제 5 장

Double and Triple integrals

Double integral, **triple integral**,

제 1 절 Double integral as volumes

For the convenience of presentation we assume the domain of a function on a rectangle R given by

$$R = \{(x, y)\colon a \le x \le b, c \le y \le d\}.$$

We also denote it by $[a, b] \times [c, d]$ and call it **Cartesian product** of $[a, b]$, $[c, d]$ If f is nonnegative then the graph of $z = f(x, y)$ is a surface over R.

Double Integral(of a nonnegative function)

The volume of the region above R and under the graph of a nonnegative function $f(x, y)$ is called the **double integral** of f over R and is denoted by

$$\iint_R f(x, y)\, dA \text{ or } \iint_R f(x, y)\, dxdy,$$

Since f has nonnegative value, we interpret it as the volume. Let V be the volume of the region between the graph of and xy-plane.

Cavalieri's Principle and reduction to iterated integral

Fig 5.3 Let $A(x)$ be the volume of cross section by a plane perpendicular to the axis, then the volume is

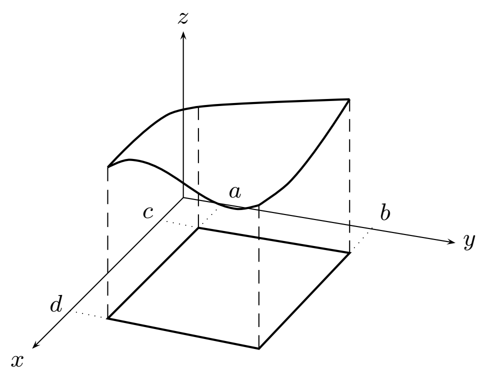


그림 5.1: Graph of a function

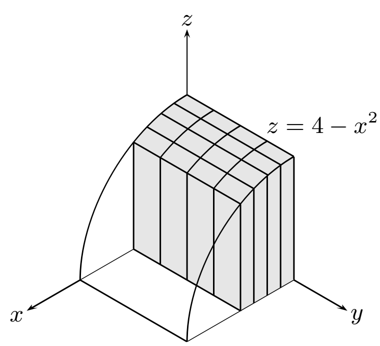


그림 5.2: $z = 4 - x^2$, $0 \le x \le 1$, $0 \le y \le 2$

$$V = \int_a^b A(x)\,dx \tag{5.1}$$

This is **Cavalieri's Principle**. To understand it. let us divide the interval $[a, b]$ by n subintervals $a = x_0 < x_1 < \cdots < x_n = b$, $\Delta x_i = x_i - x_{i-1}$. Then $A(x)\Delta x_i$ is approx. the volume of each slice. (Fig 5.3) Now the Riemann sum

$$\mathcal{R}(A, n) = \sum_{i=1}^{n} A(x_i)\Delta x$$

is approx volume. If $n \to \infty$, the limit is (5.1).

Reduction to iterated integrals

Consider the volume of a solid under f over $R = [a, b] \times [c, d]$. As in figure 5.4, the cross section along $x = x_0$ is given by the inequality $0 \le z \le f(x_0, y)$, $(c \le y \le d)$ the area of cross section is

$$\int_c^d f(x_0, y)\, dy$$

Hence Cavalieri principle says the volume is

$$\int_a^b A(x)\, dx = \int_a^b \left[\int_c^d f(x, y)\, dy\right] dx$$

This is called **iterated integral**. On the other hand, if we cut it by the plane $y = y_0$, then the volume is

$$\int_a^b A(y)\, dy = \int_c^d \left[\int_a^b f(x, y)\, dx\right] dy$$

Since these two values are equal,

$$\int_R f(x, y)\, dA = \int_a^b \left[\int_c^d f(x, y)\, dy\right] dx = \int_c^d \left[\int_a^b f(x, y)\, dx\right] dy$$

This is called **Fubini's theorem**. Notice the similarity with Cavalieri Principle.

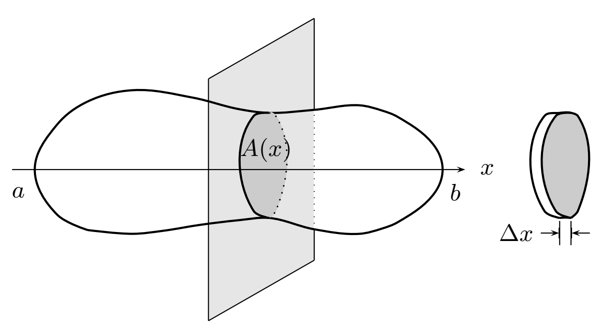


그림 5.3: Cavalieri's Principle

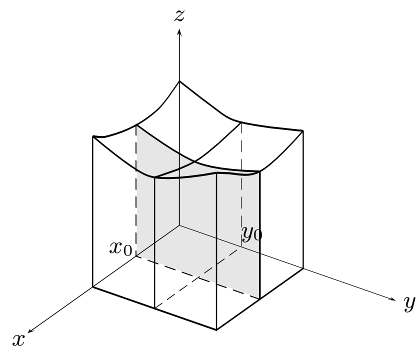


그림 5.4: Fubini's theorem by Cavalieri Principle

Example 1.1. Evaluate

$$\iint_R (x^2+y^2)\,dxdy, \quad R=[-1,1]\times[0,1]$$

sol.

$$\int_0^1 \left[\int_{-1}^1 (x^2+y^2)\right] dy = \frac{4}{3}$$

Now change the order.

