Vector Calculus

2008년 9월 4일

제 2 장

Differentiation

제 1 절 Geometry of real valued functions

Functions and Mappings

In this section, we study functions whose domain is $\mathbb{R}^n$ or its subset with values in $\mathbb{R}^m$.

Definition 1.1. If the domain of f is $\mathbb{R}^n$ or its subset and the range is $\mathbb{R}$, then f is called **n-variable scalar-valued function**. In particular, if $n \geq 2$, it is called **functions of several variables**. If the domain is $A \subset \mathbb{R}^n$ then we write $f \colon A \subset \mathbb{R}^n \to \mathbb{R}$. Sometimes $\mathbf{x} \mapsto f(\mathbf{x})$ is used. If the range is $\mathbb{R}^m$, $m \geq 2$ it is called **vector-valued function**. Use $f \colon \mathbb{R}^n \to \mathbb{R}^m$, $\mathbf{x} \in \mathbb{R}^n$. We denote $f(\mathbf{x}) = (f_1, \ldots, f_m) \in \mathbb{R}^m$, where $f_i \colon \mathbb{R}^n \to \mathbb{R}$ functions of n variables. In other words, $f(\mathbf{x}) = (f_1(\mathbf{x}), \ldots, f_n(\mathbf{x}))$ each f_i is called **i-th component** (**i-th component function**) of f.

Definition 1.2. The **graph** of a functions of several variables $f \colon A \subset \mathbb{R}^n \to \mathbb{R}$ is (**graph**) the following set.

$$graph(f) = \{(\mathbf{x}, f(\mathbf{x})) \in \mathbb{R}^{n+1} \mid \mathbf{x} \in A \subset \mathbb{R}^n\}$$

Componentwise,

$$graph(f) = \{(x_1, \cdots, x_n, , f(x_1, \cdots, x_n)) \in \mathbb{R}^{n+1} \mid \mathbf{x} \in A \subset \mathbb{R}^n\}$$

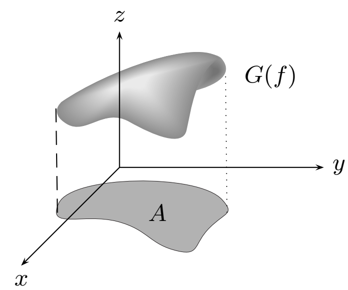


그림 2.1: Graph of two variable function

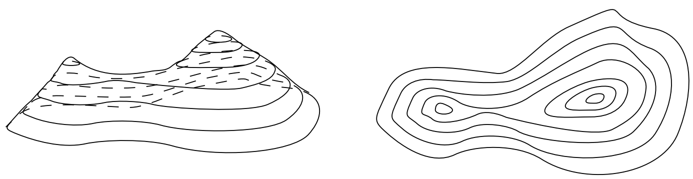


그림 2.2: Level set

Level sets, curves, surfaces

Definition 1.3. The **level set** of $f\colon \mathbb{R}^n \to \mathbb{R}$ is the set of all $\mathbf{x}$ where the function f has constant value:

$$S_c = \{\mathbf{x} \in \mathbb{R}^n \mid f(\mathbf{x}) = c, c \in \mathbb{R}\}$$

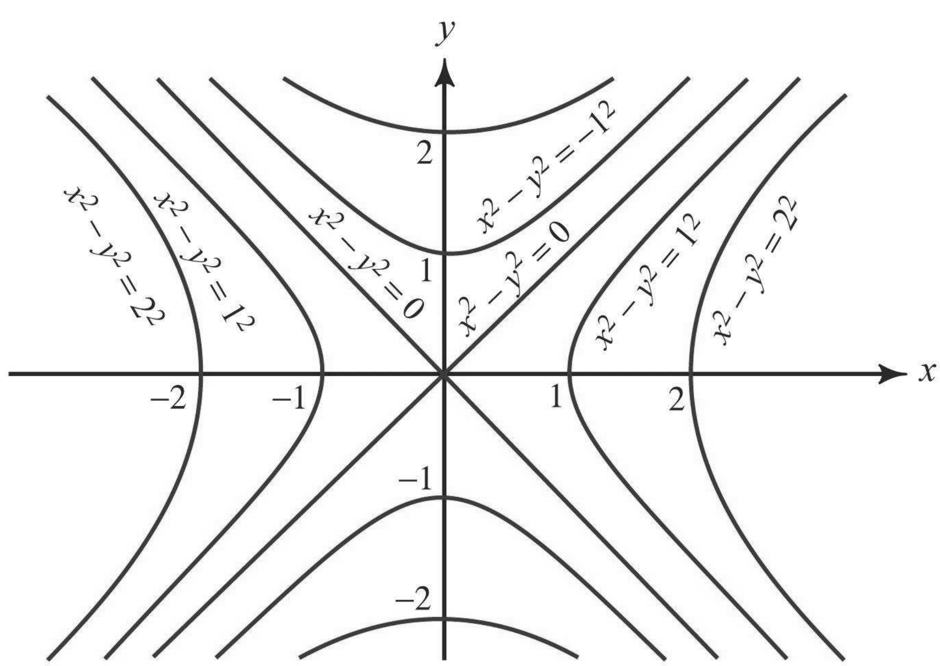
If $n = 2$, it is **level curve** and if $n = 3$, **level surface**.

Example 1.4. The graph of $f(x, y) = x^2 + y^2$ is called **paraboloid** or **paraboloid of revolution**. Draw the level sets.

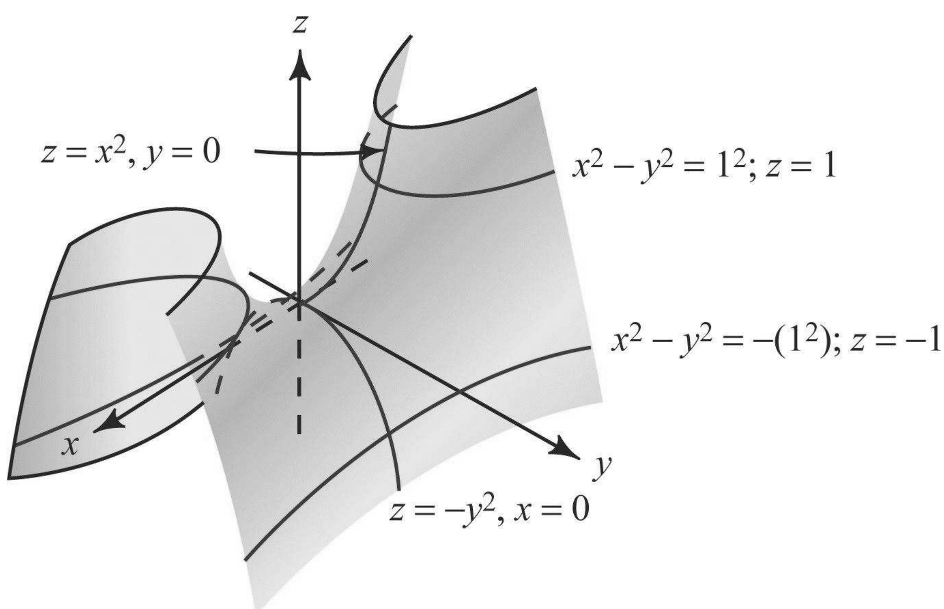
sol. The level set of $x^2 + y^2 = c$ is 0 if $c = 0$. For $c > 0$ it is a circle of radius $\sqrt{c}$. If $c < 0$, the level set is empty.

