제 4 장

Vector valued functions

제 1 절 Acceleration and Newton's law

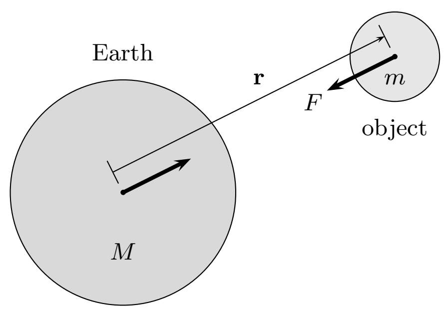


그림 4.1: Gravitational force is represented as vectors

Example 1.1. The force acting on a particle of mass m. Suppose the mass of the earth is M and G gravitational constant. If $\mathbf{r}$ is the position vector then the gravity $\mathbf{F}$ is (Figure 4.1)

$$\mathbf{F} = -\frac{GmM}{\|\mathbf{r}\|^3}\mathbf{r}$$

Definition 1.2. A **path** C is the image of a function $\mathbf{c}$ with domain I range in $\mathbb{R}^n$. We call $\mathbf{c}$ the **parametrization** of C .

A parameterized curve $\mathbf{c}$ in $\mathbb{R}^2$ or $\mathbb{R}^3$ can be written as $\mathbf{c}(t) = (x(t), y(t), z(t))$.

If $x(t)$, $y(t)$, $z(t)$ are continuous, then we say $\mathbf{c}$ is continuous, and if $x(t), y(t), z(t)$ are differentiable, then $\mathbf{c}$ is differentiable. If $x'(t)$, $y'(t)$, $z'(t)$ are continuous then we say $\mathbf{c}$ is C^1-curve.

Differentiation Rues

(1) $\frac{d}{dt}[\mathbf{b}(t) + \mathbf{c}(t)] = \mathbf{b}'(t) + \mathbf{c}'(t)$

(2) $\frac{d}{dt}[p(t)\mathbf{c}(t)] = p'(t)\mathbf{c}(t) + p(t)\mathbf{c}'(t)$

(3) $\frac{d}{dt}[\mathbf{b}(t) \cdot \mathbf{c}(t)] = \mathbf{b}'(t) \cdot \mathbf{c}(t) + \mathbf{b}(t) \cdot \mathbf{c}'(t)$

(4) $\frac{d}{dt}[\mathbf{b}(t) \times \mathbf{c}(t)] = \mathbf{b}'(t) \times \mathbf{c}(t) + \mathbf{b}(t) \times \mathbf{c}'(t)$

(5) $\frac{d}{dt}[\mathbf{c}(q(t))] = q'(t)\mathbf{c}'(q(t))$

Example 1.3. Show that if $\mathbf{c}(t)$ is a vector function such that $\|\mathbf{c}(t)\|$ is constant, then $\mathbf{c}'(t)$ is perpendicular to $\mathbf{c}(t)$ for all t.

Solution:

$\|\mathbf{c}(t)\|^2 = \mathbf{c}(t) \cdot \mathbf{c}(t)$. Derivative of constant is zero. Hence

$$0 = \frac{d}{dt}[\mathbf{c}(t) \cdot \mathbf{c}(t)] = \mathbf{c}'(t) \cdot \mathbf{c}(t) + \mathbf{c}(t) \cdot \mathbf{c}'(t) = 2\mathbf{c}(t) \cdot \mathbf{c}'(t)$$

Thus $\mathbf{c}'(t)$ is perpendicular to $\mathbf{c}(t)$.

Second derivative of a curve $\mathbf{c}(t)$ is acceleration, i.e, $\mathbf{a}(t) = d\mathbf{v}(t)/dt = \mathbf{c}''(t)$. □

Example 1.4. A particle moves with a constant acceleration $\mathbf{a}(t) = - = bk$. If the position when $t = 0$ is $(0, 0, 1)$ and velocity at $t = 0$ is $\mathbf{i} + \mathbf{j}$. Describe the motion of the particle.

Sol. Let $(x(t), y(t), z(t))$ represents the path traveled by the particle. Since the acceleration is $\mathbf{c}''(t) = -\mathbf{k}$ we see the velocity is

$$\mathbf{c}'(t) = C_1\mathbf{i} + C_2\mathbf{j} - t\mathbf{k} + C_3\mathbf{k}.$$

Hence by initial condition, $\mathbf{c}'(t) = \mathbf{i} + \mathbf{j} - t\mathbf{k}$ and so $\mathbf{c}(t) = t\mathbf{i} + t\mathbf{j} - \frac{t^2}{2}\mathbf{k} + Const\ vec$. The constant vector is $\mathbf{k}$. Hence $\mathbf{c}(t) = t\mathbf{i} + t\mathbf{j} + (1 - \frac{t^2}{2})\mathbf{k}$.

Remark 1.5. The image of C^1-curve is not necessarily "smooth". it may have sharp edges;

(1) Cycloid: $\mathbf{c}(t) = (t - \sin t, 1 - \cos t)$ has cusps when it touches x-axis. That is, when $\cos t = 1$ or when $t = 2\pi n, n = 1, 2, 3, \cdots$

(2) Hypocycloid: $\mathbf{c}(t) = (\cos^3 t, \sin^3 t)$ has cusps at four points when $\cos t = 0, \pm 1$

At all these points, we can check that $\mathbf{c}'(t) = 0$.(Roughly speaking, tangent vector has no direction.)

Definition 1.6. A differentiable path $\mathbf{c}$ is said to be **regular** if $\mathbf{c}'(t) \neq 0$ at all t. In this case, the image curve looks smooth.

Circular Orbits

Consider a particle of mass m moving at constant speed s in a circular path of radius r_0. We can represent its motion (in the plane) as

$$\mathbf{r}(t) = (r_0 \cos Ct, r_0 \sin Ct)$$

Find C. Since speed is $\|\mathbf{r}'(t)\| = |C| r_0 = s$, we get $C = s/r_0$. So the motion is described as

$$\mathbf{r}(t) = \left(r_0 \cos \frac{st}{r_0}, r_0 \sin \frac{st}{r_0}\right)$$

The quantity $\frac{s}{r_0}$ is called **frequency** denoted by ω. Thus

$$\mathbf{r}(t) = (r_0 \cos \omega t, r_0 \sin \omega t)$$

It's acceleration is

$$\mathbf{a}(t) = \mathbf{r}''(t) = -\frac{^2 st}{r_0}\left(r_0 \cos \frac{st}{r_0}, r_0 \sin \frac{st}{r_0}\right) = -\frac{s^2 t}{r_0^2}\mathbf{r}(t) = -\omega^2 \mathbf{r}(t).$$

Let us describe motion of a particle having circular motion. The centripetal force must equal to Gravitational force; By Newton's Law $\mathbf{F} = m\mathbf{a}$

$$-\frac{s^2 m}{r_0^2}\mathbf{r}(t) = -\frac{GmM}{r_0^3}\mathbf{r}(t)$$

Hence

$$s^2 = \frac{GM}{r_0}$$

If T denotes the period $s = 2\pi r_0/T$; then we obtain

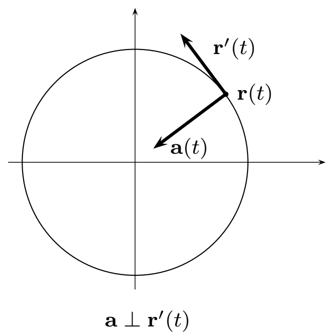


그림 4.2: acceleration and centripetal force $m\mathbf{a}$,

Kepler's Law

$$T^2 = r_0^3 \frac{(2\pi)^2}{GM}$$

Thus the square of the period is equal to the cube of the radius.