Example 1.2. Evaluate

$$\iint_S \cos x \sin y\,dxdy, \quad S=[0,\frac{\pi}{2}]\times[0,\frac{\pi}{2}]$$

sol.

$$\begin{aligned}\iint_S \cos x\sin y\,dxdy &= \int_0^{\pi/2}\left[\int_0^{\pi/2}\cos x\sin y\,dx\right]dy\\ &= \int_0^{\pi/2}\sin y\left[\int_0^{\pi/2}\cos x\,dx\right]dy = \int_0^{\pi/2}\sin y\,dy = 1\end{aligned}$$

Now change the order.

제 2 절 Double integral over a rectangle

n- (regular partition)

Subdivide two intervals $[a, b]$, $[c, d]$ by n-equal

$$a = x_0 < x_1 < \cdots < x_n = b, \quad c = y_0 < y_1 < \cdots < y_n = d$$

and

$$\Delta x = x_j - x_{j-1} = \frac{b-a}{n}, \quad \Delta y = y_k - y_{k-1} = \frac{d-c}{n}$$

Let $\Delta R_{jk} = [x_{j-1}, x_j] \times [y_{k-1}, y_k]$ its area is $\Delta x \Delta y$ and for any point c_{jk} in ΔR_{jk} consider Riemann sum

$$S_n = \mathcal{R}(f, n) = \sum_{i,j=1}^{n} f(c_{ij}) \Delta x \Delta y. \tag{5.2}$$

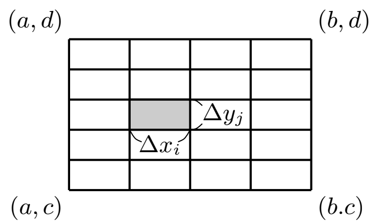


그림 5.5: partition of a rectangle

Definition 2.1. If the series $\{S_n\}$ converge to S regardless of the points c_{jk} then f is **integrable** over R and its value is denoted by

$$\int_R f, \quad \iint_R f(x, y)\, dA, \quad \iint_R f(x, y)\, dxdy$$

$$\lim_{n \to \infty} \sum_{j,k=1}^{n} f(c_{jk}) \Delta x \Delta y = \iint_R f(x, y)\, dxdy$$

Theorem 2.2. *Continuous functions over a rectangle is integrable.*

Properties of integral

Theorem 2.3. *If f, g are integrable over R, R_1, R_2*

(1) $\iint_R cf(x, y)\, dxdy = c \iint_R f(x, y)\, dxdy$, *(단, c는 상수).*

(2) $\iint_R (f(x,y) + g(x,y))\, dxdy$
$$= \iint_R f(x,y)\, dxdy + \iint_R g(x,y)\, dxdy.$$

(3) If $f(x,y) \geq 0$, $\iint_R f(x,y)\, dxdy \geq 0$.

(4) If $f(x,y) \geq g(x,y)$, $\iint_R f(x,y)\, dxdy \geq \iint_R g(x,y)\, dxdy$.

(5) If R_1 and R_2 do not meet, then for $R = R_1 \cup R_2$
$$\iint_R f(x,y)\, dxdy = \iint_{R_1} f(x,y)\, dxdy + \iint_{R_2} f(x,y)\, dxdy.$$

(6) $\left|\iint_R f dA\right| \leq \iint_R |f| dA$

Integrability of bounded function

Definition 2.4. f is called **bounded** if there is M such that $-M \leq f(x,y) \leq M$ holds for all (x,y) in R.

Theorem 2.5. *If f is bounded on R and the set of points where f is discontinuous lies on a finite union of graphs of continuous functions, then f is integrable over R.*

This is useful to define integral over general region.

Example 2.6. (1) $\{(x,y) \mid y = \sqrt{x}, 0 \leq x \leq 1\} \cup \{(x,y) \mid y = x, 0 \leq x \leq 1\}$

(2) $\{(x,y) \mid y = x^2, -1 \leq x \leq 1\} \cup \{(x,y) \mid y = 1, -1 \leq x \leq 1\}$

Theorem 2.7 (Fubini Theorem 1)**.** *Let f be continuous on $R = [a,b] \times [c,d]$. Then f satisfies*

$$\int_a^b \left[\int_c^d f(x,y)\, dy\right] dx = \int_c^d \left[\int_a^b f(x,y)\, dx\right] dy = \iint_R f(x,y)\, dA. \tag{5.3}$$

Sketch of proof first.

Let $a = x_0 < x_1 < \cdots < x_n = b, \quad c = y_0 < y_1 < \cdots < y_n = d$ be the regular partition of $[a,b]$ and $[c,d]$. Riemann sum for iterated integral in this case is

$$\sum_{j,k=1}^{n} f(c_{jk})\Delta x \Delta y = \sum_{j=1}^{n}\left(\sum_{k=1}^{n} f(c_{jk})\Delta y\right)\Delta x = \sum_{k=1}^{n}\left(\sum_{j=1}^{n} f(c_{jk})\Delta x\right)\Delta y.$$

This can be proved as follows: Let $[a_{jk}]$ be $n \times n$ array of numbers. Then

$$\sum_{j,k=1}^{n} a_{jk} = \sum_{j=1}^{n}\left(\sum_{k=1}^{n} a_{jk}\right) = \sum_{k=1}^{n}\left(\sum_{j=1}^{n} a_{jk}\right)$$

This idea will be used.