Example 1.5. Draw level sets of $f(x, y) = x^2 - y^2$. The graph is called **hyperbolic paraboloid** or **saddle**.

sol. The level sets of $f(x, y) = x^2 - y^2 = c$:
If $c = 0$, then it is $y = \pm x$, two lines through origin. If $c > 0$, the level set is a hyperbola meeting with x-axis, and if $c < 0$ level set is a hyperbola meeting with y-axis. The intersection with xz-plane is the parabola $z = x^2$, and the intersection with yz-plane is the parabola $z = -y^2$. Hence the graph of f is as in Figure **??**.



Level surface of function of three variables



Example 1.6. Study the level surface of $f(x, y, z) = x^2 + y^2 + z^2$.

sol. The set $x^2 + y^2 + z^2 = c$ becomes

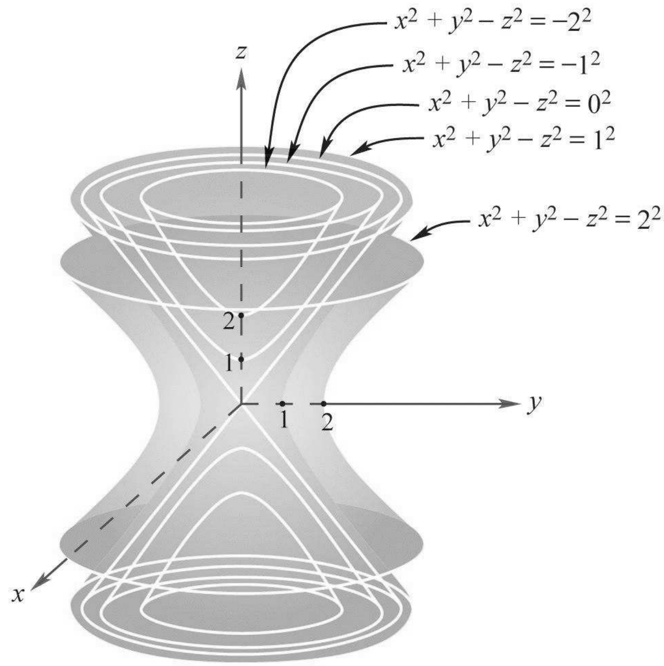
$$\begin{cases} \text{origin} & \text{if } c = 0 \\ \text{circle of radius } \sqrt{c} & \text{if } c > 0 \\ \text{empty if} & \text{if } c < 0 \end{cases}$$

To imagine the graph in $\mathbb{R}^4$, consider intersection with $\mathbb{R}^3_{z=0} = \{(x, y, z, w) \mid z = 0\}$. It is

$$\{(x, y, z, w) \mid w = x^2 + y^2, z = 0\}$$

Hence it is figure **??**.

Example 1.7. Describe the graph of $f(x, y, z) = x^2 + y^2 - z^2$.



sol. The graph of $f = x^2 + y^2 - z^2$ is a subset of 4-dim space. If we denote point in this space by (x, y, z, t), then graph is given by

$$\{(x, y, z, t)|t = x^2 + y^2 - z^2\}$$

The level surface is

$$L_c = \{(x, y, z)|x^2 + y^2 - z^2 = c\}$$

For $c = 0$, it is a cone $z = \pm\sqrt{x^2 + y^2}$. If $c = -a^2$we obtain $z = \pm\sqrt{x^2 + y^2 + a^2}$. this is a **hyperboloid of two sheets**. If $c = a^2 > 0$ we obtain $z = \pm\sqrt{x^2 + y^2 - a^2}$. This is **hyperboloid of single sheet**. On the other hand, if we consider intersection with $y = 0$; $S_{y=0} = \{(x, y, z, t) \mid y = 0\}$, the intersection with graph of f is

$$S_{y=0} \cap \text{ graph of } f = \{(x, y, z, t) \mid y = 0, t = x^2 - z^2\}$$

This is the set

$$\{(x, y, z, t) \mid t = x^2 - y^2, y = 0\}$$

This is considered to belong to (x, y, t) space and is a hyperbolic paraboloid(saddle).

□

제 2 절 Limits and Continuity

Limits using Open Sets

Definition 2.1 (Open sets)**.** Let $\mathbf{x}_0 \in \mathbb{R}^n$. The **open disk or ball** of radius r and center $\mathbf{x}_0$ is the set of all points $\mathbf{x}$ such that $\|\mathbf{x} - \mathbf{x}_0\| < r$. This is denoted by $D_r(\mathbf{x}_0)$ or $D(\mathbf{x}_0; r)$.

A set $U \subset \mathbb{R}^n$ is said to be **open** if for every point $\mathbf{x}_0 \in U$ there exists some $r > 0$ such that $D_r(\mathbf{x}_0)$ is contained in U(in symbol, $D_r(\mathbf{x}_0) \subset U$).

Theorem 2.2. $D_r(\mathbf{x}_0)$ *itself is open.*

Example 2.3. Half plane is open.

By a neighborhood of a point $\mathbf{x}_0$, we merely mean a set $D_r(\mathbf{x}_0)$ for any $r > 0$.

Definition 2.4 (Boundary)**.** Let $A \subset \mathbb{R}^n$. A point $\mathbf{x} \in \mathbb{R}^n$ is called a **boundary point** if every neighborhood of $\mathbf{x}_0$ contains at least a point in A and at least a point not in A.

A set $U \subset \mathbb{R}^n$ is said to be **open** if for every point $\mathbf{x}_0 \in U$ there exists some $r > 0$ such that $D_r(\mathbf{x}_0)$ is contained in U(in symbol, $D_r(\mathbf{x}_0) \subset U$).

Let us define the limit using open sets.

Definition 2.5 (Limit)**.** Let $f\colon A \subset \mathbb{R}^n \to \mathbb{R}^m$, where $A \subset \mathbb{R}^n$ and let $\mathbf{x}_0 \in A$ or boundary of A. Let N be an open neighborhood of $\mathbf{b} \in \mathbb{R}^m$. We say f is **eventually in** N **as x approaches** $\mathbf{x}_0$ if there is a neighborhood N of $\mathbf{x}_0$ such that for all point $\mathbf{x}$ of $U \cap A$, $\mathbf{x} \neq \mathbf{x}_0$, we have $f(\mathbf{x}) \in N$. in this case, we say f approaches $\mathbf{b}$ as $\mathbf{x}$ approaches $\mathbf{x}_0$, or in symbols we write

$$\lim_{\mathbf{x} \to \mathbf{x}_0} f(\mathbf{x}) = \mathbf{b}.$$

Example 2.6. Find $\lim_{x \to 1} f(x)$ where

$$g(x) = \frac{x - 1}{\sqrt{x} - 1}$$

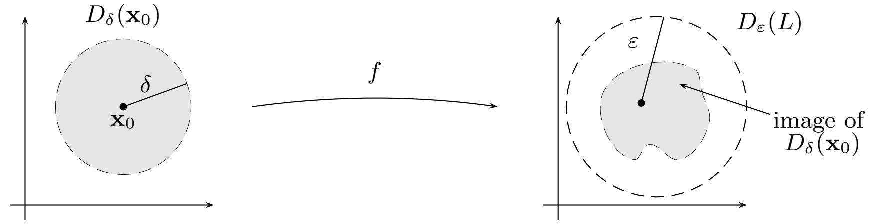


그림 2.3: Limit using neighborhood

sol.

