제 8 장

Vector Analysis

제 1 절 Green's Theorem

Given a y-simple region, D, we can divide the the boundary C into four parts C_1, C_2 B_1 and B_2. (fig 8.1) The boundary of a x-simple region can be similarly divided (fig 8.2)

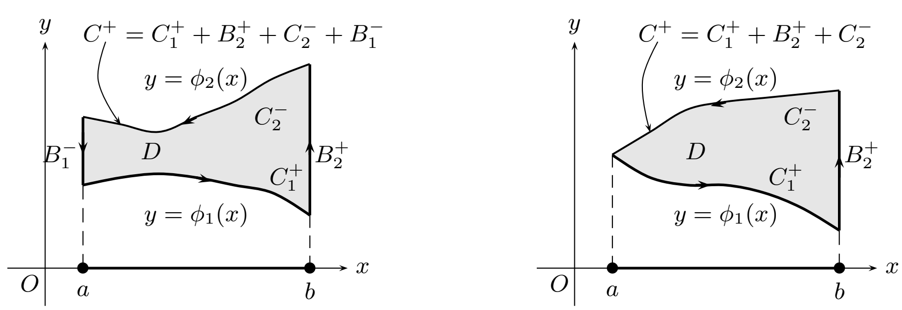


그림 8.1: y-simple region and boundary

Green's Theorem

Lemma 1.1. *Let D be a y-simple region and C^+ be its boundary with positive orientation. If P is a $\mathcal{C}^1$-function on D, then*

$$\int_{C^+} P dx = -\iint_D \frac{\partial P}{\partial y} dx dy$$

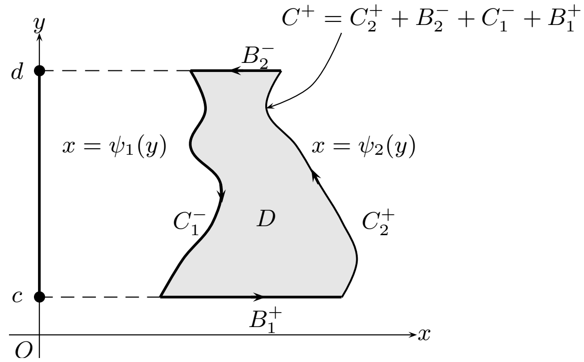


그림 8.2: x-simple region and boundary

Proof. Suppose

$$D = \{(x, y)|\ a \le x \le b, \phi_1(x) \le y \le \phi_2(x)\}$$

We decompose (fig 8.1) C^+ as $C^+ = C_1^+ + B_2^+ + C_2^- + B_1^-$ and use Fubini's theorem, we can evaluate the double integral as an iterated integral

$$\begin{aligned}\iint_D \frac{\partial P(x,y)}{\partial y} dxdy &= \int_a^b \int_{\phi_1(x)}^{\phi_2(x)} \frac{\partial P(x,y)}{\partial y} dydx \\ &= \int_a^b [P(x, \phi_2(x)) - P(x, \phi_1(x))]dx.\end{aligned}$$

On the other hand C_1^+ can be parameterized as $x \to (x, \phi_1(x)), a \le x \le b$ and C_2^+ can be parameterized as $x \to (x, \phi_2(x)), a \le x \le b$. Hence

$$\int_a^b P(x, \phi_i(x))dx = \int_{C_i^+} P(x, y)dx, \quad i = 1, 2$$

By reversing orientations

$$-\int_a^b P(x, \phi_2(x))dx = \int_{C_2^-} P(x, y)dx$$

Hence

$$\iint_D \frac{\partial P}{\partial y} dydx = -\int_{C_1^+} Pdx - \int_{C_2^-} Pdx$$

Since x is constant on B_2^+, B_1^-

$$\int_{B_2^+} Pdx = 0 = \int_{B_1^-} Pdx$$

So

$$\begin{aligned}\int_{C^+} Pdx &= \int_{C_1^+} Pdx + \int_{B_2^+} Pdx + \int_{C_2^-} Pdx + \int_{B_1^-} Pdx \\ &= \int_{C_2^+} Pdx + \int_{C_1^-} Pdx.\end{aligned}$$

Hence

$$\iint_D \frac{\partial P}{\partial y} dxdy = -\int_{C_1^+} Pdx - \int_{C_2^-} Pdx = -\int_{C^+} Pdx$$

The proof is completed. □

Lemma 1.2. *Let D be a x-simple region with boundary C^+. Then if Q is $\mathcal{C}^1$-function in D,*

$$\int_{C^+} Qdy = \iint_D \frac{\partial Q}{\partial x} dxdy$$

Proof. This is same as corollary 5.3.1 with x, y interchanged with negative direction. (figure 8.2). □

From these Lemmas we obtain Green's theorem for simple region.

Theorem 1.3. ***(Green's theorem)*** *Let D be a simple region boundary ∂D. Suppose P and $Q : D \to \mathbb{R}$ are $\mathcal{C}^1$, then*

$$\int_{\partial D} Pdx + Qdy = \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dxdy$$

Generalizing Green's theorem

In fact, Green's theorem holds for more general region. For example, Green's theorem can be used for a region with a hole. One cuts the region so that each region is simple.

Theorem 1.4. ***(Green's theorem for general region)*** *Let D be a region which can be divided into a few pieces of regions where Green's theorem apply, and let ∂D be the boundary. Suppose P and $Q : D \to \mathbb{R}$ are $\mathcal{C}^1$ then*

$$\int_{\partial D} Pdx + Qdy = \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dxdy$$

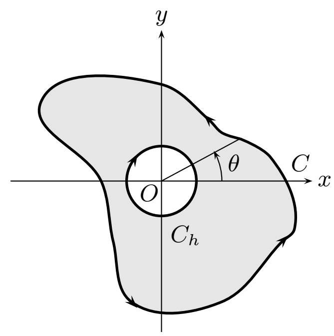


그림 8.3: Region between C_h and C

Proof. Assume D is the union of simple regions $D_i,\ i = 1, 2, \ldots, n$ whose boundary ∂D is the sum of $\partial D_i,\ i = 1, 2, \ldots, n$. In other words,

$$D = \sum_{i=1}^{n} D_i \ , \qquad \partial D = \sum_{i=1}^{n} \partial D_i.$$

So

$$\int_{\partial D} Pdx + Qdy = \sum_{i=1}^{n} \int_{\partial D_i} Pdx + Qdy$$

and

$$\iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dxdy = \sum_{i=1}^{n} \iint_{D_i} \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dxdy.$$

Since each D_i is simple, we have by corollary 5.3.1

$$\int_{\partial D_i} Pdx = \iint_{D_i} -\frac{\partial P}{\partial y} dxdy$$

and

$$\int_{\partial D_i} Qdy = \iint_{D_i} \frac{\partial Q}{\partial x} dxdy$$

We add all these terms to get the result. □

Example 1.5. Verify Green's theorem for

$$P(x, y) = \frac{-y}{x^2 + y^2}, \quad Q(x, y) = \frac{x}{x^2 + y^2}$$

on $D = \{(x, y) |\ h^2 \le x^2 + y^2 \le 1\}$, $0 < h < 1$.

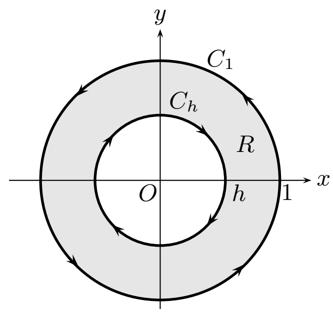


그림 8.4: Concentric region for Green's theorem

sol. The boundary of D consists of two circles(fig 8.4)

$$C_1 : x = \cos t, \qquad y = \sin t, \qquad 0 \le t \le 2\pi$$
$$C_h : x = h\cos t, \qquad y = h\sin t, \qquad 0 \le t \le 2\pi$$

The curve $\partial D = C_h \cup C_1$ is counterclockwise along C_1, and clockwise along C_h. For $(x, y) \neq (0, 0)$ P, Q are class $\mathcal{C}^1$. Since

$$\frac{\partial P}{\partial y} = \frac{(x^2+y^2)(-1)+2(2y)}{(x^2+y^2)^2} = \frac{y^2-x^2}{(x^2+y^2)^2} = \frac{\partial Q}{\partial x}$$

we have

$$\iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dxdy = \int_D 0\ dxdy = 0$$

On the other hand

$$\begin{aligned}\int_{\partial D} Pdx + Qdy &= \int_{C_1} \frac{xdy - ydx}{x^2+y^2} + \int_{C_h} \frac{xdy - ydx}{x^2+y^2} \\ &= \int_0^{2\pi} (\cos^2 t + \sin^2 t)dt + \int_{2\pi}^{0} \frac{h^2(\cos^2 t + \sin^2 t)}{h^2} dt \\ &= 2\pi - 2\pi = 0\end{aligned}$$

Hence

$$\int_{\partial D} Pdx + Qdy = 0 = \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dxdy.$$

Example 1.6. Evaluate $\int_C \frac{xdy-ydx}{x^2+y^2}$. Since the integrand is e not continuous at $(0,0)$ we cannot use Green's theorem. But use Green's theorem on the region between two two curves s (fig 8.3) to see

$$\int_C (Pdx + Qdy) = -\int_{C_h} (Pdx + Qdy)$$

Now the integral $-\int_{C_h}(Pdx + Qdy)$ can be computed by polar coordinate: From

$$\begin{aligned} x &= r\cos\theta, \qquad y = r\sin\theta, \\ dx &= -r\sin\theta d\theta + \cos\theta dr, \\ dy &= r\cos\theta d\theta + \sin\theta dr \end{aligned}$$

we see

$$\frac{xdy - ydx}{x^2 + y^2} = \frac{r^2(\cos^2\theta + \sin^2\theta)}{r^2} d\theta = d\theta$$

Hence

$$\int_C \frac{xdy - ydx}{x^2 + y^2} = 2\pi$$

Area

Theorem 1.7. *If C is a simple closed curve bounding a region D, then the area A is*

$$A = \frac{1}{2}\int_{\partial D} xdy - ydx.$$

Proof. Let $P(x, y) = -y, Q(x, y) = x$. Then

$$\begin{aligned} \frac{1}{2}\int_{\partial D} xdy - ydx &= \frac{1}{2}\iint_D \left(\frac{\partial x}{\partial x} - \frac{\partial(-y)}{\partial y}\right) dxdy \\ &= \frac{1}{2}\iint_D (1+1)dxdy = \int_D dxdy = A \end{aligned}$$

□

Example 1.8. Find the area of the region enclosed by $x^{2/3} + y^{2/3} = a^{2/3}$.

sol. Let $x = a\cos^3\theta$, $y = a\sin^3\theta$, $(0 \le \theta \le 2\pi)$. Then

$$\begin{aligned} A &= \frac{1}{2}\int_{\partial D} xdy - ydx \\ &= \frac{1}{2}\int_0^{2\pi} [(a\cos^3\theta)(3a\sin^2\theta\cos\theta) - (a\sin^3\theta)(-3a\cos^2\theta\sin\theta)]d\theta \\ &= \frac{3}{2}a^2\int_0^{2\pi} (\sin^2\theta\cos^4\theta + \cos^2\theta\sin^4\theta)d\theta \\ &= \frac{3}{8}a^2\int_0^{2\pi} \sin^2 2\theta d\theta = \frac{3}{8}\pi a^2. \end{aligned}$$

Hence area is $3\pi a^2/8$. (fig 8.5).

