제 7 장

Integrals over Paths and Surfaces

제 1 절 Path Integral

Path integral

$\mathbb{R}^2$ or $\mathbb{R}^3$. A parameterized curve $\mathbf{c}$ can be written as $\mathbf{c}(t) = (x(t), y(t), z(t))$. If $x(t)$, $y(t)$, $z(t)$ are continuous then we say $\mathbf{c}$ is continuous, and if $x(t), y(t), z(t)$ are differentiable, then $\mathbf{c}$ is differentiable. If $x'(t)$, $y'(t)$, $z'(t)$are continuous then we say $\mathbf{c}$ us C^1-curve.

Let $\mathbf{c}$ be defined on $[a,b]$ and let $P\colon a = t_0 < t_1 < \cdots < t_k = b$ be the partition of $[a,b]$. t_i^* is a point between t_{i-1} and t_i. Then the Riemann sum is

$$\sum_{i=1}^{k} f(\mathbf{c}(t_i^*))\|\mathbf{c}(t_i) - \mathbf{c}(t_{i-1})\|$$

As $\|P\|$ approaches 0 the sum approaches

$$\sum_{i=1}^{k} f(\mathbf{c}(t_i^*))\|\mathbf{c}'(t_i^*)\|(t_i - t_{i-1})$$

Definition 1.1. If $\mathbf{c}$ is defined over $I = [a,b]$ having values in $\mathbb{R}^3$- C^1-curve, f is defined over a region containing the image of $\mathbf{c}$. Then $f \circ \mathbf{c}$ is real valued function defined on I. We define **the path integral** of $\mathbf{c}$ as:

$$\int_a^b f(\mathbf{c}(t))\|\mathbf{c}'(t)\|\, dt$$

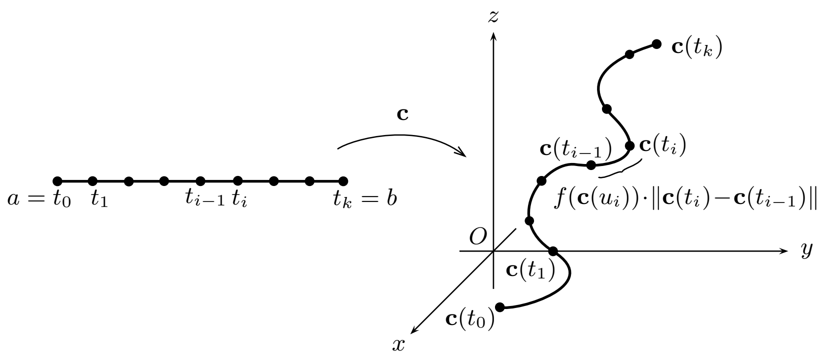


그림 7.1: Riemann sum over a path

We denote it by $\int_{\mathbf{c}} f\, ds$ or $\int_{\mathbf{c}} f(x, y, z)\, ds$. If $f = 1$ then $\int_{\mathbf{c}} 1 ds$ is the length of $\mathbf{c}$.

Example 1.2. Find path integral of $f(x, y, z) = x^2 + y^2 + z^2$ over $\mathbf{c}$.

$$\mathbf{c}(t) = (\cos t, \sin t, t), \quad t \in [0, 2\pi]$$

sol. Since $\mathbf{c}'(t) = (-\sin t, \cos t, 1)$, the line integral is

$$\begin{aligned}\int_{\mathbf{c}} f\, ds &= \int_0^{2\pi} f(\mathbf{c}(t))\|\mathbf{c}'(t)\|\, dt \\ &= \int_0^{2\pi} (1 + t^2)\sqrt{2}\, dt \\ &= \sqrt{2}\left(2\pi + 8\pi^3/3\right)\end{aligned}$$

Path integral over planar Curves

If $f(x, y)$ is a continuous function defined over a region containing the image of $\mathbf{c}$, then the path integral of f along $\mathbf{c}$ is given by

$$\int_c f(x, y)\, ds = \int_a^b f(x(t), y(t))\sqrt{x'(t)^2 + y'(t)^2} dt$$

If $f = 1$, it is nothing but the arc -length.

Example 1.3 (Tom Sawyer's fence)**.** Find the area of fence along a parameterized curve $\mathbf{c}$ in $\mathbb{R}^2$ and height is given by $f(x, y) = 1 + 3y$.

sol. $\mathbf{c}(t) = (30\cos^3 t, 30\sin^3 t)$ for $t \in [0, \pi/2]$. The area of one side is

$$\int_C f(x,y)ds$$

where $ds = \|\mathbf{c}'(t)\|dt = 90 \sin t \cos t dt$. So

$$\begin{aligned}\int_C f(x,y)ds &= \int_0^{\pi/2} \left(1 + 10\sin^3 t\right) 90 \sin t \cos t dt \\ &= 90 \int_0^{\pi/2} (\sin t + 10 \sin^4 t) \cos t \, dt = 225.\end{aligned}$$

This is half of the fence. Total area of fence(both sides) is 900 square ft. If he can get .05 dollar per square feet, he can make $900 \times 0.05 = 1.80$.

제 2 절 Line integrals

First consider work by force fields. Suppose a particle move along a curve $\mathbf{c}$ while acted upon by a force $\mathbf{F}$. If $\mathbf{c}$ is straight line segment given by the vector $\mathbf{d}$ and $\mathbf{F}$ is constant force, then the work is, by definition

$$\mathbf{F} \cdot \mathbf{d} = \text{ magnitude of force } \times \text{ displacement in the direction of force}$$

If the path is curved, we break the curve into small pieces and add the work at each piece then take the limit. So the work is defined by

$$\lim_n \sum_{i=0}^{n-1} \mathbf{F}(\mathbf{c}(t_i)) \cdot [\mathbf{c}(t + \Delta t) - \mathbf{c}(t)] = \int_a^b \mathbf{F}(\mathbf{c}(t)) \cdot \mathbf{c}'(t)dt$$

Here $\mathbf{c}'(t)dt \approx \mathbf{c}(t + \Delta t) - \mathbf{c}(t)$ represent line segment.

The work done by the force is the following sum.

$$\sum_{i=1}^{k} \mathbf{F}(\mathbf{c}(t_i)) \cdot (\mathbf{c}(t_i) - \mathbf{c}(t_{i-1}))$$

As $\|P\| \to 0$ the sum is

$$\sum_{i=1}^{k} F(\mathbf{c}(t_i)) \cdot \mathbf{c}'(t_i)\Delta t_i$$

the limit is $\int_a^b F(\mathbf{c}(t)) \cdot \mathbf{c}'(t)\, dt$.

We use the notation:

$$\Delta \mathbf{s} = \mathbf{c}(t+\Delta t) - \mathbf{c}(t) \approx \mathbf{c}'(t)\Delta t$$

In the limit this becomes $d\mathbf{s}$.

Definition 2.1. Let $\mathbf{F}$ be a vector field on $\mathbb{R}^3$ that is continuous on the C^1-path $\mathbf{c} : [a,b] \to \mathbb{R}^3$. Define the line integral

$$\int_{\mathbf{c}} \mathbf{F} \cdot d\mathbf{s} = \int_a^b \mathbf{F}(\mathbf{c}(t)) \cdot \mathbf{c}'(t) dt$$

A nice interpretation in terms of scalar integral is as follows: For $\mathbf{c}'(t) \neq 0$, we see, if $\mathbf{T}(t) = \mathbf{c}'(t)/\|\mathbf{c}'(t)\|$ is the unit tangent vector, then

$$\begin{aligned} \int_{\mathbf{c}} \mathbf{F} \cdot d\mathbf{s} &= \int_a^b \mathbf{F}(\mathbf{c}(t)) \cdot \mathbf{c}'(t) dt \\ &= \int_a^b \left[\mathbf{F}(\mathbf{c}(t)) \cdot \frac{\mathbf{c}'(t)}{\|\mathbf{c}'(t)\|} \right] \|\mathbf{c}'(t)\| dt \\ &= \int_a^b \left[\mathbf{F}(\mathbf{c}(t)) \cdot \mathbf{T}(t) \right] \|\mathbf{c}'(t)\| dt \\ &= \int_C \mathbf{F}(\mathbf{c}(t)) \cdot \mathbf{T}(t) ds. \end{aligned}$$

So the line integral is like the **path integral of the tangential component**: $\mathbf{F}(\mathbf{c}(t)) \cdot \mathbf{T}(t)$ along $\mathbf{c}$.