Example 1.7. Suppose a satellite is in circular motion about the earth over the equator. What is the radius of *geosynchronous* orbit?(It stays fixed over a point on equator) $M = 5.98 \times 10^{24}$ kg and $G = 6.67 \times 10^{-11}$ meter kg -sec.

Sol. Period must be one day: So $T = 60 \times 60 \times 24 = 86,400$ seconds. From Kepler's law,

$$r_0^3 = \frac{T^2 GM}{(2\pi)^2} \approx 7.54 \times 10^{22} m^3 \approx 42,300 km$$

제 2 절 Arc Length

The length of an arc is obtained by dividing the arc into several pieces and then summing the length of individual line segements; then take the limit.

The Riemann sum is defined as

$$\sum_{i=1}^{k} \|\mathbf{c}(t_i) - \mathbf{c}(t_{i-1})\|$$

As the partition $P \to 0$(i.e, $\|P\| \to 0$) [1] we see the sum

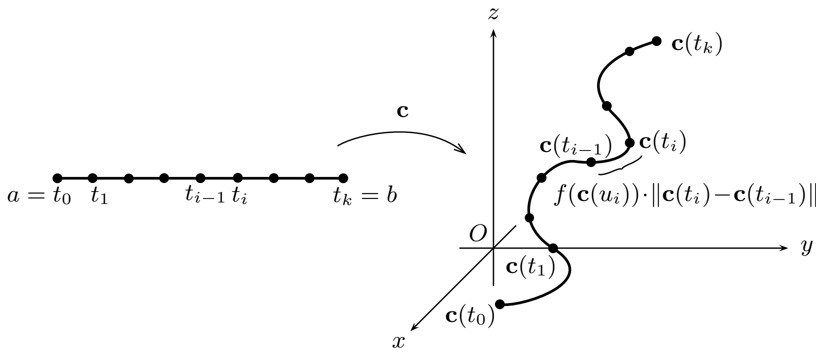


그림 4.3: Riemann sum on the curve

$$\sum_{i=1}^{k} \|\mathbf{c}'(u_i)\|(t_i - t_{i-1})$$

approaches $\int_a^b \|\mathbf{c}'(t)\|\, dt$.

Definition 2.1 (Arc Length)**.** Suppose a curve C has one-to-one C^1-parametrization $\mathbf{c}$. Then the **arc length** of C is defined by

$$L(\mathbf{c}) = \int_a^b \|\mathbf{c}'(t)\|\, dt = \int_a^b \sqrt{x'(t)^2 + y'(t)^2 + z'(t)^2} dt$$

Example 2.2. Find the arclength of $(\cos t, \sin t, t^2)$ $0 \le t \le 2\pi$.

Sol.

$$\|\mathbf{v}\| = \sqrt{1+4t^2} = 2\sqrt{t^2 + \frac{1}{4}}$$

To evaluate this integral we need a table of integrals:

$$\int \sqrt{x^2+a^2}\, dx = \frac{1}{2}[x\sqrt{x^2+a^2} + a^2 \log(x + \sqrt{x^2+a^2})] + C$$

Thus

$$L(\mathbf{c}) = \cdots$$

Example 2.3.

$$\mathbf{c}(\theta) = (2\cos^2\theta + \cos\theta,\ \sin 2\theta + \cos\theta), \quad \theta \in [0, 2\pi]$$

[1] The norm of partition $P = \{t_0, t_1, \ldots, t_n\}$ $\|P\|$ is defined as $\max_{0 \le i < n}(t_{i+1} - t_i)$.

sol. Above curve is divided into three paths.

$$\mathbf{c}_1(\theta) = (2\cos^2\theta + \cos\theta,\ \sin 2\theta + \cos\theta), \quad \theta \in [0, 2\pi/3]$$
$$\mathbf{c}_2(\theta) = (2\cos^2\theta + \cos\theta,\ \sin 2\theta + \cos\theta), \quad \theta \in [2\pi/3, 4\pi/3]$$
$$\mathbf{c}_3(\theta) = (2\cos^2\theta + \cos\theta,\ \sin 2\theta + \cos\theta), \quad \theta \in [4\pi/3, 2\pi]$$

Example 2.4. Consider the cycloid

$$\mathbf{c}(t) = (t - \sin t, 1 - \cos t)$$

Its length is

$$\|\mathbf{c}'(t)\|\sqrt{2 - 2\cos t}$$

Hence

$$\begin{aligned} L(\mathbf{c}) &= \int_0^{2\pi} \sqrt{2 - 2\cos t}dt = 2\int_0^{2\pi} \sqrt{\frac{1-\cos t}{2}}\, dt \\ &= 2\int_0^{2\pi} \sin\frac{t}{2}\, dt \\ &= 4\left(-\cos\frac{t}{2}\right)\Big|_0^{2\pi} = 8 \end{aligned}$$

Arc-Length Differential

Definition 2.5. We let the **infinitesimal displacement** of a particle following a path $\mathbf{c}(t) = x(t)\mathbf{i} + y(t)\mathbf{j} + z(t)\mathbf{k}$

$$d\mathbf{s} = dx\mathbf{i} + dy\mathbf{j} + dz\mathbf{k}$$

and its **arc length differential**

$$ds = \sqrt{dx^2 + dy^2 + dz^2} = \sqrt{x'(t)^2 + y'(t)^2 + z'(t)^2}dt$$

The arc length is defined as

$$\text{arc length } = \int_a^b ds = \int_a^b \sqrt{x'(t)^2 + y'(t)^2 + z'(t)^2}dt$$

One can also define **arc-length function** $s(t)$:

$$s(t) = \int_a^t \|\mathbf{c}'(t)\|\, dt$$

so that

$$s'(t) = \|\mathbf{c}'(t)\|$$

Definition 2.6. Suppose $\mathbf{c} : [a, b] \to \mathbb{R}^n$ is a C^1-parametrization of a curve C. Then the **arc length** of C is defined by

$$L(\mathbf{c}) = \int_a^b \|\mathbf{c}'(t)\|\, dt = \int_a^b \sqrt{x'(t)^2 + y'(t)^2 + z'(t)^2} dt.$$

Example 2.7. Suppose a function $y = f(x)$ given. Then the graph is viewed as a curve parameterized by $t = x$ and $\mathbf{c}(x) = (x, f(x))$ So the length of the graph from a to b is

$$L(\mathbf{c}) = \int_a^b \sqrt{1 + (f'(x))^2} dx$$

Path integral-See Ch 7

Definition 2.8. Suppose a curve C has one-to-one C^1-parametrization $\mathbf{c}$. Then the **path integral** of $f(x, y, z)$ on C is defined by

$$\int_{\mathbf{c}} f(x, y, z)\, ds$$

We write it as $\int_C f\, ds$ as $\int_C f(x, y, z)\, ds$. Note this integral is independent of choice of $\mathbf{c}$.

