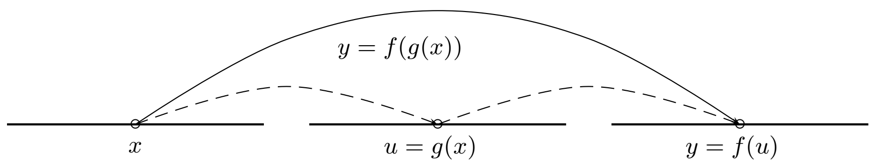


Figure 2.5: Chain rule

On the other hand, since $(1+x^4)^5$ is a composite function of $u = v^5$ and $v = 1 + x^4$, use Chain rule again

$$\frac{du}{dx} = 5v^4 \cdot 4x^3.$$

$$\frac{dy}{dx} = -\sin(1+x^4)^5 \cdot 20(1+x^4)^4 x^3. \qquad \square$$

Outside-Inside Rule

Example 2.7.3. The derivative of $|x|$ can be computed as follows: Notice that $|x| = \sqrt{x^2}$. Hence

$$\frac{d}{dx}\sqrt{x^2} = \frac{1}{2\sqrt{x^2}} \cdot 2x = \frac{x}{|x|}, \quad x \neq 0.$$

Example 2.7.4. Find slope of tangent line to $y = (1-2x)^3$ at $x = 1$.

Derivatives of Power function

Find derivative of $f(x) = u^r(x)$ for any real r and $x > 0$.

Theorem 2.7.5. *Since* $u^r = e^{r \ln u}$

$$\frac{d}{dx}u^r = ru^r \frac{d \ln u}{dx} = ur^r \frac{1}{u}\frac{du}{dx} = ru^{r-1}\frac{du}{dx}.$$

2.8 Implicit differentiation

There are situation where some relation between x and y defines some graph, but not graph of any function. Suppose x, y satisfy $y^5 + \sin xy = x^3 y$. This relation defines a function implicitly.

Implicitly defined function

Assuming y is a differentiable function of x satisfying some relation like $F(x,y) = 0$, we take derivative of $F(x,y) = 0$ w.r.t x using the chain rule. Then solving for dy/dx we find the derivative. This procedure is called an **Implicit differentiation**.

$$\frac{d}{dx}\left(y^5\right) + \frac{d}{dx}(\sin xy) = \frac{d}{dx}\left(x^3 y\right)$$
$$5y^4\frac{dy}{dx} + (\cos xy)\left(y + x\frac{dy}{dx}\right) = 3x^2 y + x^3\frac{dy}{dx}.$$

Hence

$$\frac{dy}{dx} = \frac{3x^2 y - y\cos xy}{5y^4 + x\cos xy - x^3}$$

Example 2.8.1. (1) $x^3 + y^3 = 3xy$. Find dy/dx and d^2y/dx^2

(2) Find equation of tangent line to $x^3 + y^3 = 3xy$ at $(3/2, 3/2)$

sol.

(1) Taking derivative

$$(3y^2 - 3x)\frac{dy}{dx} = 3y - 3x^2$$

Chain rule

$$\left(6y\frac{dy}{dx} - 3\right)\frac{dy}{dx} + (3y^2 - 3x)\frac{d^2y}{dx^2} = 3\frac{dy}{dx} - 6x.$$

Hence

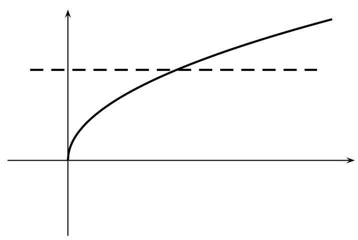
$$\frac{dy}{dx} = \frac{y - x^2}{y^2 - x},$$

$$\frac{d^2y}{dx^2} = \frac{-2\left((y-x^2)/(y^2-x)\right)^2 y + 2(y-x^2)/(y^2-x) - 2x}{y^2 - x}.$$

(2) At $(3/2, 3/2)$, $dy/dx = -1$ The tangent line is

$$y = -\left(x - \frac{3}{2}\right) + \frac{3}{2} = -x + 3.$$ □

□



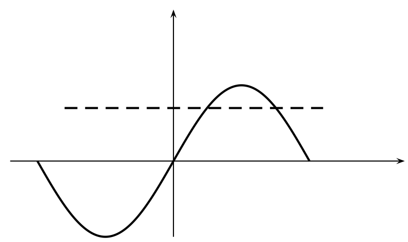


Figure 2.6: Horizontal line test

2.9 Inverse functions and Their Derivatives

Definition 2.9.1. A function f is one-to-one on a domain D if $f(x_1) \neq f(x_2)$ whenever $x_1 \neq x_2$.

Definition 2.9.2. Suppose a function f is one-to-one on a domain D with range R. The **inverse function** f^{-1} exists and is defined by

$$f^{-1}(b) = a \text{ if } f(a) = b.$$

The domain of f^{-1} is R and range is D.

$$(f^{-1} \circ f)(x) = x, \quad x \in D$$
$$(f \circ f^{-1})(y) = y, \quad y \in R$$

Horizontal line test

Derivatives of inverse function

Theorem 2.9.3. *Suppose f is one-to-one and differentiable in I. If $f'(x)$ exists and is never zero, then f^{-1} exists, differentiable. Furthermore for $a \in I$, $f(a) = b$, then*

$$(f^{-1})'(b) = \frac{1}{f'(a)}.$$

Set $y = f(x)$. Then the inverse function is $x = f^{-1}(y)$, and its derivative is

$$\left.\frac{dx}{dy}\right|_{y=f(a)} = \frac{1}{dy/dx|_{x=a}}, \quad a \in I.$$

Proof. Differentiate $x = (f^{-1} \circ f)(x)$ w.r.t x using the Chain rule

$$1 = (f^{-1})'(f(a))f'(a)$$

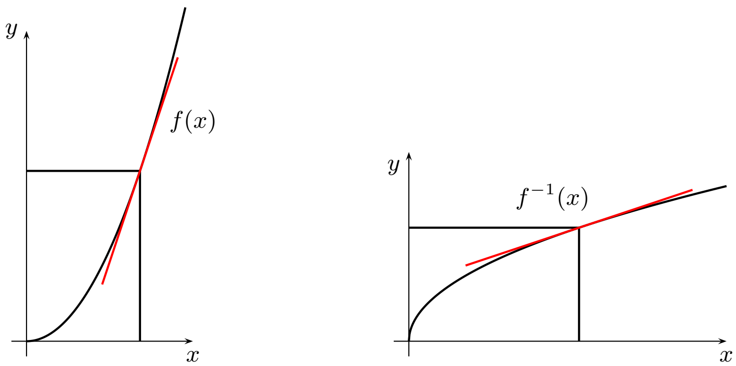


Figure 2.7: Slope of inverse function

Hence

$$(f^{-1})'(b) = \frac{1}{f'(a)}$$

□

Usually, we use the notation $y = f^{-1}(x)$. The graph of $y = f(x)$ and that of $y = f^{-1}(x)$ are symmetric w.r.t the line $y = x$.

Example 2.9.4. (1) $f(x) = x^7 + 8x^3 + 4x - 2$. Find $(f^{-1})'(-2)$.

(2) Find $\frac{d}{dx}\sin^{-1} x$.

sol. (1) Since $f' = 7x^6 + 24x^2 + 4 \geq 4$, the inverse function f^{-1} exists. Since $f(0) = -2$ we have

$$(f^{-1})'(-2) = (f^{-1})'(f(0)) = \frac{1}{f'(0)} = \frac{1}{4}.$$

(2) $y = \sin^{-1} x$ is equivalent to $x = \sin y$. Hence

$$\begin{aligned}\frac{d}{dx}\sin^{-1} x &= \frac{dy}{dx} = \frac{1}{dx/dy} = \frac{1}{(d/dy)\sin y} \\ &= \frac{1}{\cos y} = \frac{1}{\sqrt{1-\sin^2 y}} = \frac{1}{\sqrt{1-x^2}}.\end{aligned}$$

□

□

2.10 Logarithmic functions

Definition 2.10.1. The logarithmic function with base a, $y = \log_a x$ is the inverse function of $y = a^x (a > 0, a \neq 1)$

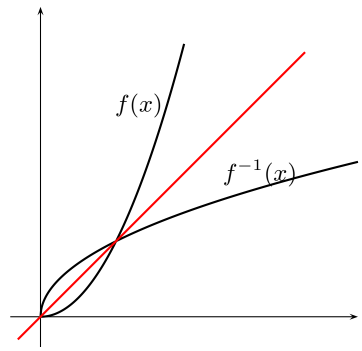


Figure 2.8: Graph of inverse function is symmetric about $y = x$

$\log_e x$ is written as $\ln x$ and called *natural logarithmic function*

$\log_{10} x$ is written as $\log x$ and called *common logarithmic function*

$$\ln x = y \Leftrightarrow e^y = x$$

Properties

(1) Product rule: $\log_a xy = \log_a x + \log_a y$

(2) Quotient rule: $\log_a \frac{x}{y} = \log_a x - \log_a y$

(3) Product rule: $\log_a \frac{1}{y} = -\log_a y$

(4) Power rule: $\log_a x^y = y \log_a x$

Inverse properties

(1) Base a: $a^{\log_a x} = x$, $\log_a(a^x) = x (a > 0, a \neq 1, x > 0)$

(2) Base e: $e^{\ln x} = x$, $\ln(e^x) = x (x > 0)$

Hence (by substituting a^x for x)

$$\begin{aligned} a^x &= e^{\ln a^x} \\ &= e^{x \ln a} \\ &= e^{(\ln a)x} \end{aligned}$$

Every exponential function can be written as power of natural exponential function:

$$a^x = e^{x \ln a}$$

Change of base

$$\log_a x = \frac{\ln x}{\ln a} \quad (a > 0, a \neq 1, x > 0)$$

Exponential growth

Example 2.10.2 (exponential growth)**.** It is used to model the amount of money to grow when the interest is compounded. Say P dollar is initially invested in the bank with interest rate of r per year, t is the time in years, then the total amount of money after t years is

$$y = Pe^{rt}$$

Example 2.10.3 (exponential decay)**.** It is used to model the amount of radioactive material. e.g, Carbon 14 case, If A is initial original amount of carbon 14.

$$y = Ae^{-1.2\times10^{-4}t}$$

Carbon 14 is used to detect how old is the dead organisms such as shells, wooden artifacts.

Half Life- Given a radioactive material, the time required for the material to decay into half is the Half life. Let t_h be the half life. Then

$$y(t_h) = Ae^{-1.2\times10^{-4}t_h} = \frac{1}{2}A$$

Thus

$$e^{-1.2\times10^{-4}t_h} = 0.5, \quad t_h = \frac{\ln 2}{1.2\times10^{-4}}$$

The formula for the half life is of a radioactive material obeying the relation $y(t) = Ae^{-rt}$ is

$$t_h = \frac{\ln 2}{r}$$

Example 2.10.4. Half life of Polonium 210. The time for radioactive substance required to decay by half is independent of the initial quantity. $y = y_0e^{-kt}$. So $t = \ln 2/k$. For Polonium 210, $k = 5\cdot10^{-3}$.

Derivatives of log function $y = \ln x$

First method: Regards $\ln x$ as an inverse of exponential function $f(x) = e^x$: Recall $f(x) = e^x$ iff $f^{-1}(x) = \ln x$.

$$\begin{aligned}(f^{-1})'(x) &= \frac{1}{f'(f^{-1}(x))} \\ &= \frac{1}{e^{f^{-1}(x)}} \quad (f'(u) = u) \\ &= \frac{1}{e^{\ln x}} \\ &= \frac{1}{x}\end{aligned}$$

Alternative way is to leave the variable as is: Write $y = \ln x$ is as $e^y = x$. Then

$$\begin{aligned}\frac{d}{dx}e^y &= 1 \\ e^y\frac{dy}{dx} &= 1 \\ \frac{dy}{dx} &= \frac{1}{e^y} = \frac{1}{x}\end{aligned}$$

Example 2.10.5.

$$\frac{d}{dx}(\ln u) = \frac{1}{u}\frac{du}{dx}$$

$$\frac{d}{dx}\ln(x^4+2) = \frac{1}{x^4+2}\cdot 4x^3$$

Derivatives of $\ln|x|$

$\ln|x| = 1/x$

Derivatives of a^u

$a^x = e^{x\ln a}$. So

$$(a^x)' = e^{x\ln a}\ln a = a^x \ln a$$

In general,

$$\frac{d}{dx}a^u = a^u \ln a\frac{du}{dx}$$

Derivatives of $\log_a u$

$$\log_a x = \frac{\ln x}{\ln a}$$

So

$$\frac{d}{dx}\log_a x = \frac{1}{x\ln a}$$

and
$$\frac{d}{dx}\log_a u = \frac{1}{u\ln a}\frac{du}{dx}$$

Logarithmic Differentiation

Find dy/dx if $y = \frac{(x^2+1)^{1/3}(x-3)^{1/2}}{x+5}$

Derivatives of Power function

We prove the following theorem which was stated earlier.