Proof. We will first prove that

$$\int_a^b \left[\int_c^d f(x,y)\,dy\right] dx = \int_R f(x,y)\,dA.$$

Let

$$F(x) = \int_c^d f(x,y)\,dy$$

Then

$$F(x) = \sum_{k=1}^{n} \int_{y_{k-1}}^{y_k} f(x,y)\,dy$$

Integral mean value theorem (with fixed x). Then for some $Y_k(x)$ in $[y_{k-1}, y_k]$

$$\int_{y_{k-1}}^{y_k} f(x,y)\,dy = f(x, Y_k(x))\Delta y$$

Thus

$$F(x) = \sum_{k=1}^{n} f(x, Y_k(x))\Delta y.$$

Then it holds by one variable integral that, for any $p_j \in [x_{j-1}, x_j]$

$$\int_a^b F(x)\,dx = \lim_{n\to\infty} \sum_{j=1}^{n} F(p_j)\Delta x$$

Let $c_{jk} = (p_j, Y_k(p_j))$. Then

$$F(p_j) = \sum_{k=1}^{n} f(p_j, Y_k(p_j))\Delta y = \sum_{k=1}^{n} f(c_{jk})\Delta y$$

holds. Hence

$$\begin{aligned}\int_a^b \left[\int_c^d f(x,y)\,dy\right] dx &= \int_a^b F(x)\,dx \\ &= \lim_{n\to\infty}\sum_{j=1}^{n} F(p_j)\Delta x \\ &= \lim_{n\to\infty}\sum_{j=1}^{n}\sum_{k=1}^{n} f(c_{jk})\Delta y\Delta x \\ &= \iint_R f(x,y)\,dA.\end{aligned}$$

By the same reasoning, we can show

$$\int_c^d \left[\int_a^b f(x,y)\,dx\right] dy = \iint_R f(x,y)\,dA.$$

□

Remark 2.8. Fubini's theorem holds if f has certain discontinuities.

Example 2.9. Find the volume of the region $0 \le x \le 1$, $0 \le y \le 1$, $0 \le z \le 2 - x - y$.

sol. Fix x then the area of cross section with a plane perpendicular to x-axis is

$$A(x) = \int_0^1 (2 - x - y)\,dy$$

So the volume is

$$\begin{aligned}V &= \int_0^1 A(x)\,dx = \int_{x=0}^{x=1}\int_{y=0}^{y=1}(2-x-y)\,dydx \\ &= \int_0^1 \left[2y - xy - \frac{y^2}{2}\right]_0^1 dx \\ &= \int_0^1 \left(\frac{3}{2} - x\right) dx = \left[\frac{3x}{2} - \frac{x^2}{2}\right]_0^1 = 1\end{aligned}$$

⌟

sol. [2] Another way: You can fix y. Then the area of cross section with a plane perpendicular to y-axis is

$$A(y) = \int_0^1 (2 - x - y)\, dx$$

Hence the volume is

$$\begin{aligned} V &= \int_0^1 A(y)\, dy = \int_{y=0}^{y=1} \int_{x=0}^{x=1} (2 - x - y)\, dxdy \\ &= \int_0^1 \left[2x - \frac{x^2}{2} - xy \right]_0^1 dy \\ &= \int_0^1 \left(\frac{3}{2} - y \right) dy = \left[\frac{3y}{2} - \frac{y^2}{2} \right]_0^1 = 1 \end{aligned}$$

Example 2.10. Compute $\iint_R (x^2 + y) dA$, where $A = [0, 1] \times [0, 1]$.

sol.

$$\iint_R (x^2+y)dA = \int_0^1 \int_0^1 (x^2+y)dxdy = \int_0^1 [\int_0^1 (x^2+y)dx]dy = \int_0^1 (\frac{1}{3}+y)dy = \frac{5}{6}.$$

Change the order.

Example 2.11. $f = y(x^3 - 12x)$. f take both positive and negative values. $-2 \le x \le 1$, $0 \le y \le 1$,

sol.

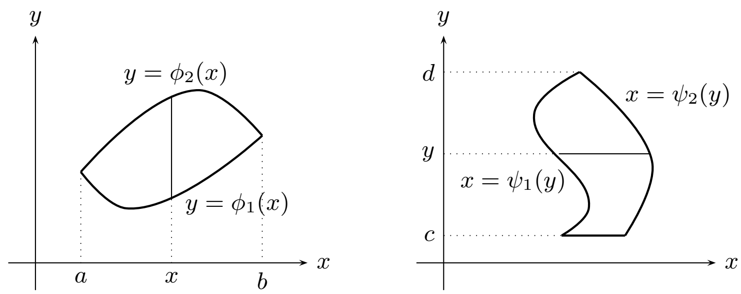
$$\iint_R y(x^3 - 12x)dxdy = \int_0^1 \left[\int_{-2}^1 y(x^3 - 12x)dx \right] dy = \frac{57}{4} \int_0^1 ydy = \frac{57}{8}.$$

제 3 절 Double integral over general regions

Elementary regions

There are three kind of elementary regions: Let $y = \phi_1(x)$, $y = \phi_2(x)$ ($\phi_1(x) \le \phi_2(x)$ for $x \in [a, b]$) be two continuous functions. The the region

$$D = \{(x, y) \mid a \le x \le b,\ \phi_1(x) \le y \le \phi_2(x)\}$$



(a) y-simple or region of 1st kind (b) x-simple or 2nd kind

그림 5.6: y-simple region, x-simple region

is y**-simple region**.

Now change the role of x, y as in figure 5.6 (b). If $x = \psi_1(y)$, $x = \psi_2(y)$, $(c \le y \le d)$ satisfies $\psi_1(y) \le \psi_2(y)$ for $y \in [c, d]$, then the region determined by

$$D = \{(x, y) \mid c \le y \le d,\ \psi_1(y) \le x \le \psi_2(y)\}$$

is x**-simple region**. The region that is both x-simple and y-simple is called **region of the third kind**. These are called **elementary regions**.