Example 2.9. Find the path integral of $f(x, y, z) = x^2y^2z^2$ on C.

$$C = \{(x, y, z) \colon x^2 + y^2 + z^2 = 2,\ x^2 + y^2 = z^2,\ z > 0\}$$

$$\int_C f\, ds = \int_{C_1} f\, ds + \int_{C_2} f\, ds + \cdots + \int_{C_k} f\, ds$$

$$\begin{aligned}
\int_{-\mathbf{c}} f\,ds &= \int_a^b f(-\mathbf{c}(t))\|(-\mathbf{c})'(t)\|\,dt \\
&= \int_a^b f(\mathbf{c}(b+a-t))\|\mathbf{c}'(b+a-t)(-1)\|\,dt \\
&= \int_a^b f(\mathbf{c}(b+a-t))\|\mathbf{c}'(b+a-t)\|\,dt \\
&= \int_b^a f(\mathbf{c}(u))\|\mathbf{c}'(u)\|(-1)\,du \\
&= \int_{\mathbf{c}} f\,ds
\end{aligned}$$

where $u = b + a - t$ was used.

Example 2.10. Find the path integral of $f(x,y,z) = x^2 + y^2 + z^2$ on C.

$$C = \{(\cos t, \sin t, t)\colon t \in [0, 2\pi]\} \cup \{(1, 0, t)\colon t \in [0, 2\pi]\}$$

sol. $C =$ is the sum of C_1 and C_2.

$$C_1 = \{(\cos t, \sin t, t)\colon t \in [0, 2\pi]\}, \quad C_2 = \{(1, 0, t)\colon t \in [0, 2\pi]\}$$

So parameterize C_1 and C_2 as follows:

$$\mathbf{c}_1 = (\cos t, \sin t, t) \quad t \in [0, 1], \quad \mathbf{c}_2 = (1, 0, t) \quad t \in [0, 2\pi]$$

Hence the integral is

$$\begin{aligned}
\int_C f\,ds &= \int_{C_1} f\,ds + \int_{C_2} f\,ds \\
&= \int_{\mathbf{c}_1} f\,ds + \int_{\mathbf{c}_2} f\,ds \\
&= \int_0^{2\pi} (1 + t^2)\sqrt{2}\,dt + \int_0^{2\pi} (1 + t^2)\,dt \\
&= (1 + \sqrt{2})\left(2\pi + 8\pi^3/3\right)
\end{aligned}$$

제 3 절 Vector Fields

Definition 3.1. Let $D \subset \mathbb{R}^n$. A vector function defined on D with values lying in $\mathbb{R}^n$ is called **vector field**.

A vector field $\mathbf{F}$ is represented by n real valued function $F_1, F_2, \ldots, F_n$.

$$\mathbf{F}(P) = (F_1(P), F_2(P), \ldots, F_n(P))$$

If $n = 3$

$$\mathbf{F}(P) = (F_1(P), F_2(P), F_3(P))$$

is written as

$$\mathbf{F}(P) = F_1(P)\mathbf{i} + F_2(P)\mathbf{j} + F_3(P)\mathbf{k}$$

Gradient vector field

Given real valued function $f(x_1, x_2, \ldots, x_n)$ we define

$$\nabla f := (\frac{\partial f}{\partial x_1}, \frac{\partial f}{\partial x_2}, \ldots, \frac{\partial f}{\partial x_n})$$

call it **gradient vector field**.

Flow lines

Example 3.2. Heat flux vector fields

$$J = -k\nabla T$$

Example 3.3. A gravitational force field. Place the origin at the center of earth. Newton's law

$$\mathbf{F} = -\frac{mMG}{r^3}\mathbf{r}$$

We see $\mathbf{F} = -\nabla V$ where the potential V is given by

$$V = -\frac{mMG}{r}$$

Note that $\mathbf{F}$ point in the direction of decreasing V.

Example 3.4. Similarly, we have Coulomb's law. The force acting on a charge e at position $\mathbf{r}$ due to a charge Q at the origin is

$$\mathbf{F} = \frac{\epsilon Qe}{r^3}\mathbf{r} = -\nabla V$$

where $V = \epsilon Qe/r$. The level sets of V are called **equipotential surface or lines** Note that the force field is orthogonal to the equipotential surfaces. We see $\mathbf{F} = -\nabla V$ where the potential V is given by

$$V = -\frac{mMG}{r}$$

Note that $\mathbf{F}$ point in the direction of decreasing V.

Example 3.5. Show the vector field $\mathbf{V}(x, y) = y\mathbf{i} - x\mathbf{j}$ is not a gradient vector field. i.e, there is no C^1 function f such that

$$\mathbf{V} = \nabla f = \frac{\partial f}{\partial x}\mathbf{i} + \frac{\partial f}{\partial y}\mathbf{j}$$

sol. Suppose there is such an f. Then $\frac{\partial f}{\partial x} = y$ and $\frac{\partial f}{\partial y} = -x$. Solving, $f(x, y) = xy + g(y)$. Then $\frac{\partial f}{\partial y} = x + g'(y) = -x$, which is impossible.

Conservation of energy

Consider a particle of mass m moving in a force field that is a potential field. ($\mathbf{F} = -\nabla V$)

$$m\mathbf{r}''(t) = -\nabla V(\mathbf{r}(t))$$

A basic fact about such a motion is the *conservation of energy*. The energy E is defined to be the sum of kinetic energy and potential energy

$$E = \frac{1}{2}m\|\mathbf{r}'(t)\|^2 + V(\mathbf{r}(t))$$

The principle of *Conservation of energy* says: E is independent of time. So $dE/dt = 0$. We can prove it simply:

$$\frac{dE}{dt} = m\mathbf{r}'(t) \cdot \mathbf{r}''(t) + (\nabla V) \cdot \mathbf{r}'(t) = \mathbf{r}' \cdot (-\nabla V + \nabla V) = 0$$

Escape Velocity

As an application of conservation of energy, we compute the velocity of a rocket to escape the earth gravitational influence. The energy(kinetic energy +potential energy) is

$$E_0 = \frac{1}{2}mv_e^2 - \frac{mMG}{R_0}.$$

The escape velocity is obtained when this energy is zero. Thus

$$v_e = \sqrt{\frac{2MG}{R_0}}.$$

Now MG/R_0^2 is gravity g, thus

$$v_e = \sqrt{2gR_0}.$$

Flow Lines

Definition 3.6. Given a vector field $\mathbf{F}$ the path $\mathbf{c}(t)$ satisfying

$$\mathbf{c}'(t) = \mathbf{F}(\mathbf{c}(t))$$

is called the **flow line** for $\mathbf{F}$. That is, $\mathbf{F}$ yields the velocity fields of the path $\mathbf{c}(t)$. **flow line** is also called as **streamlines** or **integral curves**

Example 3.7. Suppose water is flowing in a pipe as in fig 4.4. Suppose it does not depends on time. Then it is given by a vector field.