Properties of Limits

Theorem 2.7. *Let $f, g : A \subset \mathbb{R}^n \to \mathbb{R}^m$ and let $\mathbf{x}_0$ be a point of A or boundary. $\mathbf{b} \in \mathbb{R}^m$ and c a scalar.*

Then the following hold:

(1) *If $\lim_{\mathbf{x}\to\mathbf{x}_0}(f)(\mathbf{x}) = \mathbf{b}$ then $\lim_{\mathbf{x}\to c\mathbf{x}_0}(f)(\mathbf{x}) = c\mathbf{b}$.*

(2) *If $\lim_{\mathbf{x}\to\mathbf{x}_0}(f)(\mathbf{x}) = \mathbf{b}_1$ and $\lim_{\mathbf{x}\to\mathbf{x}_0}(g)(\mathbf{x}) = \mathbf{b}_2$ then $\lim_{\mathbf{x}\to\mathbf{x}_0}(f \pm g)(\mathbf{x}) = L_1 \pm L_2$.*

(3) *If $m = 1$, then $\lim_{\mathbf{x}\to\mathbf{x}_0}(fg)(\mathbf{x}) = b_1 b_2$.*

(4) *Same with division.*

(5) *If $\mathbf{f}(\mathbf{x}) = (f_1(\mathbf{x}), \cdots, f_m(\mathbf{x}))$ where $f_i : A \to \mathbb{R}^m$, are components of $\mathbf{f}$, then $\lim_{\mathbf{x}\to\mathbf{x}_0}\mathbf{f}(\mathbf{x}) = \mathbf{b} = (b_1, \cdots, b_m)$, where $b_i = f(\mathbf{x}_i)$ for $i = 1, \cdots, m$.*

Using this theorem, we can find limits of polynomials or rational functions.

Example 2.8. Find the limit.

(1) $\lim_{(x,y)\to(0,0)} x^2 + xy^3 - x^2y + 2$

(2) $\lim_{(x,y)\to(0,0)} \dfrac{x^2 + xy^3 - x^2y + 2}{xy + 3}$

sol.

$$\lim_{(x,y)\to(0,0)} x^2 + xy^3 - x^2y + 2 = 0 + 0 + 0 + 2 = 2,$$
$$\lim_{(x,y)\to(0,0)} \frac{x^2 + xy^3 - x^2y + 2}{xy + 3} = \frac{2}{3}$$

Continuity

Definition 2.9. $f\colon A \subset \mathbb{R}^n \to \mathbb{R}^m$ **continuous** at $\mathbf{x}_0 \in A$ if

$$\lim_{\mathbf{x}\to\mathbf{x}_0} f(\mathbf{x}) = f(\mathbf{x}_0)$$

f is continuous on A if it is so at all points of A.

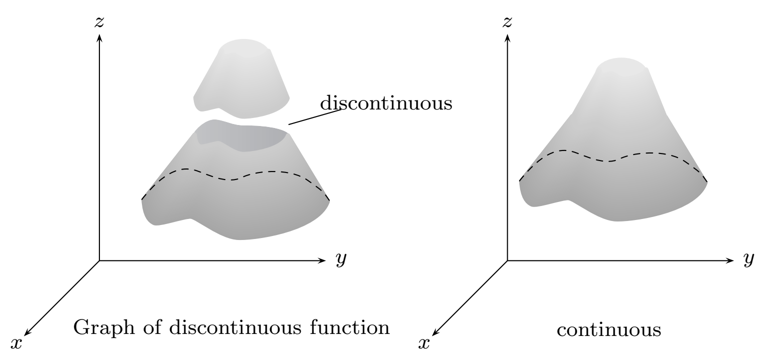


그림 2.4: continuous, discontinuous function

Example 2.10. Show that the following function is continuous at $(0,0)$.

$$f(x,y) = \begin{cases} \dfrac{xy}{\sqrt{x^2+y^2}} & (x,y) \neq (0,0)\ , \\ 0 & (x,y) = (0,0) \end{cases}$$

sol. We have seen in example 2.16 the limit of this function at $(0,0)$ is 0, and this eqauls $f(0,0)$. Hence f is continuous there.

□

The following results hold.

Theorem 2.11. *Suppose two functions $f, g : A \subset \mathbb{R}^n \to \mathbb{R}^m$ are two functions continuous at $\mathbf{x}_0 \in A$. then the followings are also continuous at $\mathbf{x}_0 \in A$.*

(1) $f \pm g$

(2) For any real k, the function kf

(3) When $m = 1$, the product fg

(4) When $m = 1$, $g(\mathbf{x}_0) \neq 0$, then f/g

(5) $\mathbf{f}$ is continuous iff each f_i is continuous, for $i = 1, \cdots, m$.

Theorem 2.12 (Composit function)**.** *Suppose $g \colon A \subset \mathbb{R}^n \to \mathbb{R}^m$ and $f \colon B \subset \mathbb{R}^m \to \mathbb{R}^p$ are given Suppose $g(A) \subset B$ so that $f \circ g$ is defined. If g is continuous at $\mathbf{x}_0 \in A$ and $f(\mathbf{x}_0) \in B$에 f is continuous at $\mathbf{y}_0 = g(\mathbf{x}_0)$ then $f \circ g$ is continuous at $\mathbf{x}_0$.*

Example 2.13. Show $f(x, y) = \cos^2((y + x^3)/(1 + x^2))$ is continuous

Limit using ε-δ

Using ε-δ.

Definition 2.14 (Limit suing ε-δ)**.** Let $\mathbf{f} \colon \mathbf{A} \subset \mathbb{R}^\mathbf{n} \to \mathbb{R}^\mathbf{m}$. We say **limit** of f at $\mathbf{x}_0 \in \mathbb{R}^n$ is $\mathbf{b}$, if for any $\varepsilon > 0$ there exists some positive δ such that for all $\mathbf{x} \in A$ satisfying $0 < \|\mathbf{x} - \mathbf{x}_0\| < \delta$ we have $\|\mathbf{f}(\mathbf{x}) - \mathbf{b}\| < \varepsilon$.

Example 2.15. The function is defined on all points except $(0, 0)$.

$$f = \frac{\sin(x^2 + y^2)}{x^2 + y^2}$$

Find the limit as $\mathbf{x} \to (0, 0)$.

sol. We know in one variable calculus that

$$\lim_{x\to 0}\frac{\sin x}{x}=1$$

So we guess

$$\lim_{(x,y)\to(0,0)}\frac{\sin(x^2+y^2)}{x^2+y^2}=\lim_{(x,y)\to(0,0)}\frac{\sin\|(x,y)\|^2}{\|(x,y)\|^2}=1$$

Since $\lim_{x\to 0}\sin x/x=1$ for any $\varepsilon>0$ there exists $\delta>0$ such that $|x|<\delta \Longrightarrow |(\sin x)/x-1|<\varepsilon$. Here we can assume $0<\delta<1$. Write $\mathbf{v}=(x,y)$. Then if $\|\mathbf{v}\|<\delta$ holds then

$$|f(x,y)-1|=\Big|\frac{\sin(\|\mathbf{v}\|^2)}{\|\mathbf{v}\|^2}\Big|<\varepsilon$$

Hence $\lim_{(x,y)\to(0,0)}f(x,y)=1$.

Example 2.16. Show

$$\lim_{(x,y)\to(0,0)}\frac{xy}{\sqrt{x^2+y^2}}=0$$

sol. Watch

$$0\le\frac{xy}{\sqrt{x^2+y^2}}\le\frac{x^2+y^2}{\sqrt{x^2+y^2}}=\sqrt{x^2+y^2}.$$

For any ε choose $\delta=\varepsilon$. Then for $\|(x,y)\|<\delta$ we have

$$\Big|\frac{xy}{\sqrt{x^2+y^2}}-0\Big|=\frac{xy}{\sqrt{x^2+y^2}}\le\sqrt{x^2+y^2}=\|(x,y)\|<\delta=\varepsilon$$

Thus limit is 0.