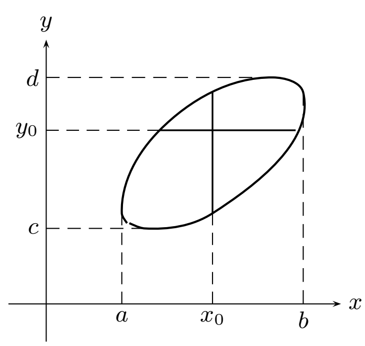


그림 5.7: Region of third kind

Double Integral by Cavalieri principle

y**-simple region**: Let $f \ge 0$. As in figure 5.6 (a) cross section with a plane perpendicular to x-axis, has area

$$A(x) = \int_{\phi_1(x)}^{\phi_2(x)} f(x,y)\,dy$$

Then by Cavalieri principle, the volume of solid between the graph of f and xy-plane over D is

$$\iint_D f(x,y)\,dA = \int_a^b A(x)\,dx = \int_a^b \int_{\phi_1(x)}^{\phi_2(x)} f(x,y)\,dydx$$

x-simple region: Now the cross section by plane perpendicular to y-axis has area

$$A(y) = \int_{\psi_1(y)}^{\psi_2(y)} f(x,y)\,dx$$

and the integral is

$$\iint_D f(x,y)\,dA = \int_c^d A(y)\,dy = \int_c^d \int_{\psi_1(y)}^{\psi_2(y)} f(x,y)\,dxdy.$$

This will be justified below by extending the definition of function and using the Fubini's theorem.

Integrals over elementary regions(by extension to 0)

We would like top define the integral of f defined on an elementary region. The idea is to extend the function to a rectangular domain. Given a continuous function f on D where D is an elementary region

$$D = \{(x,y) \mid \phi_1(x) \le y \le \phi_2(x), \quad a \le x \le b\}.$$

The idea is to find a rectangle which contains D and extend f to R outside D by zero:

$$f^*(x,y) = \begin{cases} f(x,y), & (x,y) \in D \\ 0, & (x,y) \in R \setminus D. \end{cases}$$

Then f^* has discontinuities on the graphs of $y = \phi_1(x)$, $y = \phi_2(x)$, $a \le x \le b$. Hence it is integrable by Theorem 2.5. Thus we can define

$$\iint_D f(x,y)\,dA := \iint_R f^*(x,y)\,dA$$

Definition 3.1. Assume the boundary of D is given by graphs of a finite number of continuous functions. The integral of f over D is defined as the limit of Riemann sum

$$\iint_D f\,dA = \lim_{n\to\infty} \sum_{i,j} f(c_{ij})\Delta R_{ij}$$

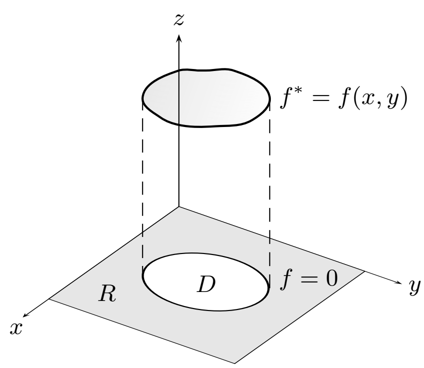


그림 5.8: Extension of a function

Theorem 3.2 (Reduction to iterated integrals)**.** *If f is continuous on $D \subset R$*

(1) If D is a domain bounded by two continuous functions ϕ_1, ϕ_2, i.e, $D = \{(x,y) : \phi_1(x) \le y \le \phi_2(x),\ a \le x \le b\ \}$ then f is integrable on D and

$$\iint_D f(x,y)\,dA = \int_a^b \left[\int_{\phi_1(x)}^{\phi_2(x)} f(x,y)\,dy\right] dx.$$

(2) Similarly if D is bounded by two continuous functions ψ_1, ψ_2, $\psi_1(y) \le x \le \psi_2(y)$ $(c \le y \le d)$ then

$$\iint_D f(x,y)\,dA = \int_c^d \left[\int_{\psi_1(y)}^{\psi_2(y)} f(x,y)\,dx\right] dy.$$

Proof. By Fubini theorem, we have

$$\begin{aligned}\iint_D f(x,y)\,dA &= \iint_D f^*(x,y)\,dA \\ &= \int_a^b \int_c^d f^*(x,y)\,dydx \\ &= \int_c^d \int_a^b f^*(x,y)\,dxdy\end{aligned}$$

If D is y-simple, we use $\int_a^b \int_c^d f^*(x,y)\,dydx$, and if If D is x-simple, we use $\int_c^d \int_a^b f^*(x,y)\,dxdy$. So we have for y-simple region,

$$\int_c^d f^*(x,y)\,dy = \int_{\phi_1(x)}^{\phi_2(x)} f(x,y)\,dy.$$

For x-simple region, we see

$$\int_a^b f^*(x,y)\,dx = \int_{\psi_1(y)}^{\psi_2(y)} f(x,y)\,dx.$$

□

Example 3.3. Find the integral on $D: 0 \le x \le 1,\ x \le y \le 1$

$$\iint_D (x+y^2)\,dxdy$$

sol. Use Fubini's theorem

$$\begin{aligned}\int_0^1 \int_x^1 (x+y^2)\,dydx &= \int_0^1 \left[xy + \frac{y^3}{3}\right]_x^1 dx \\ &= \int_0^1 \left(x + \frac{1}{3} - x^2 - \frac{x^3}{3}\right) dx \\ &= \left[\frac{x^2}{2} + \frac{x}{3} - \frac{x^3}{3} - \frac{x^4}{12}\right]_0^1 = \frac{5}{12}\end{aligned}$$