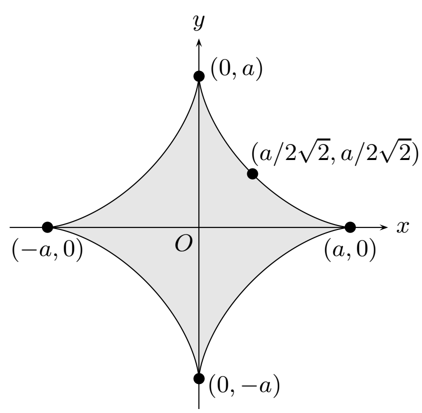


그림 8.5: 곡선 $x^{2/3} + y^{2/3} = a^{2/3}$

Vector Form using the Curl

For a vector field $\mathbf{F} = P\mathbf{i} + Q\mathbf{j}$ on $\mathbb{R}^2$, we may consider it as $\mathbf{F} = P\mathbf{i} + Q\mathbf{j} + 0\mathbf{k}$. Then curl of $\mathbf{F}$ is $(\partial Q/\partial x - \partial P/\partial y)\mathbf{k}$. Then using $d\mathbf{S} = \mathbf{k}\,dxdy$ we obtain

$$(\text{curl}\,\mathbf{F}) \cdot d\mathbf{S} = \left[\Big(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\Big)\mathbf{k}\right] \cdot \mathbf{k}\,dxdy = \Big(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\Big)dxdy$$

Hence by Green's theorem,

$$\int_{\partial D} \mathbf{F} \cdot d\mathbf{s} = \int_{\partial D} Pdx + Qdy = \iint_D \Big(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\Big)dxdy$$

This is vector form of Green's theorem.

Theorem 1.9. ***(Vector form of Green's theorem)*** *Let $D \subset \mathbb{R}^2$ be region with ∂D. if $\mathbf{F} = P\mathbf{i} + Q\mathbf{j}$ is a $\mathcal{C}^1$-vector field on D then*

$$\int_{\partial D} \mathbf{F} \cdot d\mathbf{s} = \iint_D (\operatorname{curl} \mathbf{F}) \cdot \mathbf{k}\, dxdy = \iint_D (\nabla \times \mathbf{F}) \cdot \mathbf{k}\, dxdy$$

(fig 8.6)

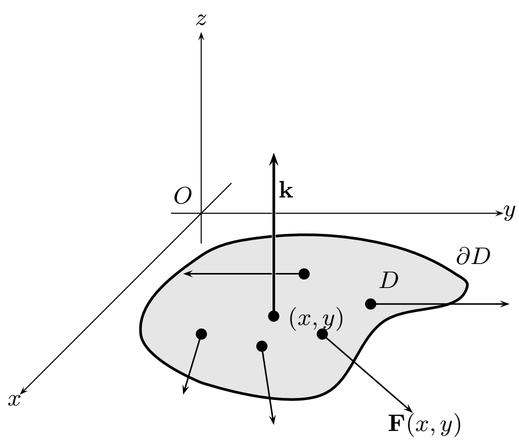


그림 8.6: vector in Green's Theorem

Example 1.10. Given a vector field $\mathbf{F}(x, y) = (y + 3x^4)\mathbf{i} + (e^y - x)\mathbf{j}$ and $C : x^2 + y^2/4 = 1$. Find $\int_C \mathbf{F} \cdot d\mathbf{s}$.

sol. Let $P(x, y) = y + 3x^4$, $Q(x, y) = e^y - x$. Then $\partial Q/\partial x = -1$, $\partial P/\partial y = 1$ and $\operatorname{curl} \mathbf{F} = (-1 - (-1))\mathbf{k} = -2\mathbf{k}$. Hence

$$\int_C \mathbf{F} \cdot d\mathbf{s} = \iint_D (\operatorname{curl} \mathbf{F}) \cdot \mathbf{k}\, dxdy = \iint_D -2\mathbf{k} \cdot \mathbf{k}\, dxdy = (-2) \cdot (\text{Are(D)}).$$

The area of $x^2/a^2 + y^2/b^2 = 1$ is πab area of D is 2π. Hence $\int_C \mathbf{F} \cdot d\mathbf{s} = -4\pi$.

We study a divergence form of Green's theorem. For this, we need **unit outward normal** Suppose $D \subset \mathbb{R}^2$ is a region with boundary ∂D. If ∂D is parameterized as $\sigma(t) = (x(t), y(t))$: $[a, b] \to \mathbb{R}^2$, $t \to \sigma(t)$ then the unit outward normal vector $\mathbf{n}$ on ∂D is defined as

$$\mathbf{n} = \frac{(y'(t), -x'(t))}{\sqrt{[x'(t)]^2 + [y'(t)]^2}}$$

Vector form of Divergence

Theorem 1.11. ***Divergence form of Green's theorem*** *If* $\mathbf{F} = P\mathbf{i} + Q\mathbf{j}$ *is a* $\mathcal{C}^1$*-vector field on* D *then*

$$\int_{\partial D} (\mathbf{F} \cdot \mathbf{n}) ds = \iint_D \operatorname{div} \mathbf{F} \, dxdy$$

(fig 8.7).

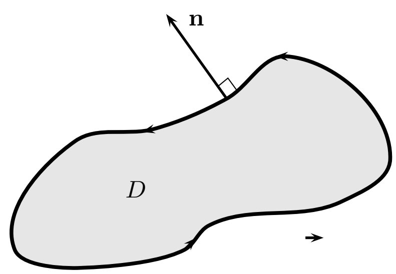


그림 8.7: $\mathbf{n}$ is the unit outward normal vector to ∂D

Proof. Since $\sigma'(t) = (x'(t), y'(t))$ is tangent to ∂D we see $\mathbf{n} \cdot \sigma' = 0$. i.e, $\mathbf{n}$ is perpendicular to the boundary. Choosing the proper sign of $\mathbf{n}$

$$\begin{aligned}\int_{\partial D} (\mathbf{F} \cdot \mathbf{n}) ds &= \int_a^b \left(\frac{P(x,y)y'(t) - Q(x,y)x'(t)}{\sqrt{[x'(t)]^2 + [y'(t)]^2}} \right) \sqrt{[x'(t)]^2 + [y'(t)]^2} dt \\ &= \int_a^b [P(x,y)y'(t) - Q(x,y)x'(t)] dt \\ &= \int_{\partial D} Pdy - Qdx.\end{aligned}$$

By Green's theorem,

$$\begin{aligned}\int_{\partial D} (\mathbf{F} \cdot \mathbf{n}) ds &= \int_{\partial D} Pdy - Qdx \\ &= \iint_D \left(\frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} \right) dxdy \\ &= \iint_D \operatorname{div} \mathbf{F} \, dxdy\end{aligned}$$

□

Example 1.12. Given a vector field $\mathbf{F} = \sin y^3 \mathbf{i} + e^{x^2} \mathbf{j}$ find $\int_C (\mathbf{F} \cdot \mathbf{n}) ds$ where C is the boundary of rectangle given by fig 8.8.

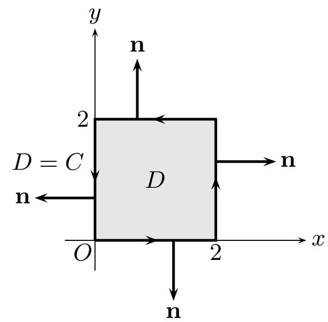


그림 8.8: Curve of Example 1.12

sol. Let $C = \partial D$. Since

$$\operatorname{div}\mathbf{F} = \frac{\partial}{\partial x}\left(\sin y^3\right) + \frac{\partial}{\partial y}\left(e^{x^2}\right) = 0 + 0 = 0$$

$$\int_C (\mathbf{F}\cdot\mathbf{n})ds = \int_{\partial D}(\mathbf{F}\cdot\mathbf{n})ds = \int_D \operatorname{div}\mathbf{F}\,dxdy = \int_D 0\,dxdy = 0.$$

제 2 절 Stokes' Theorem

The vector form of Green's theorem related the line integral of a vector field on a simple closed curve to the integral of the curl of the vector on the surface having the curve as boundary.

Stokes' theorem exactly the generalization of Green's theorem: the curve may lie in the space the the region is replaced by any surface in the space surrounded by the curve: A caution: there are many surfaces having the same curve as boundary. But as long as the vector fields are C^1 in a region, any surface play the same role.

Stokes' Theorem for Graphs

Let us recall ch7. Suppose a surface S is given as the graph of a function parameterized by

$$\begin{cases} x = u \\ y = v \\ z = f(u,v) \end{cases}$$

for (u, v) in some domain D. The integral of a vector field $\mathbf{F} = F_1\,\mathbf{i} + F_2\,\mathbf{j} + F_3\,\mathbf{k}$ over S was defined by

$$\iint_S \mathbf{F} \cdot d\mathbf{S} = \iint_D \left[F_1 \left(-\frac{\partial z}{\partial x} \right) + F_2 \left(-\frac{\partial z}{\partial y} \right) + F_3 \right] dxdy \tag{8.1}$$

Suppose $\mathbf{c}(t) : [a, b] \to \mathbb{R}$, $\mathbf{c}(t) = (x(t), y(t))$ is a parametrization of ∂D in positive direction. Then we define the **boundary curve** ∂S to be the oriented simple closed curve that is the image of the mapping $\mathbf{p} : t \to (x(t), y(t), f(x(t), y(t)))$ with the orientation induced by $\mathbf{c}$.