Example 2.17. How about

$$\lim_{(x,y)\to(0,0)}\frac{x^2}{x^2+y^2}$$

and

$$\lim_{(x,y)\to(0,0)}\frac{2x^2y}{x^2+y^2}=0$$

sol. (a) Set $y = 0$ and let $x \to 0$. Next set $x = 0$ let $y \to 0$. The limit is different!
(b) Note that

$$0 \leq \frac{2x^2y}{x^2+y^2} \leq \frac{2x^2y}{x^2} = 2|y|.$$

For any ε choose $\delta = \varepsilon/2$. Then for $\|(x,y)\| < \delta$ we have

$$\left|\frac{2x^2y}{x^2+y^2} - 0\right| < 2\delta = \varepsilon$$

Thus limit is 0.

제 3 절 Differentiation

Partial derivatives

Definition 3.1. Let $f\colon U \subset \mathbb{R}^n \to \mathbb{R}$ be a real valued function. Then the **partial derivative** with respect to i-th variable x_i denoted by $\partial f/\partial x_i$ is:

$$\begin{aligned}\frac{\partial f}{\partial x_i}(x_1,\dots,x_n) &= \lim_{h\to 0}\frac{f(x_1,x_2,\dots,x_i+h,\dots,x_n) - f(x_1,\dots,x_n)}{h}\\ &= \lim_{h\to 0}\frac{f(\mathbf{x}+h\mathbf{e}_i) - f(\mathbf{x})}{h}\end{aligned}$$

partial derivatives $\partial f/\partial x_i$ at $\mathbf{x}_0 \in \mathbb{R}^n$ is called **partial derivative**.

$$\frac{\partial f}{\partial x_i}(\mathbf{x}_0), \quad \text{or} \quad \frac{\partial f}{\partial x_i}\Big|_{\mathbf{x}_0}$$

For vector valued function $f\colon U \subset \mathbb{R}^n \to \mathbb{R}^m$, the partial derivative is the partial derivative of each component function f_j where $f = (f_1,\dots,f_m)$.

Example 3.2. Find partial derivatives of $f(x,y) = (x^2, x+y^2)$.

sol. $\partial f_1/\partial x = 2x$, $\partial f_1/\partial y = 0$, $\partial f_2/\partial x = 1$, $\partial f_2/\partial y = 2y$.

Example 3.3. Find partial derivatives of $f(x,y) = xy/\sqrt{x^2+y^2}$ at $(1,1)$.

sol. First

$$\frac{\partial f}{\partial x}(1,1) = \frac{y\sqrt{x^2+y^2} - xy(x/\sqrt{x^2+y^2})}{x^2+y^2} = \frac{y(x^2+y^2) - x^2y}{(x^2+y^2)^{3/2}} = 2^{3/2}$$

Linear approximation

Motive: Find equation of tangent plane to $z = f(x,y)$ at (x_0, y_0). Suppose the surface has a tangent plane

$$z = ax + by + c$$

at (x_0, y_0). The slope along x-direction is $\partial f/\partial x(x_0, y_0)$ and the slope along y-direction is $\partial f/\partial y(x_0, y_0)$. Hence

$$a = \frac{\partial f}{\partial x}(x_0, y_0), \quad b = \frac{\partial f}{\partial y}(x_0, y_0).$$

Also since the point $(x_0, y_0, f(x_0, y_0))$ lies in the plane, we see

$$z = f(x_0, y_0) + \frac{\partial f}{\partial x}(x_0, y_0)(x - x_0) + \frac{\partial f}{\partial y}(x_0, y_0)(y - y_0)$$

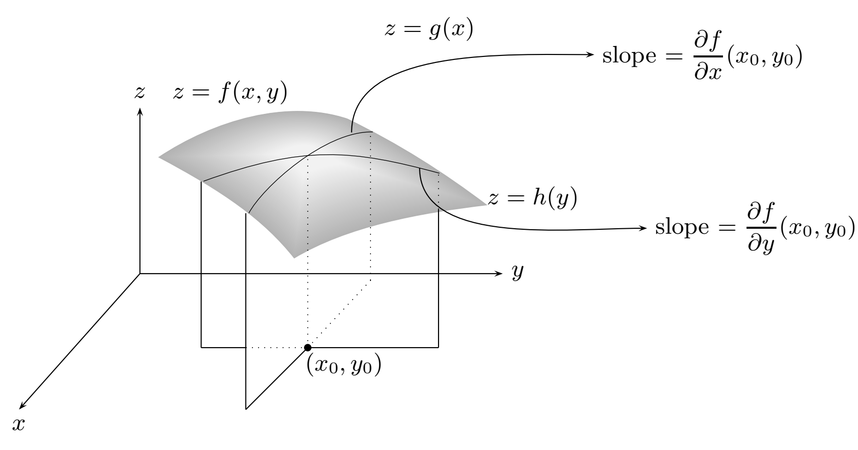


그림 2.5: Geometric meaning of partial derivative

Example 3.4. Find partial derivative of $f(x,y) = x^{1/3}y^{1/3}$ by definition,

$$\frac{\partial f}{\partial x}(0,0) = \lim_{h\to 0}\frac{f(h,0)-f(0,0)}{h} = \lim_{h\to 0}\frac{0-0}{h} = 0$$

Similarly $(\partial f/\partial y)(0,0) = 0$. But this is not differentiable. Graph of f is in Fig **??**.

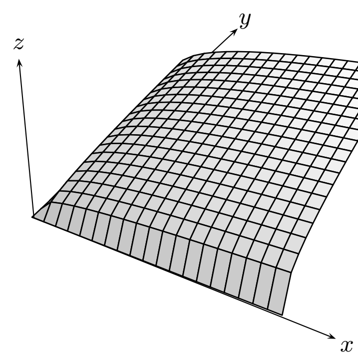


그림 2.6: Graph of $f(x,y) = x^{1/3}y^{1/3}$

Differentiation of function of several variable

For one variable differentiable function f, $f(x)$ can be approximated near x_0 by the value of tangent line at x: $f(x_0)+f'(x_0)(x-x_0)$ this is called **tangent line approximation** or **linear approximation** of $f(x)$. It satisfies (figure 2.7)

$$\lim_{x\to x_0}\frac{f(x)-f(x_0)-f'(x_0)(x-x_0)}{x-x_0} = 0 \tag{2.1}$$

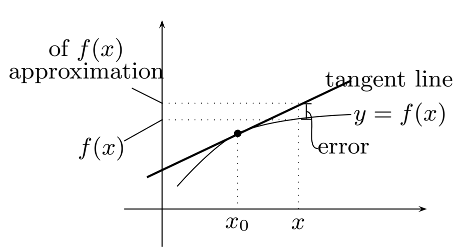


그림 2.7: tangent approximation of a function of one variable

For two variable function $f\colon \mathbb{R}^2 \to \mathbb{R}$ which has partial derivative at (x_0, y_0) and has tangent plane (x_0, y_0) then tangent plane is given by

$$z = f(x_0, y_0) + \frac{\partial f}{\partial x}(x_0, y_0)(x - x_0) + \frac{\partial f}{\partial y}(x_0, y_0)(y - y_0)$$