Theorem 2.10.6. *For any real r, $\frac{d}{dx}u^r = u^{r-1}\frac{du}{dx}$.*

Proof. Since $u^r = e^{r\ln u}$ we have
$$\frac{d}{dx}u^r = ru^r\frac{d\ln u}{dx} = ur^r\frac{1}{u}\frac{du}{dx} = ru^{r-1}\frac{du}{dx}.$$
□

Example 2.10.7. Differentiate $f(x) = x^x, x > 0$

sol. Write $f(x) = x^x = e^{x\ln x}$. So
$$\begin{aligned} f'(x) &= \frac{d}{dx}(e^{x\ln x}) \\ &= (e^{x\ln x})\frac{d}{dx}(x\ln x) \\ &= e^{x\ln x}(\ln x + x\cdot\frac{1}{x}) \\ &= x^x(\ln x + 1) \end{aligned}$$
□

The number e

In section 2.6, we saw e was defined so that it satisfies
$$\lim_{h\to 0}\frac{e^h-1}{h} = 1.$$
An important property is the following.

Theorem 2.10.8. *The number e satisfies*
$$e = \lim_{x\to 0}(1+x)^{1/x}.$$

Proof. If $f(x) = \ln x$. Then $f'(1) = \frac{1}{x}|_{x=1} = 1$. By definition,
$$1 = f'(1) = \lim_{x\to 0^+}\frac{\ln(1+x)-\ln 1}{x} = \lim_{x\to 0^+}\ln[(1+x)^{\frac{1}{x}}] = \ln[\lim_{x\to 0^+}(1+x)^{\frac{1}{x}}].$$
Now exponentiate. □

2.11 Inverse trig functions

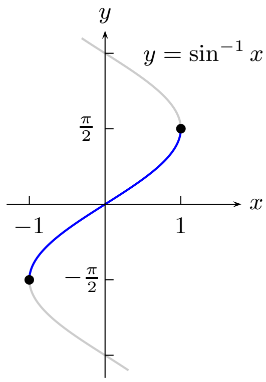


Figure 2.9: $y = \sin^{-1} x$

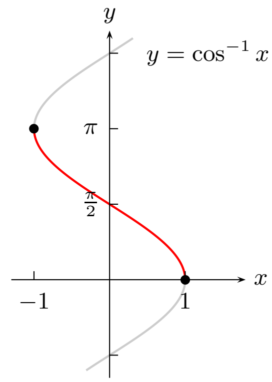


Figure 2.10: $y = \cos^{-1} x$

Inverse sine

Restrict the function $\sin x$ on $[-\pi/2, \pi/2]$. Then $\sin x\colon [-\pi/2, \pi/2] \to [-1, 1]$ is one-to - one function. So the inverse exists. Define

$$\sin^{-1} x\colon [-1, 1] \longrightarrow [-\pi/2, \pi/2].$$

whenever $x = \sin y$ for $x \in [-\pi/2, \pi/2]$. Graph is as in figure 2.9. $\sin^{-1} x$ is sometimes written as $\arcsin x$.

Example 2.11.1. (1) $\sin^{-1}(1/2) = \pi/6$

(2) $\sin^{-1}(1) = \pi/2$

Inverse cosine

Restrict $\cos x$ to $[0, \pi]$, we obtain $\cos^{-1} x$.

$$\cos^{-1} x\colon [-1, 1] \longrightarrow [0, \pi]$$

If $\cos x = y$ for any $x \in [0, \pi]$ then $\cos^{-1} y = x$ is defined and figure is in 2.10 written as $\cos^{-1} x$ or $\arccos x$.

Example 2.11.2. (1) $\cos^{-1}(1/2) = \pi/3$

(2) $\cos^{-1} 0 = \pi/2$

Example 2.11.3.

$$\sin^{-1} x + \cos^{-1} x = \frac{\pi}{2}, \quad \cos^{-1} x + \cos^{-1}(-x) = \pi$$

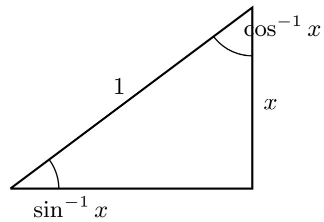


Figure 2.11: $\sin^{-1} x$

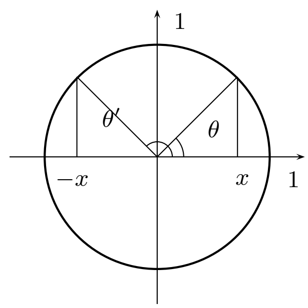


Figure 2.12: $\theta = \cos^{-1} x, \quad \theta' = \cos^{-1}(-x)$

Inverse of $\tan x$

On $(-\pi/2, \pi/2)$ $\tan x$ has inverse $\tan^{-1} x$

$$\tan^{-1} x \colon \mathbb{R} \longrightarrow (-\pi/2, \pi/2)$$

for any $x \in \mathbb{R}$, $\tan x = \alpha$ iff $\tan^{-1} \alpha = x$. See figure 2.13. It is written as $\tan^{-1} x$ or $\arctan x$.

$\tan^{-1} 1 = \pi/4$ $\tan^{-1} 0 = 0$

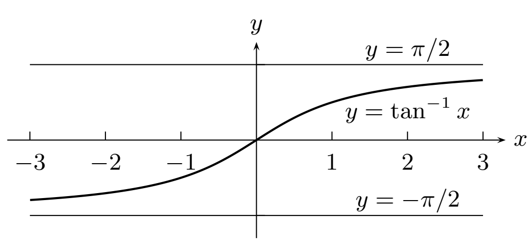


Figure 2.13: $y = \tan^{-1} x$

Example 2.11.4. Find the derivative of $\tan^{-1} x$.

From $y = f(x) = \tan x$, we see

$$\begin{aligned}(f^{-1})'(y) &= \frac{1}{f'(x)} \\ &= \frac{1}{1+\tan^2 x} \\ &= \frac{1}{1+y^2}\end{aligned}$$

Thus $(f^{-1})'(x) = \frac{1}{1+x^2}$.

Example 2.11.5. Find derivatives

(1) $y = \sin^{-1} x, \quad (|x| \le 1)$.

(2) $y = \sec^{-1} x, \quad (|x| \ge 1)$.

sol. (3) Let $y = \sec^{-1} x$. Then $x = \sec y$. Taking derivative w.r.t x we get $1 = \sec y \tan y (dy/dx)$. Thus

$$\frac{dy}{dx} = \frac{1}{\sec y \tan y}.$$

We need to change it to expression in x.

For $x > 1$, $\tan y = \sqrt{x^2-1}$. Hence, we have

$$\frac{dy}{dx} = \frac{1}{x\sqrt{x^2-1}}, \quad x > 1.$$

For $x < -1$, use $(x \to -x)$ to get

$$\frac{dy}{dx} = \frac{1}{-x\sqrt{x^2-1}}, \quad x < -1$$

Hence

$$\frac{d}{dx}\sec^{-1} x = \frac{1}{|x|\sqrt{x^2-1}}, \quad |x| > 1$$

□

Proposition 2.11.6. *Similarly, we get the derivatives of inverse trig. function*

(1) $\frac{d}{dx}\sin^{-1} x = \frac{1}{\sqrt{1-x^2}}$

(2) $\frac{d}{dx}\cos^{-1} x = -\frac{1}{\sqrt{1-x^2}}$

(3) $\dfrac{d}{dx}\tan^{-1}x = \dfrac{1}{1+x^2}$

(4) $\dfrac{d}{dx}\csc^{-1}x = -\dfrac{1}{|x|\sqrt{x^2-1}}$, $|x| > 1$

(5) $\dfrac{d}{dx}\sec^{-1}x = \dfrac{1}{|x|\sqrt{x^2-1}}$, $|x| > 1$

(6) $\dfrac{d}{dx}\cot^{-1}x = -\dfrac{1}{1+x^2}$

Other inverse trig function

Inverses of $\csc x$, $\sec x$, $\cot x$

$\csc^{-1}x : \mathbb{R} - (-1,1) \to [-\pi/2, \pi/2] - \{0\}$

$\sec^{-1}x : \mathbb{R} - (-1,1) \to [0, \pi] - \{\pi/2\}$

$\cot^{-1}x : \mathbb{R} \to (0, \pi)$

We see the following relation hold

Proposition 2.11.7. *(1)* $\cot^{-1}x = (\pi/2) - \tan^{-1}x$

(2) $\sec^{-1}x = \cos^{-1}(1/x)$

(3) $\csc^{-1}x = \sin^{-1}(1/x)$

Example 2.11.8. (1) Find $\sin(\cos^{-1}(3/5))$

(2) Simplify $\tan(\sin^{-1}a)$

sol. (1) Let $\theta = \cos^{-1}(3/5)$. Then $\cos\theta = 3/5$ and $0 \le \theta \le \pi$

$$\sin\theta = \sqrt{1 - \frac{9}{25}} = \frac{4}{5}.$$

(2) Let $\theta = \sin^{-1}a$. Then $\sin\theta = a$ and $-\pi/2 \le \theta \le \pi/2$

$$\cos\theta = \sqrt{1-a^2}.$$

Hence

$$\tan\theta = \sin\theta/\cos\theta = a/\sqrt{1-a^2}.$$ □

□

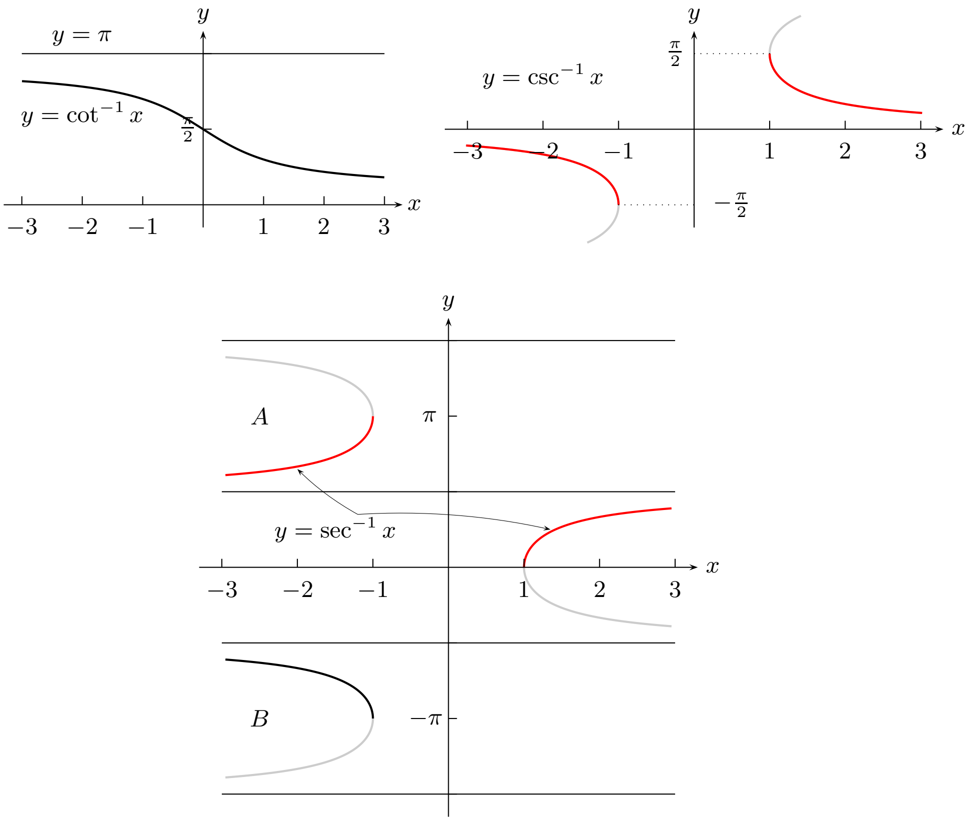


Figure 2.14:

2.12 Related rates

Related rates

y is a function of x and $x(x = x(t))$ is a function of time t the rate of change dy/dx and dy/dt satisfies $dy/dt = (dy/dx)(dx/dt)$.

Example 2.12.1. Figure 2.15. Water is being poured into a conical tank at the rate of $9\text{ft}^3/\text{min}$. Find rate of rising water level when depth of water is 6ft.

sol.

(1) V : Volume of water at t

(2) x: Radius of water surface at t

(3) y: Depth of water at t

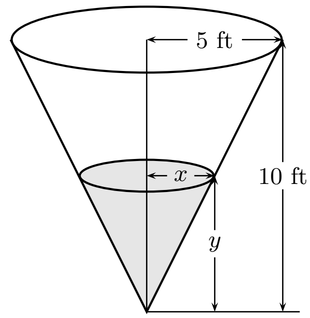


Figure 2.15: Conic Tank

We have $x = y/2$

$$V = \frac{1}{3}\pi x^2 y = \frac{\pi}{12}y^3,$$
$$\frac{dV}{dt} = \frac{\pi}{4}y^2\frac{dy}{dt}$$

Substitute $dV/dt = 9$, $y = 6$. Then from $9 = \pi/4 \cdot 36 \cdot dx/dt|_{y=6}$, we get $dx/dt|_{y=6} = \pi$.

□

Strategy solving related rate problem

(1) Draw figures and name variables(think of good representatives)

(2) Write down numerical information

(3) Write down what is asked for

(4) Write down equation

(5) Solve etc....