Another Notation for Line integral

Let us write $\mathbf{c}(t) = (x(t), y(t), z(t))$ and $\mathbf{F} = (F_1, F_2, F_3)$. Then $d\mathbf{s} = (dx, dy, dz) = (\frac{dx}{dt}, \frac{dy}{dt}, \frac{dy}{dt})dt$. So the integral

$$\begin{aligned} \int_{\mathbf{c}} \mathbf{F} \cdot d\mathbf{s} &= \int_{\mathbf{c}} (F_1, F_2, F_3) \cdot (dx, dy, dz) \\ &= \int_{\mathbf{c}} F_1 dx + F_2 dy + F_3 dz \\ &= \int_a^b \left(F_1 \frac{dx}{dt} + F_2 \frac{dy}{dt} + F_3 \frac{dy}{dt} \right) dt \end{aligned}$$

Example 2.2.

$$\int_c x^2 dx + xy dy + dz = \frac{11}{15}$$

where $\mathbf{c}(t) = (t, t^2, 1) = (x, y, z)$ on $[0, 1]$.

Example 2.3.

$$\int_c \cos z dx + e^x dy + e^y dz$$

where $\mathbf{c}(t) = (1, t, e^t)$ on $[0, 2]$.

Example 2.4.

$$\int_c (\sin z dx + \cos z dy - (xy)^{1/3} dz) = -\frac{1}{2}$$

where $x = \cos^3\theta, \quad y = \sin^3\theta, z = \theta$ on $[0, 7\pi/2]$.

Draw the curve in $\mathbb{R}^3$.

Example 2.5. Suppose $\mathbf{F}(x, y, z) = x^3\mathbf{i} + y\mathbf{j} + z\mathbf{k}$ and parameterize the circle

$$x = 0, \; y = a\cos\theta, \; z = a\sin\theta, \quad 0 \le \theta \le 2\pi$$

$$\mathbf{c}'(t) = (0, -a\sin\theta, a\cos\theta)$$

Since $\mathbf{F}(\mathbf{c}(\theta)) \cdot \mathbf{c}'(\theta) = 0$, the work must be zero. You can verify by finding the value.

Reparametrization

The line integral depends not only on $\mathbf{F}$ but also depends on the path $\mathbf{c}$. If $\mathbf{c}_1$, $\mathbf{c}_2$ are two different parametrization of the same curve, we shall see

$$\int_{\mathbf{c}_1} \mathbf{F} \cdot d\mathbf{s} = \pm \int_{\mathbf{c}_2} \mathbf{F} \cdot d\mathbf{s}$$

Definition 2.6. Let $h : I \to I_1$ be real valued C^1 curve that is one-to-one. Let $\mathbf{c} : I_1 \to \mathbb{R}^3$ be a C^1 curve. Then the composition

$$\mathbf{p} = \mathbf{c} \circ h : I \to \mathbb{R}^3$$

is called a **reparametrization** of $\mathbf{c}$.

Theorem 2.7. *Let* $\mathbf{c}\colon [a_1, b_1] \to \mathbb{R}^3$ *be a curve and* $\mathbf{p}\colon [a, b] \to \mathbb{R}^3$ *is given by* $\mathbf{p} = \mathbf{c} \circ h$ *where* $h\colon [a, b] \to [a_1, b_1]$ *satisfies*

$$p(a) = a_1, \quad h(b) = b_1 \textit{(Orientation preserving)}$$

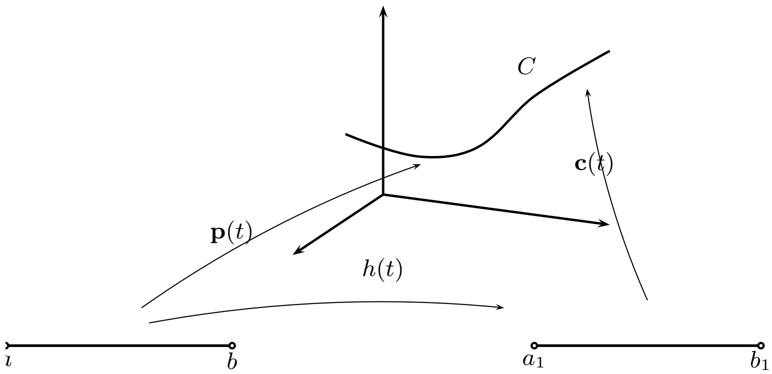


그림 7.2: Reparametrization of a curve

or

$$h(a) = b_1, \quad h(b) = a_1 \textit{(Orientation reversing)}$$

Then we have

$$\int_{\mathbf{p}} \mathbf{F} \cdot d\mathbf{s} = \pm \int_{\mathbf{c}} \mathbf{F} \cdot d\mathbf{s}$$

Here we have + sign, if $\mathbf{p}$ *is orientation preserving, and* $-$ *sign, if* $\mathbf{p}$ *is orientation reversing.*

Proof. If h is orientation preserving then $h(a) = a_1$, $h(b) = b_1$. In this case,

$$\int_{\mathbf{p}} \mathbf{F}(\mathbf{c}(s)) \cdot d\mathbf{s} = \int_a^b \mathbf{F}(\mathbf{c}(h(t))) \cdot \mathbf{c}'(h(t))h'(t)\, dt$$

Let $s = h(t)$. Then

$$\begin{aligned}
&= \int_a^b \mathbf{F}(\mathbf{c}(h(t))) \cdot \mathbf{c}'(h(t))h'(t)\, dt \\
&= \int_{h(a)}^{h(b)} \mathbf{F}(\mathbf{c}(s)) \cdot \mathbf{c}'(s)\, ds \\
&= \int_{a_1}^{b_1} \mathbf{F}(\mathbf{c}(s)) \cdot \mathbf{c}'(s)\, ds \\
&= \int_{\mathbf{c}} \mathbf{F}(\mathbf{c}(s))\, d\mathbf{s}
\end{aligned}$$

If $\mathbf{p}$ is orientation reversing, then the integral becomes

$$= \int_{b_1}^{a_1} \mathbf{F}(\mathbf{c}(s)) \cdot \mathbf{c}'(s)\, ds$$
$$= -\int_{\mathbf{c}} \mathbf{F}(\mathbf{c}(s))\, ds$$

□

Example 2.8. (1) Given $\mathbf{c}\colon [a,b] \to \mathbb{R}^n$. As a typical example, consider $\mathbf{c}_{op}\colon [a,b] \to \mathbb{R}^n$ defined by

$$\mathbf{c}_{op} = \mathbf{c}(a+b-t)\colon [a,b] \to \mathbb{R}^n.$$

$\mathbf{c}_{op}$ is called opposite path. This is orientation reversing. We see

$$\begin{aligned}\int_{\mathbf{c}_{op}} \mathbf{F} \cdot d\mathbf{s} &= \int_a^b \mathbf{F} \cdot (\mathbf{c}_{op})'(t)dt \\ &= \int_a^b \mathbf{F} \cdot \mathbf{c}'(b+a-t)(-1)dt \\ &= \int_b^a \mathbf{F} \cdot \mathbf{c}'(u)du \\ &= -\int_a^b \mathbf{F} \cdot \mathbf{c}'(u)du \\ &= -\int_{\mathbf{c}} \mathbf{F} \cdot d\mathbf{s}\end{aligned}$$

(2) The path $\mathbf{p}; [0,1] \to \mathbb{R}^3$ given by $\mathbf{p}(t) = \mathbf{c}(a+(b-a)t)$ is an orientation preserving reparametrization.

Example 2.9. Find the line integral of $f(x,y,z) = x^2y^2z^2$ on C.

$$C = \{(x,y,z)\colon x^2+y^2+z^2 = 2,\ x^2+y^2 = z^2,\ z > 0\}$$

sol. Subtracting we see $z^2 = 1$. So $z = 1$. A parametrization of C is

$$\mathbf{c}(t) = (\cos t, \sin t, 1),\ t \in [0, 2\pi]$$

$$\mathbf{c}'(t) = (-\sin t, \cos t, 0)$$

Hence the integral is

$$\begin{aligned} \int_{\mathbf{c}} f\, ds &= \int_0^{2\pi} f(\mathbf{c}(t))\|\mathbf{c}'(t)\|\, dt \\ &= \int_0^{2\pi} \cos^2 t\, \sin^2 t\, dt \\ &= \frac{1}{8}\int_0^{2\pi} (1 - \cos 4t)\, dt \\ &= \frac{\pi}{4} \end{aligned}$$

The line integral is an *oriented integral*, in the sense that change of sign occurs if the orientation is reversed. The *path integral* does not have this property.