Example 3.4. Find $\iint_D (x^3 y + \cos x)\,dA$ where D is given by $0 \le x \le \pi/2$, $0 \le y \le x$.

sol.

$$\iint_D (x^3y + \cos x)\, dA$$

$$= \int_0^{\pi/2} \int_0^x (x^3y + \cos x) dy\, dx$$
$$= \int_0^{\pi/2} \left[\frac{x^3y^2}{2} + y\cos x \right]_{y=0}^{x} dx = \int_0^{\pi/2} \left(\frac{x^5}{2} + x\cos x \right) dx$$
$$= \frac{\pi^6}{768} + \frac{\pi}{2} - 1.$$

Example 3.5. Find volume of tetrahedron bounded by the planes $y = 0, x = 0, y - x + z = 1$.

sol.

$$\iint_D (1 - y + x) dA = \int_{-1}^{0} \int_0^{1+x} (1 - y + x) dy dx = \int_{-1}^{0} \left[(1+x)y - \frac{y^2}{2} \right]_{y=0}^{1+x} dx = \frac{1}{6}.$$

Example 3.6. Let D be given by $0 \le x \le \ln 2, \quad 0 \le y \le e^x - 1$ Express the double integral

$$\iint_D f(x, y)\, dA$$

in two iterated integrals.

sol. See figure 5.9 As an x-simple region, the points of intersection is $y = 0$, $y = e^x - 1(\ 0 \le x \le \ln 2)$. Hence

$$\int_0^{\ln 2} \int_0^{e^x - 1} f(x, y)\, dydx$$

As a y-simple region, the points of intersection is $x = \ln(y+1)$, $x = \ln 2($ $0 \le y \le 2)$ So the integral is

$$\int_0^{1} \int_{\ln(y+1)}^{\ln 2} f(x, y)\, dxdy$$

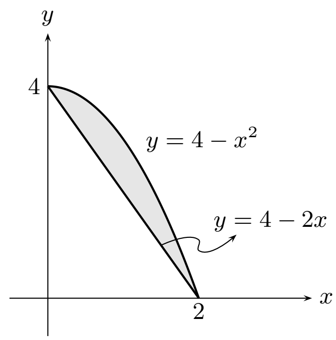


그림 5.9: $4 - 2x \le y \le 4 - x^2$

Example 3.7. Given domain D (fig. 5.10) by

$$4 - 2x \le y \le 4 - x^2, \quad 0 \le x \le 2$$

Find

$$\iint_D (1 + x)\, dA$$

sol. This region is third kind.

$$\begin{aligned}
\int_0^2 \int_{4-2x}^{4-x^2} (1 + x)\, dydx &= \int_0^2 [(1 + x)y]_{y=4-2x}^{y=4-x^2}\, dx \\
&= \int_0^2 (-x^3 + x^2 + 2x)\, dx \\
&= \left[-\frac{x^4}{4} + \frac{x^3}{3} + x^2 \right]_0^2 = \frac{8}{3}.
\end{aligned}$$

On the other hand, as a function of y $x = (4 - y)/2$, $x = \sqrt{4 - y}$. So

$$\begin{aligned}
\int_0^4 \int_{(4-y)/2}^{\sqrt{4-y}} (1 + x)\, dxdy &= \int_0^4 \left[x + \frac{x^2}{2} \right]_{x=(4-y)/2}^{x=\sqrt{4-y}} dy \\
&= \int_0^4 \left(\sqrt{4 - y} - \frac{(4 - y)^2}{8} \right) dy \\
&= \left[-\frac{2}{3}(4 - y)^{3/2} + \frac{(4 - y)^3}{24} \right]_0^4 \\
&= \frac{2}{3} 4^{3/2} - \frac{4^3}{24} = \frac{8}{3}
\end{aligned}$$

제 4 절 Change order of integration

Suppose D is both x-simple and y-simple. Thus it is give by

$$\phi_1(x) \le y \le \phi_2(x),\ a \le x \le b$$

and

$$\psi_1(y) \le x \le \psi_2(y),\ c \le y \le d$$

$$\iint_D f(x,y)dA = \int_a^b \int_{\phi_1(x)}^{\phi_2(x)} f(x,y)dydx = \int_c^d \int_{\psi_1(y)}^{\psi_2(y)} f(x,y)\,dxdy$$

Theorem 4.1. *Suppose D is given by $\psi_1(y) \le x \le \psi_2(y)$*

$$\int_a^b \int_{\phi_1(x)}^{\phi_2(x)} f(x,y)\,dydx$$

Example 4.2. Compute by change of order of integration

$$\int_0^a \int_0^{(a^2-x^2)^{1/2}} (a^2-y^2)^{1/2}\,dydx$$

sol.

$$\begin{aligned}\int_0^a \int_0^{(a^2-x^2)^{1/2}} (a^2-y^2)^{1/2}\,dydx &= \int_0^a \int_0^{(a^2-y^2)^{1/2}} (a^2-y^2)^{1/2}\,dxdy \\ &= \int_0^a [x(a^2-y^2)^{1/2}]_0^{(a^2-y^2)^{1/2}} (a^2-y^2)^{1/2}\,dy \\ &= \int_0^a (a^2-y^2)\,dy = \frac{2a^3}{3}.\end{aligned}$$

