제 6 장

Change of variables

제 1 절 Geometry of Maps from $\mathbb{R}^2$ to $\mathbb{R}^2$

If $x : [a, b] \to \mathbb{R}$ is $\mathcal{C}^1$ function and $f : \mathbb{R} \to \mathbb{R}$ is integrable, then

$$\int_a^b f(x(t))x'(t)dt = \int_{x(a)}^{x(b)} f(x)dx \tag{6.1}$$

Maps of a region to another

Let D^* be a region in $\mathbb{R}^2$. Suppose T is C^1-map $D^* \to \mathbb{R}^2$. We denote the image by $D = T(D^*)$.

$$T(D^*) = \{(x, y) \mid (x, y) = T(x^*, y^*), \quad (x^*, y^*) \in D^*\}$$

(Fig 6.1)

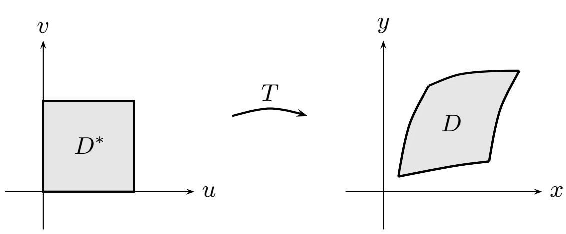

그림 6.1: Map from D^* to D

Example 1.1. Let D^* be the rectangle $D^* = [0, 1] \times [0, \pi/3]$ in (r, θ) plane. Find the image of D^* under $T = T(r, \theta) = (r\cos\theta, r\sin\theta)$.

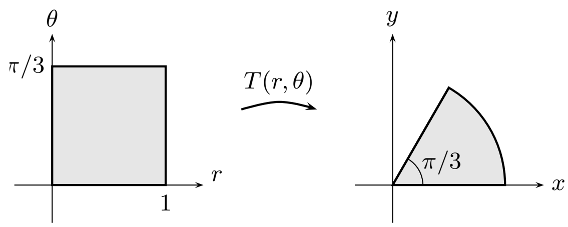

그림 6.2: Map by Polar coordinate

sol. Let $T(r,\theta) = (x, y)$. Then $x^2 + y^2 = r^2$, $0 \le r \le 1$. Thus D is a circular sector $0 \le r \le 1$, $0 \le \theta \le \pi/3$. See figure 6.2.

Example 1.2. Let T be defined by $T(u, v) = ((u + v)/2, (u - v)/2)$ and $D^* = [-1, 1] \times [-1, 1]$. Determine the image $T(D^*)$.

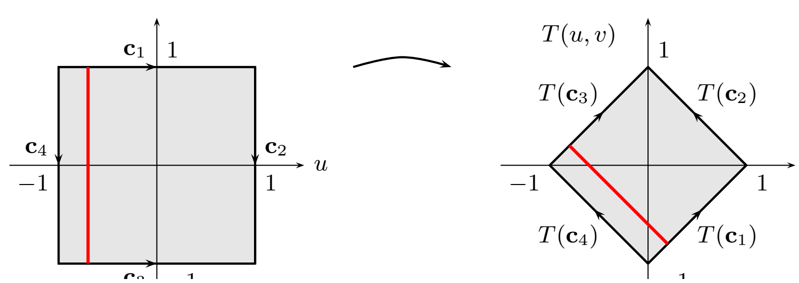

그림 6.3: Effect of linear transform

sol. Let $D = T(D^*)$(see fig 6.3). Consider the effect of T on the line $\mathbf{c}_1(t) = (t, 1)$, $-1 \le t \le 1$.

$$T(\mathbf{c}(t)) = ((t+1)/2, (t-1)/2), \quad -1 \le t \le 1$$

The image is the line segment $y = x - 1$, $-1 \le x \le 1$.
Similarly try to see the image of

$$\begin{aligned} \mathbf{c}_2(t) &= (1, t), \quad -1 \le t \le 1 \\ \mathbf{c}_3(t) &= (t, -1), \quad -1 \le t \le 1 \\ \mathbf{c}_4(t) &= (-1, t), \quad -1 \le t \le 1 \end{aligned}$$

The image seems like rotated rectangle. To prove indeed this is the case, we consider the image of the *red* line

$$\mathbf{c}(t) = (\alpha, t) - 1 \le t \le 1$$

which is

$$T(\mathbf{c}(t)) = ((\alpha+1)/2, (\alpha-1)/2), \quad -1 \le t \le 1$$

Then $x + y = \alpha$. This is the *red* line passing through inside the diamond shape.

Remark 1.3. Let $T(u, v) = (x, y) = ((u+v)/2, (u-v)/2)$. Then we have

$$x = (u+v)/2, \quad y = (u-v)/2$$

Hence the map $T : \mathbb{R}^2 \to \mathbb{R}^2$ in Example 1.2 can be represented by a matrix

$$A = \frac{1}{2}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$$

$$T(u, v) := \begin{bmatrix} x \\ y \end{bmatrix} = A \begin{bmatrix} u \\ v \end{bmatrix}$$