Theorem 2.10 (Path integral is independent of parametrization)**.** *If* $\mathbf{c}$ *and* $\mathbf{p}$ *are two parametrization of a piecewise* C^1*-curve* C*, and* f *is any real valued continuous function, then*

$$\int_{\mathbf{c}} f(x, y, z)\, ds = \int_{\mathbf{p}} f(x, y, z)\, ds$$

Let $\mathbf{c}(t) : [a, b] \to \mathbb{R}^n$. As an example, let $\mathbf{p}$ defined by $\mathbf{p}(t) = \mathbf{c}(a + b - t)$. Then

$$\begin{aligned} \int_{\mathbf{p}} f\, ds &= \int_a^b f(-\mathbf{c}(t))\|(-\mathbf{c})'(t)\|\, dt \\ &= \int_a^b f(\mathbf{c}(b + a - t))\|\mathbf{c}'(b + a - t)(-1)\|\, dt \\ &= \int_a^b f(\mathbf{c}(b + a - t))\|\mathbf{c}'(b + a - t)\|\, dt \\ &= \int_b^a f(\mathbf{c}(u))\|\mathbf{c}'(u)\|(-1)\, du \\ &= \int_{\mathbf{c}} f\, ds \end{aligned}$$

Here $u = b + a - t$.

Example 2.11. Find path integral of $f(x,y,z)=x^2+y^2+z^2$ over C.

$$C=\{(\cos t,\sin t,t)\colon t\in[0,2\pi]\}\cup\{(1,0,t)\colon t\in[0,2\pi]\}$$

sol. C is the union of C_1 and C_2.

$$C_1=\{(\cos t,\sin t,t)\colon t\in[0,2\pi]\},\quad C_2=\{(1,0,t)\colon t\in[0,2\pi]\}$$

We parameterize C_1 and C_2 as follows:

$$\mathbf{c}_1=(\cos t,\sin t,t)\quad t\in[0,1],\quad \mathbf{c}_2=(1,0,t)\quad t\in[0,2\pi]$$

Then

$$\begin{aligned}\int_C f\,ds &= \int_{C_1} f\,ds+\int_{C_2} f\,ds\\ &= \int_{\mathbf{c}_1} f\,ds+\int_{\mathbf{c}_2} f\,ds\\ &= \int_0^{2\pi}(1+t^2)\sqrt{2}\,dt+\int_0^{2\pi}(1+t^2)\,dt\\ &= (1+\sqrt{2})\left(2\pi+8\pi^3/3\right)\end{aligned}$$

Line integrals of Gradient Fields

A vector field $\mathbf{F}$ is called a **gradient vector field** if $\mathbf{F}=\nabla f$ for some real valued function f. Thus

$$\mathbf{F}=\frac{\partial f}{\partial x}\mathbf{i}+\frac{\partial f}{\partial y}\mathbf{j}+\frac{\partial f}{\partial z}\mathbf{k}.$$

Theorem 2.12. *Suppose $f:\mathbb{R}^3\to\mathbb{R}$ is class C^1 and $\mathbf{c}:[a,b]\to\mathbb{R}^3$ is smooth. Then*

$$\int_{\mathbf{c}}\nabla f\cdot d\mathbf{s}=f(\mathbf{c}(b))-f(\mathbf{c}(a)).$$

Proof. Apply chain rule to

$$f(\mathbf{c}(t))$$

$$(f\circ\mathbf{c})'(t)=\nabla f(\mathbf{c}(t))\cdot\mathbf{c}'(t)$$

So

$$\int_{\mathbf{c}} \nabla f \cdot d\mathbf{s} = \int_a^b \nabla f(\mathbf{c}(t)) \cdot \mathbf{c}'(t)dt = \int_a^b f'(u)du = f(\mathbf{c}(b)) - f(\mathbf{c}(a)).$$

So the line integral is independent of parametrization. □

Example 2.13. Let $\mathbf{c}(t) = (t^4, \sin^3(t\pi/2), 0)$, $t \in [0,1]$. Evaluate

$$\int_{\mathbf{c}} ydx + xdy$$

which means $\int_{\mathbf{c}} ydx + xdy + 0dz$.

sol. We recognize $(y, x, 0)$ as gradient of $f(x,y,z) = xy$. Hence the value is

$$f(\mathbf{c}(1)) - f(\mathbf{c}(1)) = 1 - 0$$

Definition 2.14. We say a curve is **simple** if it is the 1-1 image of piecewise C^1 map $\mathbf{c} : I \to \mathbb{R}^3$. A simple curve is one which does not intersect itself.

If $I = [a,b]$, then $\mathbf{c}(a)$, $\mathbf{c}(b)$ are called end points of the curve. Each simple curve has two orientations. (From P to Q) It is **oriented or directed curve**.

Definition 2.15. If $\mathbf{c}$ is 1-1 except at end points and, $\mathbf{c}(a) = \mathbf{c}(b)$, it is **simple closed curve** if it is $\mathbf{c}(a) = \mathbf{c}(b)$, but not 1-1, then it is a **closed curve**.

Line integrals over oriented simple curves

Suppose $\mathbf{c}$ is any orientation preserving parametrization of C, then the line integral is independent of parametrization: Hence we can define

$$\int_C \mathbf{F} \cdot d\mathbf{s} = \int_{\mathbf{c}} \mathbf{F} \cdot d\mathbf{s}$$

If C^- is the same curve as C but with the opposite orientation. Then

$$\int_C \mathbf{F} \cdot d\mathbf{s} = -\int_{C^-} \mathbf{F} \cdot d\mathbf{s}$$

Line integrals over curves with several components

Let C be an oriented curve which is made up of several oriented curves $C_i, i = 1, 2, \cdots$ Since each C_i can be parameterized separately, we can show that the integral satisfies

$$\int_C \mathbf{F} \cdot d\mathbf{s} = \int_{C_1} \mathbf{F} \cdot d\mathbf{s} + \int_{C_2} \mathbf{F} \cdot d\mathbf{s} + \cdots + \int_{C_k} \mathbf{F} \cdot d\mathbf{s}$$

Thus the following sum of oriented curves makes sense.

$$C = C_1 + C_2 + \cdots + C_k$$

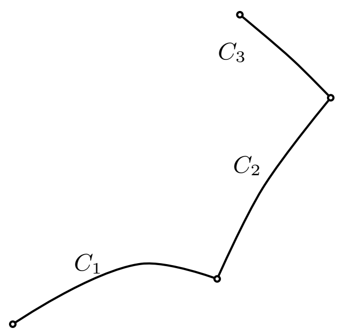


그림 7.3: Sum of several curves

Example 2.16. Find the line integral of $\mathbf{F}(x, y, z) = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$ over $\mathbf{c}$.

$$\mathbf{c}(t) = (\cos t, \sin t, t), \quad t \in [0, 2\pi]$$

sol. We have $\mathbf{c}'(t) = (-\sin t, \cos t, 1)$ and $\mathbf{F}(\mathbf{c}(t)) = \cos t\mathbf{i} + \sin t\mathbf{j} + t\mathbf{k}$. Hence $\mathbf{F}(\mathbf{c}(t)) \cdot \mathbf{c}'(t) = \cos t(-\sin t) + \sin t \cos t + t = t$

$$\int_{\mathbf{c}} \mathbf{F} \cdot d\mathbf{s} = \int_0^{2\pi} t\, dt = 2\pi^2$$

More examples in the book.

The notation $d\mathbf{r}$ for line integrals

Sometimes we use the notation for line integral:

$$\int_C \mathbf{F} \cdot d\mathbf{r}$$

Here $\mathbf{r}$ denotes the position vector $\mathbf{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$.

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \int_a^b \mathbf{F}(\mathbf{r}(t)) \cdot \frac{d\mathbf{r}}{dt} dt$$

제 3 절 Parameterized Surfaces

Graphs are too restrictive.

See the example of a surface in the book. Or simply a sphere or torus. Those are important examples of figures that arise often in real life. But those figures cannot be represented as the graphs of functions.