Theorem 2.1. *Let S be a oriented surface defined by C^2-function $z = f(x, y)$, where $(x, y) \in D$, a region to which Green's theorem holds. Let $\mathbf{F}$ be a C^1-vector fields on S. Then*

$$\iint_S \operatorname{curl} \mathbf{F} \cdot d\mathbf{S} = \iint_S (\nabla \times \mathbf{F}) \cdot d\mathbf{S} = \int_{\partial S} \mathbf{F} \cdot d\mathbf{s}.$$

Proof. If $\mathbf{F} = F_1\mathbf{i} + F_2\mathbf{j} + F_3\mathbf{k}$ then by (8.1)

$$\begin{aligned} \operatorname{curl} \mathbf{F} &= \Big(\frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z}\Big)\mathbf{i} + \Big(\frac{\partial F_1}{\partial z} - \frac{\partial F_3}{\partial x}\Big)\mathbf{j} + \Big(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\Big)\mathbf{k} \\ &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ F_1 & F_2 & F_3 \end{vmatrix} \\ &= \nabla \times \mathbf{F}. \end{aligned}$$

$$\begin{aligned} \iint_S \operatorname{curl} \mathbf{F} \cdot d\mathbf{S} \quad = \quad & \iint_D \Big[\Big(\frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z}\Big) \left(-\frac{\partial z}{\partial x}\right) \\ & + \Big(\frac{\partial F_1}{\partial z} - \frac{\partial F_3}{\partial x}\Big) \left(-\frac{\partial z}{\partial y}\right) + \Big(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\Big)\Big] dxdy \end{aligned}$$

On the other hand

$$\int_{\partial S} \mathbf{F} \cdot d\mathbf{s} = \int_{\mathbf{p}} \mathbf{F} \cdot d\mathbf{s} = \int_{\mathbf{p}} F_1 dx + F_2 dy + F_3 dz$$

If we use parametrization of $\mathbf{p} = (x(t), y(t), z(t))$ then

$$\int_{\partial S} \mathbf{F} \cdot d\mathbf{s} = \int_a^b \left(F_1 \frac{dx}{dt} + F_2 \frac{dy}{dt} + F_3 \frac{dz}{dt} \right) dt \tag{8.2}$$

Use the chain rule

$$\frac{dz}{dt} = \frac{\partial z}{\partial x}\frac{dx}{dt} + \frac{\partial z}{\partial y}\frac{dy}{dt}$$

Substituting into above

$$\begin{aligned}
\int_{\partial S} \mathbf{F} \cdot d\mathbf{s} &= \int_a^b \left[\left(F_1 + F_3\frac{\partial z}{\partial x}\right)\frac{dx}{dt} + \left(F_2 + F_3\frac{\partial z}{\partial y}\right)\frac{dy}{dt}\right] dt \\
&= \int_{\mathbf{c}} \left(F_1 + F_3\frac{\partial z}{\partial x}\right) dx + \left(F_2 + F_3\frac{\partial z}{\partial y}\right) dy \qquad (8.3) \\
&= \int_{\partial D} \left(F_1 + F_3\frac{\partial z}{\partial x}\right) dx + \left(F_2 + F_3\frac{\partial z}{\partial y}\right) dy
\end{aligned}$$

Applying Green's theorem to (8.3) yields

$$\iint_D \left[\left(\frac{\partial(F_2 + F_3\frac{\partial z}{\partial y})}{\partial x} - \frac{\partial(F_1 + F_3\frac{\partial z}{\partial x})}{\partial y}\right)\right] dxdy$$

Now use chain rule remembering F_1, F_2, F_3 are functions of x, y, and that z is also a function of x, y,

$$\begin{aligned}
&\iint_D \left[\left(\frac{\partial F_2}{\partial x} + \frac{\partial F_2}{\partial z}\frac{\partial z}{\partial x} + \frac{\partial F_3}{\partial x}\frac{\partial z}{\partial y} + \frac{\partial F_3}{\partial z}\frac{\partial z}{\partial x}\frac{\partial z}{\partial y} + F_3\frac{\partial^2 z}{\partial x \partial y}\right)\right. \\
&\left. - \left(\frac{\partial F_1}{\partial y} + \frac{\partial F_1}{\partial z}\frac{\partial z}{\partial y} + \frac{\partial F_3}{\partial y}\frac{\partial z}{\partial x} + \frac{\partial F_3}{\partial z}\frac{\partial z}{\partial y}\frac{\partial z}{\partial x} + F_3\frac{\partial^2 z}{\partial x \partial y}\right)\right] dA
\end{aligned}$$

Because mixed partials are equal last two integrals cancel and we obtain

$$\begin{aligned}
&\iint_D \left[\left(\frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z}\right)\left(-\frac{\partial z}{\partial x}\right) + \left(\frac{\partial F_1}{\partial z} - \frac{\partial F_3}{\partial x}\right)\left(-\frac{\partial z}{\partial y}\right) + \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right)\right] dxdy \\
= &\iint_S \operatorname{curl} \mathbf{F} \cdot d\mathbf{S}
\end{aligned}$$

□

Example 2.2. Let $\mathbf{F} = ye^z\mathbf{i} + xe^z\mathbf{j} + xye^z\mathbf{k}$. Compute $\int_C \mathbf{F} \cdot d\mathbf{s}$.

$$\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ ye^z & xe^z & xye^z \end{vmatrix} = 0$$

Theorem 2.3. ***(Stokes' theorem for Parameterized Surface)*** *Let S be a oriented surface defined by C^2- parametrization $\Phi : D \subset \mathbb{R}^2 \to S$. If ∂S is* *an oriented boundary of S and $\mathbf{F}$ is $\mathcal{C}^1$-vector field then*

$$\iint_S (\nabla \times \mathbf{F}) \cdot d\mathbf{S} = \int_{\partial S} \mathbf{F} \cdot d\mathbf{s}$$

Example 2.4. Show that $\int_C \mathbf{F} \cdot d\mathbf{s} = -4\pi$ for $\mathbf{F} = (x^2+y)\mathbf{i} + (x^2+2y)\mathbf{j} + 2z^3\mathbf{k}$ and $C : x^2 + y^2 = 4$.

sol. Let S be the region surrounded by $C : x^2 + y^2 = 4$. If $\mathbf{n}$ is the unit normal to S. Then $\mathbf{n} = \mathbf{k}$ and

$$\nabla \times \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ x^2 + y & x^2 + 2y & 2z^3 \end{vmatrix}$$
$$= (0-0)\mathbf{i} - (0-0)\mathbf{j} + (2x-1)\mathbf{k} = (2x-1)\mathbf{k}$$

Hence Stokes' theorem

$$\begin{aligned} \int_C \mathbf{F} \cdot d\mathbf{s} &= \iint_S (\nabla \times \mathbf{F}) \cdot \mathbf{n} dS \\ &= \iint_S (2x-1)\mathbf{k} \cdot \mathbf{k} dS = \int_{-2}^{2} \int_{-\sqrt{4-y^2}}^{\sqrt{4-y^2}} (2x-1) dx dy \\ &= -2 \int_{-2}^{2} \sqrt{4-y^2} dy = -4\pi \end{aligned}$$

이다.

Example 2.5. Evaluate

$$\int_C -y^3 dx + x^3 dy - z^3 dz$$

where C is the intersection of the cylinder $x^2 + y^2 = 1$ and plane $x + y + z = 1$.

sol. let $\mathbf{F} = -y^3\mathbf{i} + x^3\mathbf{j} - z^3\mathbf{k}$. Then $\text{curl}\,\mathbf{F} = 3(x^2+y^2)\mathbf{k}$. Here we need to compute $d\mathbf{S}$ or $dS = \|T_u \times T_v\|\, du dv$ from $(u, v, 1-u-v)$. But we can use graph form (8.1)

$$\int_S \text{curl}\,\mathbf{F} \cdot d\mathbf{S} = \int_D 3(x^2+y^2) dx dy = \frac{3\pi}{2}.$$

Writing thm 5.4.3 again. Let F_T denote the tangential component of $\mathbf{F}$, we have

$$\int_S (\operatorname{curl}\mathbf{F})\cdot \mathbf{n}dS = \int_S \operatorname{curl}\mathbf{F}\cdot d\mathbf{S} = \int_{\partial S}\mathbf{F}\cdot d\mathbf{s} = \int_{\partial S} F_T ds$$

Example 2.6. Show Stokes theorem holds for

$$\mathbf{F} = (2x-y)\mathbf{i} - yz^2\mathbf{j} - y^2z\mathbf{k}$$

when S is upper part of $x^2+y^2+z^2=1$, C is the boundary

sol. The boundary C is the unit circle in xy-plane, oriented counter-clockwise. So

$$x=\cos t,\ \ y=\sin t,\ \ z=0,\ \ (0\le t\le 2\pi).$$

$$\begin{aligned}\int_C \mathbf{F}\cdot d\mathbf{s} &= \int_C (2x-y)dx - yz^2dy - y^2zdz\\ &= \int_0^{2\pi}(2\cos t-\sin t)(-\sin t)dt = \pi.\end{aligned}$$

Since $\nabla\times\mathbf{F} = \mathbf{k}$

$$\begin{aligned}\iint_S (\nabla\times\mathbf{F})\cdot\mathbf{n}dS = \iint_S(\mathbf{k}\cdot\mathbf{n})dS &= \int_{-1}^{1}\int_{-\sqrt{1-y^2}}^{\sqrt{1-y^2}} dydx\\ &= 4\int_0^1\int_0^{\sqrt{1-y^2}} dydx = \pi.\end{aligned}$$

Hence

$$\int_S(\nabla\times\mathbf{F})\cdot\mathbf{n}dS = \pi = \int_C \mathbf{F}\cdot d\mathbf{s}.$$

Stokes' theorem holds.