In general we have

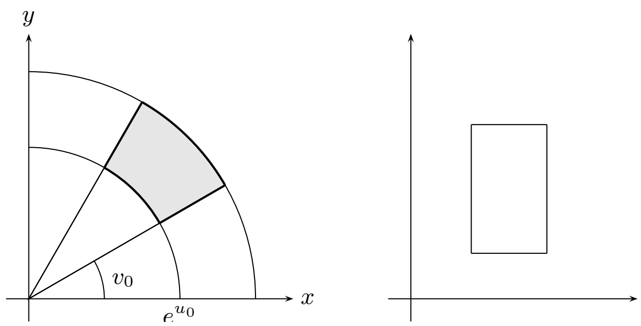

그림 6.4: Inverse image of concentric circle

Theorem 1.4. *Let A be a 2×2 matrix with non zero determinant. Let T be a linear transformation given by $T(\mathbf{x}) = A\mathbf{x}$. Then T maps parallelograms into parallelograms and vertices into vertices.*

One-to-one map

Definition 1.5. A map is called **one to one** on D^*, if for (u, v) and $(u', v') \in D^*$, $T(u, v) = T(u', v')$ implies $(u, v) = (u', v')$.

Example 1.6. Show the polar coordinate map $T = T(r, \theta) = (r\cos\theta, r\sin\theta)$ is not one-to-one. But the linear map in example 1.2 is one-to-one.

Onto map

Definition 1.7. A map T is called **onto** D, if for every point $(x, y) \in D$ there exists at least a point $(u, v) \in D$ such that $T(u, v) = (x, y)$.

Thus if T is onto then we can solve the equation $T(u, v) = (x, y)$. if in addition, T is one-to-one, the solution is unique.

Example 1.8. Let D be the region in the first quadrant lying between concentric circles $r = a, r = b$ and $\theta_1 \le \theta \le \theta_2$. Let

$$T(u, v) = (r\cos\theta, r\sin\theta)$$

be polar coordinate map. Find a region D^*(in (r, θ) coordinate plane) such that $D = T(D^*)$.

sol. In D, we see

$$a^2 \le r^2 \le b^2, \quad \theta_1 \le \theta \le \theta_2.$$

Hence

$$D^* = [a, b] \times [\theta_1, \theta_2]$$

Example 1.9. Let T be $((x+y)/2, (x-y)/2)$ and let D be the square whose vertices are $(1,0), (0,1), (-1,0), (0,-1)$. Find a D^* such that $D = T(D^*)$.

sol. Since T is linear $T(\mathbf{x}) = A\mathbf{x}$ where A is 2×2 matrix whose determinant is nonzero. T^{-1} is also a linear transform. Hence by thm 1, D^* must be a parallelogram. Then we find inverse image of vertices to find D^*. It turns out

$$D^* = [-1, 1] \times [-1, 1]$$

Remark 1.10. A linear transform from $\mathbb{R}^n$ to $\mathbb{R}^n$ given by a matrix A is **one to one** and **onto** if $\det A \neq 0$.

제 2 절 Change of variable

Given two regions D and D^*, differentiable mapping T on D^* with image D, that is $D = T(D^*)$, we would like to express $\iint_D f(x,y)dxdy$ as an integral over D^* of the composite function $f \circ T$. Let

$$T(u,v) = (x(u,v), y(u,v)) \text{ for } (u,v) \in D^*$$

As a special case, consider the case when $f = 1$. Then, in general

$$\iint_D f(x,y)dxdy = A(D) \neq A(D^*) = \iint_{D^*} f \circ T(u,v)dudv$$

Jacobian Determinant-measures change of area

We want to see how a region changes under the map.

Let $D^* = [0,1] \times [0,1]$, and $T(u,v) = (2u+v, v)$, $D = T(D^*)$. Fix $v = v_0 = 0$, then $T(u,0) = (x(u,0), y(u,0)), (0 \le u \le 1)$ is a path. Hence its tangent vector at $u = 0$ is $\frac{\partial x}{\partial u}$, which is the first column of

$$DT(0,0) = \begin{bmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{bmatrix}$$

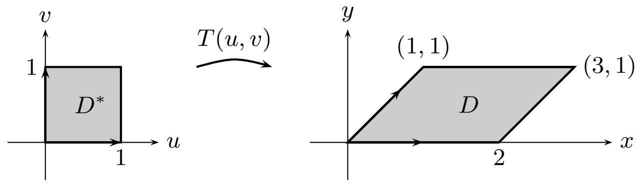

그림 6.5: image of rectangle under linear transform T

It can be explained as: The tangent vector at $(0,0)$ in D^* is mapped to the tangent vector at $T(0,0)$ by $DT(0,0)$. Hence the tangent vector $(1,0)$ is mapped to $DT(0,0)(1,0)^T$. Now fix $u=0$ and repeat the argument to see the tangent vector $(0,1)$ is mapped to $DT(0,0)(0,1)^T$. Write them as column vector $(1,0)^T$, $(0,1)^T$ they are two columns of $DT(0,0)$, $T(D^*)$ is parallelogram the area is given by cross product.

$$DT = \begin{bmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{bmatrix}$$

The two tangent vectors are

$$\frac{\partial x}{\partial u}\mathbf{i} + \frac{\partial y}{\partial u}\mathbf{j}, \quad \frac{\partial y}{\partial v}\mathbf{i} + \frac{\partial y}{\partial v}\mathbf{j}$$

Hence the area is $|J|$ where

$$J = \begin{bmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{bmatrix} = |DT|$$

(Also write $J = \partial(x,y)/\partial(u,v)$) J is **(Jacobian determinant)**. Hence the small rectangle of size Δu, Δv along u, v direction is mapped to a rectangle with area $|J|\Delta u \Delta v$.