Definition 3.5. We say $f\colon \mathbb{R}^2 \to \mathbb{R}$ **differentiable** at (x_0, y_0) if $\partial f/\partial x$ and $\partial f/\partial y$ exists and for $(x, y) \to (x_0,\, y_0)$

$$\frac{f(x,y) - f(x_0,y_0) - \frac{\partial f}{\partial x}(x_0,y_0)(x-x_0) - \frac{\partial f}{\partial y}(x_0,y_0)(y-y_0)}{\|(x,y) - (x_0,y_0)\|} \to 0$$

Definition 3.6. Suppose $f\colon \mathbb{R}^2 \to \mathbb{R}$ is differentiable at (x_0, y_0). The **tangent plane** at (x_0, y_0) is given by

$$z = f(x_0, y_0) + \frac{\partial f}{\partial x}(x_0, y_0)(x - x_0) + \frac{\partial f}{\partial y}(x_0, y_0)(y - y_0)$$

Example 3.7. Find the tangent plane of $f(x, y) = x^2 + y^2$ at $(0, 0)$.

sol. We see $(\partial f/\partial x)(0, 0) = (\partial f/\partial y)(0, 0) = 0$.

$$\begin{aligned}
&\lim_{(x,y)\to(0,0)} \frac{f(x,y) - f(0,0) - \frac{\partial f}{\partial x}(0,0)(x) - \frac{\partial f}{\partial y}(0,0)(y)}{\|(x,y) - (0,0)\|} \\
&= \lim_{(x,y)\to(0,0)} \frac{f(x,y)}{\|(x,y)\|} = \lim_{(x,y)\to(0,0)} \sqrt{x^2+y^2} \\
&= 0.
\end{aligned}$$

Hence the equation of tangent plane is

$$f(x_0, y_0) + \frac{\partial f}{\partial x}(x_0, y_0)(x - x_0) + \frac{\partial f}{\partial y}(x_0, y_0)(y - y_0) = 0$$

Differentiability of vector valued function

Let $\mathbf{f}\colon \mathbb{R}^n \to \mathbb{R}^m$. If every component of $\mathbf{f}$ is differentiable, we say $\mathbf{f}$ is differentiable. We can express the concept of differentiability of a vector function in vector notation. As an example, let $f\colon \mathbb{R}^2 \to \mathbb{R}$ and define the 1×2 matrix

$\mathbf{D}f(x_0, y_0)$ by

$$\mathbf{D}f(x_0, y_0) = \begin{bmatrix} \frac{\partial f}{\partial x}(x_0, y_0) & \frac{\partial f}{\partial y}(x_0, y_0) \end{bmatrix}$$

Then f differentiable at (x_0, y_0) if

$$\lim_{(x,y)\to(x_0,y_0)} \frac{f(x,y) - f(x_0,y_0) - \mathbf{D}f(x_0,y_0)\begin{bmatrix} x - x_0 \\ y - y_0 \end{bmatrix}}{\|(x,y) - (x_0,y_0)\|} = 0$$

Here $\mathbf{D}f(x_0, y_0)$ is called the **derivative of** f. In general we define as follows:

Definition 3.8. $\mathbf{f} = (f_1, \ldots, f_m) \colon \mathbb{R}^n \to \mathbb{R}^m$ is said to be **differentiable** at $\mathbf{x}_0$ if partial derivatives of $\mathbf{f}$ exists at $\mathbf{x}_0$ and

$$\lim_{\mathbf{x}\to\mathbf{x}_0} \frac{\|\mathbf{f}(\mathbf{x}) - \mathbf{f}(\mathbf{x}_0) - \mathbf{Df}(\mathbf{x}_0)(\mathbf{x} - \mathbf{x}_0)\|}{\|\mathbf{x} - \mathbf{x}_0\|} = 0$$

holds. Here $\mathbf{Df}(\mathbf{x}_0)$ is $m \times n$ matrix.

If $m = 1$, then

$$\begin{bmatrix} \frac{\partial f}{\partial x_1} & \cdots & \frac{\partial f}{\partial x_n} \end{bmatrix}$$

Also called a **gradient** of f and denoted by ∇f.

If we let $\mathbf{h} = \mathbf{x} - \mathbf{x}_0$, then real valued function f is differentiable at a point $\mathbf{x}_0$ if

$$\lim_{\mathbf{x}\to\mathbf{x}_0} \frac{1}{\|\mathbf{h}\|} \left| \|f(\mathbf{x}_0 + \mathbf{h}) - f(\mathbf{x}_0) - \sum_{j=1}^{n} \frac{\partial f}{\partial x_j}(\mathbf{x}_0) h_j \right| = 0$$

In general, the derivative has the following form:

$$D\mathbf{f}(\mathbf{x}_0) = \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \cdots & \frac{\partial f_1}{\partial x_n} \\ \vdots & & \vdots \\ \frac{\partial f_m}{\partial x_1} & \cdots & \frac{\partial f_m}{\partial x_n} \end{bmatrix}$$

$\mathbf{Df}(\mathbf{x}_0)(\mathbf{x} - \mathbf{x}_0)$ means the product of $m \times n$ matrix $\mathbf{D}f(\mathbf{x}_0)$ and the $n \times 1$ vector $\mathbf{x} - \mathbf{x}_0$. $\mathbf{Df}(\mathbf{x}_0)(\mathbf{x} - \mathbf{x}_0)$ is called the **derivative** of f at $\mathbf{x}_0$. Sometimes it is called the **Jacobian matrix**.

Example 3.9. Find the derivative of $\mathbf{Df}(x, y)$.

(1) $\mathbf{f}(x, y) = (xy, x + y)$

(2) $\mathbf{f}(x,y) = (e^{xy}, x^2+y^2, xe^y)$

sol. (1) $f_1 = xy,\ f_2 = x+y$. Hence

$$\mathbf{Df}(\mathbf{x}) = \begin{bmatrix} y & x \\ 1 & 1 \end{bmatrix}$$

(2) $f_1 = e^{x+y},\ f_2 = x^2+y^2,\ f_3 = xe^y$. Hence

$$\mathbf{Df}(\mathbf{x}) = \begin{bmatrix} e^{x+y} & e^{x+y} \\ 2x & 2y \\ e^y & xe^y \end{bmatrix}$$

Example 3.10. Show $f(x,y) = (xy, x+y)$ is differentiable at $(0,0)$.

sol. From example 3.9

$$\mathbf{Df}(0,0) = \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix}$$

$$\begin{aligned} &\lim_{(x,y)\to(0,0)} \frac{\left\| \mathbf{f}(x,y) - \mathbf{f}(0,0) - \mathbf{Df}(0,0)\begin{bmatrix} x \\ y \end{bmatrix} \right\|}{\|(x,y)-(0,0)\|} \\ &= \lim_{(x,y)\to(0,0)} \frac{\|(xy, x+y) - (0, x+y)\|}{\|(x,y)\|} \\ &= \lim_{(x,y)\to(0,0)} \frac{|xy|}{\sqrt{x^2+y^2}} = 0 \end{aligned}$$

Relation with continuity

Theorem 3.11. *If* $\mathbf{f} = (f_1, \ldots, f_n)\colon \mathbb{R}^n \to \mathbb{R}^m$ *has all partial derivatives* $\partial f_i/\partial x_j$ *all exist and continuous in a neighborhood of* $\mathbf{x}$ *then* $\mathbf{f}$ *is* $\mathbf{x}$ *differentiable.*

Example 3.12. $\mathbf{f}(x,y) = (e^{xy}, x^2+y^2, xe^y)$ is differentiable at all points of $\mathbb{R}^2$.

sol. Since all the partial derivatives are continuous on $\mathbb{R}^2$ f is differentiable by Theorem 3.11

Example 3.13. Given

$$f(x,y) = \begin{cases} \dfrac{xy}{\sqrt{x^2+y^2}} & (x,y) \neq (0,0) \\ 0 & (x,y) = (0,0) \end{cases}$$

(1) partial derivatives at $(0,0)$ exist.