Example 2.12.2. Rocket is rising at the speed of 300m/sec vertically. At 1000m above the ground, a camera man is watching 1000m away. Find the rate of change of this camera angle θ.

sol. From figure 2.16, we see the angle of elevation θ and the height y is related by

$$\tan\theta = \frac{y}{1000}$$

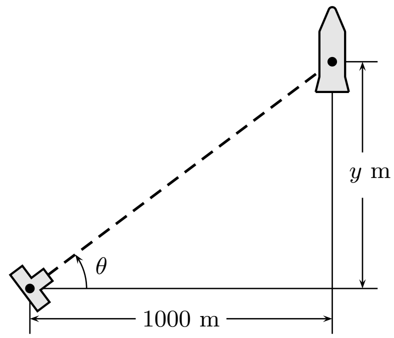


Figure 2.16: A camera following the tip of a rising rocket

Differentiating w.r.t t

$$\sec^2\theta \frac{d\theta}{dt} = \frac{1}{1000}\frac{dy}{dt}$$

When

$$y = 1000, \quad \theta = 45°, \quad \frac{dy}{dt} = 300$$

$$\left.\frac{d\theta}{dt}\right|_{y=1000} = \frac{300}{1000}\cdot\frac{1}{2}\cdot\frac{180}{\pi} \approx 8.59(\text{degree/sec})$$

□

2.13 Linearization and differential

When we study complicated functions, we can find its derivative at a point and study the tangent line instead. Thus, a tangent approximation is meaningful and call it linearization. Note that any nice curve, if enlarged, will look like a line.

Definition 2.13.1.

$$L(x) = f(a) + f'(a)(x-a)$$

is called the **linearization** of f at a.

Example 2.13.2. (1) Find linearization of $\cos x$ at $\pi/2$

(2) Find approx value of $\sqrt{1.003}$ using linearization of $\sqrt{1+x}$ at $x=0$.

(3) Find linearization of $\frac{1}{\sqrt[3]{x^4+1}}$ at $x=0$

(4) Find approx value of $\sqrt{4.8}$

(5) Find linearization of $\cos x$ at $\pi/2$. Ans $-x+\pi/2$.

(6) Show the linearization of $(1+x)^k$ is $1+kx$.

(7) Compare the linearization of $\sqrt{1+x^2}$ at $a=0$ and the linearization of $\sqrt{x}$ at $a=1$ and $\Delta x = x^2$.

Let $f(x) = \sqrt{1+x^2}$. Then $f'(x) = \frac{1}{2}\frac{2x}{\sqrt{1+x^2}}|_0 = 0$. Thus the linearization is $L(x) = 1$. But if we let $g(x) = \sqrt{x}$, then $g'(x) = \frac{1}{2\sqrt{x}}|_1 = \frac{1}{2}$. So the linearization is $L(x) = 1 + \frac{1}{2}x$. Thus $\sqrt{1+x^2} \approx 1 + \frac{1}{2}x^2$.

Differential

Definition 2.13.3. Let $y = f(x)$ be differentiable. The differential dx is an independent variable. The quantity dy defined by

$$dy := f'(x)dx$$

is called the **differential** of f.

The geometric meaning of differential is given in Figure 3.8.

$$\Delta y = f(a+dx) - f(a), \quad f(a+dx) = f(a) + \Delta y \approx f(a) + dy$$

We see that dy is precisely the change of the tangent line as x changes by an amount of $dx = \Delta x$. In other words, dy is an approximation of exact change Δy.

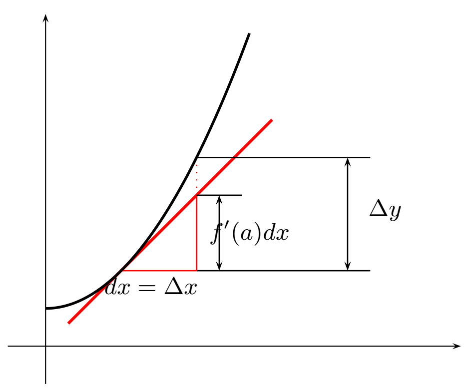


Figure 2.17: Differential $dy = f'(a)dx$ and Δy

Example 2.13.4. Find differential of

(1) $y = x^3 - \sin x$

(2) $y = \sin u(x)$

(3) $\tan(3x)$

(4) $d(\frac{x}{1+x})$.

Estimating with differentials

Radius of a circle is enlarged from 10 to 10.1. Use dA to estimate the increase in area. Compare with exact increase.

$A = \pi r^2$,

$$dA = 2\pi r dr = 2\pi(10)(0.1) = 2\pi$$

Actual increase is $A(10.1) - A(10) = \pi((10.1)^2 - 100) = 0.01\pi$.

Error in differential approximation

We estimate the change in y in more detail.

Theorem 2.13.5. *We have*

$$\Delta f = f'(a)\Delta x + \epsilon\Delta x$$

where $\epsilon \to 0$ as $\Delta x \to 0$.

Proof.

$$\begin{aligned}
\text{approximation error} &= \Delta f - df \\
&= \Delta f - f'(a)\Delta x \\
&= f(a+\Delta x) - f(a) - f'(a)\Delta x \\
&= \left(\frac{f(a+\Delta x) - f(a)}{\Delta x} - f'(a)\right)\Delta x \\
&= \epsilon\Delta x
\end{aligned}$$

Since f is differentiable, we know $\epsilon := \left(\frac{f(a+\Delta x)-f(a)}{\Delta x} - f'(a)\right)$ approaches 0 as Δx approaches 0. Thus

$$\overset{\text{true change}}{\Delta f} = \overset{\text{estimated change}}{f'(a)\Delta x} + \overset{\text{error}}{\epsilon\Delta x}$$

□

Proof of Chain rule

Assume $y = f(u)$ is a diff'ble function of u and $u = g(x)$ is a diff'ble function of x. Then the composite function $y = f(g(x))$ is diff'ble and by above theorem there exist ϵ_1, ϵ_2 which approach 0 as $\Delta u, \Delta x$ approaches 0(resp.) in such a way that

$$\begin{aligned}
\Delta y &= f'(u_0)\Delta u + \epsilon_1\Delta u \\
\Delta u &= g'(x_0)\Delta x + \epsilon_2\Delta x
\end{aligned}$$

Hence

$$\Delta y = (f'(u_0) + \epsilon_1)(g'(x_0) + \epsilon_2)\Delta x$$
$$\frac{\Delta y}{\Delta x} = (f'(u_0) + \epsilon_1)(g'(x_0) + \epsilon_2)$$

Let $\Delta x \to 0$. Then we obtain the Chain rule.

Example 2.13.6. Converting mass to energy: The Newton's law

$$F = m\frac{dv}{dt} = ma$$

is not exactly true when an object is moving at very high speed, because the mass increases with velocity. In Einstein's correction, the mass is

$$m = \frac{m_0}{\sqrt{1 - v^2/c^2}} \approx m_0(1 + \frac{v^2}{2c^2})$$

So the new mass is

$$m \approx m_0 + \frac{mv^2}{2c^2}$$

By multiplying c^2

$$(m - m_0)c^2 \approx \frac{1}{2}mv^2 = \frac{1}{2}mv^2 - \frac{1}{2}m \cdot 0^2 = \Delta(KE)$$

Thus the change in the energy by the mass corresponds to the change in the Kinetic Energy.

Power function

Derivative of a power function$y = u^r(x)$ for rational number r.

Theorem 2.13.7. *For any rational number r*

$$\frac{d}{dx}u^r = ru^{r-1}\frac{du}{dx}.$$

Proof. Assume $r > 0$. We have $r = p/q$ for some positive integer p and q. Hence $y = u^r = r^{p/q}$ can be written as $y^q = u^p$. Taking derivative w.r.t. x we obtain

$$qy^{q-1}\frac{dy}{dx} = pu^{p-1}\frac{du}{dx}$$
$$\frac{dy}{dx} = \frac{pu^{p-1}}{qy^{q-1}}\frac{du}{dx} = \frac{p}{q}\frac{y}{u}\frac{du}{dx} = ru^{r-1}\frac{du}{dx}$$

□

Example 2.13.8. Find the equation of tangent and normal line to the curve $y = (1 + x^{1/3})^{2/3}$ at $(-8, 1)$.

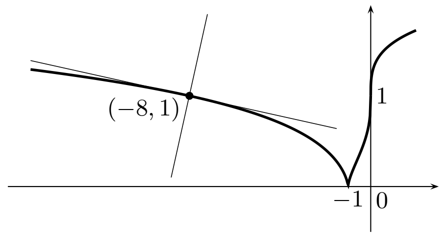


Figure 2.18: Tangent and normal

sol.

$$y'\Big|_{x=-8} = \frac{2}{3}(1+x^{1/3})^{-1/3}\cdot\frac{1}{3}x^{-2/3}\Big|_{x=-8} = -\frac{1}{18}$$

Tangent line is $y = -1/18(x+8)+1 = -x/18+5/9$ normal line is $y = 18(x+8)+1 = 18x+145$.

□

Example 2.13.9 (Slope of tangent in Polar coordinate)**.** Express dy/dx for the equation given in polar coordinate $r = f(\theta)$ in θ.

sol. Using the relation $x = r\cos\theta, y = r\sin\theta$, we see $x = f(\theta)\cos\theta, y = f(\theta)\sin\theta$. Hence the slope of tangent is

$$\frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta} = \frac{f'(\theta)\sin\theta + f(\theta)\sin\theta}{f'(\theta)\cos\theta - f(\theta)\sin\theta}$$

□

Chapter 3

Applications of Derivatives

3.1 Absolute Maximum, Minimum

Definition 3.1.1. Let f be defined on a domain D. Then f has **absolute(global) maximum** at c if

$$f(x) \le f(c), \quad \text{for all } x \in D.$$

It has **absolute(global) minimum** at c if

$$f(x) \ge f(c), \quad \text{for all } x \in D.$$

Theorem 3.1.2. *[Extreme Value Theorem] If f is continuous on a closed interval $[a, b]$. Then f assumes both absolute maximum M and absolute minimum m in $[a, b]$. In other words, there are numbers x_1, x_2 in $[a, b]$ such that $f(x_1) = m$ and $f(x_2) = M$, and $m \le f(x) \le M$ for all $x \in [a, b]$.*

Maximum or minimum are called **extreme values.**

Definition 3.1.3. Let f be defined on a domain D. Then f has **relative(local) maximum** at c if

$$f(x) \le f(c), \quad \text{for all } x \text{ in some interval containing } c.$$

It has **relative(local) minimum** at c if

$$f(x) \ge f(c), \quad \text{for all } x \text{ in some interval containing } c.$$

Theorem 3.1.4 (First derivative theorem)**.** *Suppose f is differentiable and if f has local max(min) at an interior point c, then $f'(c) = 0$.*

Proof. Suppose $f'(c) > 0$. Then there is an interval $I = (c - \delta, c + \delta)$ near c such that for all x in I

$$\frac{f(x) - f(c)}{x - c} > 0$$

hold. If $x \in I$ and $x > c$, then

$$f(x) - f(c) = (x - c)\frac{f(x) - f(c)}{x - c} > 0$$

Hence $f(x) > f(c)$ and f cannot have a maximum at c. Similarly, if $x < c$, then

$$f(x) - f(c) = (x - c)\frac{f(x) - f(c)}{x - c} < 0$$

Thus f cannot have minimum at c. This contradiction shows that f cannot have local extreme at c. The case $f'(c) < 0$ is dealt similarly. Hence we must have $f'(c) = 0$. □

Remark 3.1.5. This is not a necessary condition for a function to have a local extreme. Often, a function has an extreme value where f is not differentiable! So the points where f is not differentiable is also point of interest. Thus, we define

Definition 3.1.6. If $f'(c) = 0$ or $f'(c)$ does not exists, we say c is **critical point** of f.

Remark 3.1.7. How to find Absolute max(min) on I

(1) Evaluate f at all critical points

(2) Check all end points and compare

Example 3.1.8. Find max(min) of $f(x) = \left|4 - x^2\right|$ on $[-3, 3]$.

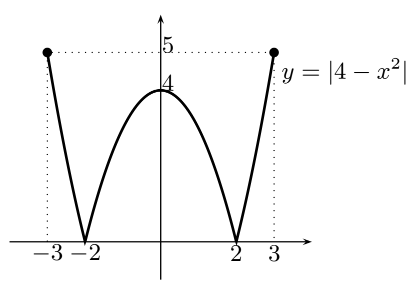


Figure 3.1: $y = |4 - x^2|$

sol. By Theorem 3.1.2, f has absolute minimum and maximum. Its graph is as in Figure 3.1. The critical points of f are -2, 0, 2. $f(-2) = f(2) = 0$, $f(0) = 4$ while at end points f assumes $f(-3) = f(3) = 5$. Hence maximum is 5 (at $x = -3$ or $x = 3$) and minimum is 0 (at $x = -2$ or $x = 2$).

□

Example 3.1.9. Find absolute extrema of $f(x) = 10x(2 - \ln x)$ on $[1, e^2]$.

sol. $f'(x) = 10(1 - \ln x)$. So critical point is e. Fig 3.1 Check end points $1, e^2$. We see maximum is

□

Example 3.1.10. Find absolute extrema of $f(x) = x^{2/3}$ on $[-2, 3]$.

sol. Derivative at 0 does not exist. By drawing graph we see $x = 0$ is cusp but local and absolute minimum.