Change of variable

Similar idea is used when T is a $\mathcal{C}^1$ mapping from a subset of $\mathbb{R}^2$ to $\mathbb{R}^2$. Let $R^* = [u, u+\Delta u] \times [v, v+\Delta v]$ and the image under T is denoted by R. Then

$$T_u = DT(u,v)\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{bmatrix}\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} \frac{\partial x}{\partial u} \\ \frac{\partial y}{\partial u} \end{bmatrix}$$

and

$$T_v = DT(u,v)\begin{bmatrix}0\\1\end{bmatrix} = \begin{bmatrix}\frac{\partial x}{\partial u} & \frac{\partial x}{\partial v}\\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v}\end{bmatrix}\begin{bmatrix}0\\1\end{bmatrix} = \begin{bmatrix}\frac{\partial x}{\partial v}\\ \frac{\partial y}{\partial v}\end{bmatrix}$$

Two tangent vectors at $T(u,v)$ are

$$T_u\Delta u, \quad T_v\Delta v$$

These form a parallelogram approximating the region D(figure 6.6). The area of the parallelogram is

$$\begin{bmatrix}\frac{\partial x}{\partial u}\Delta u & \frac{\partial x}{\partial v}\Delta v\\ \frac{\partial y}{\partial u}\Delta u & \frac{\partial y}{\partial v}\Delta v\end{bmatrix} = \begin{bmatrix}\frac{\partial x}{\partial u} & \frac{\partial x}{\partial v}\\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v}\end{bmatrix}\Delta u\Delta v = \frac{\partial(x,y)}{\partial(u,v)}\Delta u\Delta v$$

$$\|T_u \times T_v\|\Delta u\Delta v = |J|\Delta u\Delta v$$

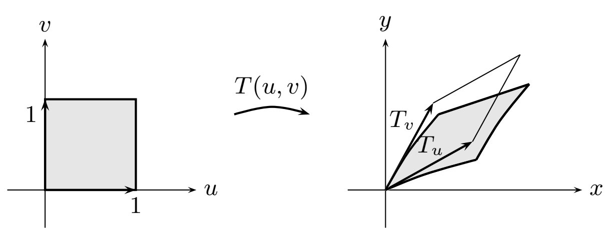

그림 6.6: approx T

Hence

Theorem 2.1. *If T is $D^* \to D \subset \mathbb{R}^2$, $\mathcal{C}^1$ one to one, onto function, then the area of D is*

$$\iint_D dxdy = \iint_{D^*} |J|\,dudv \tag{6.2}$$

If f is a continuous function on D, the integral of f is given by

$$\iint_D f(x,y)\,dxdy = \iint_{D^*} f(x(u,v),y(u,v))\left|\frac{\partial(x,y)}{\partial(u,v)}\right|\Delta u\Delta v\,dudv \tag{6.3}$$

Remark 2.2. Since $|\det DT| = \|T_u \times T_v\|$, equation (6.2) becomes

$$\iint_D dxdy = \iint_{D^*} \|T_u \times T_v\|dudv. \tag{6.4}$$

As a special case, we may consider a polar coordinate. The shaded region of $D^* = \Delta r \times \Delta\theta$ is mapped by T to a part of circular sector between r and $r + \Delta r$, θ to $\theta + \Delta\theta$. The area of this sector is $r\Delta r\Delta\theta$. So the area under polar coordinate change is

$$\iint_{T(D^*)} dxdy = \iint_D r\,drd\theta$$

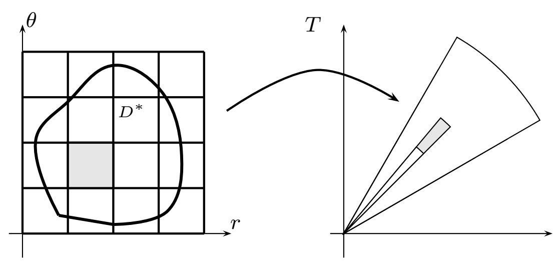

그림 6.7: approximate $T(D^*)$-polar coordinate

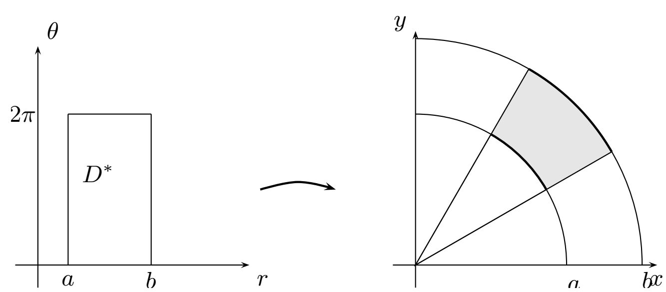

그림 6.8: polar Coordinate

Example 2.3. Change to polar coordinate.

sol. Since $x = r\cos\theta$, $y = r\sin\theta$, we can let $T(r,\theta) = (r\cos\theta, r\sin\theta)$. Then Jacobian is

$$\begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} \end{vmatrix} = \begin{vmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{vmatrix} = r$$