(2) partial derivatives at $(0,0)$ are not continuous

(3) f is not differentiable at $(0,0)$

sol. (1) From definition

$$\frac{\partial f}{\partial x}(0,0) = \lim_{(x,y)\to(0,0)} \frac{f(x,0)-f(0,0)}{x} = 0$$

$$\frac{\partial f}{\partial y}(0,0) = \lim_{(x,y)\to(0,0)} \frac{f(0,y)-f(0,0)}{y} = 0$$

For $(x,y) \neq (0,0)$

$$\begin{aligned} \frac{\partial f}{\partial x} &= \frac{y\sqrt{x^2+y^2} - 2x(xy)/2\sqrt{x^2+y^2}}{x^2+y^2} \\ &= \frac{y}{\sqrt{x^2+y^2}} - \frac{x^2 y}{(x^2+y^2)^{3/2}} \end{aligned}$$

this does not have limit at $(0,0)$.
(2) Suppose f is differentiable at $(0,0)$. Then

$$\mathbf{D}f = \begin{bmatrix} \dfrac{\partial f}{\partial x} & \dfrac{\partial f}{\partial y} \end{bmatrix}\Big|_{(0,0)} = \begin{bmatrix} 0 & 0 \end{bmatrix}$$

hence we must have

$$\lim_{(x,y)\to(0,0)} \frac{f(x,y)-f(0,0)}{\|(x,y)\|} = \lim_{(x,y)\to(0,0)} \frac{xy}{x^2+y^2}$$

But $\lim_{(x,y)\to(0,0)} xy/(x^2+y^2)$ does not exists. Contradiction.

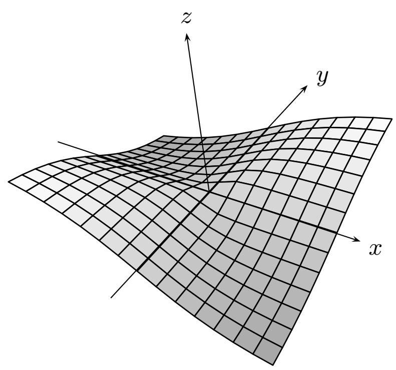


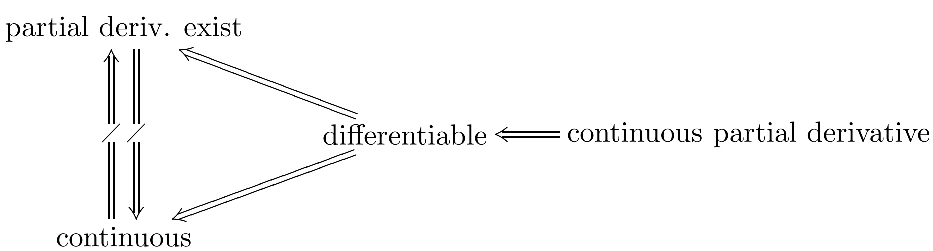
그림 2.8: Graph of example 3.13

Theorem 3.14. *If* $\mathbf{f} = (f_1, \ldots, f_n) \colon \mathbb{R}^n \to \mathbb{R}^m$ *is differentiable at* $\mathbf{x}_0$ *then* f는 $\mathbf{x}_0$ *is continuous.*

Converse is not true.

Example 3.15. Suppose $f \colon \mathbb{R}^2 \to \mathbb{R}$ is given as follows.

$$f(x, y) = \begin{cases} 1 & x = 0 \text{ or } y = 0 \\ 0 & \text{otherwise} \end{cases}$$



제 4 절 Paths and Curves

Use parameter to express a curve $\mathbf{c}(t)$. Eq. of line

$$\mathbf{c}(t) = \mathbf{x}_0 + t\mathbf{v}$$

Eq. of Circle $C : x^2 + y^2 = 1$ $\mathbf{c}(r) : \mathbb{R} \to \mathbb{R}^2$ is given by

$$\mathbf{c}(t) = (\cos t, \sin t), 0 \le t \le 2\pi$$

Cycloid:

$$\mathbf{c}(t) = (t - \sin t, 1 - \cos t)$$

Study more general cycloid described in the book.

Velocity and tangent to Paths

Definition 4.1. If $\mathbf{c}$ is a differentiable path, then the velocity is

$$\mathbf{c}'(t) = \lim_{h \to 0} \frac{\mathbf{c}(t+h) - \mathbf{c}(t)}{h}$$

If $\mathbf{c}(t) = (x(t), y(t))$ then $\mathbf{c}'(t) = (x'(t), y'(t))$. The speed of the path is $s = \|\mathbf{c}'(t)\|$.

Example 4.2. Find the velocity of the path $\mathbf{c}(t) = (t, t^2, e^t)$ at $t = 0$.

Example 4.3 (Helix)**.** $\mathbf{c}(t) = (\cos t, \sin t, t)$ at $t = \pi/2$.

Eq. of tangent line to $\mathbf{c}(t)$ when $\mathbf{c}'(t_0) \neq 0$:

$$\ell(t) = \mathbf{c}(t_0) + (t - t_0)\mathbf{c}'(t_0)$$

Example 4.4. Find the velocity of the path $\mathbf{c}(t) = (e^t, e^{-t}, \cos t)$ at $t = 1, 3$.

제 5 절   Chain rule

Some rules

Proposition 5.1 (Rules)**.** *Suppose* $f : \mathbb{R}^n \to \mathbb{R}$ *and* $g : \mathbb{R}^n \to \mathbb{R}$ *is differentiable at* $\mathbf{x}_0$*.*

*(1) [**constant multiple rule**] For all constant* c*,* cf *is differentiable at* $\mathbf{x}_0$*.*

$$\mathbf{D}(cf)(\mathbf{x}_0) = c\mathbf{D}f(\mathbf{x}_0)$$

*(2) [**sum rule**] Sum* $f + g$ *differentiable at* $\mathbf{x}_0$

$$\mathbf{D}(f + g)(\mathbf{x}_0) = \mathbf{D}f(\mathbf{x}_0) + \mathbf{D}g(\mathbf{x}_0)$$

*(3) **[product rule]** Product fg differentiable at $\mathbf{x}_0$.*

$$\mathbf{D}(fg)(\mathbf{x}_0) = g(\mathbf{x}_0)\mathbf{D}f(\mathbf{x}_0) + f(\mathbf{x}_0)\mathbf{D}g(\mathbf{x}_0)$$

*(4) **[quotient rule]** If $g(\mathbf{x}_0) \neq 0$, then f/g differentiable at $\mathbf{x}_0$.*

$$\mathbf{D}\left(\frac{f}{g}\right)(\mathbf{x}_0) = \frac{g(\mathbf{x}_0)\mathbf{D}f(\mathbf{x}_0) - f(\mathbf{x}_0)\mathbf{D}g(\mathbf{x}_0)}{(g(\mathbf{x}_0))^2}$$

Rule (1) and (2) also hold when f and g are vector functions $\mathbb{R}^n \to \mathbb{R}^m$.