□

Example 3.1.11. Find exrtema of $f(x) = x^{2/3}(1 - x)^{3/2}$ $(x \le 1)$.

sol. Since

$$\begin{aligned} f'(x) &= \frac{2}{3}x^{-1/3}(1-x)^{3/2} - \frac{2}{3}x^{2/3}(1-x)^{1/2} \\ &= x^{-1/3}(1-x)^{1/2}\left(\frac{2}{3}(1-x) - \frac{3}{2}x\right) \\ &= \frac{\sqrt{1-x}(4-13x)}{6x^{1/3}} \end{aligned}$$

Extreme points are $x = 0$, $x = 4/13$. Here $f(0) = 0$ is local min and $f(4/13) = (4/13)^{2/3}(9/13)^{3/2}$ is local max. (Refer to Fig 3.2)

□

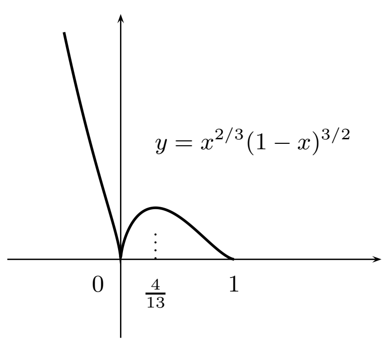


Figure 3.2: $y = x^{2/3}(1 - x)^{3/2}$

Remark 3.1.12. In above example the point $(0,0)$ satisfies

$$\lim_{x\to 0^+} f'(x) = +\infty, \quad \lim_{x\to 0^-} f'(x) = -\infty$$

This kind of point is called a **cusp** .

Example 3.1.13. Piping from Oil well from the see to the refinery on the shore.

Underwater pipe cost 500,000 per mile while land pipe cost 300,000 per mile

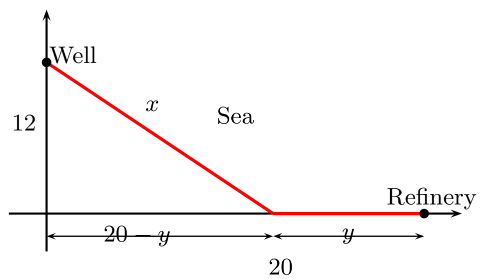


Figure 3.3: Refinery and hole in the sea

sol. cost is $c = 500,000x + 300,000y$. But $x^2 = 12^2 + (20-y)^2$. Thus $R(y)\cdot 10^{-5} = 5\sqrt{144+(20-y)^2}$

$$R'(y) = \frac{-5(20-y)}{\sqrt{144+(20-y)^2}} + 3$$

$y = 11$.

□

3.2 Mean Value Theorem

Theorem 3.2.1 (Rolle's Theorem)**.** *If f is continuous on $[a,b]$ and differentiable on (a,b). Suppose $f(a) = f(b)$ holds. Then there exists a point $c \in (a,b)$ such that $f'(c) = 0$.*

Proof. By theorem 3.1.2, f must attain maximum and minimum at some point c. Then there are two possibilities:

(1) max or min occurs at interior point.

(2) both max or min occur at end points.

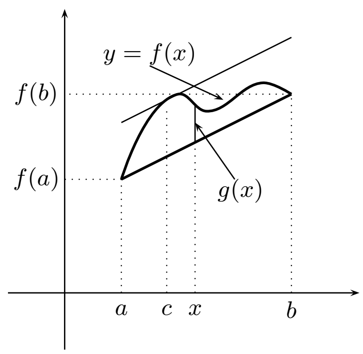


Figure 3.4:

In the first case, by theorem 3.1.4 there is a point where $f'(c) = 0$. In the second case, the maximum is equal to minimum since $f(a) = f(b)$. Hence f is constant and $f'(c) = 0$ for any $c \in (a, b)$. □

Theorem 3.2.2 (Mean Value Theorem)**.** *If f is continuous on $[a, b]$ and differentiable on (a, b). Then there exists a $c \in (a, b)$ such that*

$$f'(c) = \frac{f(b) - f(a)}{b - a}$$

Proof. Consider the line given by the equation

$$g(x) = \left(f(a) + \frac{f(b) - f(a)}{b - a}(x - a) \right)$$

Then (3.4 .)

$$h(x) = f(x) - g(x) = f(x) - \left(f(a) + \frac{f(b) - f(a)}{b - a}(x - a) \right)$$

h continuous on $[a, b]$ diff'ble in (a, b) and $h(a) = h(b)$. Hence by Thm 3.3.9 (Rolle's) there exists $c \in (a, b)$ such that $h'(c) = 0$. That is

$$h'(c) = f'(c) - \frac{f(b) - f(a)}{b - a}$$

□

Application of MVT

Theorem 3.2.3. *(1) If f satisfies $f'(x) = 0$ on an interval I, then f is constant on I.*

(2) If $f'(x) = g'(x)$ on I, then $f(x) = g(x) + C$, $x \in I$

Proof. (1) By Mean Value Theorem 3.2.2, we have for any $x, z \in I$,

$$\frac{f(z) - f(x)}{z - x} = f'(c)$$

for some $c \in (x, z) \subset I$. Since $f'(c) = 0$, we have $f(z) = f(x)$. This holds for any x, z, hence f is constant.

□

Definition 3.2.4. Suppose F defined on I. satisfies $F'(x) = f(x)$, F is and **anti-derivative** of f.

Example 3.2.5. Show that $\pi/4 + 3/25 < \tan^{-1} 4/3 < \pi/4 + 1/6$ holds.

sol. Apply Thm 3.2.2(MVT) to

$$f(x) = \tan^{-1} \text{ on } [1, 4/3]$$

Since $f'(x) = 1/(1 + x^2)$, there is a point $c \in (1, 4/3)$ such that

$$\frac{\tan^{-1}(4/3) - \pi/4}{4/3 - 1} = \frac{1}{1 + c^2}$$

holds. Hence

$$\frac{1}{1 + (4/3)^2} < 3\left(\tan^{-1}\frac{4}{3} - \frac{\pi}{4}\right) < \frac{1}{1 + 1^2}$$

$$\frac{3}{25} < \tan^{-1}\frac{4}{3} - \frac{\pi}{4} < \frac{1}{6}.$$

□

□

Proof of Log rule $\ln bx = \ln b + \ln x$

Consider

$$\frac{d}{dx}\ln(bx) = \frac{1}{x} = \frac{d}{dx}\ln x$$

So by above result,

$$\ln(bx) = \frac{1}{x} = \ln x + C$$

Place $x = 1$ to see $C = \ln b$.

Proof of Log rule $\ln^r x = r \ln x$

Consider

$$\frac{d}{dx}\ln^r x = \frac{1}{x^r}\frac{d}{dx}(x^r) = \frac{1}{x^r} r x^{r-1} = \frac{r}{x} = \frac{d}{dx}(r \ln x)$$

Thus $\ln^r x$ and $r \ln x$ have same derivative.

3.3 Monotonic function and derivative

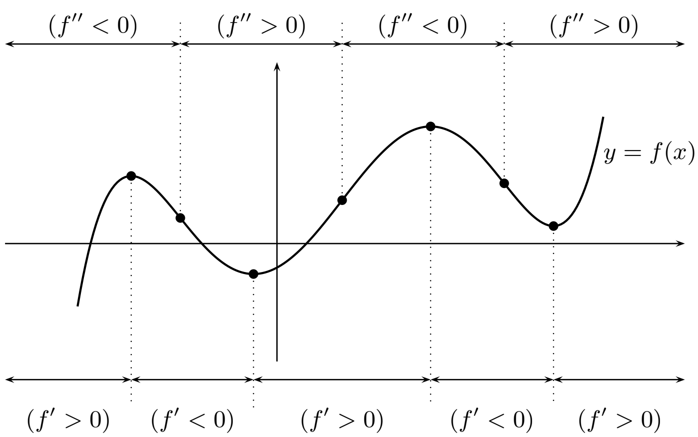


Figure 3.5: · ·

Definition 3.3.1. f is said to be a **increasing function** if $f(x) \le f(z)$ holds for all all x, z $(x < z)$.

First derivative test

Theorem 3.3.2. *(1) Suppose $f'(x) > 0$ for all x, then f is increasing on I.*

(2) Suppose $f'(x) < 0$ for all x, then f is decreasing.

Proof.

$$f'(c) = \frac{f(z) - f(x)}{z - x}$$

for some $c \in (x, z)$. Since $f'(c) \ge 0$ and $z > x$ we see $f(z) \ge f(x)$. If f is not one-to-one, there exists two point $v, w \in I$, $v < w$ such that $f(v) = f(w)$. Then $f'(x) = 0$ on (v, w). This contradicts to $f'(x) = 0$ at finite points. □

Example 3.3.3. Investigate the increase and decrease of $f(x) = x^5 - 5x^4 + 5x^3 + 1$.

sol. $f'(x) = 5x^4 - 20x^3 + 15x^2 = 5x^2(x-1)(x-3)$ We obtain the following table.

x	$\cdots$	1	$\cdots$	0	$\cdots$	3	$\cdots$
f'	+	0	−	0	−	0	+
f	$\nearrow$	2	$\searrow$	1	$\searrow$	−26	$\nearrow$

Hence f is mon. inc. on $x \le 1$ and mon. dec on $1 \le x \le 3$, mon. inc. on $3 \le x$.

□

First derivative test for local extrema

Theorem 3.3.4. *(1) If f' changes from negative to positive at c then f has local minimum at c*

(2) If f' changes from positive to negative at c then f has local maximum at c

Proof. f is decreasing on $[c-\delta, c]$ and increasing on $[c, c+\delta]$. Hence, $f(x)$ has local minimum at c. □

Example 3.3.5. (1) $f(x) = (x^2-3)e^x$

(2) $x^{1/3}(x-4)$

Second derivative test for extreme values

Theorem 3.3.6. *Suppose $f'(c) = 0$. Then*

(1) $f(c)$ is local maximum if $f''(c) < 0$.

(2) $f(c)$ is local minimum if $f''(c) > 0$.

Proof. (1) Since

$$f''(c) = \lim_{x \to c} \frac{f'(x)}{x-c} < 0$$

there is a δ such that

$$\frac{f'(x)}{x-c} < 0$$

for all $x \in (c-\delta, c+\delta)$ $(x \ne c)$. Hence if $c-\delta < x < c$ then $f'(x) > 0$, and if $c < x < c+\delta$ then $f'(x) < 0$. Hence f' change from positive to negative at c. By theorem 3.3.4, $f(c)$ is a local maximum. □

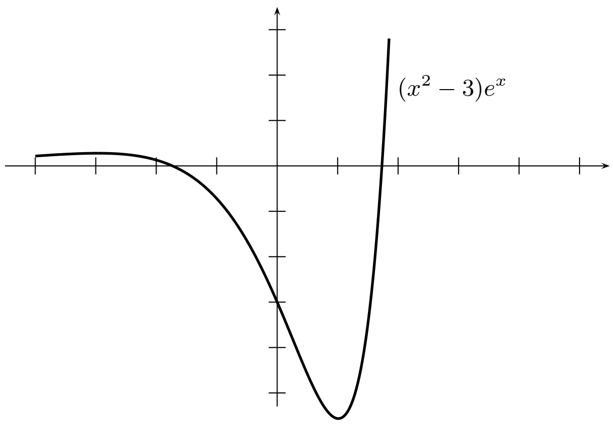


Figure 3.6: $y = (x^2-3)e^x$

3.4 Concavity and sketching

Definition 3.4.1. Assume f is differentiable. The graph of $y = f(x)$ is

(1) **concave up** if f' is increasing on I,

(2) **concave down** if f' is decreasing on I,

Theorem 3.4.2. *Suppose f'' exists on I. Then*

(1) If $f''(x) > 0$ for all $x \in I$, the graph of f is concave up.

(2) If $f''(x) < 0$ for all $x \in I$, the graph of f is concave down.