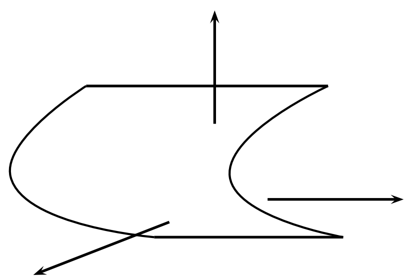


그림 7.4: A surface that is not the graph of a function

Definition 3.1. A parameterization of a Surface is a function $\Phi\colon D \subset \mathbb{R}^2 \to \mathbb{R}^3$ where D is a domain in $\mathbb{R}^2$. The surface S corresponding to the function Φ is the image $S = \Phi(D)$.

$$\Phi(u, v) = (x(u, v), y(u, v), z(u, v))$$

If Φ is differentiable or C^1, then we say S is differentiable or C^1-surface.

The graph of a function is a special case.

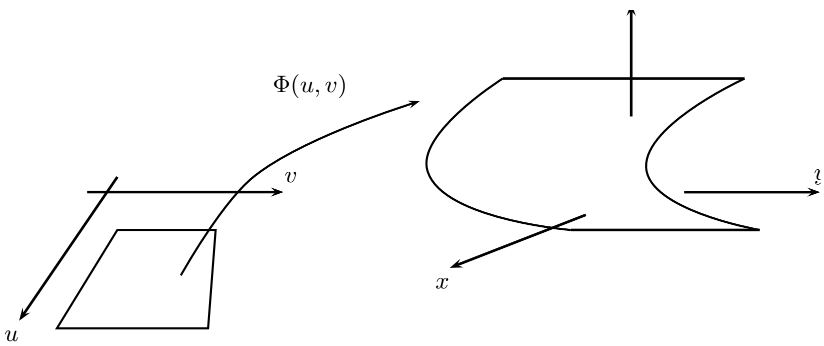


그림 7.5: A parametrization

Tangent Vectors and Tangent Plane to a Surface

Consider the mapping $\Phi\colon D \to \mathbb{R}^3$ through $(x,y,z) = \Phi(u,v)$. First look at the case the surface is the graph of $f\colon D \to R$ we can write $\Phi\colon D \to \mathbb{R}^3$ as

$$\Phi(x,y) = (x, y, f(x,y))$$

First fix $x = x_0$ and then $y = y_0$. Then tangent vector along y-axis and x-axis at $\Phi(x_0,y_0) = (x_0, y_0, f(x_0,y_0))$ is

$$\Phi_x(x_0,y_0) = \mathbf{i} + f_x(x_0,y_0)\mathbf{k}, \quad \Phi_y(x_0,y_0) = \mathbf{j} + f_y(x_0,y_0)\mathbf{k}$$

Hence tangent plane is perpendicular to the normal vector given by the cross product

$$\begin{aligned}\Phi_x(x_0,y_0) \times \Phi_y(x_0,y_0) &= (\mathbf{i} + f_x(x_0,y_0)\mathbf{k}) \times (\mathbf{j} + f_y(x_0,y_0)\mathbf{k}) \\ &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 0 & f_x(x_0,y_0) \\ 0 & 1 & f_y(x_0,y_0) \end{vmatrix} \\ &= -f_x(x_0,y_0)\mathbf{i} - f_y(x_0,y_0)\mathbf{j} + \mathbf{k}\end{aligned}$$

In general, we see two tangent vectors are

$$\mathbf{T}_u = \frac{\partial \Phi}{\partial u} = \frac{\partial x}{\partial u}\mathbf{i} + \frac{\partial y}{\partial u}\mathbf{j} + \frac{\partial z}{\partial u}\mathbf{k}\Big|_{(u_0,v_0)}$$

This is obtained by fixing u_0 Similarly,

$$\mathbf{T}_v = \frac{\partial \Phi}{\partial v} = \frac{\partial x}{\partial v}\mathbf{i} + \frac{\partial y}{\partial v}\mathbf{j} + \frac{\partial z}{\partial v}\mathbf{k}\Big|_{(u_0,v_0)}$$

If the normal vector

$$\mathbf{n} = \Phi_u \times \Phi_v = \frac{\partial \Phi}{\partial u} \times \frac{\partial \Phi}{\partial v}$$

is nonzero, then we see the surface is smooth. So call it *regular*.

Definition 3.2. Since $\mathbf{n}$ is normal to tangent plane the equation of **tangent plane** at $\Phi(u_0, v_0) = (x_0, y_0, z_0)$is given by

$$\mathbf{n} \cdot (x - x_0, y - y_0, z - z_0) = 0$$

Or if $\mathbf{n} = (n_1, n_2, n_3)$ then the **tangent plane** is

$$n_1(x - x_0) + n_2(y - y_0) + n_3(z - z_0) = 0$$

Example 3.3 (Paraboloid)**.** Consider

$$x = u\cos v, \quad y = u \sin v, \quad z = u^2 + v^2$$

Find tangent plane at $\Phi(1, 0)$.

sol. $\Phi(u, v) = (u\cos v, u\sin v, u^2 + v^2)$. So

$$\mathbf{T}_u = (\cos v, \sin v, 2u), \quad \mathbf{T}_v = (-u\sin v, u\cos v, 2v)$$

So $\mathbf{T}_u \times \mathbf{T}_v = (-2u^2\cos v + 2v\sin v, -2u^2\sin v - 2v\cos v, u)$. At $\Phi(1, 0) = (1, 0, 1)$,

$$\mathbf{n} = \mathbf{T}_u \times \mathbf{T}_v(1, 0) = (-2, 0, 1)$$

Example 3.4 (Cone)**.** Consider

$$\Phi(u, v) = (u\cos v, u\sin v, u), \quad u \le 0$$

Is ti regular, differentiable ?

sol.

$$\mathbf{T}_u = (\cos v, \sin v, 1), \quad \mathbf{T}_v = (-u\sin v, u\cos v, 0)$$

So $\mathbf{T}_u \times \mathbf{T}_v = 0$ at $(0, 0, 0)$.

Example 3.5. Find a parametrization of the following hyperboloid of one sheet.

$$x^2 + y^2 - z^2 = 1$$

sol. Since the graph is symmetric in x and y it is natural to use polar coordinate

$$x = \cos\theta, \quad y = \sin\theta$$

Then we have

$$r^2 - z^2 = 1$$

Thus we use

$$r = \cosh u, \quad z = \sinh u$$

to get

$$x = \cosh u \cos\theta, \quad y = \cosh u \sin\theta, \quad z = \sinh u$$

so

$$\begin{aligned}\Phi(u,\theta) &= (x(u,\theta), y(u,\theta), z(u,\theta)) \\ &= (\cosh u \cos\theta, \cosh u \sin\theta, \sinh u)\end{aligned}$$

Here

$$0 \le \theta \le 2\pi, \quad -\infty < u < \infty$$

$\Phi(u,\theta)$ belongs to the surface since

$$\cos^2\theta + \sin^2\theta = 1, \quad \cosh^2\theta - \sinh^2\theta = 1$$

Example 3.6. Find equation of tangent plane to Φ at $(1,0,1)$.

$$\Phi(u,v) = (u\cos v, u\sin v, u^2), \quad 0 \le v \le 2\pi, \quad -\infty < u < \infty$$

sol. $\Phi(1,0) = (1,0,1)$. Find normal vector $\mathbf{n}$ at $(1,0,1)$.

$$\begin{aligned}\mathbf{n} &= \Phi_u(1,0) \times \Phi_v(1,0)\\ &= (\cos v\mathbf{i} + \sin v\mathbf{j} + 2u\mathbf{k}) \times (-u\sin v\mathbf{i} + u\cos v\mathbf{j})\,|_{(u,v)=(1,0)}\\ &= (\mathbf{i} + 2\mathbf{k}) \times \mathbf{j}\\ &= -2\mathbf{i} + \mathbf{k}\end{aligned}$$

Then equation is

$$-2(x-1) + (z-1) = 0$$

제 4 절 Area of Parameterized Surface

Find the area of $U = \Phi(D)$ where $\Phi\colon D \to \mathbb{R}^3$ is a surface parametrization. Divide D into small rectangles. Consider small rectangle $R = [u, u+\Delta u] \times [v, u+\Delta v]$ The area of small rectangle under Φ is approximated by parallelogram by $\Phi(u,v)$, $\Phi(u+\Delta u, v)$, $\Phi(u, v+\Delta v)$, $\Phi(u+\Delta u, v+\Delta v)$. Two sides are given by $\Phi_u(u,v)\Delta u$ and $\Phi_v(u,v)\Delta v$. (Fig 7.6)

$$\begin{aligned}T_u &= \frac{\partial\Phi}{\partial u} = \frac{\partial x}{\partial u}\mathbf{i} + \frac{\partial y}{\partial u}\mathbf{j} + \frac{\partial z}{\partial u}\mathbf{k}\\ T_v &= \frac{\partial\Phi}{\partial v} = \frac{\partial x}{\partial v}\mathbf{i} + \frac{\partial y}{\partial v}\mathbf{j} + \frac{\partial z}{\partial v}\mathbf{k}\end{aligned}$$

Hence the area of $\Phi(R)$ is

$$\|\mathbf{T}_u \times \mathbf{T}_v\|\Delta u\Delta v$$

Hence Area of surface is the limit of the following

$$\sum \|\mathbf{T}_u \times \mathbf{T}_v\|\Delta u\Delta v$$

Definition 4.1. We define the surface area $A(S)$ of a parameterized surface S by

$$A(S) = \iint_D \|\mathbf{T}_u \times \mathbf{T}_v\| du dv$$

Later we shall also denote it by $\iint_{\mathbf{T}} dS$.