Proof. (3) Suppose $\mathbf{x} \to \mathbf{x}_0$ and we need to show that

$$\frac{g(\mathbf{x})f(\mathbf{x}) - g(\mathbf{x}_0)f(\mathbf{x}_0) - [g(\mathbf{x})\mathbf{D}f(\mathbf{x}) + g(\mathbf{x})\mathbf{D}f(\mathbf{x})](\mathbf{x} - \mathbf{x}_0)}{\|\mathbf{x} - \mathbf{x}_0\|} \to 0$$

Numerator is

$$\begin{aligned} g(\mathbf{x})f(\mathbf{x}) - g(\mathbf{x})f(\mathbf{x}_0) + g(\mathbf{x})f(\mathbf{x}_0) - g(\mathbf{x}_0)f(\mathbf{x}_0) \\ - [g(\mathbf{x}_0)\mathbf{D}f(\mathbf{x}_0) + g(\mathbf{x}_0)\mathbf{D}f(\mathbf{x}_0)](\mathbf{x} - \mathbf{x}_0) \\ = [g(\mathbf{x})f(\mathbf{x}) - g(\mathbf{x})f(\mathbf{x}_0) - g(\mathbf{x}_0)\mathbf{D}f(\mathbf{x}_0)(\mathbf{x} - \mathbf{x}_0)] \\ + [g(\mathbf{x})f(\mathbf{x}_0) - g(\mathbf{x}_0)f(\mathbf{x}_0) - f(\mathbf{x}_0)\mathbf{D}g(\mathbf{x}_0)(\mathbf{x} - \mathbf{x}_0)] \end{aligned}$$

Let A be the terms in the first bracket and B be the terms in the second bracket. Then

$$\begin{aligned} A &= g(\mathbf{x})f(\mathbf{x}) - g(\mathbf{x})f(\mathbf{x}_0) - g(\mathbf{x}_0)\mathbf{D}f(\mathbf{x}_0)(\mathbf{x} - \mathbf{x}_0) \\ &= g(\mathbf{x})f(\mathbf{x}) - g(\mathbf{x})f(\mathbf{x}_0) - g(\mathbf{x})\mathbf{D}f(\mathbf{x}_0)(\mathbf{x} - \mathbf{x}_0) \\ &\qquad + g(\mathbf{x})\mathbf{D}f(\mathbf{x}_0)(\mathbf{x} - \mathbf{x}_0) - g(\mathbf{x}_0)\mathbf{D}f(\mathbf{x}_0)(\mathbf{x} - \mathbf{x}_0) \\ &= g(\mathbf{x})[f(\mathbf{x}) - f(\mathbf{x}_0) - \mathbf{D}f(\mathbf{x}_0)(\mathbf{x} - \mathbf{x}_0)] \\ &\qquad + [g(\mathbf{x}) - g(\mathbf{x}_0)]\mathbf{D}f(\mathbf{x}_0)(\mathbf{x} - \mathbf{x}_0). \end{aligned}$$

Similar expression for B. Now using the definition of derivative and continuity we can show

$$\lim_{\mathbf{x} \to \mathbf{x}_0} \frac{A}{\|\mathbf{x} - \mathbf{x}_0\|} = 0, \qquad \lim_{\mathbf{x} \to \mathbf{x}_0} \frac{B}{\|\mathbf{x} - \mathbf{x}_0\|} = 0$$

□

Chain rule

Theorem 5.2 (Chain rule-simple). *(1) Suppose* $\mathbf{c}(t) = (x(t), y(t)) \colon \mathbb{R} \to \mathbb{R}^2$ *differentiable at* t_0 *and* $f \colon \mathbb{R}^2 \to \mathbb{R}$ *differentiable at* $\mathbf{x}_0 = \mathbf{c}(t_0)$ *then the composite* $h = f \circ \mathbf{c} \colon \mathbb{R} \to \mathbb{R}$ *(* $h(t) = f(x(t), y(t))$*) is differentiable at* t_0 *and its derivative* $dh/dt(t_0)$ *is*

$$\frac{dh}{dt}(t_0) = \frac{\partial f}{\partial x}(\mathbf{x}_0)\frac{dx}{dt}(t_0) + \frac{\partial f}{\partial y}(\mathbf{x}_0)\frac{dy}{dt}(t_0)$$

(2) Suppose $\mathbf{g} \colon \mathbb{R}^2 \to \mathbb{R}^2$, $\mathbf{g}(x, y) = (u(x, y), v(x, y))$ *differentiable at* $\mathbf{x}_0$ *and* $f \colon \mathbb{R}^2 \to \mathbb{R}$ *differentiable at* $g(\mathbf{x}_0) = \mathbf{y}_0$*, then the composite function* $h = f \circ \mathbf{g}$ *differentiable at* $\mathbf{x}_0$ *and* $\mathbf{D}h(\mathbf{x}_0) = \Big[\partial h/\partial x(\mathbf{x}_0), \quad \partial h/\partial y(\mathbf{x}_0)\Big]$ *is given by*

$$\begin{aligned}\frac{\partial h}{\partial x}(\mathbf{x}_0) &= \frac{\partial f}{\partial u}(\mathbf{y}_0)\frac{\partial u}{\partial x}(\mathbf{x}_0) + \frac{\partial f}{\partial v}(\mathbf{y}_0)\frac{\partial v}{\partial x}(\mathbf{x}_0)\\ \frac{\partial h}{\partial y}(\mathbf{x}_0) &= \frac{\partial f}{\partial u}(\mathbf{y}_0)\frac{\partial u}{\partial x}(\mathbf{x}_0) + \frac{\partial f}{\partial v}(\mathbf{y}_0)\frac{\partial v}{\partial y}(\mathbf{x}_0)\end{aligned}$$

Wire it in matrix form, $\mathbf{D}h = \mathbf{D}f \circ \mathbf{D}g$*, where*

$$\mathbf{D}f = [\frac{\partial f}{\partial u}, \frac{\partial f}{\partial v}], \quad \textit{and } \mathbf{D}g = \begin{bmatrix} \frac{\partial u}{\partial x}, \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x}, \frac{\partial v}{\partial y} \end{bmatrix}$$

Proof. (1) From

$$\frac{dh}{dt}(t_0) = \lim_{t \to t_0} \frac{h(t) - h(t_0)}{t - t_0}$$

we have

$$\begin{aligned}\frac{h(t) - h(t_0)}{t - t_0} &= \frac{f(x(t), y(t)) - f(x(t_0), y(t_0))}{t - t_0}\\ &= \frac{f(x(t), y(t)) - f(x(t_0), y(t)) + f(x(t_0), y(t)) - f(x(t_0), y(t_0))}{t - t_0}\end{aligned}$$

Since partial derivatives of f differentiable, we have by mean value theorem there exists c between $x(t)$ and $x(t_0)$ such that

$$f(x(t), y(t)) - f(x(t_0), y(t)) = \Big(\frac{\partial f}{\partial x}(c, y(t))\Big)(x(t) - x(t_0))$$

holds. Similarly,

$$\frac{h(t)-h(t_0)}{t-t_0} = \Big(\frac{\partial f}{\partial x}(c, y(t))\Big)\frac{x(t)-x(t_0)}{t-t_0} + \Big(\frac{\partial f}{\partial y}(x(t_0), d)\Big)\frac{y(t)-y(t_0)}{t-t_0}.$$

Let t approach t_0.