Definition 3.4.3. Assume f has a tangent line at a point c and if the concavity of $f(x)$ changes across c then it is a **inflection point**

Example 3.4.4. Sketch $f(x) = x + \sin x$.

sol. $f'(x) = 1 + \cos x$, $f''(x) = -\sin x$

x	$\cdots$	$-\pi$	$\cdots$	0	$\cdots$	π	$\cdots$	2π	$\cdots$	3π	$\cdots$
f'	$\cdots$	0	+	+	+	0	+	+	+	0	$\cdots$
f''	$\cdots$	0	+	0	$-$	0	+	0	$-$	0	$\cdots$
f	$\cdots$		$\nearrow$		$\nearrow$		$\nearrow$		$\nearrow$		$\cdots$

Hence f is increasing for all x and for integer n concave down on $(2n\pi, 2n\pi+\pi)$ concave up on $(2n\pi+\pi, 2n\pi+2\pi)$. Points of inflection are $(n\pi, n\pi)$. (n is integer)

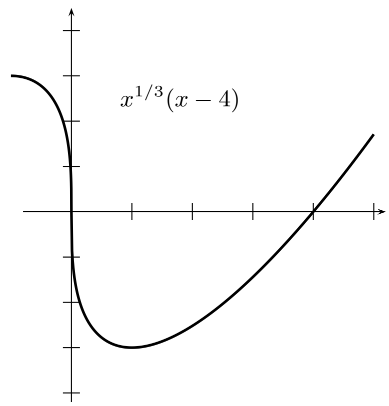


Figure 3.7: $y = x^{1/3}(x-4)$ and its inflection point

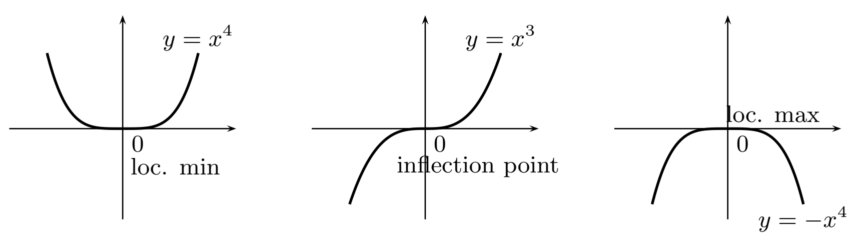


Figure 3.8: $y = x^4,\ y = x^3,\ y = -x^4$

□

Example 3.4.5. Show $f(\pi/4)$ is a local maximum when $f(x) = \sin^4 x \sin 4x + \cos^4 x \cos 4x$.

sol. Compute f' and f'' at $x = \pi/4$.

$$f'(x) = 4(\sin^3 x - \cos^3 x)\sin 5x,$$
$$f''(x) = 12 \sin x \cos x(\sin x + \cos x)\sin 5x + 20 \cos 5x(\sin^3 x - \cos^3 x)$$

Since $f'(\pi/4) = 0$ and $f''(\pi/4) = -6 < 0$. Hence $f(\pi/4)$ is local maximum.

□

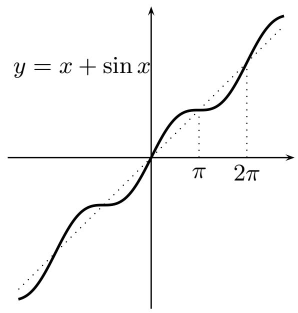


Figure 3.9: $y = x + \sin x$

Example 3.4.6. For $y = x^4$, an inflection point may not exists when $y'' = 0$.

Example 3.4.7. For $y = x^{1/3}$ an inflection point may exist even if y'' does not exist.

Example 3.4.8. Sketch the graph of $y = x^4 - 4x^3 + 10$.

Example 3.4.9. Sketch the graph of $f(x) = \frac{(x+1)^2}{1+x^2}$. Check inflection point and horizontal asymptote. Show details.

$$f'(x) = \frac{2(1-x^2)}{(1+x^2)^2}$$

Possible critical points are ± 2.

$$f"(x) = \frac{4x(x^2-3)}{(1+x^2)^3}$$

So possible point of inflection are $-\sqrt{3}, 0, \sqrt{3}$.

Example 3.4.10. Sketch the graph of $f(x) = e^{1/2x}$. Check inflection point and horizontal asymptote.

sol. Since f is not defined at $x = 0$, we investigate the behavior as $x \to^{\pm}$. Next we compute the derivative of $f = e^{\frac{1}{2x}}$. Since $f' = -\frac{1}{2x^2}e^{\frac{1}{2x}}$ we see f is decreasing function where it is defined. Now check second derivative:

$$f'' = \frac{1}{2x^3}\frac{4x+1}{2x}e^{\frac{1}{2x}}$$

Thus $x = -1/4$ is a point of inflection.

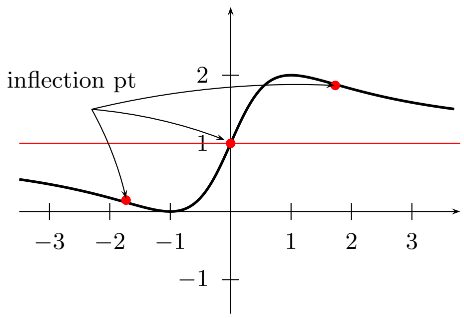
□

Theorem 3.4.11. *Suppose f'' continuous on I.*

(1) If $f'(c) = 0$ and $f''(c) < 0$ then f has local maximum

(2) If $f'(c) = 0$ and $f''(c) > 0$ then f has local minimum

(3) If $f'(c) = 0$ and $f''(c) = 0$ then the test fails. We need more information.



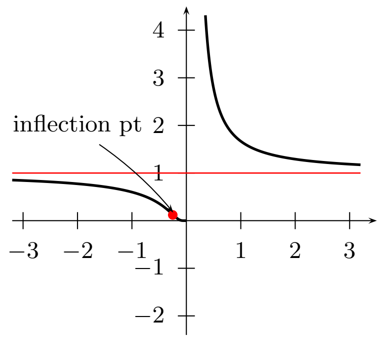


Figure 3.10: $y = \frac{(x+1)^2}{1+x^2}$ and $y = e^{\frac{1}{2x}}$

3.5 Parametrization

Suppose x, y are given by the equations

$$x = f(t), \quad y = g(t)$$

for $t \in I$. Then the set of points $(x, y) = (f(t), g(t))$ are called **parametric curve** and these are called **parametric equations.**

Example 3.5.1. (1) $x = \cos t,\ y = \sin t, 0 \le t \le 2\pi$.

(2) $x = \sqrt{t},\ y = t$.

Example 3.5.2. Sketch the path traced by the point $P(x, y)$ where
$x = t + \frac{1}{t}$, $y = t - \frac{1}{t}$.

Slopes of parametrized curves

If, f and g are differentiable and $f' \neq 0$. Then $t = f^{-1}(x)$ exists and $y(x) = (g \circ f^{-1})(x)$ is well defined. By implicit function theorem, the derivative of y w.r.t x is given by

$$\frac{dy}{dx} = \frac{dy}{dt} \cdot \frac{dt}{dx} = \frac{dy/dt}{dx/dt}$$

Second derivative for parametric equation

If $x = f(t)$, $y = g(t)$ define y as a twice differentiable function of x at the point where $dx/dt \neq 0$, then

$$\begin{aligned}\frac{d^2y}{dx^2} &= \frac{d}{dt}\left(\frac{dy}{dx}\right)/\frac{dx}{dt} \\ &= \frac{dy'/dt}{dx/dt}\end{aligned}$$

Example 3.5.3. Compute dy/dx and d^2y/dx^2 when $x = t - t^2$, $y = t - t^3$.

sol. Chain rule and implicit differentiation

$$\begin{aligned}\frac{dy}{dx} &= \frac{dy/dt}{dx/dt} = \frac{1-3t^2}{1-2t} \\ \frac{d^2y}{dx^2} &= \frac{d}{dt}\left(\frac{dy}{dx}\right)/\frac{dx}{dt} \\ &= \frac{d}{dt}\left(\frac{1-3t^2}{1-2t}\right)/(1-2t) \\ &= \frac{2-6t+6t^2}{(1-2t)^3}.\end{aligned}$$

□

3.6 Applied Optimziation

3.7 Intermediate form and L'Hopital's Rule

L'Hopital's Rule

When $f(a) = g(a) = 0$ or $f(a) = g(a) = \infty$, the limit

$$\lim_{x \to a} \frac{f(x)}{g(x)}$$

cannot be found by substituting a

Theorem 3.7.1 (L'Hopital's Rule: First form)**.** *Suppose that $f(a) = g(a) = 0$, that $f'(a)$, $g'(a)$ exist and $g'(a) \neq 0$. Then*

$$\lim_{x \to a} \frac{f(x)}{g(x)} = \frac{f'(a)}{g'(a)}.$$

Proof. $\lim_{x\to a}\frac{f(x)}{g(x)} = \frac{\lim_{x\to a}(f(x)-f(a))/(x-a)}{\lim_{x\to a}(g(x)-g(a))/(x-a)} = \frac{f'(a)}{g'(a)}$. □

Example 3.7.2. (1) $\lim_{x\to 0}\frac{\sqrt{1+x}-1}{x}\left(\frac{0}{0}\right) = \frac{1/2\sqrt{1+x}}{1}\Big|_{x=0} = \frac{1}{2}$.

(2) $\lim_{x\to 1}\frac{x^2-1}{x-1}\left(\frac{0}{0}\right) = \frac{2x}{1}\Big|_{x=1} = 2$.

(3) $\lim_{x\to(\pi/2)}\frac{\cos x}{\sin x - 1}\left(\frac{0}{0}\right) = \lim_{x\to(\pi/2)}\frac{-\sin x}{\cos x} = -\infty$.

But the first form of L'Hopital's rule cannot be used for the following case

$$\lim_{x\to\infty}\frac{x-\sin x}{x\sin x}\left(\frac{0}{0}\right)$$

because the limit of the ratio of the derivative $\frac{1-\cos x}{\sin x + x\cos x}$ is again $\left(\frac{0}{0}\right)$ form. Fortunately we can use the following result.

Theorem 3.7.3 (L'Hopital's Rule: Stronger form)**.** *Suppose that $f(a) = g(a) = 0$ and f, g are differentiable on (a, b). (The case $f'(c) = g'(c) = 0$ is allowed) and that $g'(x) \neq 0$ for $x \neq a$. Then*

$$\lim_{x\to a}\frac{f(x)}{g(x)} = \lim_{x\to a}\frac{f'(x)}{g'(x)}$$

as long as the rhs limit exists.

The proof is based on

Theorem 3.7.4 (Cauchy's Mean value theorem)**.** *Suppose f and g are conti in $[a, b]$, diff'ble in (a, b). If $g' \neq 0$ on (a, b) then $g(b) \neq g(a)$ and there exist $c \in (a, b)$ such that*

$$\frac{f(b)-f(a)}{g(b)-g(a)} = \frac{f'(c)}{g'(c)}$$

Proof. Suppose $g(b) = g(a)$ then by thm 3.2.2

$$g'(c) = \frac{g(b)-g(a)}{b-a} = 0$$

for some $c \in (a, b)$. This contradict to $g' \neq 0$. So, $g(b) \neq g(a)$. Next consider the function F defined by

$$F(x) = f(x) - f(a) - \left(\frac{f(b)-f(a)}{g(b)-g(a)}\right)(g(x)-g(a))$$

We apply Rolle's theorem 3.2.1 to F. F satisfies the condition of Rolle's thm. Hence there exist $c \in (a,b)$ such that $F'(c) = 0$. Since

$$F'(c) = f'(c) - \left(\frac{f(b)-f(a)}{g(b)-g(a)}\right) g'(c) = 0$$

we have

$$\frac{f'(c)}{g'(c)} = \frac{f(b)-f(a)}{g(b)-g(a)}$$

□

Proof. First show

$$\lim_{x\to c^+} \frac{f(x)}{g(x)} = \lim_{x\to c^+} \frac{f'(x)}{g'(x)}$$

When $c < x < b$ use thm 3.7.4(Cauchy' MVT) on $[c,x]$. Then there is $d \in (c,x)$ s.t.

$$\frac{f'(d)}{g'(d)} = \frac{f(x)-f(c)}{g(x)-g(c)} = \frac{f(x)}{g(x)}$$

and $d \to c^+$ as as $x \to c^+$

$$\begin{aligned}\lim_{x\to c^+} \frac{f(x)}{g(x)} &= \lim_{d\to c^+} \frac{f'(d)}{g'(d)} \\ &= \lim_{x\to c^+} \frac{f'(x)}{g'(x)}\end{aligned}$$

The following can be shown the same way.

$$\lim_{x\to c^-} \frac{f(x)}{g(x)} = \lim_{x\to c^-} \frac{f'(x)}{g'(x)}$$

□

Example 3.7.5. (1) $\lim_{x\to 0}\left(\frac{1}{\sin x} - \frac{1}{x}\right) (\infty - \infty) = \lim_{x\to\infty} \frac{x-\sin x}{x\sin x} \left(\frac{0}{0}\right)$

$$= \lim_{x\to 0} \frac{1-\cos x}{\sin x + x\cos x}\left(\frac{0}{0}\right) = \lim_{x\to 0}\frac{\sin x}{2\cos x - x\sin x} = \frac{0}{2} = 0.$$

Intermediate forms ∞/∞, $\infty\cdot 0$, $\infty - \infty$

Example 3.7.6.