Example 4.3. Find

$$\int_0^\pi \int_x^\pi \frac{\sin y}{y}\,dydx$$

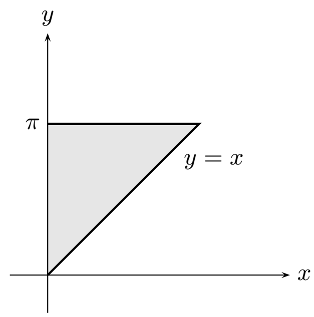


그림 5.10: $0 \le x \le \pi,\ x \le y \le \pi$

sol. It is not to find the integral as the given form. But if we change the order of integration (fig 5.11)

$$\int_0^\pi \int_x^\pi \frac{\sin y}{y}\,dydx = \int_0^\pi \int_0^y \frac{\sin y}{y}\,dxdy$$
$$= \int_0^\pi \left[\frac{\sin y}{y}x\right]_{x=0}^{x=y} dy$$
$$= \int_0^\pi \sin y\,dy = [-\cos y]_0^\pi = 2$$

Example 4.4. Find

$$\int_0^{\ln 2}\int_0^{e^x-1} e^{-x}\sqrt{1+\ln(y+1)}\,dydx + \frac{1}{2}\int_0^1 \sqrt{1+\ln(y+1)}\,dy$$

sol. Again change the order(Fig 5.9) as in example 3.6 we see

$$\int_0^1\int_{\ln(y+1)}^{\ln 2} e^{-x}\sqrt{1+\ln(y+1)}\,dxdy + \frac{1}{2}\int_0^1 \sqrt{1+\ln(y+1)}\,dy$$
$$= -\int_0^1 (e^{-\ln 2} - e^{-\ln(y+1)})\sqrt{1+\ln(y+1)}\,dy$$
$$+\frac{1}{2}\int_0^1 \sqrt{1+\ln(y+1)}\,dy$$
$$= \int_0^1 \frac{\sqrt{1+\ln(y+1)}}{y+1}\,dy = \left[\frac{2}{3}(1+\ln(y+1))^{3/2}\right]_0^1$$
$$= \frac{2}{3}[(1+\ln 2)^{3/2} - 1].$$

4.1 Mean value inequality

Theorem 4.5. *Suppose $f : D \to \mathbb{R}$ is continuous on an elementary region D and $m \le f(x, y) \le M$. Then we have*

$$mA(D) \le \iint_D f\,dA \le MA(D) \tag{5.4}$$

Example 4.6. Estimate

$$\int_D \frac{1}{\sqrt{1 + x^6 + y^7}} dxdy$$

where D is the unit square.

$$\frac{1}{\sqrt{3}} \le \frac{1}{\sqrt{1 + x^6 + y^7}} \le 1$$

Theorem 4.7. *If f is continuous over D then*

$$\iint_D f\,dA = f(x_0, y_0)A(D)$$

Proof. Divide (5.4) by $A(D)$ to get

$$m \le \frac{1}{A(D)} \iint_D f\,dA \le M$$

Since $m \le f \le M$ and f is continuous, there is a point (x_0, y_0) such that

$$f(x_0, y_0) = \frac{1}{A(D)} \iint_D f$$

which is precisely the conclusion. □

제 5 절 Triple integral

Assume $f(x, y, z)$ is defined on $B = [a, b] \times [c, d] \times [p, q]$

We partition the box into small n^3 boxes as in 5.12, and denote each volume as ΔV_{ijk} $(i, j, k = 1, \ldots, n)$. Then the Riemann sum becomes

$$\mathcal{R}(f, n) = S_n = \sum_{i,j,k=1}^{n} f(c_{ijk})\Delta V_{ijk}.$$

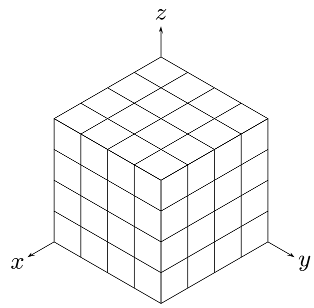


그림 5.11: partition of box

Here c_{ijk} is any point in the subbox B_{ijk}.

Definition 5.1. Let f be a bounded function on B. If $\lim_n S_n = S$ exists and is independent of the choice of c_{ijk}, then we say f is integrable in B and call S the **triple integral** and we write it by

$$\iiint_D f dV, \quad \iiint_D f(x,y,z)dV, \text{ or } \quad \iiint_D f(x,y,z)dxdydz$$

Reduction to iterated integral

Theorem 5.2 (Fubini's theorem)**.** . *Suppose f is continuous on $D = [a,b] \times [c,d] \times [p,q]$ Then the triple integral $\iiint_D f(x,y,z)dxdydz$ equals with any of the following integrals.*

$$\int_p^q \int_c^d \int_a^b f(x,y,z)\, dxdydz, \quad \int_p^q \int_a^b \int_c^d f(x,y,z)\, dydxdz$$
$$\int_a^b \int_c^d \int_p^q f(x,y,z)\, dzdydx$$

Example 5.3. $B = [0,1] \times [-\frac{1}{2}, 0] \times [0, \frac{1}{3}]$.

$$\iiint_B (x+2y+3z)^2 dxdydz = \frac{1}{12}$$

Change the order and compute again to see the value does not change.

Example 5.4.

$$\int_B e^{x+y+z} dV$$

where B is unit cube at origin.