Curl as Circulation per Unit area

Suppose $\mathbf{V}$ represent the velocity of a fluid. Consider a point P and unit normal vector $\mathbf{n}$. If S_ρ is a disk centered at P with radius ρ perpendicular to

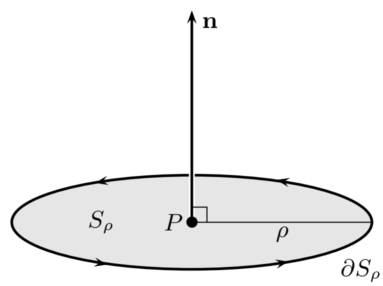


그림 8.9: **n** unit normal to S_ρ

n (fig 8.9)then by Stokes' theorem,

$$\int_{S_\rho} \operatorname{curl} \mathbf{V} \cdot d\mathbf{S} = \int_{S_\rho} (\operatorname{curl} \mathbf{V}) \cdot \mathbf{n} dS = \int_{\partial S_\rho} \mathbf{V} \cdot d\mathbf{s}.$$

holds. Here ∂S_ρ has the orientation according to **n**. If $A(S_\rho) = \pi\rho^2$ denote the area of S_ρ, $\operatorname{curl} \mathbf{V}(Q)$ is an average $\operatorname{curl} \mathbf{V}$ on Q에서의 , $\mathbf{n}(Q)$, we have by MVT

$$\int_{S_\rho} (\operatorname{curl} \mathbf{V}) \cdot \mathbf{n} dS = [\operatorname{curl} \mathbf{V}(Q) \cdot \mathbf{n}(Q)] A(S_\rho)$$

for some point Q in S_ρ. Hence

$$\begin{aligned} \lim_{\rho \to 0} \frac{1}{A(S_\rho)} \int_{\partial S_\rho} \mathbf{V} \cdot d\mathbf{s} &= \lim_{\rho \to 0} \frac{1}{A(S_\rho)} \int_{S_\rho} (\operatorname{curl} \mathbf{V}) \cdot \mathbf{n} dS \\ &= \lim_{\rho \to 0} (\operatorname{curl} \mathbf{V}(Q)) \cdot \mathbf{n}(Q) \\ &= (\operatorname{curl} \mathbf{V}(P)) \cdot \mathbf{n}(P). \end{aligned}$$

$$(\operatorname{curl} \mathbf{V}(P)) \cdot \mathbf{n}(P) = \lim_{\rho \to 0} \frac{1}{A(S_\rho)} \int_{\partial S_\rho} \mathbf{V} \cdot d\mathbf{s}. \qquad (8.4)$$

Now consider physical meaning of $\int_C \mathbf{V} \cdot d\mathbf{s}$ (fig 8.10).

Assume **V** is tangent to C and $\int_C \mathbf{V} \cdot d\mathbf{s} > 0$ then an object on C rotates along the direction of C. If $\int_C \mathbf{V} \cdot d\mathbf{s} < 0$, it rotates counter-clockwise on C.

Also, if $\mathbf{V} \perp C$ then object on C does not rotate and

$$\int_C \mathbf{V} \cdot d\mathbf{s} = 0$$

In general the integral of tangential component of a fluid vector field **V** $\int_C \mathbf{V} \cdot d\mathbf{s}$

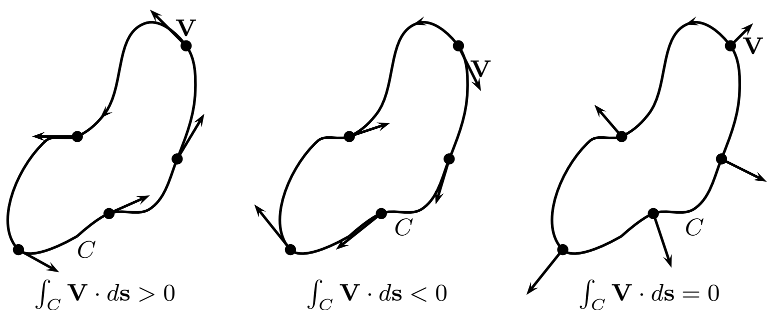


그림 8.10: Meaning of $\int_C \mathbf{V}\cdot d\mathbf{s}$

represent **the net amount of turning** around C. Thus,

$$\int_C \mathbf{V}\cdot d\mathbf{s}$$

is called the **circulation** of $\mathbf{V}$ around C. (fig 8.11).

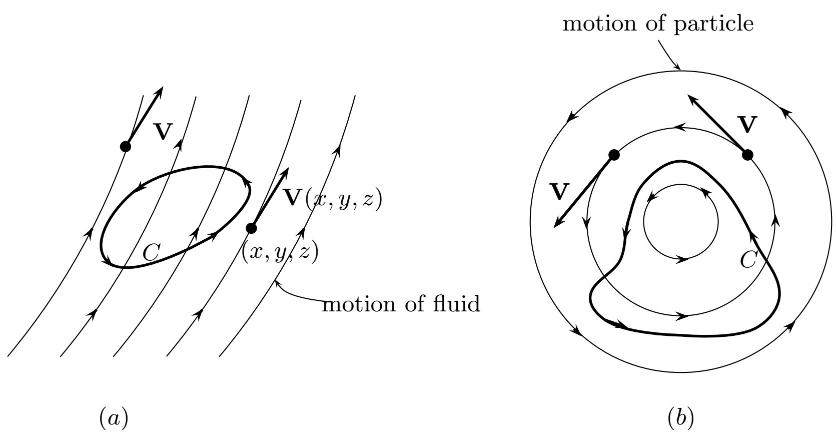


그림 8.11: Circulation of a vector field; (a) 0 circulation (b) nonzero circulation

The circulation $\int_{\partial S_\rho} \mathbf{V}\cdot d\mathbf{s}$ is the net velocity of a fluid around는 ∂S_ρ, and $(\mathrm{curl}\,\mathbf{V})\cdot\mathbf{n}$ is the circulation of V per unit area on a surface perpendicular to $\mathbf{n}$

Observe that $(\mathrm{curl}\,\mathbf{V})\cdot\mathbf{n}$ is maximized when $\mathbf{n} = \mathrm{curl}\,\mathbf{V}/||\mathrm{curl}\,\mathbf{V}||$. So the rotating effect is maximized about an axis parallel to $\mathrm{curl}\,\mathbf{V}/||\mathrm{curl}\,\mathbf{V}||$.

Example 2.7. p 541. Use physical interpretation to compute $\nabla\times\mathbf{F}\cdot\mathbf{e}_r$. Let $\mathbf{e}_r$, $\mathbf{e}_\theta$, $\mathbf{e}_z$ associate to cylindrical coordinates as fig **??**. Let $\mathbf{F} = F_r\mathbf{e}_r + F_\theta\mathbf{e}_\theta + F_z\mathbf{e}_z$.

Find a formula for $\mathbf{e}_r$ component of $\nabla \times \mathbf{F}$ in cylindrical coordinate.

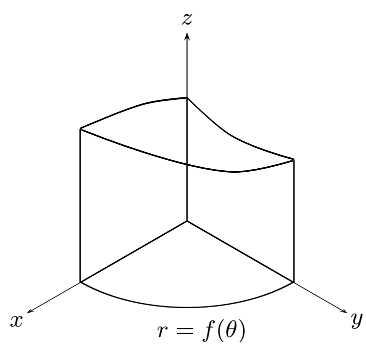


그림 8.12: cylindrical coordinate

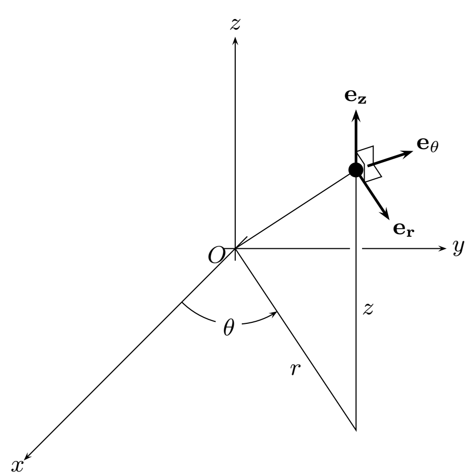
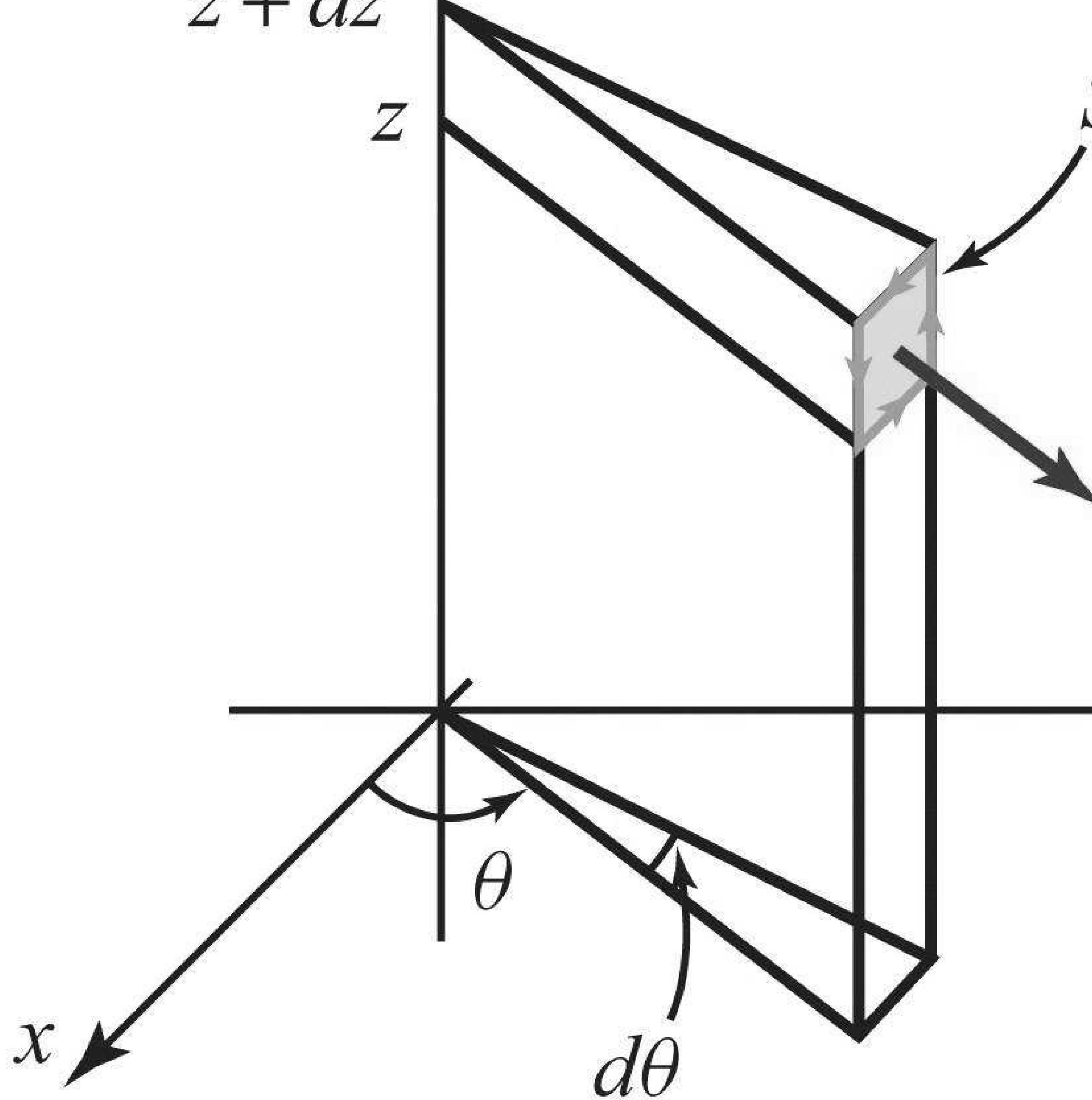


그림 8.13: unit orthogonal vectors in cylindrical coordinate $\mathbf{e}_r, \mathbf{e}_\theta, \mathbf{e}_z$



sol.