(1) $\lim_{x\to 0} x \sin \frac{1}{x}$

(2) $\lim_{x\to 0^+} \sqrt{x}\ln x$

(3) $\lim_{x\to(\pi/2)^-} \frac{\tan x}{1+\tan x}\left(\frac{\infty}{\infty}\right) = \lim_{x\to(\pi/2)^-}\frac{\sec^2 x}{\sec^2 x} = 1.$

(4) $\lim_{x\to\infty} \frac{\pi/2 - \tan^{-1} x}{1/x} \left(\frac{0}{0}\right) = \lim_{x\to\infty} \frac{-1/(1+x^2)}{-1/x^2}$

$= \lim_{x\to\infty} \frac{x^2}{1+x^2} \left(\frac{\infty}{\infty}\right) = \lim_{x\to\infty} \frac{2x}{2x} = 1.$

(5) $\lim_{x\to\infty} \frac{\sqrt{9x+1}}{\sqrt{4x+1}} \left(\frac{\infty}{\infty}\right) = \lim_{x\to\infty} \frac{9/(2\sqrt{9x+1})}{4/(2\sqrt{4x+1})} \left(\frac{0}{0}\right) \qquad = \lim_{x\to\infty} \frac{-(81/4)(9x+1)^{-3/2}}{-4(4x+1)^{-3/2}} \left(\frac{0}{0}\right).$

(6) $\lim_{x\to\pi/2} \frac{\sec x}{1+\tan x}$

(7) $\lim_{x\to\infty} \frac{\ln x}{2\sqrt{x}}$

Wrong use of L'hopital's rule

$\lim_{x\to\infty} \frac{\pi/2 - \tan^{-1} x}{1/x} \left(\frac{0}{0}\right) = \lim_{x\to\infty} \frac{-1/(1+x^2)}{-1/x^2}$
$= \lim_{x\to\infty} \frac{x^2}{1+x^2} \left(\frac{\infty}{\infty}\right) = \lim_{x\to\infty} \frac{2x}{2x} = 1.$

In this case we can find limit as follows:

$$\lim_{x\to\infty} \frac{\sqrt{9x+1}}{\sqrt{4x+1}} = \sqrt{\lim_{x\to\infty} \frac{9x+1}{4x+1}} = \sqrt{\frac{9}{4}} = \frac{3}{2}.$$

$$\lim_{x\to 0} \frac{1-\cos x}{x+x^2} = \lim_{x\to 0} \frac{\sin x}{1+2x} = 0$$

But do not continue.

Intermediate forms 0^∞, ∞^0, $\infty - \infty$

Example 3.7.7. Use continuity

If $\lim_{x\to a} \ln f(x) = L$ then

$$\lim_{x\to a} f(x) = \lim_{x\to a} e^{\ln f(x)} = e^L.$$

Here a may be either finite or infinite.

(1) $\lim_{x\to 0^+} (1+x)^{1/x}$

(2) $\lim_{x\to\infty} x^{1/x}$

(3) $\lim_{x\to 0} \left(\frac{1}{\sin x} - \frac{1}{x}\right)$

Example 3.7.8. Sketch the graph of $x^{1/x}$. To do this, we first investigate the behavior of the function as $x \to 0^+$ and $x \to \infty$. To study the limit, we take the logarithm:

$$\lim_{x\to\infty} \ln x^{1/x} = \lim_{x\to\infty} \frac{\ln x}{x} = 0$$

Hence

$$\lim_{x\to\infty} x^{1/x} = \exp(\lim_{x\to\infty} \frac{\ln x}{x}) = e^0 = 1$$

Meanwhile

$$\lim_{x\to 0^+} \ln x^{1/x} = \lim_{x\to 0^+} \frac{\ln x}{x} = -\infty.$$

Hence

$$\lim_{x\to 0^+} x^{1/x} = \exp(\lim_{x\to 0^+} \frac{\ln x}{x}) = e^{-\infty} = 0.$$

To see the local extrema, take the derivative and find the critical point. $f'(x) = (1 - \ln x)/x^2 = 0$ for $x = e$. By checking the sign of $f'(x)$ near $x = e$, we conclude $x = e$ is a point of local maximum.

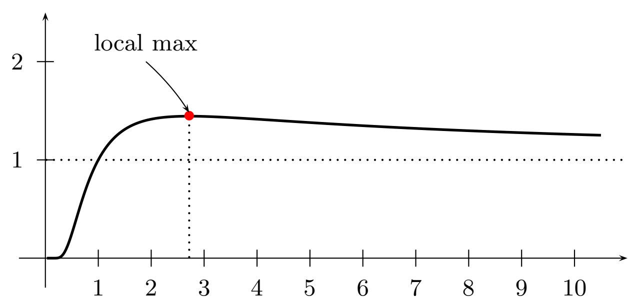


Figure 3.11: Graph of $y = x^{1/x}$

3.8 Newton's Methods

Let us now derive the Newton's method for solving $f(x) = 0$ using the tangent approximation. Given an approximate root x_n, we consider

$$y = f(x_n) - f'(x_n)(x - x_n)$$

The root of the linear equation $y = 0$ is

$$x = x_n - \frac{f(x_n)}{f'(x_n)}.$$

Hence we define

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}, \text{ when } f'(x_n) \neq 0.$$

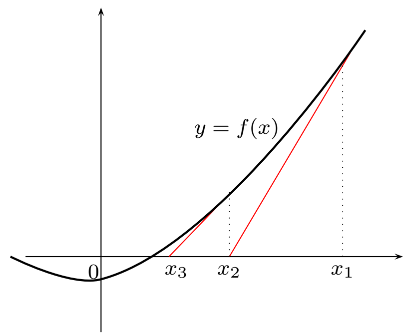


Figure 3.12: Newton's method

3.9 Hyperbolic functions

Definitions

For any function $f(x)$ we have

$$f(x) = \underbrace{\frac{f(x)+f(-x)}{2}}_{\text{Even function}} + \underbrace{\frac{f(x)-f(-x)}{2}}_{\text{Odd function}}$$

In particular, e^x has the form

$$e^x = \frac{e^x+e^{-x}}{2} + \frac{e^x-e^{-x}}{2} \tag{3.1}$$

Definition 3.9.1. A (hyperbolic function) is defined as

$$\textbf{hyperbolic cosine } \cosh x = \frac{e^x+e^{-x}}{2},$$
$$\textbf{hyperbolic sine } \sinh x = \frac{e^x-e^{-x}}{2},$$
$$\textbf{hyperbolic tangent } \tanh x = \frac{\sinh x}{\cosh x} = \frac{e^x-e^{-x}}{e^x+e^{-x}},$$
$$\textbf{hyperbolic cotangent } \coth x = \frac{1}{\tanh x} = \frac{e^x+e^{-x}}{e^x-e^{-x}},$$
$$\textbf{hyperbolic secant } \operatorname{sech} x = \frac{1}{\cosh x} = \frac{2}{e^x+e^{-x}},$$
$$\textbf{hyperbolic cosecant } \operatorname{csch} x = \frac{1}{\sinh x} = \frac{2}{e^x-e^{-x}}.$$

Proposition 3.9.2.

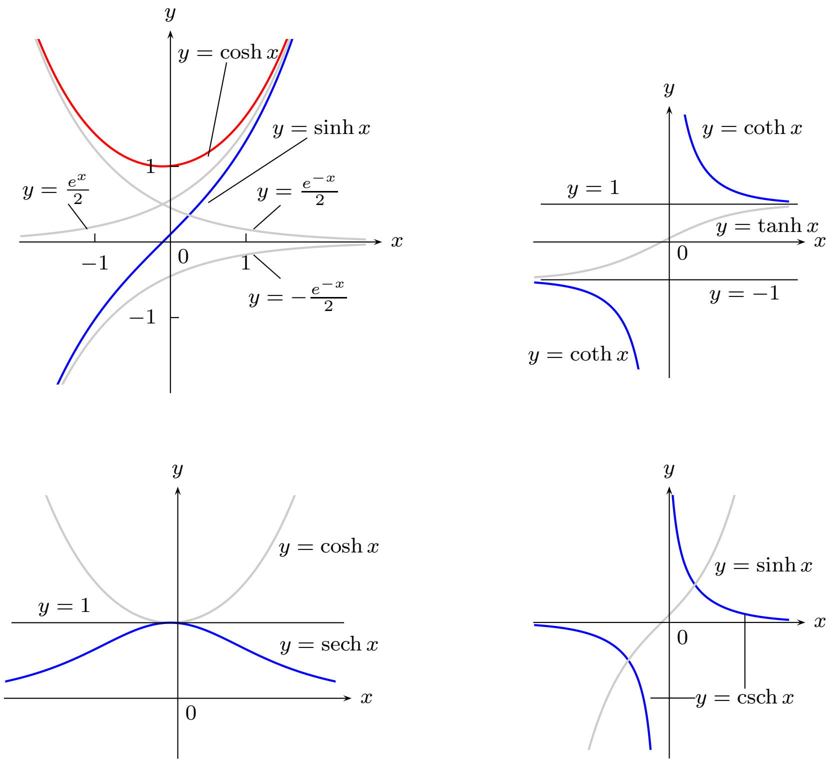


Figure 3.13: hyperbolic functions

(1) $\sinh 2x = 2 \sinh x \cosh x$

(2) $\cosh 2x = \cosh^2 x + \sinh^2 x$

(3) $\sinh^2 x = \dfrac{\cosh 2x - 1}{2}$

(4) $\cosh^2 x = \dfrac{\cosh 2x + 1}{2}$

(5) $\cosh^2 x - \sinh^2 x = 1$

(6) $\tanh^2 x = 1 - \operatorname{sech}^2 x$

(7) $\coth^2 x = 1 + \operatorname{csch}^2 x$

Derivatives of hyperbolic functions

Proposition 3.9.3.

(1) $\dfrac{d}{dx}(\sinh u) = \cosh u \dfrac{du}{dx}$

(2) $\dfrac{d}{dx}(\coth u) = -\operatorname{csch}^2 u \dfrac{du}{dx}$

(3) $\dfrac{d}{dx}(\cosh u) = \sinh u \dfrac{du}{dx}$

(4) $\dfrac{d}{dx}(\operatorname{sech} u) = -\operatorname{sech} u \tanh u \dfrac{du}{dx}$

(5) $\dfrac{d}{dx}(\tanh u) = \operatorname{sech}^2 u \dfrac{du}{dx}$

(6) $\dfrac{d}{dx}(\operatorname{csch} u) = -\operatorname{csch} u \coth u \dfrac{du}{dx}$

Proposition 3.9.4.

(1) $\displaystyle\int \sinh u\, du = \cosh u + C$

(2) $\displaystyle\int \cosh u\, du = \sinh u + C$

(3) $\displaystyle\int \operatorname{sech}^2 u\, du = \tanh u + C$

(4) $\displaystyle\int \operatorname{csch}^2 u\, du = -\coth u + C$

(5) $\displaystyle\int \operatorname{sech} u \tanh u\, du = -\operatorname{sech} u + C$

(6) $\displaystyle\int \operatorname{csch}\ u \coth u du = -\operatorname{csch} u + C$

Example 3.9.5. (1) The integral of $\sinh^2 x$ is obtained in a similar method to that of $\sin^2 x$.

$$\begin{aligned}\int_0^1 \sinh^2 x\, dx &= \int_0^1 \frac{\cosh 2x - 1}{2}\, dx \\ &= \frac{1}{2}\left[\frac{\sinh 2x}{2} - x\right]_0^1 \\ &= \frac{\sinh 2}{4} - \frac{1}{2}.\end{aligned}$$

(2) By definition of $\sinh x$

$$\begin{aligned}\int_0^{\ln 2} 4e^x \sinh x\, dx &= \int_0^{\ln 2} 4e^x \frac{e^x - e^{-x}}{2}\, dx = \int_0^{\ln 2} (2e^{2x} - 2)\, dx \\ &= \left[e^{2x} - 2x\right]_0^{\ln 2} \\ &= 3 - 2\ln 2.\end{aligned}$$

Inverse hyperbolic functions

$y = \sinh x$ is a one-to -one function from $(-\infty, \infty)$ onto $(-\infty, \infty)$. Thus its inverse function $y = \sinh^{-1} x$ is well defined on all of $(-\infty, \infty)$. Thus the **inverse hyperbolic sine** is

$$\sinh^{-1} x : (-\infty, \infty) \to (-\infty, \infty).$$

For $y = \cosh x$, we restrict the domain to $x \geq 0$. Then $y = \cosh^{-1} x$ exists on $[1, \infty)$. Thus the **inverse hyperbolic cosine** is

$$y = \cosh^{-1} : [1, \infty) \to [0, \infty).$$

Likewise if we restricted $y = \operatorname{sech} x$ to $x \geq 0$, then inverse function $y = \operatorname{sech}^{-1} x$ exists on $(0, 1]$. $y = \tanh x$, $y = \coth x$, $y = \operatorname{csch} x$ are all one-to-one on $(-\infty, \infty)$. Hence inverse functions

$$\begin{aligned} y &= \tanh^{-1} x : (-1, 1) \to (-\infty, \infty), \\ y &= \coth^{-1} x : |x| > 1 \to (-\infty, \infty) \\ y &= \operatorname{csch}^{-1} x : (-\infty, \infty) \setminus \{0\} \to (-\infty, \infty). \end{aligned}$$

Proposition 3.9.6.