Hence

$$\iint f(x,y)\,dxdy = \iint f(r\cos\theta, r\sin\theta)\,rdrd\theta$$

Example 2.4. D is between two concentric circles: $x^2 + y^2 = 4$, $x^2 + y^2 = 1 (x, y \geq 0)$. Find the integral

$$\iint_D \log(x^2 + y^2)dxdy$$

sol. Use polar coordinate. Since two concentric circles are described by $r = 1, 2$, $0 \leq \theta \leq \pi/2$, we let $D^* = [1,2] \times [0, \pi/2]$ and $T(r,\theta) = (r\cos\theta, \sin\theta)$. Then $T(D^*) = D$ and

$$\begin{aligned}
\iint_D \log(x^2 + y^2)dxdy &= \iint_{D^*} (\log r^2)rdrd\theta \\
&= \int_1^2 \int_0^{\pi/2} 2r\log r d\theta dr \\
&= \int_1^2 \pi r \log r dr \\
&= \pi \left[\frac{r^2}{2}\log r - \frac{r^2}{4}\right]_1^2 \\
&= \pi(2\log 2 - \frac{3}{4}).
\end{aligned}$$

Example 2.5. $D^* = \{(u,v)\colon 1 \leq u^2 + v^2 \leq 4\}$ and T is given by

$$T(u,v) = \left(\frac{u}{u^2+v^2}, -\frac{v}{u^2+v^2}\right)$$

Find the area of $D = T(D^*)$.

sol. First compute T_u, T_v

$$T_u = \left(\frac{-u^2+v^2}{(u^2+v^2)^2}, \frac{2uv}{(u^2+v^2)^2} \right)$$
$$T_v = \left(\frac{-2uv}{(u^2+v^2)^2}, \frac{-u^2+v^2}{(u^2+v^2)^2} \right)$$

So

$$|J| = \|T_u \times T_v\| = \frac{1}{(u^2+v^2)^2}$$

Hence area is

$$\begin{aligned}\iint_{T(D^*)} dxdy &= \iint_{D^*} |J|\, dudv \\ &= \iint_{D^*} \frac{dudv}{(u^2+v^2)^2} \\ &= \int_0^{2\pi}\int_1^2 \frac{1}{r^3}\, drd\theta \\ &= \frac{3\pi}{4}\end{aligned}$$

The circle of radius r in D^* is $u^2+v^2=r^2$. Hence the image satisfies

$$x^2+y^2 = \left(\frac{u}{u^2+v^2}\right)^2 + \left(\frac{-v}{u^2+v^2}\right)^2 = \frac{1}{u^2+v^2} = \frac{1}{r^2}$$

It is circle of radius $1/r$ If $u = \alpha v$, $\alpha > 0$ represent a line through origin, the image is $x = -\alpha y$

Example 2.6 (The Gaussian integral)**.** Show that

$$\int_{-\infty}^{\infty} e^{-x^2} dx = \sqrt{\pi}.$$

To compute this, let us first observe

$$\int_{-\infty}^{\infty} e^{-x^2} dx \int_{-\infty}^{\infty} e^{-y^2} dy = \int_{-\infty}^{\infty} e^{-(x^2+y^2)} dxdy$$

compute

$$\int_{D_a} e^{-(x^2+y^2)} dxdy = \sqrt{\pi}$$

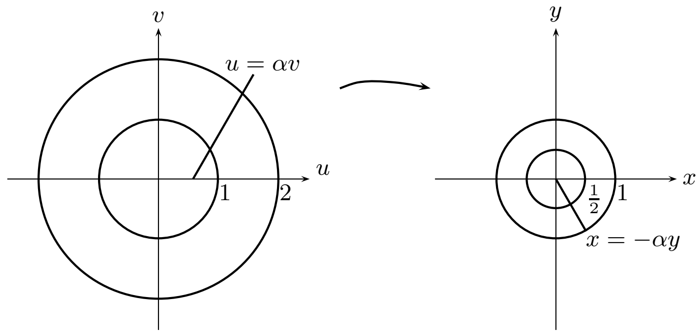

그림 6.9: $T(u,v) = (u/(u^2+v^2), -v/(u^2+v^2))$

by change of variables.

$$\begin{aligned}\int_{D_a} e^{-(x^2+y^2)}dxdy &= \int_0^{2\pi}\int_0^a e^{-r^2} r\,drd\theta = \int_0^{2\pi}\left(-\frac{1}{2}e^{-r^2}\right)\Big|_0^a \\ &= -\frac{1}{2}\int_0^{2\pi}(e^{-a^2}-1)d\theta = \pi(1-e^{-a^2}).\end{aligned}$$

Let $a \to \infty$ then we obtain the result.

Change of Variables in Triple Integrals

As with 2-D case, the change of variable in a subregion of $\mathbb{R}^3$.