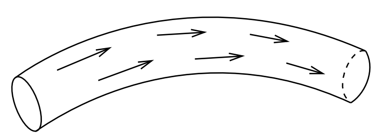


그림 4.4: Water flow in a pipe

Let $\mathbf{F} = P\mathbf{i} + Q\mathbf{j} + R\mathbf{k}$ and $\mathbf{c}(t) = (x(t), y(t), z(t))$ A flow line may be viewed as the solution of system of DE. Indeed from $\mathbf{c}'(t) = \mathbf{F}(\mathbf{c}(t))$, we see

$$\begin{aligned} x'(t) &= P(x(t), y(t), z(t)) \\ y'(t) &= Q(x(t), y(t), z(t)) \\ z'(t) &= R(x(t), y(t), z(t)) \end{aligned}$$

Example 3.8. Show $\mathbf{c}(t) = (\cos t, \sin t)$ is a flow line of $\mathbf{F} = -y\mathbf{i} + x\mathbf{j}$. can you find others?

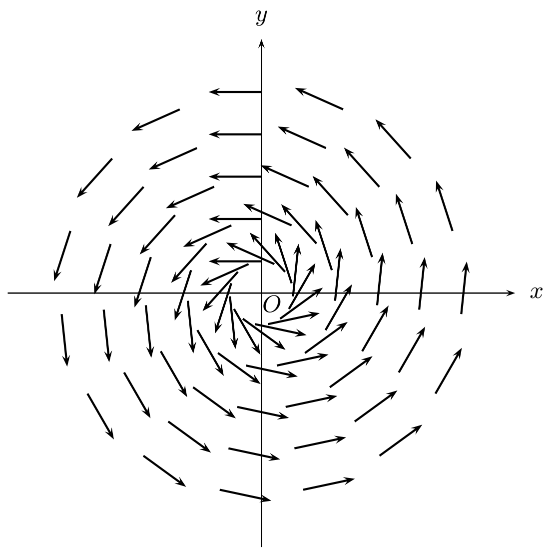


그림 4.5: The vector field $\mathbf{F}(x,y) = (-y\mathbf{i} + x\mathbf{j})/\sqrt{x^2+y^2}$

sol. Verify

$$\mathbf{c}'(t) = \mathbf{F}(\mathbf{c}(t))$$

Others may be

$$\mathbf{c}(t) = (r\cos(t-t_0), r\sin(t-t_0))$$

Example 3.9. Draw the vector fields **F**

$$\mathbf{F}(x,y) = \frac{-y\mathbf{i} + x\mathbf{j}}{x^2+y^2}, (x,y) \neq (0,0).$$

It describes the flow of water in a sink(bathtub).

Example 3.10. Draw the vector fields **F** and its integral curve (Flow line) (Fig 4.5)

$$\mathbf{F}(x,y) = \frac{-y\mathbf{i} + x\mathbf{j}}{\sqrt{x^2+y^2}}, (x,y) \neq (0,0).$$

Let $\mathbf{c}(t) = (x(t), y(t))$ be the flow line. Then $\mathbf{c}'(t) = (x'(t), y'(t))$ must be $\mathbf{F}(\mathbf{c}(t))$. Hence

$$x'(t) = \frac{-y(t)}{\sqrt{x(t)^2 + y(t)^2}} \tag{4.1}$$

$$y'(t) = \frac{x(t)}{\sqrt{x(t)^2 + y(t)^2}} \tag{4.2}$$

Multiply first by $x(t)$ and second by $y(t)$. Then adding we get

$$x'(t)x(t) + y'(t)y(t) = 0$$

Integrating

$$x(t)^2 + y(t)^2 = r$$

This is equation for circle. So we can parameterize it by trig function.

$$\mathbf{c}(t) = (x(t), y(t)) = (r\cos\theta(t), r\sin\theta(t))$$

Hence

$$x'(t) = -r\theta'(t)\sin\theta(t) \tag{4.3}$$

$$y'(t) = r\theta'(t)\cos\theta(t) \tag{4.4}$$

From (4.2) (4.4)

$$-r\theta'(t)\sin\theta(t) = -\sin\theta(t)$$

Hence

$$\theta'(t) = \frac{1}{r}$$

So the flow line $\mathbf{c}(t)$ is

$$\mathbf{c}(t) = (x(t), y(t)) = (r\cos\frac{t}{r}, r\sin\frac{t}{r})$$

The period of $\mathbf{c}(t)$ is $2\pi r$.

Example 3.11. Show that $\mathbf{F}(x, y) = x\mathbf{i} - y\mathbf{j}$ is a gradient field and find flow line sol. Suppose $\mathbf{F}$ is a gradient field of $f(x,y)$ then

$$\frac{\partial f}{\partial x}\mathbf{i}+\frac{\partial f}{\partial y}\mathbf{j}=x\mathbf{i}-y\mathbf{j}$$

Find $f(x,y)$ such that

$$\frac{\partial f}{\partial x}=x, \quad \frac{\partial f}{\partial y}=-y$$

Hence $f(x,y)=\frac{1}{2}(x^2-y^2)$

$$\frac{1}{2}(x^2-y^2)=\mathrm{c}$$

Suppose $\mathbf{F}(x,y)$ is given by

$$\mathbf{F}(x,y)=(F_1(x,y),F_2(x,y)).$$

Then the flow line $(x(t),y(t))$ satisfies

$$\begin{aligned}x'(t)&=F_1(x(t),y(t))\\ y'(t)&=F_2(x(t),y(t)).\end{aligned}$$

제 4 절 Divergence and curl

For divergence and curl operations,(PROCESS) we make use of **del operator** defined by

$$\nabla=\mathbf{i}\frac{\partial}{\partial x}+\mathbf{j}\frac{\partial}{\partial y}+\mathbf{k}\frac{\partial}{\partial z}$$

It works like this: For gradient

$$\nabla f=\left(\mathbf{i}\frac{\partial}{\partial x}+\mathbf{j}\frac{\partial}{\partial y}+\mathbf{k}\frac{\partial}{\partial z}\right)f=\mathbf{i}\frac{\partial f}{\partial x}+\mathbf{j}\frac{\partial f}{\partial y}+\mathbf{k}\frac{\partial f}{\partial z}$$

Divergence

Definition 4.1 (Divergence)**.** If $\mathbf{F}=F_1\mathbf{i}+F_2\mathbf{j}+F_3\mathbf{k}$ is a vector field then the **divergence** is the scalar field

$$\begin{aligned}\operatorname{div}\mathbf{F} &= \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z} \\ &= \Big(\frac{\partial}{\partial x}\mathbf{i} + \frac{\partial}{\partial y}\mathbf{j} + \frac{\partial}{\partial z}\mathbf{k}\Big)\cdot\Big(F_1\mathbf{i} + F_2\mathbf{j} + F_3\mathbf{k}\Big) \\ &= \nabla\cdot\mathbf{F}\end{aligned}$$

Similarly for n-dim

$$\operatorname{div}\mathbf{F} = \sum_{i=1}^{n}\frac{\partial F_i}{\partial x_i} = \frac{\partial F_1}{\partial x_1} + \frac{\partial F_2}{\partial x_2} + \cdots + \frac{\partial F_n}{\partial x_n}$$

Example 4.2. Find the divergence of $\mathbf{F} = (e^x \sin y,\, e^x \cos y,\, yz^2)$.

sol. Since $\operatorname{div}\mathbf{F} = \nabla\cdot\mathbf{F}$

$$\begin{aligned}\operatorname{div}\mathbf{F} &= \frac{\partial}{\partial x}(e^x\sin y) + \frac{\partial}{\partial y}(e^x\cos y) + \frac{\partial}{\partial z}(yz^2) \\ &= e^x\sin y + (-e^x\sin y) + 2yz = 2yz.\end{aligned}$$

Hence $\operatorname{div}\mathbf{F} = 2yz$.