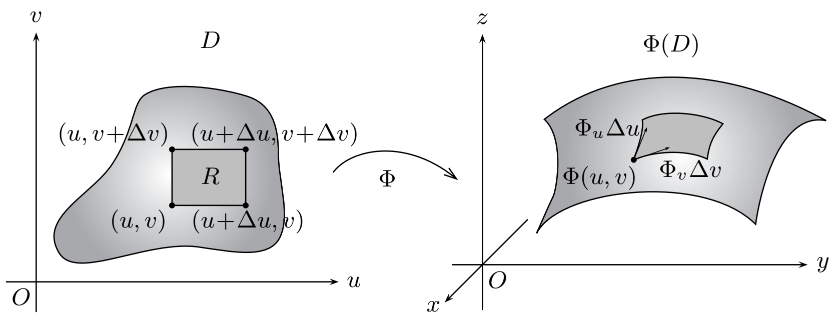


그림 7.6: Area of surface

Let $\Phi(u,v) = (x(u,v), y(u,v), z(u,v))$. We define dS by

$$dS = \|\Phi_u \times \Phi_v\| dudv,$$

and call the **(area element)**. Recall the notation

$$\frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix}$$

Then we see (Not trivial. Need justification).[1]

$$\begin{aligned} \iint_\Phi dS &= \iint_D \|\Phi_u \times \Phi_v\| dudv \\ &= \iint_D \sqrt{\left[\frac{\partial(y,z)}{\partial(u,v)}\right]^2 + \left[\frac{\partial(z,x)}{\partial(u,v)}\right]^2 + \left[\frac{\partial(x,y)}{\partial(u,v)}\right]^2}\, dudv \end{aligned}$$

Area of surface is independent of parametrization.

Example 4.2 (Cone)**.** Let D be given by

$$x = r\cos\theta, \quad y = r\sin\theta, \quad z = r.$$

sol. Either use formula above or $\|\mathbf{T}_r \times \mathbf{T}_\theta\| dr d\theta$. $\|\mathbf{T}_r \times \mathbf{T}_\theta\| = r\sqrt{2}$

[1] Φ is assumed to be 1-1.

Example 4.3 (Helicoid)**.** Let the helicoid given by

$$x = r\cos\theta, \quad y = r\sin\theta, \quad z = \theta.$$

and let D be the region where $0 \le \theta \le 2\pi$ and $0 \le r \le 1$. Find the area.

$\|\mathbf{T}_r \times \mathbf{T}_\theta\| = \sqrt{r^2+1}$ Need table to see

$$\pi[\sqrt{2} + \log(1+\sqrt{2})].$$

Surface Area of a Graph

When a surface U is given by the graph of function $z = f(x, y)$ on D, we see U is parameterized by $\Phi(x, y) = (x, y, f(x, y))$. Find Φ_x Φ_y by

$$\Phi_x = \mathbf{i} + \frac{\partial f}{\partial x}\mathbf{k}, \qquad \Phi_y = \mathbf{j} + \frac{\partial f}{\partial y}\mathbf{k}$$

Since

$$\Phi_x \times \Phi_y = (\mathbf{i} + \frac{\partial f}{\partial x}\mathbf{k}) \times (\mathbf{j} + \frac{\partial f}{\partial y}\mathbf{k}) = -\frac{\partial f}{\partial x}\mathbf{i} - \frac{\partial f}{\partial y}\mathbf{j} + \mathbf{k},$$

Area is

$$\iint_\Phi dS = \iint_D \left((\partial f/\partial x)^2 + (\partial f/\partial y)^2 + 1\right)^{1/2} dxdy$$

The unit normal vector $\mathbf{n}(x, y, z)$ on U is

$$\mathbf{n} = \mathbf{n}(x, y, z) = -\frac{\partial f}{\partial x}\mathbf{i} - \frac{\partial f}{\partial y}\mathbf{j} + \mathbf{k}$$

We can find the formula using the angle between $\mathbf{n}$ and $\mathbf{k}$. Let φ be the angle between $\mathbf{n}$ and $\mathbf{k}$. Then $\cos\varphi$ satisfies

$$\cos\varphi = \frac{\mathbf{n}\cdot\mathbf{k}}{\|\mathbf{n}\|} = \frac{1}{\sqrt{(\partial f/\partial x)^2 + (\partial f/\partial y)^2 + 1}}$$

Hence

$$dS = \sqrt{(\partial f/\partial x)^2 + (\partial f/\partial y)^2 + 1}dxdy = \frac{dxdy}{\cos\varphi},$$

and we get

$$\iint_\Phi dS = \iint_D \frac{dxdy}{\cos\varphi}$$

Example 4.4. Find the surface area of a unit ball.

sol. From $x^2 + y^2 + z^2 = 1$, we let $z = f(x, y) = \sqrt{1 - x^2 - y^2}$.

$$\frac{\partial f}{\partial x} = \frac{-x}{\sqrt{1 - x^2 - y^2}}, \quad \frac{\partial f}{\partial y} = \frac{-y}{\sqrt{1 - x^2 - y^2}}$$

Area of half sphere is

$$\begin{aligned} \iint_{\Phi} dS &= \iint_D \frac{1}{\sqrt{1 - x^2 - y^2}} dxdy \\ &= \int_0^{2\pi} \int_0^1 \frac{r}{\sqrt{1 - r^2}} \, drd\theta \\ &= 2\pi \end{aligned}$$

Surface of revolution

The lateral surface area generated by revolving the graph $y = f(x) \geq 0$ is

$$A = 2\pi \int_a^b y\sqrt{1 + (f'(x))^2} dx$$

Example 4.5. Use parametrization to express the area generated by revolving the graph $y = f(x)$. We can choose the parametrization

$$\Phi(u, v) = (x, y, z) = (u, f(u) \cos v, f(u) \sin v)$$

over the region

$$a \leq u \leq b, \quad 0 \leq v \leq 2\pi.$$

sol. Find the derivatives

$$\frac{\partial(y,z)}{\partial(u,v)} = f(u)f'(u), \ \frac{\partial(z,x)}{\partial(u,v)} = f(u)\cos v, \ \frac{\partial(x,y)}{\partial(u,v)} = -f(u)\sin v$$

Hence the area is

$$\begin{aligned}\iint_\Phi dS &= \iint_D |f(u)| \sqrt{\left[\frac{\partial(y,z)}{\partial(u,v)}\right]^2 + \left[\frac{\partial(z,x)}{\partial(u,v)}\right]^2 + \left[\frac{\partial(x,y)}{\partial(u,v)}\right]^2}\, dudv \\ &= \iint_D f(u)\sqrt{1+[f'(u)]^2}\, dudv \\ &= \int_a^b \int_0^{2\pi} f(u)\sqrt{1+[f'(u)]^2}\, dvdu \\ &= 2\pi \int_a^b f(u)\sqrt{1+[f'(u)]^2}\, du\end{aligned}$$

This formula coincide with earlier formula.