Definition 2.7. Let $T : \mathbb{R}^3 \to \mathbb{R}^3$ be given by

$$T(u,v,w) = (x(u,v,w), y(u,v,w), z(u,v,w))$$

The the derivative DT of T is called **Jacobian determinant** or **Jacobian** and denoted by J or $J(T)$.

$$J = \frac{\partial(x,y,z)}{\partial(u,v,w)} = \det\begin{bmatrix}\frac{\partial x}{\partial u}, & \frac{\partial x}{\partial v}, & \frac{\partial x}{\partial w}\\ \frac{\partial y}{\partial u}, & \frac{\partial y}{\partial v}, & \frac{\partial y}{\partial w}\\ \frac{\partial z}{\partial u}, & \frac{\partial z}{\partial v}, & \frac{\partial z}{\partial w}\end{bmatrix}$$

The absolute value of this determinant is equal to the volume of parallelepiped determ'd by three vectors

$$\begin{aligned} \mathbf{T}_u &= \frac{\partial x}{\partial u}\mathbf{i} + \frac{\partial x}{\partial v}\mathbf{j} + \frac{\partial x}{\partial w}\mathbf{k} \\ \mathbf{T}_v &= \frac{\partial y}{\partial u}\mathbf{i} + \frac{\partial y}{\partial v}\mathbf{j} + \frac{\partial y}{\partial w}\mathbf{k} \\ \mathbf{T}_w &= \frac{\partial z}{\partial u}\mathbf{i} + \frac{\partial z}{\partial v}\mathbf{j} + \frac{\partial z}{\partial w}\mathbf{k} \end{aligned}$$

Theorem 2.8. *T is a $\mathcal{C}^1$ map from D^* onto D in $\mathbb{R}^3$ and $f : D \subset \mathbb{R}^3 \to \mathbb{R}$ is continuous, then*

$$\iiint_D dxdydz = \iiint_{D^*} |J|\, dudvdw \tag{6.5}$$

$$\iiint_D f(x,y,z)\, dxdydz = \iiint_{D^*} f(T(u,v,w))|J|\, dudvdw \tag{6.6}$$

where

$$J = \frac{\partial(x,y,z)}{\partial(u,v,w)}$$

Cylindrical Coordinate

$$x = r\cos\theta, \quad y = r\sin\theta, \quad z = z$$

Then Jacobian is r. (Compute) So

$$\iiint_D f(x,y,z)\, dxdydz = \iiint_{D^*} f(r\cos\theta, r\sin\theta, z)r\, drd\theta dz \tag{6.7}$$

Spherical coordinate

sol. Spherical coordinate is given by

$$x = \rho \sin\phi \cos\theta, \quad y = \rho \sin\phi \sin\theta, \quad z = \rho \cos\phi$$

The Jacobian of the mapping $(\rho, \theta, \phi) \to (x, y, z)$ is

$$\begin{aligned}
\frac{\partial(x,y,z)}{\partial(\rho,\theta,\phi)} &= \begin{vmatrix} \frac{\partial x}{\partial \rho} & \frac{\partial x}{\partial \theta} & \frac{\partial x}{\partial \phi} \\ \frac{\partial y}{\partial \rho} & \frac{\partial y}{\partial \theta} & \frac{\partial y}{\partial \phi} \\ \frac{\partial z}{\partial \rho} & \frac{\partial z}{\partial \theta} & \frac{\partial z}{\partial \phi} \end{vmatrix} \\
&= \begin{vmatrix} \sin\phi\cos\theta & -\rho\sin\phi\sin\theta & \rho\cos\phi\cos\theta \\ \sin\phi\sin\theta & \rho\sin\phi\cos\theta & \rho\cos\phi\sin\theta \\ \cos\phi & 0 & -\rho\sin\phi \end{vmatrix} \\
&= \cos\phi \begin{vmatrix} -\rho\sin\phi\sin\theta & \rho\cos\phi\cos\theta \\ \rho\sin\phi\cos\theta & \rho\cos\phi\sin\theta \end{vmatrix} \\
&- \rho\sin\phi \begin{vmatrix} \sin\phi\cos\theta & -\rho\sin\phi\sin\theta \\ \sin\phi\sin\theta & \rho\sin\phi\cos\theta \end{vmatrix} \\
&= -\rho^2 \sin\phi(\cos^2\phi + \sin^2\phi) = -\rho^2\sin\phi
\end{aligned}$$

Hence

$$\iiint_D f(x,y,z)\,dxdydz = \iiint_{D^*} F(\rho,\theta,\phi)\rho^2 \sin\phi\, d\rho\, d\theta\, d\phi$$

Here $F(\rho, \phi, \theta)$ means $f(x(\rho,\theta,\phi), y(\rho,\theta,\phi), z(\rho,\theta,\phi))$. This agrees with earlier formula derived by geometric insight.

Spherical Coordinate-Intuitive Derivation

The surfaces given by $\rho = c_1$, $\theta = c_2$, $\phi = c_3$ are a sphere, vertical plane, and a cone, (figure 6.10.) (figure 6.11)

$$\rho_0 \le \rho \le \rho_0 + \Delta\rho, \quad \theta_0 \le \theta \le \theta_0 + \Delta\theta, \quad \phi_0 \le \phi \le \phi_0 + \Delta\phi$$

The region is obtained when we cut by two planes $\theta = \theta_0$, $\theta = \theta_0 + \Delta\theta$, two cones $\phi = \phi_0$, $\phi = \phi_0 + \Delta\phi$ and two spheres of radius ρ, $\rho + \Delta\rho$.