Example 4.3.

$$\mathbf{F} = x^2y\mathbf{i} + z\mathbf{j} + xyz\mathbf{k}$$

Meaning of divergence

Suppose $\mathbf{F}$ represent the velocity of a gas or fluid. Then divergence represents **the rate of expansion per unit volume**: If $\operatorname{div}\mathbf{F}(P) > 0$ then it is expanding. If ($\operatorname{div}\mathbf{F}(P) < 0$) then it is compressing.

It is

$$\frac{1}{V(0)}\frac{d}{dt}V(t)\Big|_{t=0} \approx \operatorname{div}\mathbf{F}(\mathbf{x}_0)$$

Example 4.4. Draw flow lines of

$$F = x\mathbf{i} + y\mathbf{j}$$

Div is positive

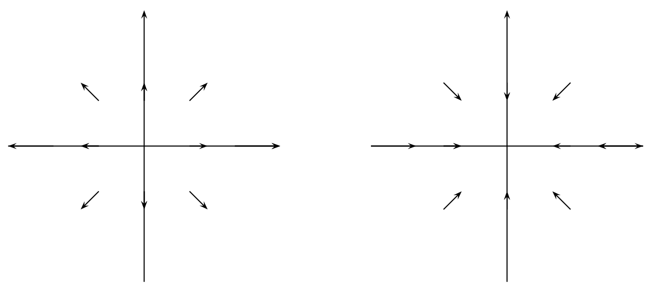
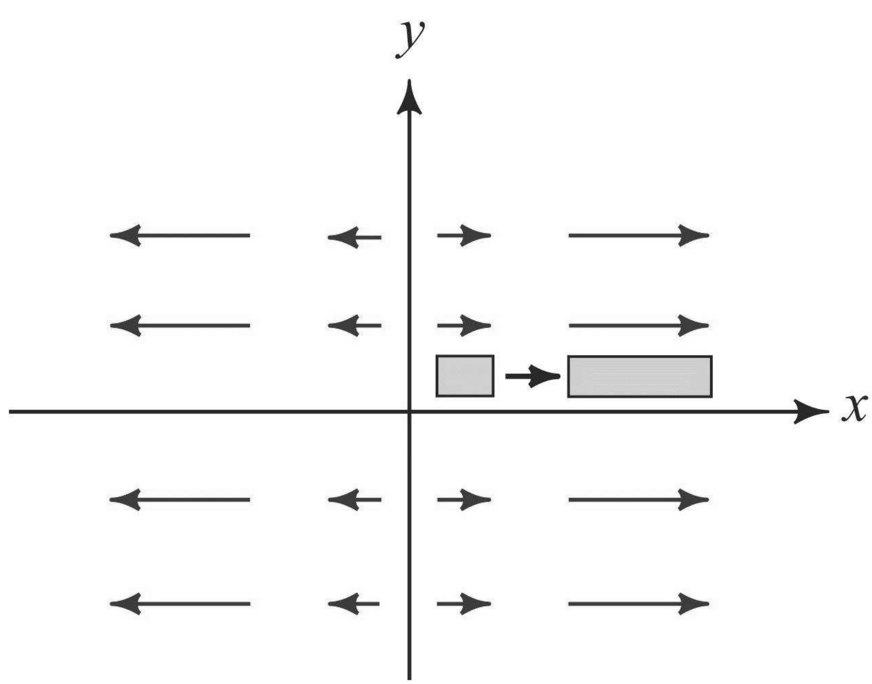


그림 4.6: vector field (x, y) and $-(x, y)$

Example 4.5. Graph of

$$F = x\mathbf{i}$$

Div is positive(Expanding)



Example 4.6. Graph of

$$F = -x\mathbf{i} - y\mathbf{j}$$

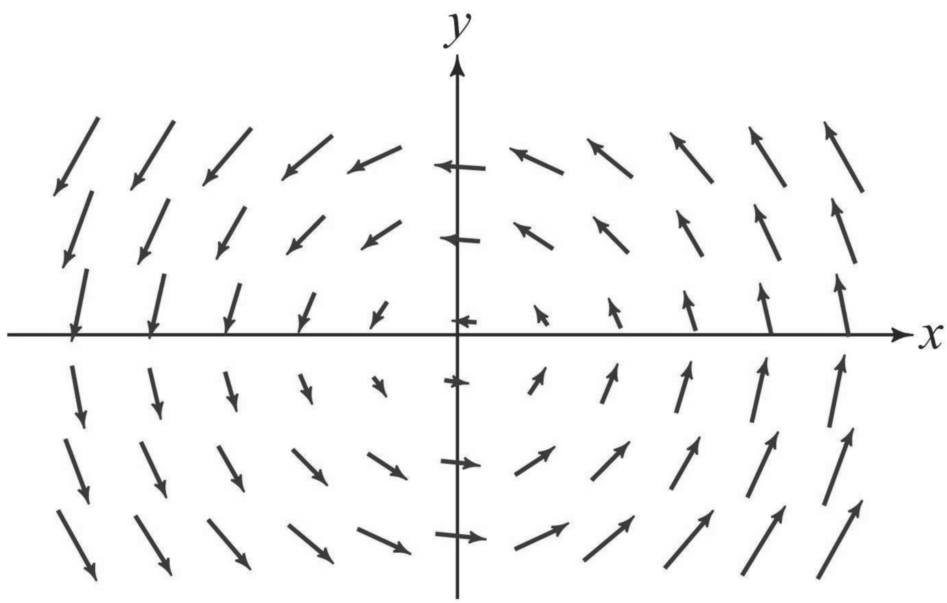


그림 4.7: Incompressible $\mathbf{F} = -y\mathbf{i} + x\mathbf{j}$

Example 4.7. The flow lines of

$$F = -y\mathbf{i} + x\mathbf{j}$$

are concentric circles. From this we guess the fluid is neither expanding or compressing. Div is zero. Fig 4.7.

Example 4.8. The vector field $F = x\mathbf{i} - y\mathbf{j}$ is divergence free. The flow lines are as in figure.