Elementary regions

Definition 5.5. A region D is **elementary regions** if a variable lies between two continuous functions of the other variables, and the domain of these functions is elementary(i.e, x-simple or y-simple). If f is continuous on D, then we extend f on box E

$$f^*(x,y,z) = \begin{cases} f(x,y,z), & (x,y,z) \in D \\ 0, & (x,y,z) \in E \setminus D \end{cases}$$

and define

$$\int_D f dV = \int_E f^* dV$$

Suppose R is an elementary region in xy-plane and there are continuous functions $\gamma_1(x,y)$, $\gamma_2(x,y)$

$$\{(x,y,z) \mid \gamma_1(x,y) \le z \le \gamma_2(x,y), \quad (x,y) \in R\} \tag{5.5}$$

Sometimes called **elementary region of first kind**. In other words, there exist two functions $\phi_1(x)$, $\phi_2(x)$ such that

$$R = \{(x,y) \mid \phi_1(x) \le y \le \phi_2(x), \quad a \le x \le b\}$$

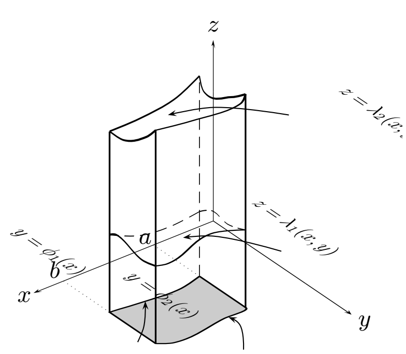


그림 5.12: 1st kind

If roles of x, z are interchanged, i.e, there exist $\psi_1(y)$, $\psi_2(y)$ such that

$$R = \{(x, y) \mid \psi_1(y) \le x \le \psi_2(y), \quad c \le y \le d\}$$

and (5.5) holds then it is called **elementary region of second kind**.

Example 5.6. Describe the unit ball as an elementary region.

sol. The domain of defining function is described by

$$-\sqrt{1-x^2} \le y \le \sqrt{1-x^2}, \quad -1 \le x \le 1$$

while the functions are

$$-\sqrt{1-x^2-y^2} \le z \le \sqrt{1-x^2-y^2}, \quad \text{on unit disk}$$

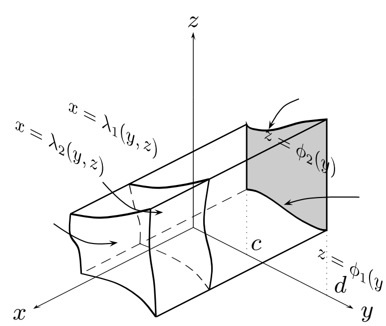


그림 5.13: 2nd kind

Integrals over Elementary regions

Given a y-simple region in xy-plane

$$R = \{(x, y) \mid \phi_1(x) \le y \le \phi_2(x), \quad a \le x \le b\}$$

D is defined by

$$D = \{(x, y, z) \mid \gamma_1(x, y) \le z \le \gamma_2(x, y), \quad (x, y) \in R\}$$

the integral is given by

$$\begin{aligned}\iiint_D f\,dV &= \int_R \int f(x,y,z)\,dzdA \\ &= \int_a^b \int_{\phi_1(x)}^{\phi_2(x)} \int_{\gamma_1(x,y)}^{\gamma_2(x,y)} f(x,y,z)\,dzdydx\end{aligned}$$

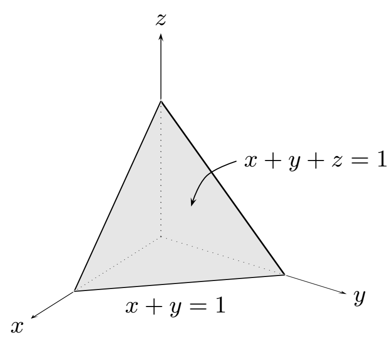


그림 5.14: $x+y+z=1$

Example 5.7. Find the volume of radius 1.

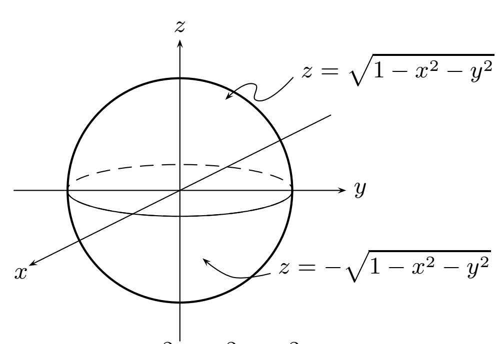


그림 5.15: $x^2+y^2+z^2=1$

sol. Unit ball is described by $x^2+y^2+z^2 \le 1$. The volume is (fig 5.16)

$$\int_D 1\,dV, \quad D = \{(x,y,z) \mid x^2+y^2+z^2 \le 1\}$$

Here we can take $R = \{(x,y) \mid x^2+y^2 \le 1\}$ and $D = \{-\sqrt{1-x^2-y^2} \le z \le \sqrt{1-x^2-y^2}, (x,y) \in R\}$. Hence

$$\begin{aligned}\int_R \int dzdydx &= \int_R \int_{z=-\sqrt{1-x^2-y^2}}^{z=\sqrt{1-x^2-y^2}} 1\,dzdydx \\ &= 2\int_R \sqrt{1-x^2-y^2}\,dydx \\ &= 2\int_{-1}^{1}\int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} \sqrt{1-x^2-y^2}\,dydx\end{aligned}$$

Let $\sqrt{1-x^2} = a$. The inner integral is area of semi circle or radius a

$$2\int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} \sqrt{1-x^2-y^2}\,dy = 2\int_{-a}^{a} \sqrt{a^2-y^2}dy = a^2\pi = (1-x^2)\pi$$

Hence

$$\begin{aligned}2\int_{-1}^{1}\int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} \sqrt{1-x^2-y^2}\,dydx &= \int_{-1}^{1}(1-x^2)\pi\,dx \\ &= \left[(x-\frac{x^3}{3})\pi\right]_{-1}^{1} = 2(1-\frac{1}{3})\pi = \frac{4}{3}\pi\end{aligned}$$

□

Other type of elementary regions can be described similarly. If a region can be described in all three ways we call these regions **symmetric elementary regions**