제 5 절 Integrals of Scalar functions over Surface

Integrals of Scalar functions over Surface

Let $\Phi\colon D \to \mathbb{R}^3$ be a parameterized surface $S = \Phi(D)$ and let $f\colon S \to R$ be a real valued function defined on Φ. First, if $f = 1$, it represents the area.

$$\iint_S dS = \iint_D \|\Phi_u \times \Phi_v\| dudv,$$

In general, the integral of f on $\iint_S f\, dS$ is defined by is

Definition 5.1.

$$\iint_S f\, dS = \iint_D f(\Phi(u,v))\|\Phi_u \times \Phi_v\| dudv$$

If the surface is parameterized by $\Phi(u,v) = (x(u,v), y(u,v), z(u,v))$ then the integral becomes

$$\iint_S f\, dS = \iint_D f(\Phi(u,v))\|T_u \times T_v\|\, dudv$$

$$\iint_D f(x(u,v), y(u,v), z(u,v)) \sqrt{\left[\frac{\partial(y,z)}{\partial(u,v)}\right]^2 + \left[\frac{\partial(z,x)}{\partial(u,v)}\right]^2 + \left[\frac{\partial(x,y)}{\partial(u,v)}\right]^2}\, dudv$$

If $\Phi =$ the sum of $\Phi_1, \Phi_2, \ldots, \Phi_m$,

$$\iint_{\Phi} f\, dS = \iint_{\Phi_1} f\, dS + \iint_{\Phi_2} f\, dS + \cdots + \iint_{\Phi_m} f\, dS$$

Surface integrals over graphs

Suppose S is the graph of a C^1 function $z = g(x, y)$. Then we parameterize it by

$$x = u, \quad y = v, \quad z = g(u, v)$$

and

$$\|T_u \times T_v\| = \sqrt{1 + (\frac{\partial g}{\partial u})^2 + (\frac{\partial g}{\partial v})^2}$$

So

$$\iint_S f(x, y, z)\, dS = \iint_D f(x, y, g(x, y)) \sqrt{1 + (\frac{\partial g}{\partial x})^2 + (\frac{\partial g}{\partial y})^2} dxdy$$

Example 5.2. Let S be defined by $z = x^2 + y$, where D is $0 \le x \le 1, \quad -1 \le y \le 1$. Find $\iint_S x\, dS$

sol.

$$\begin{aligned} \iint_S x\, dS &= \iint_D x \sqrt{1 + (\frac{\partial g}{\partial x})^2 + (\frac{\partial g}{\partial y})^2} dxdy = \int_{-1}^{1} \int_0^1 x\sqrt{1 + 4x^2 + 1} dxdy \\ &= \frac{1}{8} \int_{-1}^{1} \left[\int_0^1 (2 + 4x^2)^{1/2} (8xdx) \right] dy = \frac{2}{3} \frac{1}{8} \int_{-1}^{1} \left[(2 + 4x^2)^{3/2} \right] \Big|_0^1 dy \\ &= \sqrt{6} - \frac{\sqrt{2}}{3} \end{aligned}$$

□

Example 5.3. Evaluate $\iint_S z^2 dS$ when S is the unit sphere.

sol. Use spherical coordinate for $x^2+y^2+z^2=1$. Then $\rho=1$ and $z^2=\cos^2\phi$.

$$\iint_S z^2 dS = \iint_D \cos^2\phi \|T_\theta \times T_\phi\| d\theta d\phi$$

Since

$$\|T_\theta \times T_\phi\| = \sin\phi$$

$$\begin{aligned}\iint_S f(x,y,z)\,dS &= \int_0^{2\pi}\int_0^{\pi} \cos^2\phi \sin\phi \, d\phi d\theta \\ &= \frac{4\pi}{3}\end{aligned}$$

Integral over Graphs

We show

$$\iint_S f(x,y,z)dS = \iint_D f(\Phi(u,v))\|T_u \times T_v\| dudv = \iint_D \frac{f(x,y,g(x,y))}{\cos\theta} dxdy$$

where θ is the angle between normal vector and $\mathbf{k}$ vector. $\Delta S = \Delta A/\cos\theta$.

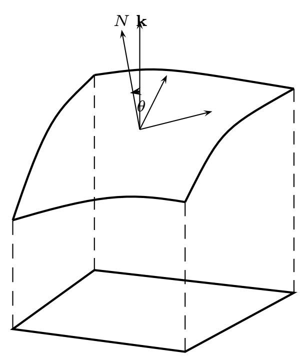
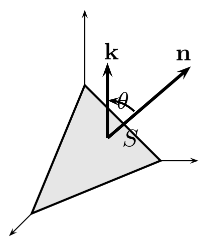


그림 7.7: Ratio between two surface

$$\cos\theta = \frac{\mathbf{N}\cdot\mathbf{k}}{\|\mathbf{N}\|} = \frac{1}{\sqrt{(\partial g/\partial x)^2 + (\partial g/\partial y)^2 + 1}}$$

Example 5.4. Compute $\iint_S x dS$ where S is triangle with vertices $(1,0,0)$, $(0,1,0)$ and $(0,0,1)$.



sol. $\cos\theta = \mathbf{n}\cdot\mathbf{k} = 1/\sqrt{3}$.

$$\sqrt{3}\iint_D x\,dx\,dy = \sqrt{3}\int_0^1\int_0^{1-x} x\,dy\,dx = \frac{\sqrt{3}}{6}$$

Example 5.5. Let $\Phi = (r\cos\theta, r\sin\theta, \theta)$ be the helicoid where $0 \le r \le 1$. Suppose S has density m equal to twice the distance to the central axis, $m = 2\sqrt{x^2+y^2} = 2r$. Total mass of the surface.

sol.

$$M = \iint_S 2r\,dS = 2\iint_D r\|T_r \times T_\theta\|\,dr\,d\theta$$

But $\|T_r \times T_\theta\| = \sqrt{1+r^2}$. Hence

$$M = \int_0^{2\pi}\int_0^1 2r\sqrt{1+r^2}\,dr\,d\theta = \frac{4}{3}\pi(2^{3/2}-1).$$

제 6 절 Surfaces Integrals of vector Fields

This section we develop the notion of integral of a vector field over a surface.

Recall the line integral of a vector field has a physical interpretation: **Work**. Similarly, the notion of integral of a vector field over a surface is a **Flux**.

Suppose $\mathbf{F}$ represent the velocity of a fluid.(like river) Then you place a net into the water and imagine the amount of water that passes through your net per unit time(= **rate at which water pass through the net**)

Let $\mathbf{F}\colon V \to \mathbb{R}^3$ defined over the surface $S = \Phi(D)$, $\Phi\colon D \to \mathbb{R}^3$ we define $\iint_\Phi \mathbf{F}\cdot d\mathbf{S}$

Definition 6.1.

$$\iint_{\Phi} \mathbf{F} \cdot d\mathbf{S} = \iint_D \mathbf{F}(\Phi(u,v)) \cdot (\Phi_u \times \Phi_v) du dv$$

Note we used the notation $\iint_{\Phi}$ instead of $\iint_S$.

If we let $\mathbf{n} = \Phi_u \times \Phi_v / \|\Phi_u \times \Phi_v\|$ be the unit normal vector to the surface, then

$$\iint_{\Phi} \mathbf{F} \cdot d\mathbf{S} = \iint_D (\mathbf{F} \cdot \mathbf{n}) \cdot (\Phi_u \times \Phi_v) du dv = \iint_S \mathbf{F} \cdot \mathbf{n} dS.$$

Hence if $\mathbf{F}$ has same direction as $\mathbf{n} = \Phi_u \times \Phi_v / \|\Phi_u \times \Phi_v\|$, i.e, ($\mathbf{F}(x,y,z) = f(x,y,z)\mathbf{n}$ for some scalar function f) then

$$\iint_{\Phi} f(x,y,z) dS = \iint_{\Phi} \mathbf{F}(x,y,z) \cdot d\mathbf{S}$$

Example 6.2.

Example 6.3 (Spherical coordinate)**.** Let S be the unit sphere parameterized by

$$D\colon 0 \le \theta \le 2\pi, \quad 0 \le \phi \le \pi$$
$$\Phi\colon D \to \mathbb{R}^3, \quad \Phi(\theta, \phi) = (\cos\theta \sin\phi, \sin\theta \sin\phi, \cos\phi)$$

Compute $\iint_S \mathbf{r} \cdot d\mathbf{S}$ where $\mathbf{r} = x\mathbf{i} + y\mathbf{i} + z\mathbf{k}$ denotes the position vector.

sol.

$$\Phi_\theta = -\sin\theta\sin\phi\mathbf{i} + \cos\theta\sin\phi\mathbf{j}$$
$$\Phi_\phi = \cos\theta\cos\phi\mathbf{i} + \sin\theta\cos\phi\mathbf{j} - \sin\phi\mathbf{k}$$

Now compute $\Phi_\theta \times \Phi_\phi$:

$$\begin{aligned}\Phi_\theta \times \Phi_\phi &= -\sin\phi(\cos\theta\sin\phi\mathbf{i} + \sin\theta\sin\phi\mathbf{j} + \cos\phi\mathbf{k}) \\ &= (-\sin\phi)\,\mathbf{r}(\Phi(\theta,\phi))\end{aligned}$$

Hence

$$\begin{aligned}\iint_\Phi \mathbf{r}\cdot d\mathbf{S} &= \iint_D (-\sin\phi)\mathbf{r}\cdot\mathbf{r}\,d\theta d\phi \\ &= \iint_D (-\sin\phi)\,d\theta d\phi = -4\pi\end{aligned}$$

Orientation

As with line integral, the surface integral also has the notion of direction. First we need to define the **orientation** of a surface S. It depends on the particular parametrization.