Curl

Use the symbol ∇ to see

$$\begin{aligned} \operatorname{curl} \mathbf{F} = \nabla \times \mathbf{F} &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ F_1 & F_2 & F_3 \end{vmatrix} \\ &= \Big(\frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z}\Big)\mathbf{i} + \Big(\frac{\partial F_1}{\partial z} - \frac{\partial F_3}{\partial x}\Big)\mathbf{j} + \Big(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\Big)\mathbf{k} \end{aligned}$$

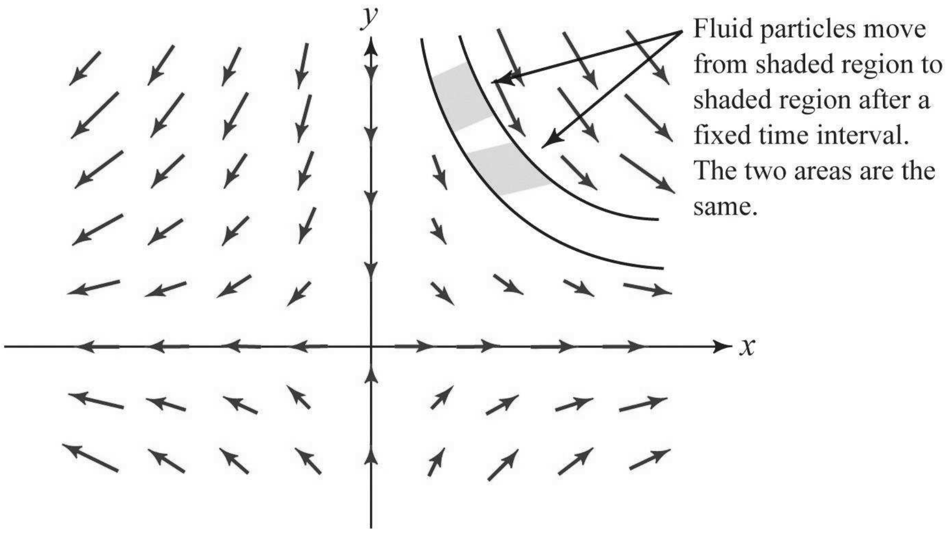


그림 4.8: $\mathbf{F} = x\mathbf{i} - y\mathbf{j}$ Incompressible: Volume preserving

Example 4.9. Let $\mathbf{F} = x\mathbf{i} + xy\mathbf{j} + \mathbf{k}$. Find $\nabla \times \mathbf{F}$.

$$\nabla \times \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ x & xy & 1 \end{vmatrix} = 0\mathbf{i} - 0\mathbf{j} + y\mathbf{k}$$

Example 4.10. Let $\mathbf{F} = xy\mathbf{i} - \sin z\mathbf{j} + \mathbf{k}$. Find $\nabla \times \mathbf{F}$.

$$\begin{aligned} \nabla \times \mathbf{F} &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ xy & -\sin z & 1 \end{vmatrix} \\ &= \begin{vmatrix} \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ -\sin z & 1 \end{vmatrix} \mathbf{i} - \begin{vmatrix} \frac{\partial}{\partial z} & \frac{\partial}{\partial z} \\ xy & 1 \end{vmatrix} \mathbf{j} + \begin{vmatrix} \frac{\partial}{\partial x} & \frac{\partial}{\partial y} \\ xy & \sin z \end{vmatrix} \mathbf{k} \\ &= \cos z\mathbf{i} - x\mathbf{k} \end{aligned}$$

Meaning of curl

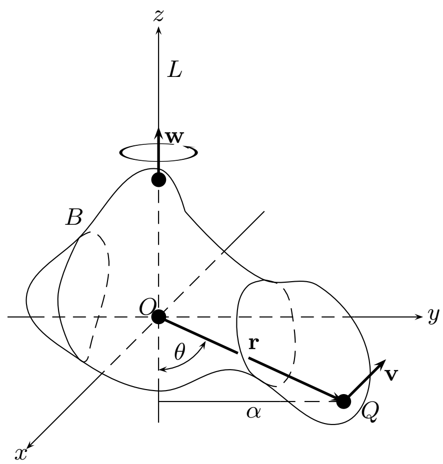


그림 4.9: velocity $\mathbf{v}$ and angular velocity $\mathbf{w}$ has relation $\mathbf{v} = \mathbf{w} \times \mathbf{r}$.

Consider a rigid body B rotating about an axis L. (Fig 4.9). **The rotational motion of B can be described by a vector along axis of rotation $\mathbf{w}$**. Let $\mathbf{w}$ the vector along z-axis s.t. $\omega = ||\mathbf{w}||$. The vector $\mathbf{w}$ is called the **angular velocity vector** and ω is angular speed.

$\omega = ||\mathbf{w}||$

Assume L is z-axis Q is any point on the body B, α is distance from Q to L. Then $\alpha = ||\mathbf{r}|| \sin\theta$ ($\mathbf{r}$ points to Q). Consider the tangent vector $\mathbf{v}$ at Q. Since Q moves around a circle of radius α and parallel to xy-plane (counterclockwise), we see,

$$||\mathbf{v}|| = \omega\alpha = \omega||\mathbf{r}|| \sin\theta = ||\mathbf{w}||||\mathbf{r}|| \sin\theta,$$

Then by definition of cross product,

$$\mathbf{v} = \mathbf{w} \times \mathbf{r}.$$

Since $\mathbf{w} = \omega\mathbf{k}$, $\mathbf{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$ we see from the property of cross product,

$$\mathbf{v} = \mathbf{w} \times \mathbf{r} = -\omega y\mathbf{i} + \omega x\mathbf{j}.$$

So $\operatorname{curl} \mathbf{v} = 2\omega\mathbf{k} = 2\mathbf{w}$. Hence for the rotation of a rigid body, the curl is a vector field whose direction is along the axis of rotation and magnitude is twice the angular speed.

Curl and rotational flow

$\nabla \times \mathbf{F}$ represents twice the angular velocity: So if it is 0, then we have irrotational fluid.

Example 4.11. Find curl $\mathbf{F}$ when $\mathbf{F}(x, y, z) = (y\mathbf{i} - x\mathbf{j})/(x^2 + y^2)$ in $\mathbb{R}^3$.

sol. Write $\mathbf{F}(x, y, z) = \dfrac{y}{x^2 + y^2}\mathbf{i} + \dfrac{-x}{x^2 + y^2}\mathbf{j} + 0\mathbf{k}$. Then we see

$$
\begin{aligned}
\nabla \times \mathbf{F} &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ \dfrac{y}{x^2+y^2} & \dfrac{-x}{x^2+y^2} & 0 \end{vmatrix} \\
&= \left[\frac{\partial}{\partial x}\left(\frac{-x}{x^2+y^2}\right) - \frac{\partial}{\partial y}\left(\frac{y}{x^2+y^2}\right)\right]\mathbf{k} \\
&= \left[\frac{-(x^2+y^2) - (-x)(2x)}{(x^2+y^2)^2} - \frac{(x^2+y^2) - (y)(2y)}{(x^2+y^2)^2}\right]\mathbf{k} \\
&= \left[\frac{x^2-y^2}{(x^2+y^2)^2} - \frac{x^2-y^2}{(x^2+y^2)^2}\right]\mathbf{k} \\
&= \mathbf{0}.
\end{aligned}
$$

Gradients are curl Free

Theorem 4.12. *For any C^2 function*

$$\nabla \times (\nabla f) = 0$$

Proof.

$$
\begin{aligned}
\nabla \times \nabla f &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} & \frac{\partial f}{\partial z} \end{vmatrix} \\
&= \left(\frac{\partial^2 f}{\partial y \partial z} - \frac{\partial^2 f}{\partial z \partial y}\right)\mathbf{i} + \left(\frac{\partial^2 f}{\partial z \partial z x} - \frac{\partial^2 f}{\partial x \partial z}\right)\mathbf{j} + \left(\frac{\partial^2 f}{\partial x \partial y} - \frac{\partial^2 f}{\partial y \partial x}\right)\mathbf{k}
\end{aligned}
$$

□

Example 4.13. Show $\mathbf{F}(x, y, z) = y\mathbf{i} - x\mathbf{j}$ is not a gradient field. i.e, there does not exists $\mathcal{C}^2$-function f s.t $\nabla f = \mathbf{F}$ holds.