Example 5.8. Let D be the region bounded by $x+y+z=1$, $x=0$, $y=0$, $z=0$. Find

$$\iiint_D (1+2z)dxdydz$$

sol. Let $R = \{(x, y) \mid 0 \le y \le 1 - x, \quad 0 \le x \le 1\}$. Then D is described by

$$D = \{(x, y, z) \mid 0 \le z \le 1 - x - y, \quad (x, y) \in R\}$$

and integrate along z direction.

$$\begin{aligned}\iiint_D (1 + 2z)\, dxdydz &= \iint_R \left[z + z^2\right]_0^{1-x-y} dxdy \\ &= \int_0^1 \int_{y=0}^{y=1-x} (1 - x - y + (1 - x - y)^2)\, dydx \\ &= \int_0^1 \left[-\frac{(1-x-y)^2}{2} - \frac{(1-x-y)^3}{3}\right]_{y=0}^{y=1-x} dx \\ &= \int_0^1 \left(\frac{(1-x)^2}{2} + \frac{(1-x)^3}{3}\right) dx = \frac{1}{4}\end{aligned}$$

Example 5.9. Let W $x = 0, y = 0 z = 2$ and the surface $z = x^2 + y^2$ and $x \ge 0, y \ge 0$ Find $\iiint_W x\, dxdydz$.

sol. Method1. We describe the region by

$$0 \le x \le \sqrt{2}, \quad 0 \le y \le \sqrt{2 - x^2}, \quad x^2 + y^2 \le z \le 2.$$

$$\begin{aligned}\iiint_W x\, dxdydz &= \int_0^{\sqrt{2}} \left[\int_0^{\sqrt{2-x^2}} (\int_{x^2+y^2}^2 x\, dz) dy\right] dx \\ &= \frac{8\sqrt{2}}{15}\end{aligned}$$

Method2. We describe the region by solving for x, i.e, $0 \le x \le (z - y^2)^{1/2}$ and (y, z) in R where R is given by the relation

$$0 \le z \le 2, \quad 0 \le y \le z^{1/2}.$$

Then

$$\begin{aligned}\iiint_W x\, dxdydz &= \iint_R \left(\int_0^{(z-y^2)^{1/2}} xdx\right) dydz \\ &= \frac{8\sqrt{2}}{15}\end{aligned}$$

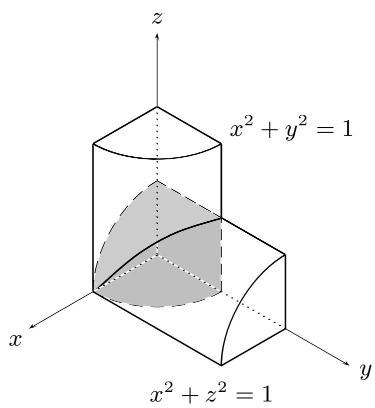


그림 5.16: common region of two cylinders

Example 5.10.

$$\int_0^1 \int_0^x \int_{x^2+y^2}^2 dzdydx$$

Sketch region first.

sol. Sketch

Example 5.11. Find the common region of two cylinders (figure 5.17) $x^2 + y^2 \leq 1$, $x^2 + z^2 \leq 1$ $(z \geq 0)$.

sol.

$$\begin{aligned}\iint_{x^2+y^2\leq 1}\int_0^{\sqrt{1-x^2}} dzdxdy &= \int_{-1}^1 \int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} \sqrt{1-x^2}dydx \\ &= 2\int_{-1}^1 (1-x^2)dx \\ &= 2\left[x - \frac{x^3}{3}\right]_{-1}^1 = 4(1-\frac{1}{3}) = \frac{8}{3}\end{aligned}$$

Example 5.12. Find the region bounded by two parabolids $z = x^2 + y^2$ and $z = 2 - 3x^2 - y^2$.(figure 5.18) sol. The intersection is the curve $x^2 + y^2 = 2 - 3x^2 - y^2$, i.e, $2x^2 + y^2 = 1$. If we let $R = \{(x, y) : 2x^2 + y^2 \le 1\}$ this region is 1st kind on R. Hence

$$\iiint_D dxdydz = \iint_{2x^2+y^2\le 1} (2 - 3x^2 - y^2) - (x^2 + y^2)\, dxdy$$
$$= \iint_{2x^2+y^2\le 1} (2 - 4x^2 - 2y^2)\, dxdy$$

Now use polar coordinate $x = r/2\cos\theta, y = r/\sqrt{2}\sin\theta$. Then $dxdy = r/(2\sqrt{2})drd\theta$. Hence

$$\frac{1}{2\sqrt{2}}\int_0^{2\pi}\int_{r\le\sqrt{2}} (2 - r^2) r\, drd\theta$$
$$= \frac{1}{2\sqrt{2}}\int_0^{2\pi}\left[r^2 - \frac{r^4}{4}\right]_0^{\sqrt{2}} d\theta = \frac{\pi}{\sqrt{2}}$$

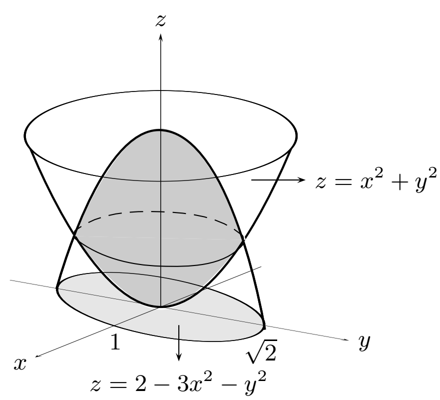


그림 5.17: $z = x^2 + y^2$, $z = 2 - 3x^2 - y^2$