See figure 8.13. The area of S is $r d\theta\, dz$. The integral of $\mathbf{F}$ around the edges of S is

$$\begin{aligned}[F_\theta(r,\theta,z) - F_\theta(r,\theta,z+dz)]r d\theta + [F_z(r,\theta+d\theta,z) - F_z(r,\theta,z)]dz \\ \approx -\frac{\partial F_\theta}{\partial z} dz\, r d\theta + \frac{\partial F_z}{\partial \theta} d\theta\, dz\end{aligned}$$

Dividing by the area, we get circulation per unit area.

제 3 절 Conservative Field and Fundamental Theorem of Calculus

Let us summarize theorems so far.

- Fundamental Theorem of Calculus:

$$\int_\sigma \nabla f \cdot d\mathbf{s} = f(\sigma(b)) - f(\sigma(a))$$

- Green's Theorem :

$$\int_{\partial D} Pdx + Qdy = \int_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dxdy$$

- Divergence Theorem for Plane:

$$\int_{\partial D} (\mathbf{F} \cdot \mathbf{n}) ds = \int_D \operatorname{div} \mathbf{F} \, dxdy$$

- Stokes' Theorem:

$$\int_S \operatorname{curl} \mathbf{F} \cdot d\mathbf{S} = \int_{\partial S} \mathbf{F} \cdot d\mathbf{s}$$

- Gauss' Divergence Theorem:

$$\int_\Omega \operatorname{div} \mathbf{F} dV = \int_{\partial\Omega} \mathbf{F} \cdot d\mathbf{S}$$

Look at (1):

$$\int_\sigma \nabla f \cdot d\mathbf{s} = f(\sigma(b)) - f(\sigma(a)) = f(A) - f(B)$$

The line integral of a gradient is independent of path:

For example, if $V = -f$ represents a potential energy(Gravitational, electrical) the $\mathbf{F} = \nabla f$ is a force. For $f = \frac{GmM}{r}$, the force $\mathbf{F} = -\frac{GmM}{r^2}\mathbf{r} = -\frac{GmM}{r^2}\mathbf{n}$ is the gravitational force.

What vectors are Gradient?

Theorem 3.1. ***(Conservative Field*** *Let* $\mathbf{F}$ *be a* $\mathcal{C}^1$*-vector field in* $\mathbb{R}^3$ *except finite number of points. Then the following conditions are equivalent:*

(1)

(2) For any oriented simple closed curve C, $\int_C \mathbf{F} \cdot d\mathbf{s} = 0$.

(3) For any two oriented simple curve C_1, C_2 having same end points,

$$\int_{C_1} \mathbf{F} \cdot d\mathbf{s} = \int_{C_2} \mathbf{F} \cdot d\mathbf{s}.$$

(4) $\mathbf{F}$ is the gradient of some function f, i.e, $\mathbf{F} = \nabla f$.

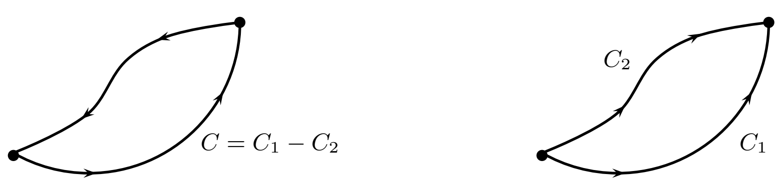
(5) $\operatorname{curl} \mathbf{F} = \mathbf{0}$.

*If a vector field $\mathbf{F}$ satisfy one of these conditions we say **conservative field**.*

Proof. We use the sequence of implication: (i)⇒(2)⇒(3)⇒(4)⇒(1).

(1)⇒(2): Suppose we have two curves C_1, C_2 having same end points, we can form a closed curve by $C = C_1 - C_2$(fig 8.14) Hence by (1)

$$0 = \int_C \mathbf{F} \cdot d\mathbf{s} = \int_{C_1 - C_2} \mathbf{F} \cdot d\mathbf{s} = \int_{C_1} \mathbf{F} \cdot d\mathbf{s} - \int_{C_2} \mathbf{F} \cdot d\mathbf{s}$$



riented simple closed curve $C = C_1 - C_2$ (b) two oriented simple curves C_1

그림 8.14: Constructing oriented simple closed curve from two oriented simple curve C_1, C_2

(2)⇒(3): Fix a point (x_0, y_0, z_0). Given any point (x, y, z), choose any curve C connecting two points we define. Given $\mathbf{F} = (F_1, F_2, F_3)$ define

$$f(x, y, z) = \int_C \mathbf{F} \cdot d\mathbf{s} = \int_C F_1 dx + F_2 dy + F_3 dz$$

Here f is well-defined, since it is defined independent of the choice of C. So we choose C consisting of edges of rectangular pipe.

In particular, choose $C = C_1 + C_2 + C_3 + C_4$ (fig 8.15). Then

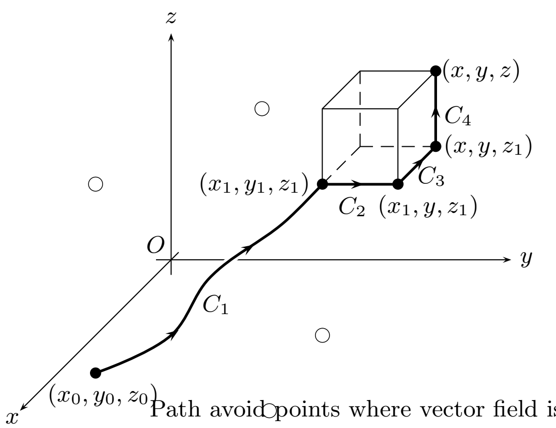


그림 8.15: A path from (x_0, y_0, z_0) to (x, y, z) is $C = C_1 + C_2 + C_3 + C_4$

$$\begin{aligned} f(x,y,z) &= \int_C \mathbf{F}\cdot d\mathbf{s} \\ &= \int_{C_1} \mathbf{F}\cdot d\mathbf{s} + \int_{C_2} \mathbf{F}\cdot d\mathbf{s} + \int_{C_3} \mathbf{F}\cdot d\mathbf{s} + \int_{C_4} \mathbf{F}\cdot d\mathbf{s} \\ &= \int_{C_1} \mathbf{F}\cdot d\mathbf{s} + \int_{y_1}^{y} F_2(x_1, t, z_1)dt \\ &\qquad + \int_{x_1}^{x} F_1(t, y, z_1)dt + \int_{z_1}^{z} F_3(x, y, t)dt. \end{aligned}$$

From this we see $\partial f/\partial z = F_3$. Similarly by choosing different path(i.e, choosing a path whose last path is along x-direction) we have

$$f(x,y,z) = \int_{C_1} \mathbf{F}\cdot d\mathbf{s} + \int_{y_1}^{y} F_2(x_1, t, z_1)dt$$
$$+ \int_{z_1}^{z} F_3(x_1, y, t)dt + \int_{x_1}^{x} F_1(t, y, z)dt$$

so $\partial f/\partial x = F_1$. Similarly, we have $\partial f/\partial y = F_2$. Thus $\mathbf{F} = \nabla f$.

(3)⇒(4): By theorem 5.1.2, we have $\nabla \times \nabla f = 0$. So $\operatorname{curl}\mathbf{F} = \nabla \times \mathbf{F} = 0$.

(4)⇒(1): Let S be a surface having C as boundary. Then

$$\int_C \mathbf{F}\cdot d\mathbf{s} = \int_S \operatorname{curl}\mathbf{F}\cdot d\mathbf{S}$$

Since $\operatorname{curl}\mathbf{F} = \mathbf{0}$ the integral $\int_C \mathbf{F}\cdot d\mathbf{s} = 0$. □

Physical Interpretation of $\int_C \mathbf{F}\cdot d\mathbf{s}$

(1) $\int_C \mathbf{F}\cdot d\mathbf{s}$ is work done be $\mathbf{F}$ along C.

(2) Circulation: $\mathbf{F} \cdot \Delta\mathbf{s}$ is tangential component $\dot{\Delta}\mathbf{s}$. Thus $\int_C \mathbf{F} \cdot d\mathbf{s}$ is net tangential component of $\mathbf{F}$ along C.

Example 3.2.

$$\mathbf{F}(x, y, z) = y\mathbf{i} + (z \cos yz)\mathbf{j} + (y \cos yz)\mathbf{k}$$

Show $\mathbf{F}$ is irrotational and find a scalar potential.

Example 3.3.

$$\mathbf{F}(x, y) = -y\mathbf{i} + x\mathbf{j}$$

We see $\mathbf{F}(0, 0) = 0$ but $\operatorname{curl} \mathbf{F} = 2$. So if a paddle is place at this point, it rotates even if it stay there. Show $\mathbf{F}$ is irrotational and find a scalar potential.

sol.

$$\begin{aligned}\nabla \times \mathbf{F} &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ y & z \cos yz & y \cos yz \end{vmatrix} \\ &= (\cos yz - yz \sin yz - \cos yz + yz \sin yz)\mathbf{i} - (0 - 0)\mathbf{j} + (1 - 1)\mathbf{k} = 0\end{aligned}$$

So $\mathbf{F}$ is irrotational. To find a potential
Method 1:

$$\begin{aligned}f(x, y, z) &= + \int_0^x F_1(t, 0, 0)dt + \int_0^y F_2(x, t, 0)dt + \int_0^y F_3(x, t, 0)dt \\ &= \int_0^x 0dt + \int_0^y x\,dt + \int_0^y y \cos yt\,dt \\ &= 0 + xy + \sin yz\end{aligned}$$

One easily check that $\nabla f = \mathbf{F}$.