(1) $\sinh^{-1} x = \ln\bigl(x + \sqrt{x^2 + 1}\bigr), \quad -\infty < x < \infty$

(2) $\cosh^{-1} x = \ln\bigl(x + \sqrt{x^2 - 1}\bigr), \quad x \geq 1$

(3) $\tanh^{-1} x = \dfrac{1}{2} \ln \dfrac{1 + x}{1 - x}, \quad |x| < 1$

(4) $\operatorname{sech}^{-1} x = \ln\left(\dfrac{1 + \sqrt{1 - x^2}}{x}\right), \quad 0 < x \leq 1$

(5) $\operatorname{csch}^{-1} x = \ln\left(\dfrac{1}{x} + \dfrac{\sqrt{1 + x^2}}{|x|}\right), \quad x \neq 0$

(6) $\coth^{-1} x = \dfrac{1}{2} \ln \dfrac{x + 1}{x - 1}, \quad |x| > 1$

Proof. We prove for $\sinh^{-1} x$ only.

$$\begin{gathered} y = \sinh x = \frac{e^x - e^{-x}}{2}, \\ e^x - e^{-x} = 2y, \\ e^{2x} - 2ye^x - 1 = 0. \end{gathered}$$

Solving this equation for e^x we have

$$e^x = y + \sqrt{y^2 + 1}.$$

Since $y - \sqrt{y^2 + 1}$ is negative, we choose positive sign. So $x = \ln(y + \sqrt{y^2 + 1})$ and hence $y = \ln(x + \sqrt{x^2 + 1})$ is the inverse of $\sinh^{-1} x$. $\square$

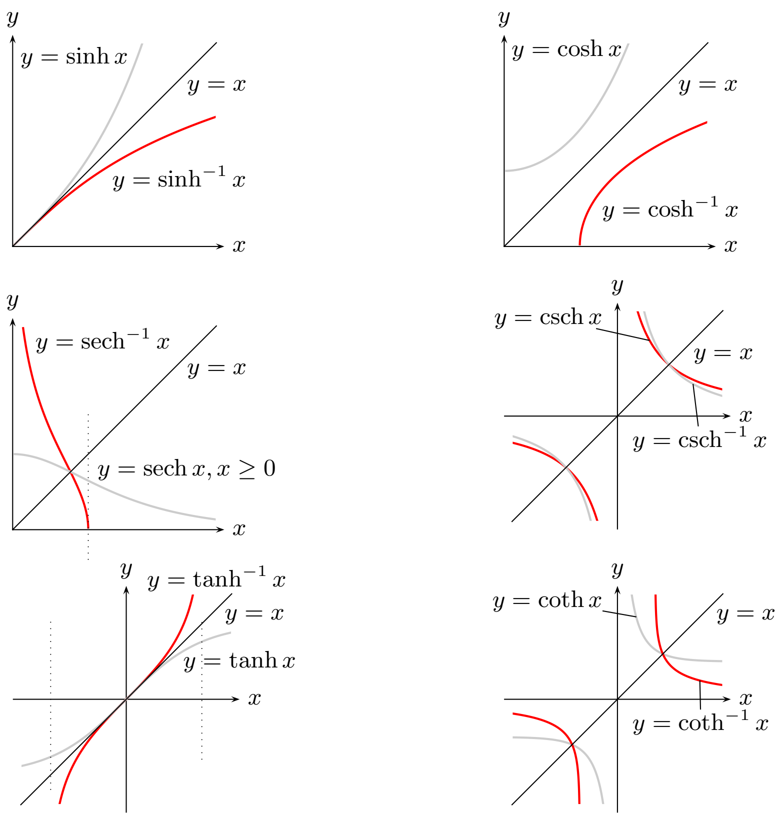


Figure 3.14: Inverse hyperbolic functions

Properties of inverse hyperbolic functions

Proposition 3.9.7.

(1) $\operatorname{sech}^{-1} x = \cosh^{-1} \dfrac{1}{x}$

(2) $\operatorname{csch}^{-1} x = \sinh^{-1} \dfrac{1}{x}$

(3) $\coth^{-1} x = \tanh^{-1} \dfrac{1}{x}$

Proposition 3.9.8.

(1) $\dfrac{d(\sinh^{-1} u)}{dx} = \dfrac{1}{\sqrt{1+u^2}} \dfrac{du}{dx}$

(2) $\dfrac{d(\cosh^{-1} u)}{dx} = \dfrac{1}{\sqrt{u^2-1}} \dfrac{du}{dx}, \quad u > 1$

(3) $\dfrac{d(\tanh^{-1} u)}{dx} = \dfrac{1}{1-u^2} \dfrac{du}{dx}, \quad |u| < 1$

(4) $\dfrac{d(\coth^{-1} u)}{dx} = \dfrac{1}{1-u^2}\dfrac{du}{dx}, \quad |u| > 1$

(5) $\dfrac{d(\operatorname{sech}^{-1} u)}{dx} = \dfrac{-du/dx}{u\sqrt{1-u^2}}, \quad 0 < u < 1$

(6) $\dfrac{d(\operatorname{csch}^{-1} u)}{dx} = \dfrac{-du/dx}{|u|\sqrt{1+u^2}}, \quad u \neq 0$

Proposition 3.9.9.

(1) $\displaystyle\int \frac{du}{\sqrt{1+u^2}} = \sinh^{-1} u + C$

(2) $\displaystyle\int \frac{du}{\sqrt{u^2-1}} = \cosh^{-1} u + C, \quad u > 1$

(3) $\displaystyle\int \frac{du}{1-u^2} = \begin{cases} \tanh^{-1} u + C, & |u| < 1 \\ \coth^{-1} u + C, & |u| > 1 \end{cases},$

(4) $\displaystyle\int \frac{du}{u\sqrt{1-u^2}} = -\operatorname{sech}^{-1}|u| + C = -\cosh^{-1}\left(\frac{1}{|u|}\right) + C$

(5) $\displaystyle\int \frac{du}{u\sqrt{1+u^2}} = -\operatorname{csch}^{-1}|u| + C = -\sinh^{-1}\left(\frac{1}{|u|}\right) + C$

Exercise 3.9.10. (1) Show

(a) $\sinh(x+y) = \sinh x \cosh y + \cosh x \sinh y$
(b) $\cosh(x+y) = \cosh x \cosh y + \sinh x \sinh y$

(2) Find derivatives

(a) $x - \tanh x^2$
(b) $\ln(\operatorname{sech} x)$
(c) $\frac{1}{2}\ln|\tanh x|$
(d) $\tan^{-1}(\sinh x)$
(e) $(x^2+1)\operatorname{sech}(\ln x)$
(f) $(1-x)\tanh^{-1} x$
(g) $\sinh^{-1}(\tan x)$
(h) $(1-x^2)\coth^{-1} x$
(i) $\tan^{-1}(\sin x)$, $-\pi/2 < x < \pi/2$
(j) $\operatorname{sech}^{-1}(\sin x)$, $0 < x < \pi/2$

(3) Find the following integrals

(a) $\displaystyle\int_{-1}^{0} \cosh(2x+1)\,dx$

(b) $\displaystyle\int_{-\pi}^{\pi} \tanh 2x\,dx$

(c) $\displaystyle\int_{0}^{1/2} 4e^{-x}\sinh x\,dx$

(d) $\displaystyle\int \frac{\cosh(\ln x)}{x}\,dx$

(e) $\displaystyle\int_{0}^{\ln 2} \tanh^2 x\,dx$

(f) $\displaystyle\int_{1}^{4} \frac{\cosh\sqrt{x}}{\sqrt{x}}\,dx$

(g) $\displaystyle\int_{\ln 2}^{\ln 3} \cosh^2 x\,dx$

(h) $\displaystyle\int_{-\ln 2}^{\ln 2} \sqrt{\cosh(2x-1)}\,dx$

(i) $\displaystyle\int \operatorname{sech}^3 5x\tanh 5x\,dx$

(j) $\displaystyle\int \tanh^3 x\,dx$

(4) Prove theorem 3.9.6.

(5) Find the integral.

(a) $\displaystyle\int_{0}^{1} \frac{dx}{\sqrt{1+2x^2}}$

(b) $\displaystyle\int_{4/5}^{12/13} \frac{dx}{x\sqrt{1-x^2}}$

(c) $\displaystyle\int_{5/4}^{5/3} \frac{dx}{\sqrt{x^2-1}}$

(d) $\displaystyle\int_{0}^{1/2} \frac{dx}{\sqrt{1-x^2}}$

(e) $\displaystyle\int_{5/4}^{2} \frac{dx}{\sqrt{1-x^2}}$

(f) $\displaystyle\int_{0}^{2\sqrt{3}} \frac{dx}{\sqrt{4+x^2}}$

(g) $\displaystyle\int_{1}^{2} \frac{dx}{x\sqrt{4+x^2}}$

(h) $\displaystyle\int_0^{\pi} \frac{\cos x dx}{\sqrt{1+\sin^2 x}}$

(6) Find the volume of the region when the graph $y = \operatorname{sech} x$ is rotated about x axis between $-\ln\sqrt{3} \le x \le \ln\sqrt{3}$.

(7) Find the centroid of the volume obtained when the region between $y = \tanh x$, $y = 1$, $x = 0$, $x = \ln\sqrt{199}$ is rotated about $y = 1$.

(8) Find the solution of differential equation.

$$x\frac{d^2y}{dx^2} = \sqrt{1+\left(\frac{dy}{dx}\right)^2}, \quad y(1) = 0, \quad \frac{dy}{dx}(1) = 0$$

Chapter 4

Integration

4.1 Anti-derivatives

If $F'(x) = f(x)$ then F is an anti-derivative. In general, $F(x) + C$ is an anti-derivative.

D.E. Find the solution of $y' = x^2$

Definition 4.1.1. The set of all anti-derivative of f is called **indefinite integral** and denoted by

$$\int f(x)\, dx.$$

4.2 Estimating with finite sums

4.3 Sigma and limit

4.4 Definite integral

Suppose $f(x)$ is a positive(temporarily) continuous function on $[a, b]$. Let A be the region under the graph of $y = f(x)$, between $x = a$, $x = b$. To find the area of the region, we divide $[a, b]$ into small subintervals. (See Figure 4.1). For example, uniform n-subintervals $\{x_0, x_1, \ldots, x_n\}$ are obtained if

$$x_i = a + i(b - a)/n, \quad i = 0, 1, \cdots, n.$$

Now the area A is approximated by

$$S_n = \sum_{i=0}^{n-1} f(c_i)(x_{i+1} - x_i).$$

In general, nonuniform intervals allowed provided that $\max_i |x_{i+1} - x_i| \to 0$ as $n \to \infty$.

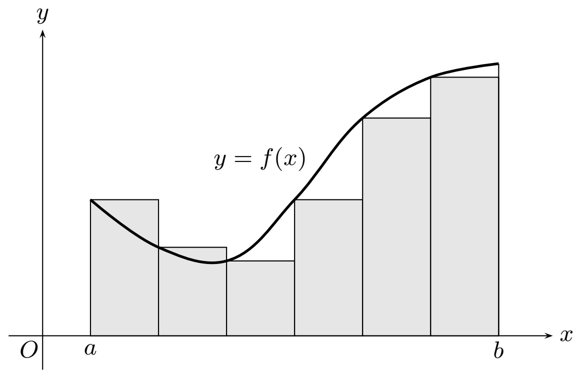


Figure 4.1: S_6

The set of points $\{x_0, x_1, \ldots, x_n\}$ such that

$$a = x_0 < x_1 < \cdots < x_{n-1} < x_n = b$$

is called a **partition** of $[a, b]$. Suppose $f(x)$ is defined on $[a, b]$ and $P = \{x_0, x_1, \ldots, x_n\}$ is any partition of $[a, b]$. For any set of points $\{c_1, c_2, \ldots, c_n\}$ satisfying $x_i \le c_i \le x_{i+1}$, we define the **Riemann sum** $R(f, P)$ of $f(x)$ w.r.t P (Figure 4.2) as

$$R(f, P) = \sum_{i=1}^{n} f(c_i)(x_{i+1} - x_i).$$

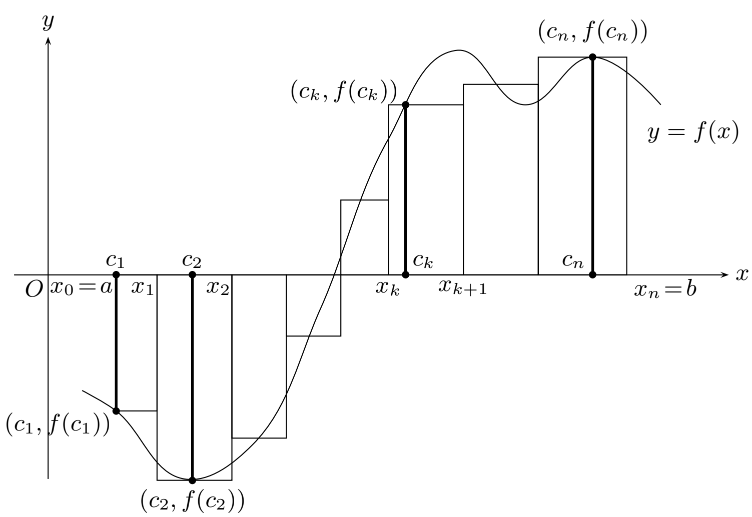


Figure 4.2: $R(f, P)$

Now as we can expect the Riemann sum is a good estimate for the area as long as all the subintervals are small. Thus we need a measure to control the length of subintervals.