Definition 6.4 (Oriented Surface)**.** Oriented Surface is two sided surface with one side specified as outside(or positive side) At each point there are two unit normal vectors $\mathbf{n}_1$ and $\mathbf{n}_2$, where $\mathbf{n}_1 = -\mathbf{n}_2$. Each of these normals can be associated with one side of the surface. **(orientation) orientable surface** . For orientable surface, there are two possible normal vectors. There are nonorientable surface.(Mobius strip)

Let $\Phi\colon D \to \mathbb{R}^3$ represent a oriented surface. If $\mathbf{n}(\Phi)$ is the unit normal to S, then

$$\mathbf{n}(\Phi) = \pm\frac{\Phi_u \times \Phi_v}{\|\Phi_u \times \Phi_v\|}$$

A parametrization is called **Orientation-preserving** if

$$\frac{T_u \times T_v}{\|T_u \times T_v\|} = +\mathbf{n}(\Phi)$$

Otherwise, it is **Orientation-reversing**.

Example 6.5. The parametrization of sphere by spherical coordinate is orientation-reversing. By changing the order of θ and ϕ, we can get orientation-preserving parametrization.

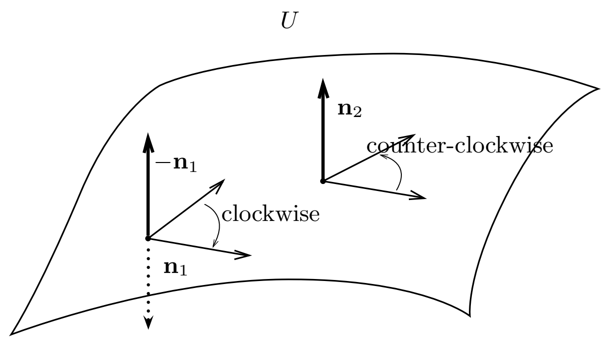


그림 7.8: clockwise, counter-clockwise ($\mathbf{n}_1$ and $\mathbf{n}_2$ are normals the orientation points)

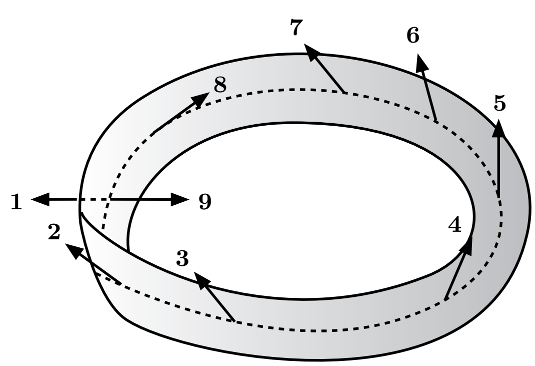


그림 7.9: Möbius strip

Orientation of graph

Example 6.6. Let S be the graph of a function $z = g(x, y)$. Then the unit normal

The normal is given by

$$\mathbf{n} = \frac{-\frac{\partial g}{\partial x}\mathbf{i} - \frac{\partial g}{\partial y}\mathbf{j} + \mathbf{k}}{\sqrt{1 + \left(\frac{\partial g}{\partial x}\right)^2 + \left(\frac{\partial g}{\partial y}\right)^2}} dxdy$$

We can give orientation of such surface by taking the positive side to be the side away from which $\mathbf{n}$ points. (Fig 7.7)

Independence of parametrization

Theorem 6.7. *Let S be an oriented surface and let Φ_1, Φ_2 be two regular orientation preserving parametrization, then for continuous $\mathbf{F}$ defined on S, then*

$$\iint_{\Phi_1} \mathbf{F} \cdot d\mathbf{S} = \iint_{\Phi_2} \mathbf{F} \cdot d\mathbf{S}$$

If one of them is orientation reversing, then

$$\iint_{\Phi_1} \mathbf{F} \cdot d\mathbf{S} = -\iint_{\Phi_2} \mathbf{F} \cdot d\mathbf{S}$$

For scalar f, we have for any Φ_1, Φ_2

$$\iint_{\Phi_1} f dS = \iint_{\Phi_2} f dS$$

If Φ consists of Φ_1, Φ_2, $\ldots$, Φ_m

$$\iint_{\Phi} \mathbf{F} \cdot d\mathbf{S} = \iint_{\Phi_1} \mathbf{F} \cdot d\mathbf{S} + \iint_{\Phi_2} \mathbf{F} \cdot d\mathbf{S} + \cdots + \iint_{\Phi_m} \mathbf{F} \cdot d\mathbf{S}$$

Hence we can define the sum of surfaces as

$$\Phi = \Phi_1 + \Phi_2 + \cdots + \Phi_m$$

$$\mathbf{n} = \frac{-\partial f/\partial x\,\mathbf{i} - \partial f/\partial y\,\mathbf{j} + \mathbf{k}}{\sqrt{(\partial f/\partial x)^2 + (\partial f/\partial y)^2 + 1}}$$

Surface U is given by $\Phi(x, y) = (x, y, f(x, y))$. If $\mathbf{F} = F_1\,\mathbf{i} + F_2\,\mathbf{j} + F_3\,\mathbf{k}$ is a vector field then $\iint_S \mathbf{F}\cdot d\mathbf{S}$ is expressed as

$$\begin{aligned}\iint_\Phi \mathbf{F}\cdot d\mathbf{S} &= \iint_D \mathbf{F}\cdot(\Phi_x\times\Phi_y)dxdy\\ &= \iint_D \left[F_1\left(-\frac{\partial f}{\partial x}\right)+F_2\left(-\frac{\partial f}{\partial y}\right)+F_3\right]dxdy\end{aligned}$$

Relation with scalar integrals

Recall

Definition 6.8.

$$\iint_\Phi \mathbf{F}\cdot d\mathbf{S} = \iint_D \mathbf{F}(\Phi(u,v))\cdot(\Phi_u\times\Phi_v)dudv$$

If we write

$$\mathbf{n} = (\Phi_u\times\Phi_v)/\|\Phi_u\times\Phi_v\|,\quad d\mathbf{S} = (\Phi_u\times\Phi_v)dudv,\quad d\mathbf{S} = \mathbf{n}\,dS$$

then it can be written as

$$\iint_\Phi \mathbf{F}\cdot d\mathbf{S} = \iint_\Phi \mathbf{F}\cdot\mathbf{n}\,dS$$

Physical Interpretation of Surface Integrals

Consider the parallelogram determined by three vectors $\mathbf{F}$, $T_u\Delta u$ and $T_v\Delta v$. The volume is

$$\mathbf{F}\cdot(T_u\Delta u\times T_v\Delta v) = \mathbf{F}\cdot(T_u\times T_v)\Delta u\Delta v$$

If $\mathbf{F}$ is velocity of a fluid, the volume is *the amount of fluid to flow ourward the surface per unit time*. Hence

$$\iint_S \mathbf{F}\cdot d\mathbf{S}$$

is the *net quantity of fluid to flow across the surface per unit time*, i.e, **the rate of fluid flow**. It is also called **flux of F across** S.