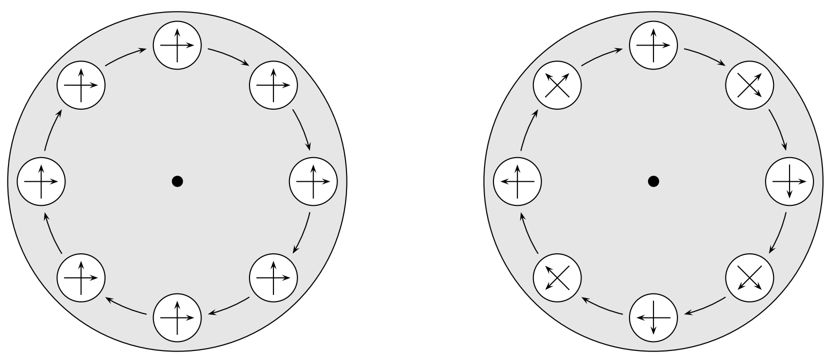
sol. If $\mathbf{F}$ is the gradient of f then by Thm 4.12

$$\operatorname{curl}\mathbf{F} = \nabla \times \mathbf{F} = \nabla \times \nabla f = \mathbf{0}$$

Let $\mathbf{F}(x, y, z) = y\mathbf{i} - x\mathbf{j} + 0\mathbf{k}$. Then

$$\begin{aligned}\nabla \times \mathbf{F} &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y & -x & 0 \end{vmatrix} \\ &= \left[\frac{\partial}{\partial x}\Big(-x\Big) - \frac{\partial}{\partial y}\Big(y\Big)\right]\mathbf{k} = -2\mathbf{k}\end{aligned}$$

$\operatorname{curl}\mathbf{F} = -2\mathbf{k} \neq \mathbf{0}$, a contradiction.



(a) $\mathbf{F} = (y\mathbf{i} - x\mathbf{j})/(x^2 + y^2)$ (b) $\mathbf{F} = y\mathbf{i} - x\mathbf{j}$

그림 4.10: Movement of small paddle in vector fields

Remark 4.14. Vector field $\mathbf{F}(x, y, z) = (y\mathbf{i} - x\mathbf{j})/(x^2 + y^2)$(It describes flow in a tub) does not rotate about any point except z-axis. When small paddle is placed in the fluid, it will follow the flow line(a circle in this case), but it does not rotate about its own axis. Such a field is called **irrotational**.

But the vector field $\mathbf{F}(x, y, z) = y\mathbf{i} - x\mathbf{j}$ has nonzero rotation. (fig 4.10(b)).

Scalar curl

As a special case, if $\mathbf{F} = P\mathbf{i} + Q\mathbf{j}$ then

$$\nabla \times \mathbf{F} = \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)\mathbf{k}$$

$(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y})$ is called a scalar curl.

Example 4.15. Find scalar curl of $\mathbf{V} = -y^2\mathbf{i} + x\mathbf{j}$,

$$\nabla \times \mathbf{V} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ -y^2 & x & 0 \end{vmatrix} = (1 + 2y)\mathbf{k}$$

Curls are divergence free

Theorem 4.16. *For any C^2 vector field* $\mathbf{F}$

$$\mathrm{div}\,\mathrm{curl}\mathbf{F} = \nabla \cdot (\nabla \times \mathbf{F}) = 0$$

Physical meaning of divergence

Let

$$\mathbf{F}(x, y, z) = (F_1, F_2, F_3) = F_1\mathbf{i} + F_2\mathbf{j} + F_3\mathbf{k}$$

be a velocity vector field of some fluid in $\mathbb{R}^3$.

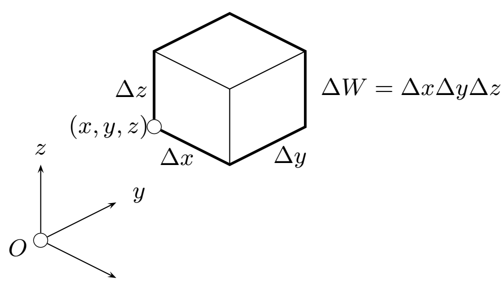


그림 4.11: Geometric meaning of divergence

Fig 4.11. Consider a box W with dimension $\Delta x, \Delta y, \Delta z$ Then volume of W is $\Delta W = \Delta x \Delta y \Delta z$. Consider the loss of fluid across W per unit time. First consider fluid loss through left side of W whose area is $\Delta x \Delta z$. (Consider F_2 only). The outflux is

$$\mathbf{F}(x, y, z) \cdot (-\mathbf{j})\Delta x \Delta z = -F_2(x, y, z)\Delta x \Delta z$$

influx is

$$\mathbf{F}(x, y + \Delta y, z) \cdot \mathbf{j}\Delta x \Delta z = F_2(x, y + \Delta y, z)\Delta x \Delta z$$

$$\Big(F_2(x,y+\Delta y,z)-F_2(x,y,z)\Big)\Delta x\Delta z \approx \Big(\frac{\partial F_2}{\partial y}\Delta y\Big)\Delta x\Delta z$$

Considering all the direction, the change in fluid across W per unit time is(total flux)

$$\Big(\frac{\partial F_1}{\partial x}+\frac{\partial F_2}{\partial y}+\frac{\partial F_3}{\partial z}\Big)\Delta x\Delta y\Delta z$$

Now divide by volume ΔW

$$\text{density of flux/time} = \frac{\text{Flux across boundary}}{\text{vol}} \approx \Big(\frac{\partial F_1}{\partial x}+\frac{\partial F_2}{\partial y}+\frac{\partial F_3}{\partial z}\Big)$$

Let $\Delta x, \Delta y, \Delta z \to 0$ Then fluid density of $\mathbf{F}$ is $\operatorname{div}\mathbf{F}$. If $\mathbf{F}$ is gas, then $\operatorname{div}\mathbf{F}$ represents the rate of expansion of gas per unit time per unit volume. If $\mathbf{F}(x,y,z) = x\mathbf{i}+y\mathbf{j}+z\mathbf{k}$ then $\operatorname{div}\mathbf{F} = 3$ and this means the gas is expanding three times per unit time.

Laplace operator

$$\nabla^2 f = \nabla\cdot(\nabla f) = \frac{\partial^2 f}{\partial x^2}+\frac{\partial^2 f}{\partial y^2}+\frac{\partial^2 f}{\partial z^2}.$$

$\nabla^2 f = 0$ is called Laplace equation.