□

sol. Method 2: If such f exists, it satisfies

$$\frac{\partial f}{\partial x} = y, \quad \frac{\partial f}{\partial y} = x + z\cos yz, \quad \frac{\partial f}{\partial z} = y\cos yz. \tag{8.5}$$

Thus we obtain

(1) $f(x, y, z) = xy + h_1(y, z)$

(2) $f(x, y, z) = \sin yz + xy + h_2(x, z)$

(3) $f(x, y, z) = \sin yz + h_3(x, y)$

Substitute this into (8.5) we find

$$\frac{\partial h_1(y, z)}{\partial z} = y\cos yz$$

or

$$h_1(y, z) = \int y\cos yz\, dz + g(y) = \sin yz + g(y).$$

Substituting back to (1) (2) we see $g(y) = h_2(x, z)$. Then this must be constant.

Potential function

If $\mathbf{F}$ satisfies $\operatorname{curl}\mathbf{F} = \nabla \times \mathbf{F} = \mathbf{0}$ then it is given by $\mathbf{F} = \nabla f$ for some f¿ This f is called **potential function** of $\mathbf{F}$.

Theorem 3.4. *If $\mathbf{F}$ is a C^1 vector field with $div\,\mathbf{F} = 0$ then there is a C^1 field $\mathbf{G}$ with $\mathbf{F} = \operatorname{curl}\mathbf{G}$.*

Example 3.5. By Newton's law, the force acting to an object of mass M at $\mathbf{r} = (x, y, z)$ is

$$\mathbf{F}(x, y, z) = -GM\mathbf{r}/r^3$$

Show $\mathbf{F}$ is irroataitonal and find potential for it.

sol. First show $\nabla \times \mathbf{F} = \mathbf{0}$.

$$\mathbf{r} \times \mathbf{r} = \mathbf{0}, \nabla\left(\frac{1}{r^3}\right) = -3\mathbf{r}/r^5,$$

$$\nabla \times \mathbf{r} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x & y & z \end{vmatrix} = \mathbf{0},$$

$$\nabla \times \mathbf{F} = -GM\left\{\nabla\left(\frac{1}{r^3}\right) \times \mathbf{r} + \frac{1}{r^3}\nabla \times \mathbf{r}\right\} = \mathbf{0}.$$

From exercise we can show $\nabla(r^n) = nr^{n-2}\mathbf{r}$ $\mathbf{F} = -\nabla\phi$, The function $\phi(x, y, z) = -GM/r$ is the gravitational potential function.

Planar Case

Suppose $\mathbf{F}$ is a $\mathcal{C}^1$-vector field of the form $\mathbf{F} = P\mathbf{i} + Q\mathbf{j}$ then we have

$$\nabla \times \mathbf{F} = \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)\mathbf{k}$$

Hence

$$\operatorname{curl} \mathbf{F} = \nabla \times \mathbf{F} = \mathbf{0}$$

is equivalent to

$$\partial Q/\partial x = \partial P/\partial y$$

Using this we can study conservative field $\mathbb{R}^2$. In $\mathbb{R}^3$ vector field may have a few points where function is undefined. But in planar case, the vector field must be defined everywhere.

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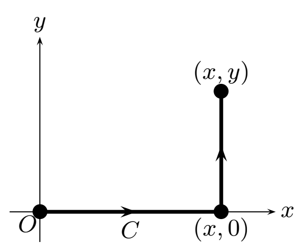

그림 8.16: A path from $(0, 0)$ to (x, y)

Example 3.6. The vector field $\mathbf{F}(x, y, z) = (y\mathbf{i} - x\mathbf{j})/(x^2 + y^2)$ satisfies (a) $\nabla \times \mathbf{F} = \mathbf{0}$ but (b) $\mathbf{F}$ is not conservative.

sol. (a) We have seen $\nabla \times \mathbf{F} = \mathbf{0}$ before.
(b) To show $\mathbf{F}$ is not conservative we need to show the line integral $\int_C \mathbf{F} \cdot d\mathbf{s}$ is nonzero for some closed curve C. Let C be

$$x = \cos t, \quad y = \sin t, \quad 0 \le t \le 2\pi$$

Then

$$\begin{aligned}
\int_C \mathbf{F} \cdot d\mathbf{s} &= \int_C \frac{y}{x^2+y^2}dx + \frac{-x}{x^2+y^2}dy \\
&= \int_0^{2\pi} \left[\frac{\sin t}{\cos^2 t + \sin^2 t}(-\sin t dt) + \frac{-\cos t}{\cos^2 t + \sin^2 t}(\cos t dt)\right] \\
&= -\int_0^{2\pi} \frac{\sin^2 t + \cos^2 t}{\cos^2 t + \sin^2 t} dt = -\int_0^{2\pi} dt = -2\pi \neq 0.
\end{aligned}$$

Hence $\mathbf{F}$ is not conservative.

Example 3.7. (a) If $\mathbf{F} = e^{xy}\mathbf{i} + e^{x+y}\mathbf{j}$ then $P(x, y) = e^{xy}, Q(x, y) = e^{x+y}$, $\partial P/\partial y = xe^{xy}$, $\partial Q/\partial x = e^{x+y}$. So no potential exists.

(b) For $\mathbf{F} = (2x \cos y)\mathbf{i} - (x^2 \sin y)\mathbf{j}$ we see $\partial P/\partial y = -2x \sin y = \partial Q/\partial x$. Hence $\mathbf{F}$ has a potential f. To find it f we see

$$\frac{\partial f}{\partial x} = 2x \sin y, \quad \frac{\partial f}{\partial y} = -x \sin y$$

$$\begin{aligned}
f(x, y) &= x^2 \cos y + h_1(y) \\
f(x, y) &= x^2 \cos y + h_2(x).
\end{aligned}$$

we can set $h_1 = h_2 = 0$. So $f(x, y) = x^2 \cos y$.

Example 3.8. For a path $\sigma\colon [1, 2] \to \mathbb{R}^2$ $x = e^{t-1}, y = \sin(\pi/t)$ find the line integral of $\mathbf{F} = 2x \cos y\mathbf{i} - x^2 \sin y\mathbf{j}$.

$$\int_\sigma \mathbf{F} \cdot d\mathbf{s} = \int_\sigma 2x \cos y dx - x^2 \sin y dy$$

sol. Since $\sigma(1)=(1,0), \sigma(2)=(e,1), \partial(2x\cos y)/\partial y = \partial(-x^2\sin y)/\partial x$ $\mathbf{F}$ is irrotataional. Hence we can replace the path by another $\mathcal{C}^1$ having same end points. Choose from $(1,0)$ to $(e,0)$. Next from $(e,0)$ to $(e,1)$. Then the integral is

$$\begin{aligned}\int_\sigma \mathbf{F}\cdot d\mathbf{s} &= \int_1^e 2t\cos 0 dt + \int_0^1 -e^2 \sin t dt\\ &= (e^2-1) + e^2(\cos 1 - 1)\\ &= e^2\cos 1 - 1.\end{aligned}$$

On the other hand $f(x,y)=x^2\cos y$ is a potential of $\mathbf{F}$. Hence

$$\begin{aligned}\int_\sigma 2x\cos y dx - x^2 \sin y dy &= \int_\sigma \nabla f\cdot d\mathbf{s}\\ &= f(\sigma(2)) - f(\sigma(1)) = e^2\cos 1 - 1.\end{aligned}$$

The latter integral is easier.

제 5 절 Gauss' Theorem

Some Elementary Regions

Gauss' Divergence Theorem

The flux of a vector field $\mathbf{F}$ across Ω is equal to the sum of $\operatorname{div}\mathbf{F}$ in Ω.s

Theorem 5.1. ***Gauss' Divergence Theorem*** *Let Ω be an elementary region in $\mathbb{R}^3$ and $\partial\Omega$ be oriented closed surface bounding it. Let $\mathbf{F}$ be a $\mathcal{C}^1$ vector field on W. Then*

$$\iiint_\Omega \operatorname{div}\mathbf{F} dV = \iint_{\partial\Omega} \mathbf{F}\cdot d\mathbf{S}$$

Proof. Suppose W is an elementary region of 4th kind and $\mathbf{F} = P\mathbf{i}+Q\mathbf{j}+R\mathbf{k}$ Then

$$\operatorname{div}\mathbf{F} = \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z}$$

and

$$\iiint_\Omega \operatorname{div}\mathbf{F} dV = \iiint_\Omega \frac{\partial P}{\partial x} dV + \iiint_\Omega \frac{\partial Q}{\partial y} dV + \iiint_\Omega \frac{\partial R}{\partial z} dV.$$

On the other hand, the surface integral is

$$\iint_{\partial\Omega}(\mathbf{F}\cdot\mathbf{n})dS = \iint_{\partial\Omega}(P\mathbf{i}+Q\mathbf{j}+R\mathbf{k})\cdot\mathbf{n}dS$$
$$= \iint_{\partial\Omega} P\mathbf{i}\cdot\mathbf{n}dS + \iint_{\partial\Omega} Q\mathbf{j}\cdot\mathbf{n}dS + \iint_{\partial\Omega} R\mathbf{k}\cdot\mathbf{n}dS.$$

If we show the following

$$\iint_{\partial\Omega} P\mathbf{i}\cdot\mathbf{n}dS = \iiint_{\Omega}\frac{\partial P}{\partial x}dV,$$

$$\iint_{\partial\Omega} Q\mathbf{j}\cdot\mathbf{n}dS = \iiint_{\Omega}\frac{\partial Q}{\partial y}dV,$$
$$\iint_{\partial\Omega} R\mathbf{k}\cdot\mathbf{n}dS = \iiint_{\Omega}\frac{\partial R}{\partial z}dV,$$

then the proof is complete. We prove only one of them.

Suppose Ω is given by two functions $z = f_1(x,y)$, $z = f_2(x,y)$ defined on a region D in xy-plane such that $\Omega = \{(x,y,z)|f_1(x,y) \le z \le f_2(x,y), (x,y) \in D\}$ (fig 8.17).