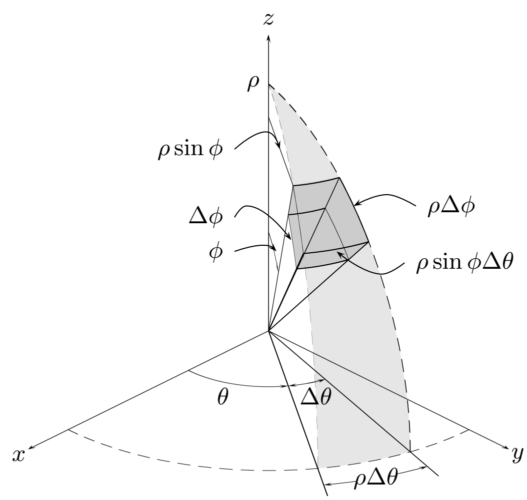

그림 6.10: Partition in spherical coordinate

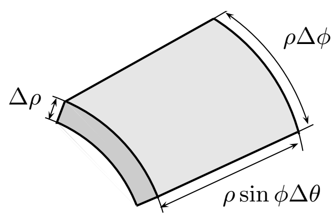

그림 6.11: A piece

First let us find the area of the region bounded by $\theta_0 \le \theta \le \theta_0 + \Delta\theta$, $\phi_0 \le \phi \le \phi_0 + \Delta\phi$ on the sphere ρ. The distance from a point on the surface to the z-axis is $\rho\sin\phi$. And when $\Delta\rho$, $\Delta\theta$ are small, this region can be approximated by rectangle whose area is (base $\rho\sin\phi\Delta\theta$ height $\rho\Delta\phi$)

$$\rho^2 \sin\phi\Delta\phi\Delta\theta$$

Now consider a solid with some thickness $\Delta\rho$. Then the volume is

$$\rho^2 \sin\phi\Delta\rho\Delta\phi\Delta\theta$$

Hence the volume of D is

$$\iiint_D dV = \int\int\int \rho^2 \sin\phi \, d\rho \, d\theta \, d\phi \tag{6.8}$$

and if a continuous function f is given, the integral is defined as

$$\iiint_D f dV = \int\int\int f(\rho,\phi,\theta)\rho^2 \sin\phi \, d\rho \, d\theta \, d\phi \tag{6.9}$$

Example 2.9. When the region D is given as

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} \le 1$$

Find

$$\iiint_D |xyz| dxdydz$$

sol. Let $T(u, v, w) = (au, bv, cw)$. Then T maps the unit ball $D^* = \{(u, v, w) \mid u^2 + v^2 + w^2 \le 1\}$ to D one-to-one, onto fashion. Since $J(T) = abc$ we have

$$\begin{aligned}\iiint_D |xyz| dxdydz &= \iiint_{D^*} (abc)^2 |uvw| \, du \, dv \, dw \\ &= 8 \iiint_{D^*_+} (abc)^2 uvw \, du \, dv \, dw.\end{aligned}$$

Here D^*_+ denotes the region among D^* $u \ge 0, v \ge 0, w \ge 0$. Now use spherical coordinate,

$$\begin{aligned}& 8 \iiint_{D^*_+} (abc)^2 uvw \, du \, dv \, dw \\ = \;& 8(abc)^2 \int_0^{\pi/2} \int_0^{\pi/2} \int_0^1 \rho^5 \sin^3\phi \cos\phi \sin\theta \cos\theta \, d\rho \, d\phi \, d\theta \\ = \;& 8(abc)^2 \int_0^{\pi/2} \int_0^{\pi/2} \left[\frac{\rho^6}{6}\right]_0^1 \sin^3\phi \cos\phi \sin\theta \cos\theta \, d\phi \, d\theta \\ = \;& \frac{4}{3}(abc)^2 \int_0^{\pi/2} \left[\frac{\sin^4\phi}{4}\right]_0^{\pi/2} \sin\theta \cos\theta \, d\theta \\ = \;& \frac{1}{3}(abc)^2 \int_0^{\pi/2} \sin\theta \cos\theta d\theta \\ = \;& \frac{1}{3}(abc)^2 \left[\frac{\sin^2\theta}{2}\right]_0^{\pi/2} = \frac{1}{6}(abc)^2.\end{aligned}$$

Example 2.10.

$$\iiint_W \exp(x^2 + y^2 + z^2)^{3/2} dV$$

W is unit ball.

sol.

$$\iiint_W \exp(x^2+y^2+z^2)^{3/2} dV = \iiint_{W^*} \rho^2 e^{\rho^3} \sin\phi d\theta\, d\phi\, d\rho$$

$$\begin{aligned} & \int_0^1 \int_0^\pi \int_0^{2\pi} \rho^2 e^{\rho^3} \sin\phi d\theta\, d\phi\, d\rho \\ = \;& 2\pi \int_0^1 \int_0^\pi \rho^2 e^{\rho^3} \sin\phi d\phi\, d\rho \\ = \;& 4\pi \int_0^1 \int_0^\pi \rho^2 e^{\rho^3} d\rho = \frac{4}{3}\pi(e-1) \end{aligned}$$

제 3 절 Application

Average

In $\mathbb{R}^n$,$(n = 1, 2, 3)$ the average of a function is defined as

$$f_{av} = \frac{\int_a^b f(x)dx}{b-a} \tag{6.10}$$

$$f_{av} = \frac{\iint_D f(x,y)dxdy}{\iint_D dxdy} \tag{6.11}$$

$$f_{av} = \frac{\iiint_D f(x)dx}{\iiint_D dxdydz} \tag{6.12}$$

Example 3.1. Find average of $f(x, y) = x \sin^2(xy)$ over $D = [0, \pi] \times [0, \pi]$.

Example 3.2. The temperature at points in the cube is proportional to the square of distance from the origin.