Example 4.17. Find $\nabla^2 f$ when

$$f(x,y) = \frac{1}{r} = \frac{1}{\sqrt{x^2+y^2+z^2},} \quad (x,y,z,) \neq 0$$

where $\mathbf{r} = x\mathbf{i}+y\mathbf{j}+z\mathbf{k}$ and $r = \|\mathbf{r}\|$.

sol.

$$\frac{\partial f}{\partial x} = \frac{-x}{(x^2+y^2+z^2)^{3/2}}, \quad \frac{\partial f}{\partial y} = \frac{-y}{(x^2+y^2+z^2)^{3/2}}, \quad \frac{\partial f}{\partial z} = \frac{-z}{(x^2+y^2+z^2)^{3/2}}.$$

Second derivatives, we find that

$$\begin{aligned}
\frac{\partial^2 f}{\partial x^2} &= \frac{3x^2}{(x^2+y^2+z^2)^{5/2}} - \frac{1}{(x^2+y^2+z^2)^{3/2}} \\
\frac{\partial^2 f}{\partial y^2} &= \frac{3y^2}{(x^2+y^2+z^2)^{5/2}} - \frac{1}{(x^2+y^2+z^2)^{3/2}} \\
\frac{\partial^2 f}{\partial z^2} &= \frac{3z^2}{(x^2+y^2+z^2)^{5/2}} - \frac{1}{(x^2+y^2+z^2)^{3/2}}
\end{aligned}$$

Thus

$$\nabla^2 f = 0.$$

Example 4.18. For given $f(x,y) = x^3 - 3xy^2 + 7z$, compute $\nabla^2 f$.

sol. First compute $\partial f/\partial x$, $\partial f/\partial y$ and $\partial f/\partial z$.

$$\frac{\partial f}{\partial x} = 3x^2 - 3y^2, \quad \frac{\partial f}{\partial y} = -6xy, \quad \frac{\partial f}{\partial z} = 7.$$

Hence

$$\frac{\partial^2 f}{\partial x^2} = 6x, \quad \frac{\partial^2 f}{\partial y^2} = -6x, \quad \frac{\partial^2 f}{\partial z^2} = 0.$$

and

$$\nabla^2 f = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2} = 6x + (-6x) + 0 = 0.$$

Vector differential identities

Vector differential identities

(1) $\nabla(f+g) = \nabla f + \nabla g$

(2) $\nabla(cf) = c\nabla f$, for constant c.

(3) $\nabla(fg) = f\nabla g + g\nabla f$

(4) $\nabla(f/g) = (g\nabla f - f\nabla g)/g^2, \quad g \neq 0$

(5) $\operatorname{div}(\mathbf{F}+\mathbf{G}) = \operatorname{div}\mathbf{F} + \operatorname{div}\mathbf{G}$

(6) $\operatorname{curl}(\mathbf{F}+\mathbf{G}) = \operatorname{curl}\mathbf{F} + \operatorname{curl}\mathbf{G}$

(7) $\operatorname{div}(f\mathbf{F}) = f\operatorname{div}\mathbf{F} + \mathbf{F}\cdot\nabla f$

(8) $\operatorname{div}(\mathbf{F}\times\mathbf{G}) = \mathbf{G}\cdot\operatorname{curl}\mathbf{F} - \mathbf{F}\cdot\operatorname{curl}\mathbf{G}$

(9) $\operatorname{div}\operatorname{curl}\mathbf{F} = 0$

(10) $\operatorname{curl}(f\mathbf{F}) = f\operatorname{curl}\mathbf{F} + \nabla f\times\mathbf{F}$

(11) $\operatorname{curl}(\mathbf{F}\times\mathbf{G}) = \mathbf{F}\operatorname{div}\mathbf{G} - \mathbf{G}\operatorname{div}\mathbf{F} + (\mathbf{G}\cdot\nabla)\mathbf{F} - (\mathbf{F}\cdot\nabla)\mathbf{G}$

(12) $\operatorname{curl}\operatorname{curl}\mathbf{F} = \operatorname{grad}\operatorname{div}\mathbf{F} - \nabla^2\mathbf{F}$

(13) $\operatorname{curl}\nabla f = \mathbf{0}$

(14) $\nabla(\mathbf{F}\cdot\mathbf{F}) = 2(\mathbf{F}\cdot\nabla)\mathbf{F} + 2\mathbf{F}\times(\operatorname{curl}\mathbf{F})$

(15) $\nabla^2(fg) = f\nabla^2 g + g\nabla^2 f + 2(\nabla f\cdot\nabla g)$

(16) $\nabla(\mathbf{F}\cdot\mathbf{G}) = (\mathbf{F}\cdot\nabla)\mathbf{G} + (\mathbf{G}\cdot\nabla)\mathbf{F} + \mathbf{F}\times\operatorname{curl}\mathbf{G} + \mathbf{G}\times\operatorname{curl}\mathbf{F}$

(17) $\operatorname{div}(\nabla f\times\nabla g) = 0$

(18) $\nabla\cdot(f\nabla g - g\nabla f) = f\nabla^2 g - g\nabla^2 f$

(19) $\mathbf{H}\cdot(\mathbf{F}\times\mathbf{G}) = \mathbf{G}\cdot(\mathbf{H}\times\mathbf{F}) = \mathbf{F}\cdot(\mathbf{G}\times\mathbf{H})$

(20) $\mathbf{H}\cdot((\mathbf{F}\times\nabla)\times\mathbf{G}) = ((\mathbf{H}\cdot\nabla)\mathbf{G})\cdot\mathbf{F} - (\mathbf{H}\cdot\mathbf{F})(\nabla\cdot\mathbf{G})$

(21) $\mathbf{F}\times(\mathbf{G}\times\mathbf{H}) = (\mathbf{F}\cdot\mathbf{H})\mathbf{G} - \mathbf{H}(\mathbf{F}\cdot\mathbf{G})$

Example 4.19. Prove (7).

sol.

$$\operatorname{div}(f\mathbf{F}) = \frac{\partial}{\partial x}(fF_1) + \frac{\partial}{\partial y}(fF_2) + \frac{\partial}{\partial z}(fF_3)$$

By product rule,

$$\begin{aligned}\operatorname{div}(f\mathbf{F}) &= f\left(\frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z}\right) + F_1\frac{\partial f}{\partial x} + F_2\frac{\partial f}{\partial y} + F_3\frac{\partial f}{\partial z} \\ &= f(\nabla\cdot\mathbf{F}) + \mathbf{F}\cdot\nabla f.\end{aligned}$$

Example 4.20. Show $\nabla^2(1/r) = 0$ for $\mathbf{r}\neq 0$.

sol. As before $\nabla(1/r) = -\mathbf{r}/r^3$. In general, $\nabla(r^n) = nr^{n-2}\mathbf{r}$. By (7),

$$\begin{aligned}-\nabla^2(1/r) &= \nabla\cdot(\mathbf{r}/r^3) = \frac{1}{r^3}\nabla\cdot\mathbf{r} + \mathbf{r}\cdot\nabla(\frac{1}{r^3}) \\ &= \frac{3}{r^3} + \mathbf{r}\cdot(\frac{-3\mathbf{r}}{r^5}) = 0.\end{aligned}$$