Definition 4.4.1. The **norm** of a partition $P = \{x_0, x_1, \ldots, x_n\}$ is defined by

$$\|P\| = \max_{0 \le n-1} (x_{i+1} - x_i).$$

If $\|P\| \to 0$, then we can assure that the lengths of all the subintervals approach zero. And in this case we can consider the limit of the Riemann sum: Suppose the Riemann sum $R(f, P)$ of $f(x)$ approaches some value I as the norm $\|P\|$ of the partition P approaches 0. Then this limit is defined as the area under the graph. A precise definition using an argument similar to ϵ-δ used to define the limit of a function is given below:

Definition 4.4.2 (Definite Integral as the limit of a Riemann Sum)**.** Let $f(x)$ be defined on $[a, b]$. We say the value I is the **definite integral of** f **over** $[a, b]$ if the following holds:

> For any $\epsilon > 0$, there corresponds a $\delta > 0$ such that for any partition $P = \{x_0, x_1, \ldots, x_n\}$ satisfying $\|P\| < \delta$ and for any choice of c_i, $(x_i \le c_i \le x_{i+1}, i = 0, \cdots, n-1)$, we have $\|R(f, P) - I\| < \epsilon$.

This value I is denoted by $\int_a^b f(x)\,dx$ and called the **definite integral** of $f(x)$ on $[a, b]$. In this case, we say $f(x)$ is **integrable** on $[a, b]$.

We write it as

$$\int_a^b f(x)\,dx = \lim_{\|P\| \to 0} \sum_{i=0}^{n-1} f(c_i)(x_{i+1} - x_i)$$

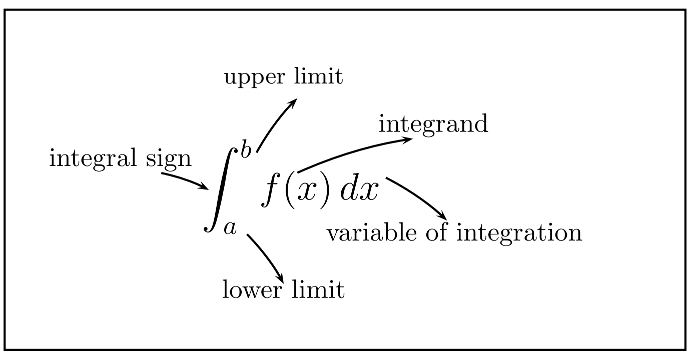


Figure 4.3: Integral of f from a to b

Non-integrable function

$$f(x) = \begin{cases} 0, & x \text{ rational number} \\ 1, & x \text{ irrational number} \end{cases}$$

Theorem 4.4.3 (Definite integral)**.** *The following holds:*

(1) $\int_a^b (f(x)+g(x))\,dx = \int_a^b f(x)\,dx + \int_a^b g(x)\,dx$

(2) $\int_a^b kf(x)\,dx = k\int_a^b f(x)\,dx$

(3) $\int_a^c f(x)\,dx = \int_a^b f(x)\,dx + \int_b^c f(x)\,dx,\ (a \le b \le c)$

(4) *If* $f(x) \ge 0$ *on* $[a,b]$, *then* $\int_a^b f(x)\,dx \ge 0$

(5) $\min_{x\in[a,b]} f(x)\cdot(b-a) \le \int_a^b f(x)\,dx \le \max_{x\in[a,b]} f(x)\cdot(b-a)$

(6) $\left|\int_a^b f(x)\,dx\right| \le \int_a^b |f(x)|\,dx$

Definition 4.4.4.

$$\int_a^a f(x)\,dx = 0, \qquad \int_b^a f(x)\,dx = -\int_a^b f(x)\,dx.$$

Example 4.4.5. Find $\int_0^a x^2\,dx$ by definition.

sol. Suppose $P = \{x_0, x_1, \ldots, x_n\}$ is a uniform partition of $[0,a]$. Then we have $x_i = i\,a/n$ and

$$\begin{aligned} \sum_{i=1}^n x_i^2(x_i - x_{i-1}) &= \sum_{i=1}^n \left(\frac{i\cdot a}{n}\right)^2 \frac{a}{n} \\ &= \frac{a^3}{n^3}\sum_{i=1}^n i^2 \\ &= \frac{a^3(n+1)(2n+1)}{6n^2} \end{aligned}$$

So

$$\begin{aligned} \int_0^a x^2\,dx &= \lim_{n\to\infty}\sum_{i=1}^n x_i^2(x_i - x_{i-1}) \\ &= \lim_{n\to\infty}\frac{a^3(n_+1)(2n+1)}{6n^2} \\ &= \frac{a^3}{3} \end{aligned}$$

□

□

Theorem 4.4.6 (Mean value theorem for integral)**.** *Let $a < b$. If $f(x)$ is conti. on closed interval $[a, b]$ then there is a c in $[a, b]$*

$$f(c) = \frac{1}{b-a}\int_a^b f(x)\,dx$$

This value is called the average of f on $[a, b]$ denoted by $av(f)$.

Proof. Since $f(x)$ is continuous on $[a, b]$ there are min and max;

$$f(x_0) = \min_{x\in[a,b]} f(x), \quad f(x_1) = \max_{x\in[a,b]} f(x)$$

for some x_0, x_1 in $[a, b]$.

First, if $x_0 = x_1$ then $f(x)$ is constant and the equality holds for all $c \in [a, b]$. Suppose $x_0 < x_1$. Then by intermediate value theorem, $f(x)$ assumes all values between $f(x_0)$ and $f(x_1)$ in $[x_0, x_1]$. Since

$$f(x_0) \le \frac{1}{b-a}\int_a^b f(x)\,dx \le f(x_1)$$

So there is a c such that

$$f(c) = \frac{1}{b-a}\int_a^b f(x)\,dx$$

for some $c \in [x_0, x_1]$.

The case $x_0 < x_1$ is the same. □

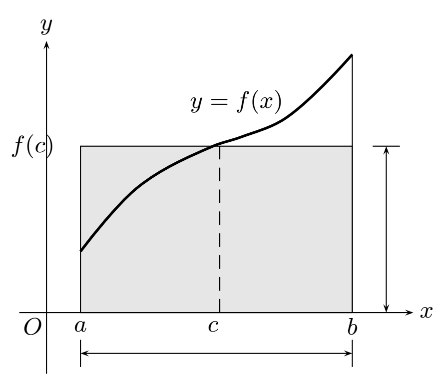


Figure 4.4: MVT for Integral

4.5 Fundamental theorem of Calculus

If f is integrable on I, the integral from a fixed point a to another point x defines a new function $F(x) = \int_a^x f(t)\,dt$. We have the following theorem.

Theorem 4.5.1 (Fundamental theorem of Calculus I)**.** *Suppose $f(x)$ is continuous on $[a, b]$. Then the function $F(x)$ defined by*

$$F(x) = \int_a^x f(t)\,dt$$

is differentiable on (a, b) and

$$F'(x) = \frac{d}{dx}\int_a^x f(t)\,dt = f(x)$$

Proof. By definition,

$$F'(x) = \lim_{h\to 0}\frac{F(x+h)-F(x)}{h}$$

$$\begin{aligned}\frac{F(x+h)-F(x)}{h} &= \frac{1}{h}\left(\int_a^{x+h} f(t)\,dt - \int_a^x f(t)\,dt\right)\\ &= \frac{1}{h}\int_x^{x+h} f(t)\,dt\end{aligned}$$

By the integral MVT there is c between x and $x+h$ s.t.

$$\frac{1}{h}\int_x^{x+h} f(t)\,dt = f(c).$$

Now $f(c)$ approaches $f(x)$ as $h \to 0$. Hence

$$\begin{aligned}F'(x) &= \lim_{h\to 0}\frac{F(x+h)-F(x)}{h}\\ &= \lim_{h\to 0}\frac{1}{h}\int_x^{x+h} f(t)\,dt\\ &= f(x)\end{aligned}$$

□

Theorem 4.5.2 (Fundamental theorem of Calculus II)**.** *Suppose $f(x)$ is conti on $[a, b]$. If $F(x)$ is the anti-derivative of $f(x)$, then*

$$\int_a^b f(x)\,dx = F(b) - F(a).$$

Proof. Set $G(x) = \int_a^x f(t)\,dt$. Then by (1), $G(x)$ is anti-derivative of $f(x)$ on $[a, b]$. Since $F(x)$ is also an anti-derivative of $f(x)$, we have

$$G(x) = F(x) + C.$$

But $G(a) = F(a) + C = \int_a^a f(t)\,dt = 0$, hence $C = -F(a)$. Hence $G(b) = \int_a^b f(t)\,dt = F(b) - F(a)$. □

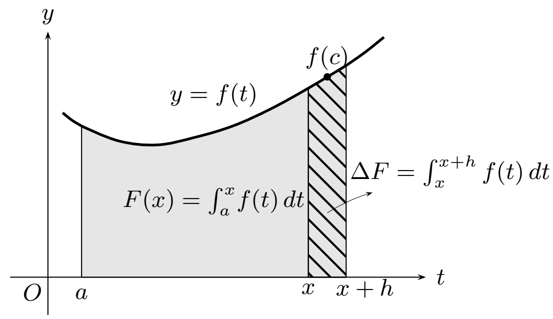


Figure 4.5: $\frac{\Delta F}{\Delta x} \approx f(c) \to f'(x)$

Example 4.5.3. (1) $\displaystyle \frac{d}{dx}\int_1^{x^2} \cos t\, dt.$

(2) $\displaystyle \frac{d}{dx}\int_{3+x^2}^{9} \frac{1}{1+e^t}\, dt.$

Example 4.5.4. Find derivative of $A(x)$ when $h(t)$ is continuous and $u(x)$, $v(x)$ are differentiable. Find the derivative of

$$A(x) = \int_{u(x)}^{v(x)} h(t)\, dt$$

sol. Let $H(t)$ be an antiderivative of $h(t)$. Then $A(x) = H(v(x)) - H(u(x))$ and $A'(x) = h(v(x))v'(x) - h(u(x))u'(x)$.

□

In other words,

$$\frac{d}{dx}\int_{u(x)}^{v(x)} h(t)\, dt = h(v(x))v'(x) - h(u(x))u'(x).$$

Example 4.5.5. $\displaystyle \int_a^b e^x\, dx = e^b - e^a.$

Total area.

4.6 Indefinite integrals and substitution

Recall

$$\frac{d}{dx}\left(\frac{u^{n+1}}{n+1}\right) = u^n \frac{du}{dx}$$

So we have

$$\int u^n du = \frac{u^{n+1}}{n+1} + C$$

Example 4.6.1. $\int \sqrt{1+x^2}2xdx = \int u^{1/2}du =$

chain rule

$$\frac{d}{dx}F(u(x)) = \frac{d}{du}F(u)\frac{d}{dx}u(x)$$

$$\int \frac{d}{du}F(u)\frac{d}{dx}u(x)\,dx = F(u(x)) + C$$

$$\int \frac{d}{du}F(u)\,du = F(u) + C$$

$$\int \frac{d}{du}F(u)\frac{d}{dx}u(x)\,dx = \int \frac{d}{du}F(u)\,du$$

Proposition 4.6.2. $\displaystyle\int f(g(x)) \cdot g'(x)\,dx = \int f(u)\,du.$

Proof. Let $F(u)$ be an anti-derivative of $f(u)$.

$$\frac{d}{dx}F(g(x)) = f(g(x))\,g'(x)$$

$$\int f(g(x))\,g'(x)\,dx = \int f(u)\,du$$

□

Example 4.6.3. $\displaystyle\int_0^{\pi/2} e^{\sin x}\cos x\,dx = \int_0^1 e^u\,du.$

Example 4.6.4. Find

$$\int \sec x\,dx$$

The idea is to multiply $\sec x + \tan x$ both the numerator and denominator:

$$\begin{aligned}\int \sec x\,dx &= \int \sec x \cdot \frac{\sec x + \tan x}{\sec x + \tan x}\,dx \\ &= \int \frac{\sec^2 x + \sec x \tan x}{\sec x + \tan x}\,dx \\ &= \int \frac{du}{u} \\ &= \ln|\sec x + \tan x| + C\end{aligned}$$

Similarly, we obtain

$$\int \csc x\,dx = -\ln|\csc x + \cot x| + C$$

Example 4.6.5.

$$\int \sin^2 x\,dx = \int \frac{1-\cos 2x}{2}dx$$

4.7 Area between curves

If $f(x) > 0$, $\int_a^b f(x)\,dx$ is the area defined by

$$y = f(x), \quad a \leq x \leq b$$

In general, when $f(x) \leq g(x)$ on $[a, b]$, the area defined by

$$\int_a^b (g(x) - f(x))\,dx$$

Use of symmetry

If there is any symmetry it is useful to take advantage of it.

Proposition 4.7.1.

$$\textit{For even function,}\ \int_{-a}^{a} f(x)\,dx = 2\int_0^a f(x)\,dx$$

$$\textit{For odd function,}\ \int_{-a}^{a} f(x)\,dx = 0$$

Integration w.r.t y

When the region is determined by functions of y, we need to integrate w.r.t. y.

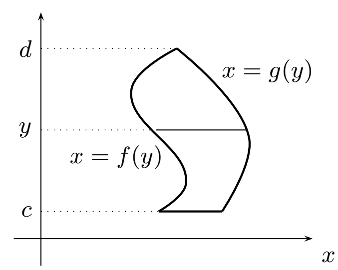


Figure 4.6: Region by between functions of y

$$A = \int_c^d (g(y) - f(y))\,dy$$