(1) Find average temperature

(2) At which point is the temperature equal to the average temperature?

sol. $T = c(x^2 + y^2 + z^2)$. So $[T]_{av} = \frac{1}{8} \iiint_W T dV = c$

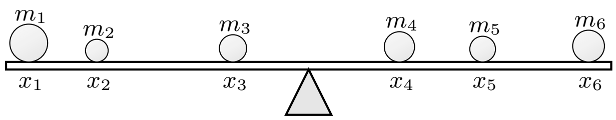

그림 6.12: Moment

Center of Mass

If masses $m_1, \cdots, m_n$ are placed at points $x_1, \cdots, x_n$ on x-axis, the center of mass is defined to be

$$\bar{x} = \frac{\sum m_i x_i}{\sum m_i}$$

The center of mass is a point where the total moment w.r.t that point is zero. i.e, $\sum(x_i - \bar{x}) = 0$.

When material is placed continuously on an axis, density is $\delta(x)$, the mass on $[x, x + \Delta x]$ is $\delta(x)\Delta x$ and moment is

$$\int x\delta(x)\,dx$$

Moment w.r.t $\bar{x}$ is

$$\int (x - \bar{x})\delta(x)\,dx$$

If this is zero $\bar{x}$ is center of mass.

3.1 Moment in 2D

There are three kind of Moment, i.e, Moment w.r.t x-axis, Moment w.r.t y-axis and Moment w.r.t the origin.

Let Δm_i be the mass whose distance from the axis of our interest is x_i. Then the moment is

$$\sum x_i \Delta m_i$$

In the limit,

$$\iint x dm.$$

In particular, we have

Definition 3.3 (Moment, center of mass)**.** Let $\delta(x, y)$ be the density. The moment of some material occupying $[x, x+\Delta x] \times [y, y+\Delta y]$ w.r.t x-axis is

$$\iint y\delta(x,y)dydx$$

The moment w.r.t x-axis is

$$M_x = \iint_R y\delta(x,y)\,dxdy,$$

The moment w.r.t y-axis is

$$M_y = \iint_R x\delta(x,y)\,dxdy,$$

If the moment w.r.t some point is zero then that point is called the **center of mass** .

$$\begin{aligned} M_{\bar{x}} &= \iint_R (x-\bar{x})\delta(x,y)\,dxdy = 0 \\ M_{\bar{y}} &= \iint_R (x-\bar{y})\delta(x,y)\,dxdy = 0 \end{aligned}$$

From this we get

$$\begin{aligned} \bar{x} = \frac{M_{\bar{y}}}{M} &= \frac{\iint_R x\delta(x,y)\,dxdy}{\iint_R \delta(x,y)\,dxdy} \\ \bar{y} = \frac{M_{\bar{x}}}{M} &= \frac{\iint_R y\delta(x,y)\,dxdy}{\iint_R \delta(x,y)\,dxdy} \end{aligned}$$

Moment is also called **first moment**. Also **Second moment (moment of inertia)** about x-axis, y-axis are defined as

$$I_x = \iint_R y^2\delta(x,y)\,dxdy, \quad I_y = \iint_R x^2\delta(x,y)\,dxdy$$

Finally, the moment of inner w.r.t. origin is

$$I_0 = \iint_R \|(x,y)-(0,0)\|\delta(x,y)\,dxdy = \iint_R (x^2+y^2)\delta(x,y)\,dxdy = I_x + I_y$$

Moment is proportional to the force(Torque). Second moment represents the energy of rotating body. Suppose an object is rotating in xy-plane about z-axis. Take a small piece and call the mass Δm_i. We we denote the distance of the object to the axis is by r_i, the angle by θ, then the angular velocity is

$\omega = d\theta/dt$. Hence the speed along the curve(circle) is

$$v_i = \frac{d}{dt}(r_i\theta) = r_i\frac{d\theta}{dt} = r_i\omega$$

The kinetic energy is

$$\frac{1}{2}\Delta m_i v_i^2 = \frac{1}{2}\omega^2 r_i^2 \Delta m_i$$

Integrating this, we get the energy

$$\frac{1}{2}\omega^2 \iint r^2 dm$$

Here dm is $\delta(x,y)dxdy$. So total kinetic energy is

$$\frac{1}{2}\omega^2 \iint (x^2+y^2)\delta\, dxdy = \frac{1}{2}I_0\omega^2$$

Example 3.4. Density of a body is given by $\delta(x,y) = x$. The body occupies the region between $y = x$, $y = x^2$. Find the mass and M_x

sol. Mass is

$$\begin{aligned} M &= \int_0^1 \int_{x^2}^x x\, dydx = \int_0^1 x\, [y]_{y=x^2}^{y=x}\, dx \\ &= \int_0^1 (x^2 - x^3)dx = \left[\frac{x^3}{3} - \frac{x^4}{4}\right]_0^1 = \frac{1}{12} \end{aligned}$$

and M_x is

$$\begin{aligned} M_x &= \int_0^1 \int_{x^2}^x yxdydx = \int_0^1 x\left[\frac{y^2}{2}\right]_{y=x^2}^{y=x} dx \\ &= \int_0^1 \frac{x}{2}(x^2 - x^4)dx = \left[\frac{x^4}{8} - \frac{x^6}{12}\right]_0^1 = \frac{1}{24} \end{aligned}$$