Example 6.9 (Heat flow)**.** Let T denote the temperature at a point. Then

$$\nabla T = \frac{\partial T}{\partial x}\mathbf{i}+\frac{\partial T}{\partial y}\mathbf{j}+\frac{\partial T}{\partial z}\mathbf{k}$$

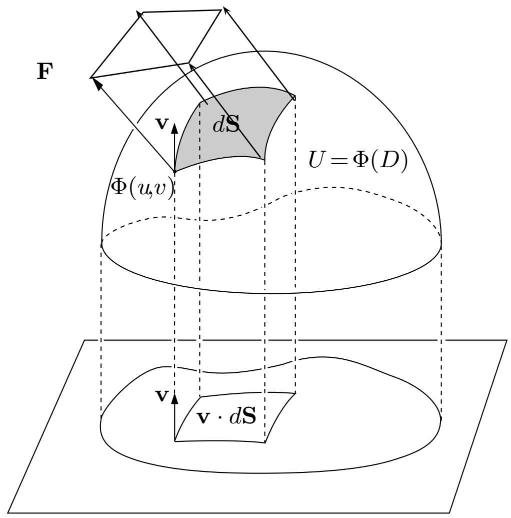


그림 7.10: Area of shadow region and flux across S

represent the temperature gradient and heat "flows" with the vector field $-k\nabla T$.

Example 6.10. Suppose temperature is $T = x^2 + y^2 + z^2$ on S where

$$S\colon x^2 + y^2 + z^2 = 1.$$

Find the flux across S if $k = 1$

sol. We have $\mathbf{F} = -2\mathbf{r}$. Then $\mathbf{r}\cdot\mathbf{n} = -2$. So

$$\iint_S \mathbf{F}\cdot d\mathbf{S} = -2\iint_S dS = -8\pi$$

Example 6.11 (Gauss Law)**.** The sum of the flux of an electric field $\mathbf{E}$ over a closed surface S is the net charge Q contained in the surface. Namely,

$$\iint_S \mathbf{E}\cdot d\mathbf{S} = Q$$

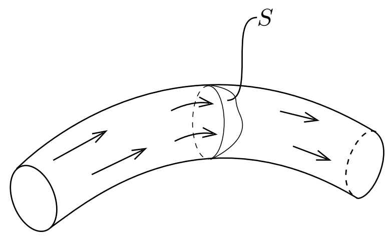


그림 7.11: Water through a pipe and a surface S

Suppose $\mathbf{E} = E\mathbf{n}$(constant multiple of normal vector) then

$$\iint_S \mathbf{E}\cdot d\mathbf{S} = \iint_S E dS = Q = EA(S)$$

So

$$E = \frac{Q}{A(S)}$$

If S is sphere of radius R then

$$E = \frac{Q}{4\pi R^2} \tag{7.1}$$

Surface Integral over Graphs

Suppose S is the graph of $z = g(x, y)$. We show

$$\iint_S \mathbf{F}\cdot d\mathbf{S} = \iint_D \mathbf{F}\cdot(T_x \times T_y)dxdy = \iint_D \left[F_1(-\frac{\partial g}{\partial x}) + F_1(-\frac{\partial g}{\partial y}) + F_3\right] dxdy$$

We parameterize the surface S by $\Phi(x, y) = (x, y, g(x, y))$ and compute

$$T_x = \mathbf{i} + \frac{\partial g}{\partial x}\mathbf{k}, \quad T_y = \mathbf{j} + \frac{\partial g}{\partial y}\mathbf{k}$$

Hence

$$T_x \times T_y = -(\frac{\partial g}{\partial x})\mathbf{i} - (\frac{\partial g}{\partial y})\mathbf{j} + \mathbf{k}$$

and we proved the formula.

Example 6.12. The equation

$$z = 12, \quad x^2 + y^2 \le 25$$

describes a disk of radius 5 lying on the plane $z = 12$. Compute $\iint_S \mathbf{r} \cdot d\mathbf{S}$ when

$$\mathbf{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}.$$

sol.

$$T_x \times T_y = \mathbf{i} \cdot \mathbf{j} = \mathbf{k}$$

So $\mathbf{r} \cdot (T_x \times T_y) = z$ and

$$\iint_S \mathbf{r} \cdot d\mathbf{S} = \iint_D z dx dy = 12A(D) = 300\pi.$$

Summary

(1) Parameterized Surface $\Phi(u, v)$

(a) Integral of a scalar f:

$$\iint_\Phi f dS = \iint_D f(\Phi(u, v)) \|T_u \times T_v\| du dv$$

(b) Scalar surface element:

$$dS = \|T_u \times T_v\| du dv$$

(c) Integral of a vector field:

$$\iint_\Phi \mathbf{F} \cdot d\mathbf{S} = \iint_D \mathbf{F}(\Phi(u, v)) \cdot (T_u \times T_v) du dv$$

(d) Vector surface element:

$$d\mathbf{S} = (T_u \times T_v) du dv = \mathbf{n}\, dS$$

(2) Graph $z = g(x, y)$

(a) Integral of a scalar f:

$$\iint_S f dS = \iint_D \frac{f(x, y, g(x, y))}{\cos\theta} dx dy$$

(b) Scalar surface element:

$$dS = \frac{dx\,dy}{\cos\theta} = \sqrt{\left(\frac{\partial g}{\partial x}\right)^2 + \left(\frac{\partial g}{\partial y}\right)^2 + 1}\, dxdy$$

(c) Integral of a vector field:

$$\iint_{\mathbf{S}} \mathbf{F} \cdot d\mathbf{S} = \iint_D \left(-F_1 \frac{\partial g}{\partial x} - F_2 \frac{\partial g}{\partial y} + F_3\right) dxdy$$

(d) Vector surface element:

$$d\mathbf{S} = \mathbf{n}\, dS = \left(-\frac{\partial g}{\partial x}\mathbf{i} - \frac{\partial g}{\partial y}\mathbf{j} + \mathbf{k}\right) dxdy$$

(3) Sphere $x^2 + y^2 + z^2 = R^2$

(a) Scalar surface element:

$$dS = R^2 \sin\phi\, d\phi d\theta$$

(b) Vector surface element:

$$d\mathbf{S} = (x\mathbf{i} + y\mathbf{j} + z\mathbf{k})R\sin\phi\, d\phi d\theta = \mathbf{r}R\sin\phi\, d\phi d\theta = \mathbf{n}R^2 \sin\phi\, d\phi d\theta$$

제 7 절 Application to Differential Geometry

Curvature

Let $\Phi \colon D \to \mathbb{R}^3$ be a surface parametrization of S. Then

$$T_u = \frac{\partial \Phi}{\partial u}, \quad T_v = \frac{\partial \Phi}{\partial v}$$

are tangent vectors. Assume the normals are well defined so that $\mathbf{T}_u \times \mathbf{T}_v \not 0$.
Let

$$E = \|\mathbf{T}_u\|^2, \quad F = \mathbf{T}_u \cdot \mathbf{T}_v, \quad G = \| \times \mathbf{T}_v\|^2$$

We can show that

$$\|\mathbf{T}_u \times \mathbf{T}_v\|^2 = EG - F^2$$

If we let

$$\mathbf{N} = \frac{\mathbf{T}_u \times \mathbf{T}_v}{\|\mathbf{T}_u \times \mathbf{T}_v\|} = \frac{\mathbf{T}_u \times \mathbf{T}_v}{\sqrt{EG - F^2}}$$

denote the unit normal vector to the surface at point $p = \Phi(u, v)$. We define "Gauss curvature" $K(p)$ and "mean curvature" $H(p)$. To define these first define three new functions ℓ, m, n on S:

$$\begin{aligned} \ell(p) &= \mathbf{N}(u,v) \cdot \tfrac{\partial^2 \Phi}{\partial u^2} = \mathbf{N}(u,v)\Phi_{uu} \\ m(p) &= \mathbf{N}(u,v) \cdot \tfrac{\partial^2 \Phi}{\partial u \partial v} = \mathbf{N}(u,v)\Phi_{uv} \\ n(p) &= \mathbf{N}(u,v) \cdot \tfrac{\partial^2 \Phi}{\partial v^2} = \mathbf{N}(u,v)\Phi_{vv} \end{aligned} \tag{7.2}$$

The "Gauss curvature" $K(p)$ is given by

$$K(p) = \frac{\ell n - m^2}{EG - F^2} \tag{7.3}$$

and "mean curvature" $H(p)$ is given by

$$H(p) = \frac{G\ell + En - 2Fm}{2(EG - F^2)} \tag{7.4}$$

7.1 Gauss-Bonnet Theorem

The Gauss curvature to the sphere of radius R is $1/R^2$. The Gauss curvature is in general a function. We consider

$$\iint_S K \, dA$$

For the sphere, we have

$$\frac{1}{R^2} \iint_S dA = 4\pi$$

Gauss-Bonnet found out that this quantity is preserved for any sphere like object. This is a topological invariant. Now consider a torus like object. (This has one handle) In general consider an object with g-handles. Then

$$\iint_S K \, dA = 4\pi(1 - g)$$