Hence

$$\iiint_{\Omega}\frac{\partial R}{\partial z}dV = \iint_D\left(\int_{z=f_1(x,y)}^{z=f_2(x,y)}\frac{\partial R}{\partial z}dz\right)dxdy$$

$$\iiint_{\Omega}\frac{\partial R}{\partial z}dV = \iint_D[R(x,y,f_2(x,y)) - R(x,y,f_1(x,y))]dxdy. \tag{8.6}$$

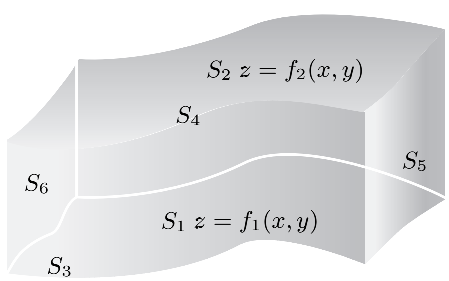

그림 8.17: For elementary region Ω $\int_{\partial\Omega} R\mathbf{k}\cdot d\mathbf{S} = \int_{\Omega}(\partial R/\partial z)dV$ and Four sides S_3, S_4, S_5, S_6 of $\partial\Omega$ are perpendicular to z.

The boundary of Ω consists of two surface $S_2:$ the graph of $z = f_2(x,y), (x,y) \in D$ and the S_1: the graph of $z = f_1(x,y), (x,y) \in D$. The other four sides of

$\partial\Omega$ are S_3, S_4, S_5, S_6 unit normal vectors to them are perpendicular to z-axis. Hence

$$\iint_{\partial\Omega} R\mathbf{k}\cdot\mathbf{n}dS = \iint_{S_1} R\mathbf{k}\cdot\mathbf{n}_1 dS + \iint_{S_2} R\mathbf{k}\cdot\mathbf{n}_2 dS + \sum_{i=3}^{6}\iint_{S_i} R\mathbf{k}\cdot\mathbf{n}_i dS.$$

Since the unit vector $\mathbf{n}_i$, $i = 3,4,5,6$ S_i are perpendicular to $\mathbf{k}$, we have $\mathbf{n}_i\cdot\mathbf{k} = 0$. Hence

$$\iint_{\partial\Omega} R\mathbf{k}\cdot\mathbf{n}dS = \iint_{S_1} R\mathbf{k}\cdot\mathbf{n}_1 dS + \iint_{S_2} R\mathbf{k}\cdot\mathbf{n}_2 dS. \qquad (8.7)$$

Bottom surface S_1 is given by $z = f_1(x,y), (x,y)\in D$

$$\mathbf{n}_1 = \frac{\frac{\partial f_1}{\partial x}\mathbf{i} + \frac{\partial f_1}{\partial y}\mathbf{j} - \mathbf{k}}{\sqrt{\left(\frac{\partial f_1}{\partial x}\right)^2 + \left(\frac{\partial f_1}{\partial y}\right)^2 + 1}}.$$

Hence

$$\mathbf{k}\cdot\mathbf{n}_1 = \frac{-1}{\sqrt{\left(\frac{\partial f_1}{\partial x}\right)^2 + \left(\frac{\partial f_1}{\partial y}\right)^2 + 1}},$$

$$\int_{S_1} R\mathbf{k}\cdot\mathbf{n}_1 dS = -\int_D R(x,y,f_1(x,y))dxdy. \qquad (8.8)$$

Similarly on S_2

$$\mathbf{k}\cdot\mathbf{n}_2 = \frac{-1}{\sqrt{\left(\frac{\partial f_2}{\partial x}\right)^2 + \left(\frac{\partial f_2}{\partial y}\right)^2 + 1}}. \qquad (8.9)$$

Hence

$$\iint_{S_2} R\mathbf{k}\cdot\mathbf{n}_2 dS = \iint_D R(x,y,f_2(x,y))dxdy.$$

Compare (5.11), (5.12) and substitute in (5.10) (5.9) we obtain

$$\iint_{\partial\Omega} R\mathbf{k}\cdot\mathbf{n}dS = \iint_{\Omega}\frac{\partial R}{\partial z}dV.$$

The other relations can be similarly shown, □

Example 5.2. S is the unit sphere $x^2 + y^2 + z^2 = 1$ and vector field is

$\mathbf{F} = 2x\mathbf{i} + y^2\mathbf{j} + z^2\mathbf{k}$ find $\iint_S \mathbf{F}\cdot\mathbf{n}dS$

sol. Let Ω be the region inside S. By Gauss theorem, it holds

$$\iint_S \mathbf{F}\cdot\mathbf{n}dS = \iiint_\Omega \operatorname{div}\mathbf{F}dV$$

Since $\operatorname{div}\mathbf{F} = \nabla\cdot(2x\mathbf{i} + y^2\mathbf{j} + z^2\mathbf{k}) = 2(1+y+z)$, the rhs is

$$2\iiint_\Omega (1+y+z)dV = 2\iiint_\Omega 1dV + 2\iiint_\Omega ydV + 2\iiint_\Omega zdV$$

and

$$\iiint_\Omega ydV = \iiint_\Omega zdV = 0$$

$$\iiint_S \mathbf{F}\cdot\mathbf{n}dS = 2\iiint_\Omega (1+y+z)dV = 2\iiint_\Omega 1dV = \frac{8}{3}\pi.$$

Hence $\iint_S \mathbf{F}\cdot\mathbf{n}dS = 8\pi/3$.

Generalizing Gauss' theorem

Example 5.3. Show Gauss' theorem holds for $\mathbf{F} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$ in Ω : $x^2+y^2+z^2 \le a^2$.

sol. First compute $\operatorname{div}\mathbf{F} = \nabla\cdot\mathbf{F}$

$$\operatorname{div}\mathbf{F} = \frac{\partial x}{\partial x} + \frac{\partial y}{\partial y} + \frac{\partial z}{\partial z} = 3$$

So

$$\iiint_\Omega (\operatorname{div}\mathbf{F})dV = \iiint_\Omega 3\,dV = 3\Big(\frac{4}{3}\pi a^3\Big) = 4\pi a^3.$$

We must find unit normal $\mathbf{n}$ on $\partial\Omega$. Set

$$f(x,y,z) = x^2+y^2+z^2-a^2$$

$$\mathbf{n} = \pm\frac{\nabla f}{||\nabla f||}.$$

The gradient of f is $\nabla f = 2(x\mathbf{i}+y\mathbf{j}+z\mathbf{k})$. hence

$$\mathbf{n} = \frac{2(x\mathbf{i}+y\mathbf{j}+z\mathbf{k})}{\sqrt{4(x^2+y^2+z^2)}} = \frac{x\mathbf{i}+y\mathbf{j}+z\mathbf{k}}{a}$$

(fig 8.18).

So when $(x,y,z)\in\partial\Omega$

$$\mathbf{F}\cdot\mathbf{n} = \frac{x^2+y^2+z^2}{a} = \frac{a^2}{a} = a$$

and

$$\iint_{\partial\Omega}\mathbf{F}\cdot\mathbf{n}dS = \iint_{\partial\Omega} a\,dS = a(4\pi a^2) = 4\pi a^3.$$

Hence

$$\iiint_\Omega(\operatorname{div}\mathbf{F})dV = 4\pi a^3 = \iint_{\partial\Omega}\mathbf{F}\cdot\mathbf{n}dS.$$

Gauss' theorem holds

Example 5.4. Region Ω is given as $x^2+y^2+z^2\le 1$ Find $\iint_{\partial\Omega}(x^2+4y-5z)dS$ by Gauss' theorem

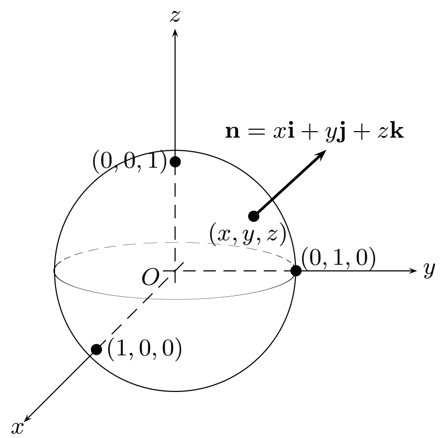

그림 8.18: The unit normal vector $\mathbf{n} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k} = \mathbf{r}$ to the unit ball

sol. To use Gauss' theorem, we need a vector field $\mathbf{F} = F_1\mathbf{i} + F_2\mathbf{j} + F_3\mathbf{k}$ such that $\mathbf{F} \cdot \mathbf{n} = x^2 + 4y - 5z$.

$\mathbf{n}$ is $\mathbf{n} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$. Hence $\mathbf{F}$ satisfies $\mathbf{F} \cdot \mathbf{n} = F_1 x + F_2 y + F_3 z$. Hence if we set $F_1 x = x^2, F_2 y = 4y, F_3 z = -5z$ and compute F_1, F_2, F_3, then $\mathbf{F} = x\mathbf{i} + 4\mathbf{j} - 5\mathbf{k}$, $\operatorname{div} \mathbf{F} = 1 + 0 + (-0) = 1$. Now by Gauss theorem

$$\begin{aligned}\iint_{\partial\Omega} (x^2 + 4y - 5z) dS &= \iint_{\partial\Omega} (x\mathbf{i} + 4\mathbf{j} - 5\mathbf{k}) \cdot \mathbf{n} dS \\ &= \iint_{\partial\Omega} \mathbf{F} \cdot \mathbf{n} dS = \iiint_{\Omega} \operatorname{div} \mathbf{F} dV \\ &= \iiint_{\Omega} 1 \; dV = \frac{4}{3}\pi.\end{aligned}$$

Hence $\iint_{\partial\Omega} (x^2 + 4y - 5z) dS = 4\pi/3$.