Example 3.5. A plate is lying over a disk of radius one on the origin. $\delta(x,y) = 1$. Find the second moment about the origin. Also find it when $\delta(x,y) = \sqrt{x^2+y^2}$.

sol. Let $U = \{(x,y) \mid x^2+y^2 \le 1\}$. Then

$$\int_U (x^2+y^2)\,dxdy = \int_0^{2\pi}\int_0^1 r^3\,drd\theta = \frac{\pi}{2}$$

If $\delta = \sqrt{x^2+y^2}$ we have

$$\int_U (x^2+y^2)^{3/2}\delta(x,y)\,dxdy = \int_0^{2\pi}\int_0^1 r^4\,drd\theta = \frac{2\pi}{5}$$

For constant density we can set $\delta = 1$ to find the center of mass. **centroid**.

Example 3.6. Find the centroid of the region bounded by $y = x$, $y = x^2$.

sol.

$$M = \int_0^1\int_{x^2}^x 1dydx = \int_0^1 [y]_{x^2}^x\,dx = \int_0^1 (x-x^2)dx = \frac{1}{6},$$
$$M_x = \int_0^1\int_{x^2}^x ydydx = \int_0^1 \left[\frac{y^2}{2}\right]_{x^2}^x dx = \int_0^1 (\frac{x^2}{2}-\frac{x^4}{2})dx = \frac{1}{15},$$
$$M_y = \int_0^1\int_{x^2}^x xdydx = \int_0^1 x\,[y]_{x^2}^x\,dx = \int_0^1 (x^2-x^3)dx = \frac{1}{12}.$$

Hence

$$\bar{x} = \frac{1/12}{1/6} = \frac{1}{2},\quad \bar{y} = \frac{1/15}{1/6} = \frac{2}{5}$$

제 4 절 Improper Integrals

We study the integrals in which the function is unbounded or the region of integration is unbounded.

One Variable Improper Integrals

Two Variable Improper Integrals

For the sake of easy presentation, we assume the domain D is y-simple and f assume nonnegative values. So assume D is given by

$$a \le x \le b, \phi_1(x) \le y \le \phi_2(x) \tag{6.13}$$

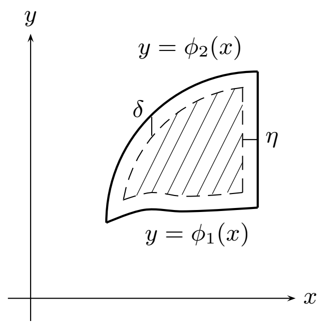

그림 6.13: Region of improper integrals

We choose some η, δ so that

$$D_{\eta,\delta} = \{(x, y)\colon a + \eta \le x \le b - \eta, \quad \phi_1(x) + \delta \le y \le \phi_2(x) - \delta\} \tag{6.14}$$

$D_{\eta,\delta} \subset D$ η, δ (fig 6.13) If the limit

$$\lim_{(\eta,\delta)\to(0,0)} \iint_{D_{\eta,\delta}} f\, dA \tag{6.15}$$

exists, we say it is the **improper integral** of f on D and write

$$\int_D f\, dA \tag{6.16}$$

Since f is integrable in $D_{\eta,\delta}$. we may use Fubini's theorem **??**, following holds.

$$\iint_{D_{\eta,\delta}} f\, dA = \int_{a+\eta}^{b-\eta} \int_{\phi_1(x)+\delta}^{\phi_2(x)-\delta} f(x, y)\, dydx \tag{6.17}$$

If the improper integral exists, it holds that

$$\iint_D f\, dA = \lim_{(\eta,\delta)\to(0,0)} \int_{a+\eta}^{b-\eta} \int_{\phi_1(x)+\delta}^{\phi_2(x)-\delta} f(x, y)\, dydx \tag{6.18}$$

Note that the integral independent of the way (η, δ) approaches $\to (0, 0)$. Now change the order of integration. Then the limit of (6.18)

$$\lim_{\eta\to 0} \int_{a+\eta}^{b-\eta} \left(\lim_{\delta\to 0} \int_{\phi_1(x)+\delta}^{\phi_2(x)-\delta} f(x, y) dy \right) dx \tag{6.19}$$

$$\int_a^b \int_{\phi_1(x)}^{\phi_2(x)} f(x,y)\,dydx$$

This is called **iterated improper integral**.

Theorem 4.1. *Suppose f is nonnegative, continuous on a x-simple region, or y-simple region D and one of the following integral exists.*

$$\int_a^b \int_{\phi_1(x)}^{\phi_2(x)} f(x,y)\,dydx, \quad \int_c^d \int_{\psi_1(y)}^{\psi_2(y)} f(x,y)\,dxdy$$

Then If the iterated integral (6.19) exists, then the improper integral (6.16) exists and have same value.

Example 4.2. Find

$$\int_D \frac{1}{\sqrt{xy}}\,dA$$

when $D = [0,1] \times [0,1]$.

sol. Since function is nonnegative, we only need to check iterated integral.

$$\int_0^1 \int_0^1 \frac{1}{\sqrt{xy}}\,dydx = \int_0^1 \left[2x^{1/2}\right]_0^1 \frac{1}{\sqrt{y}}\,dy$$
$$= 2 \cdot 2 = 4$$

□

Remark 4.3. When the domain is unbounded, we can similarly define improper integral.