Divergence as flux per unit Volume

As we have seen before that $\operatorname{div} \mathbf{F}(P)$ is the *rate of change of total flux* at P per unite volume. Let Ω_ρ be a ball of radius ρ center at P. Then for some Q in Ω_ρ

$$\iint_{\partial\Omega_\rho} \mathbf{F} \cdot \mathbf{n} dS = \iiint_{\Omega_\rho} \operatorname{div} \mathbf{F} dV = \operatorname{div} \mathbf{F}(Q) \cdot (\operatorname{Vol}(\Omega_\rho))$$

$$\operatorname{div} \mathbf{F}(P) = \lim_{\rho \to 0} \operatorname{div} \mathbf{F}(Q) = \lim_{\rho \to 0} \frac{1}{\operatorname{Vol}(\Omega_\rho)} \iint_{\partial\Omega_\rho} \mathbf{F} \cdot \mathbf{n} dS. \tag{8.10}$$

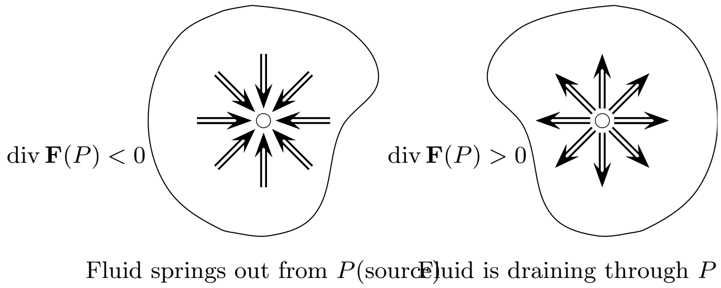

그림 8.19: Physical meaning of divergence

If div $\mathbf{F}(P) > 0$ we say P is a **source** of $\mathbf{F}$ if div $\mathbf{F}(P) < 0$ it is called **sink** of $\mathbf{F}$(fig 8.19).

If div $\mathbf{F} = 0$ then by Gauss theorem, total flux of $\mathbf{F}$ through S is $\int_S \mathbf{F} \cdot d\mathbf{S} = 0$. Thus we call this vector field **incompressible field**.

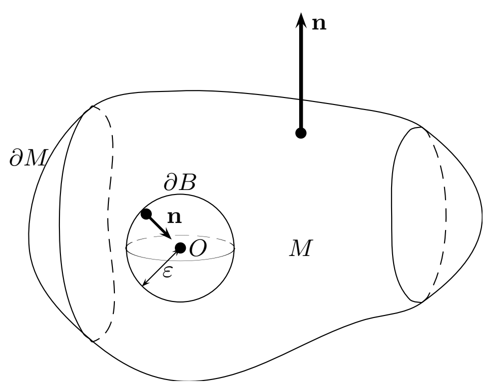

그림 8.20: Unit outward normal vector $\mathbf{n}$ to M and Gauss' Law

Example 5.5. Find $\iint_S \mathbf{f} \cdot d\mathbf{S}$ where $\mathbf{F} = xy^2\mathbf{i} + x^2y\mathbf{j} + y\mathbf{k}$ and S is the surface of the cylinder $x^2 + y^2 = 1$, bounded by $z = 1, z = -1$ including the top and bottom lid.

sol. By divergence theorem,

$$\begin{aligned}\iiint_W (\operatorname{div}\mathbf{F})dV &= \iiint_W (x^2+y^2)dxdydz \\ &= \int_{-1}^{1}\left(\iint_{x^2+y^2\le 1}(x^2+y^2)dxdy\right)dz \\ &= 2\iint_{x^2+y^2\le 1}(x^2+y^2)dxdy\end{aligned}$$

Now use polar coordinate,

$$2\iint_{x^2+y^2\le 1}(x^2+y^2)dxdy = 2\int_0^{2\pi}\int_0^1 r^3 dr d\theta = \pi$$

Gauss' Law

Now apply Gauss' theorem to more general region. (A region with a hole)

Theorem 5.6. ***(Gauss' Law)*** *Let M be a region in $\mathbb{R}^3$ and $O \notin \partial M$*

$$\iint_{\partial M}\frac{\mathbf{r}\cdot\mathbf{n}}{r^3}dS = \begin{cases}0 & \text{if } O \notin M \ , \\ 4\pi & \text{if } O \in M\end{cases}$$

Here $\mathbf{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$ and $r = \sqrt{x^2+y^2+z^2}$.

Proof. First suppose $O \notin M$. Then $\mathbf{r}/r^3$ is a C^1-vector field on M and ∂M. Hence

$$\iint_{\partial M}\frac{\mathbf{r}\cdot\mathbf{n}}{r^3}dS = \iiint_M \nabla\cdot\left(\frac{\mathbf{r}}{r^3}\right)dV$$

For $r \neq 0$, $\nabla\cdot(\mathbf{r}/r^3) = 0$. Thus the following holds.

$$\iint_{\partial M}\frac{\mathbf{r}}{r^3}dS = 0$$

Next if $O \in M$, $\mathbf{r}/r^3$ is not continuous on M. Then we remove small ball B of radius ε(fig 8.20). Then if we let W be the region between M and B, then the boundary of W is $S = \partial B \cup \partial M$. Also note that the unit vector to W is opposite direction of the normal to to B. We see in $\nabla\cdot(\mathbf{r}/r^3) = 0$ in W. Hence by Gauss theorem

$$\iint_S \frac{\mathbf{r}\cdot\mathbf{n}}{r^3}dS = \iiint_\Omega \nabla\cdot\left(\frac{\mathbf{r}}{r^3}\right)dV = 0.$$

When $\mathbf{n}$ is unit outward normal on S,

$$\iint_S \frac{\mathbf{r}\cdot\mathbf{n}}{r^3}dS = \iint_{\partial M} \frac{\mathbf{r}\cdot\mathbf{n}}{r^3}dS + \iint_{\partial B} \frac{\mathbf{r}\cdot\mathbf{n}}{r^3}dS$$

Hence

$$\iint_{\partial M} \frac{\mathbf{r}\cdot\mathbf{n}}{r^3}dS = -\iint_{\partial B} \frac{\mathbf{r}\cdot\mathbf{n}}{r^3}dS.$$

Now on ∂B, we know $\mathbf{n} = -\mathbf{r}/r$, $r = \varepsilon$, and ∂B is a sphere of radius ε.

$$-\iint_{\partial B} \frac{\mathbf{r}\cdot\mathbf{n}}{r^3}dS = \iint_{\partial B} \frac{\varepsilon^2}{\varepsilon^4}dS = \frac{1}{\varepsilon^2}\iint_{\partial B} dS.$$

Hence $\iint_{\partial B} dS = 4\pi\varepsilon^2$ and $\iint_{\partial M} \mathbf{r}\cdot\mathbf{n}/r^3 dS = 4\pi$. □

Physical Interpretation of Gauss' Law

Let

$$\phi(x,y,z) = \frac{Q}{4\pi r} = \frac{Q}{4\pi\sqrt{x^2+y^2+z^2}}$$

be the potential to a point charge O at $(0,0,0)$. Then the electric field is

$$\mathbf{E} = -\nabla\phi = \frac{Q}{4\pi}\left(\frac{\mathbf{r}}{r^3}\right).$$

The total electric flux is $\int_{\partial M} \mathbf{E}\cdot d\mathbf{S}$ if the charge lies in M, 0 otherwise.

For a charge with density ρ, the field $\mathbf{E}$ is related by

$$\operatorname{div}\mathbf{E} = \nabla\cdot\mathbf{E} = \rho$$

Thus by Gauss' theorem

$$\int_S \mathbf{E}\cdot d\mathbf{S} = \int_\Omega \rho dV = \int_\Omega \frac{Q}{V}dV = Q.$$

In other words, total flux through a surface equals total charge inside.

Divergence in Spherical Coordinate

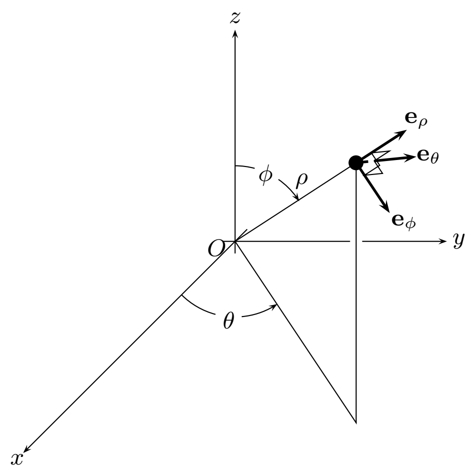

그림 8.21: unit normal vectors in spherical coordinate $\mathbf{e}_\rho, \mathbf{e}_\phi, \mathbf{e}_\theta$

See figure 8.21, **??**. We use Gauss' theorem to derive

$$\mathrm{d}iv\,\mathbf{F} = \frac{1}{\rho^2}\frac{\partial}{\partial\rho}(\rho^2 F_\rho) + \frac{1}{\rho\sin\phi}\frac{\partial}{\partial\phi}(\sin\phi F_\phi) + \frac{1}{\rho\sin\phi}\frac{\partial F_\theta}{\partial\theta}$$

Since

$$\mathrm{d}iv\,\mathbf{F} = \lim\frac{1}{W}\iint_{\partial W}\mathbf{F}\cdot\mathbf{n}\,dS$$

we compute the right hand side by inspection. The surface integral is

The area of S is $rd\theta\,dz$. The integral of $\mathbf{F}$ around the edges of S is

$$\begin{aligned} &F_\rho(\rho+d\rho,\phi,\theta)\cdot\ \text{area outer face}\ - F_\rho(\rho,\phi,\theta)\cdot\ \text{area inner face} \\ &\approx F_\rho(\rho+d\rho,\phi,\theta)(\rho+d\rho)^2\sin\phi\,d\phi d\theta - F_\rho(\rho,\phi,\theta)(\rho)^2\sin\phi\,d\phi d\theta \\ &\qquad\approx \frac{\partial}{\partial F_\rho}(\rho^2\sin\phi)\,d\rho d\phi d\theta \end{aligned}$$

Dividing by the area, we get flux per unit volume.

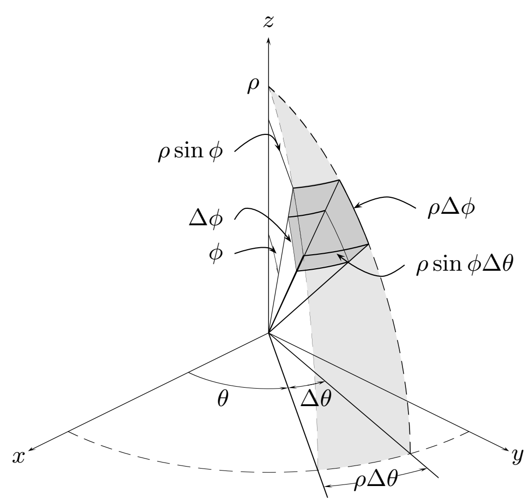

그림 8.22: Partition in spherical coordinate