(2) Treat y as constant and $\partial h/\partial x$ as function of x only.

$$\frac{\partial h}{\partial x}(\mathbf{x}_0) = \frac{\partial f}{\partial u}(\mathbf{y}_0)\frac{\partial u}{\partial x}(\mathbf{x}_0) + \frac{\partial f}{\partial v}(\mathbf{y}_0)\frac{\partial v}{\partial x}(\mathbf{x}_0)$$

Similarly

$$\frac{\partial h}{\partial y}(\mathbf{x}_0) = \frac{\partial f}{\partial u}(\mathbf{y}_0)\frac{\partial u}{\partial x}(\mathbf{x}_0) + \frac{\partial f}{\partial v}(\mathbf{y}_0)\frac{\partial v}{\partial y}(\mathbf{x}_0)$$

□

Example 5.3. Show Chain rule holds for $f(x, y) = e^{xy}$ and $\mathbf{g}(t) = (x(t), y(t))$, $x(t) = t^2$, $y(t) = 2t$.

sol. Since $h(t) = f \circ \mathbf{g}(t) = f(x(t), y(t)) = e^{2t^3}$, we have $dh/dt = 6t^2 e^{2t^3}$. On the other hand, by chain rule, we have

$$\frac{dh}{dt} = ye^{xy}\cdot 2t + xe^{xy}\cdot 2 = 6t^2 e^{2t^3}$$

Theorem 5.2 Chain rule. (1) becomes

$$\begin{aligned}\frac{dh}{dt} &= \begin{bmatrix}\frac{\partial f}{\partial x} & \frac{\partial f}{\partial y}\end{bmatrix}(\mathbf{g}(t_0))\begin{bmatrix}\frac{dx}{dt}\\ \frac{dy}{dt}\end{bmatrix}(t_0)\\ &= \mathbf{D}f(f(t_0))\cdot \mathbf{D}\mathbf{g}(t_0)\end{aligned}$$

is a product of matrix. For (2)

$$\mathbf{D}h(\mathbf{x}_0) = \begin{bmatrix}\frac{\partial f}{\partial u} & \frac{\partial f}{\partial v}\end{bmatrix}(\mathbf{y}_0)\begin{bmatrix}\frac{\partial u}{\partial x} & \frac{\partial u}{\partial y}\\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y}\end{bmatrix}(\mathbf{x}_0)$$

Example 5.4. Show Chain rule holds for $f(u, v, w) = u^2 + v^2 - w$, where

$$u(x, y, z) = x^2 y, \quad v = y^2, \quad z = e^{-xz}$$

sol. Let

$$h(x,y,z) = f(u(x,y,z), v(x,y,z), w(x,y,z))$$

Chain rule

$$\begin{aligned}\frac{\partial h}{\partial x} &= \frac{\partial f}{\partial u}\frac{\partial u}{\partial x} + \frac{\partial f}{\partial v}\frac{\partial v}{\partial x} + \frac{\partial f}{\partial w}\frac{\partial w}{\partial x} \\ &= 2u(2xy) + 2v \cdot 0 + (-1)(-ze^{-xz})\end{aligned}$$

Theorem 5.5 (Chain rule-General case)**.** *Suppose* $\mathbf{g}\colon \mathbb{R}^n \to \mathbb{R}^m$ *differentiable at* $\mathbf{x}_0$ *and* $\mathbf{f}\colon \mathbb{R}^m \to \mathbb{R}^p$ *differentiable at* $\mathbf{g}(\mathbf{x}_0) = \mathbf{y}_0$ *Then* $\mathbf{h} = \mathbf{f}\circ\mathbf{g}$ *differentiable at* $\mathbf{x}_0$ *and*

$$\mathbf{Dh}(\mathbf{x}_0) = \mathbf{Df}(\mathbf{y}_0)\mathbf{Dg}(\mathbf{x}_0)$$

Suppose $\mathbf{g}\colon \mathbb{R}^n \to \mathbb{R}^m$ and $\mathbf{f}\colon \mathbb{R}^m \to \mathbb{R}^p$ are given

$$\begin{aligned}\mathbf{g}(x_1,\cdots,x_n) &= (y_1(x_1,\cdots,x_n), y_2(x_1,\cdots,x_n),\cdots, y_m(x_1,\cdots,x_n)) \\ \mathbf{f}(u,v,w) &= (f_1(y_1,\cdots,y_m),\cdots, f_p(y_1,\cdots,y_m))\end{aligned}$$

Let the composite of $\mathbf{f}$ and $\mathbf{g}$ as h

$$\mathbf{h}(x_1,\cdots,x_n) = \mathbf{f}\circ\mathbf{g}(x_1,\cdots,x_n)$$

Then applying simple case to each component of $\mathbf{h} = [f_1\circ\mathbf{g},\cdots,f_p\circ\mathbf{g}]^T$ (Column vector) so that $h_i = f_i \circ \mathbf{g}$. and

$$\begin{aligned}Dh_1 &= Df_1 \circ D\mathbf{g} \\ Dh_2 &= Df_2 \circ D\mathbf{g} \\ &= \cdots \\ Dh_p &= Df_p \circ D\mathbf{g}\end{aligned}$$

Now just write in matrix form.

Example 5.6. Given the vector functions $\mathbf{f}$, $\mathbf{g}$ consider composite function $\mathbf{h} = (k,l) = \mathbf{f}\circ\mathbf{g}$. Find the partials $\partial k/\partial x$ and $\partial l/\partial y$.

$$\mathbf{g}(x,y,z) = (xyz, x^2+y^2+z^2, e^{xyz}), \quad \mathbf{f}(u,v,w) = (u^2 - uv, u+v+w)$$

sol. Use chain rule

$$\begin{aligned}\frac{\partial k}{\partial x} &= \frac{\partial k}{\partial u}\frac{\partial u}{\partial x} + \frac{\partial k}{\partial v}\frac{\partial v}{\partial x} + \frac{\partial k}{\partial w}\frac{\partial w}{\partial x} \\ &= (2u - v)(yz) + (-u)(2x) + 0 \\ &= (2xyz - x^2 - y^2 - z^2)(yz) - (xyz)(2x) \\ &= 2xyz - 3x^2yz - y^3 - yz^2, \\ \frac{\partial l}{\partial y} &= \frac{\partial l}{\partial u}\frac{\partial u}{\partial y} + \frac{\partial l}{\partial v}\frac{\partial v}{\partial y} + \frac{\partial l}{\partial w}\frac{\partial w}{\partial y} \\ &= 1 \cdot \frac{\partial u}{\partial y} + 1 \cdot \frac{\partial u}{\partial y} + 1 \cdot \frac{\partial w}{\partial y} \\ &= xz + 2y + xze^{xyz}\end{aligned}$$

Check it using matrix product.

Example 5.7. Use Chain rule to find the derivative of composite function

$$\mathbf{h}(t) = (h_1(t), h_2(t), h_2(t)) = \mathbf{f} \circ \mathbf{g}(t)$$

where $\mathbf{g}(t) = (x(t), y(t), z(t))$ and $\mathbf{f} \colon = (f_1(x, y, z), f_2(x, y, z), f_3(x, y, z))$.

Note that $h_i(t) = f_i(\mathbf{g}(t))$. Use Chain rule for special case(to each component)

$$\frac{dh_i}{dt} = \frac{\partial f_i}{\partial x}\frac{dx}{dt} + \frac{\partial f_i}{\partial y}\frac{dy}{dt} + \frac{\partial f_i}{\partial z}\frac{dz}{dt}$$

Use Chain rule as a whole

$$D\mathbf{f} = \begin{bmatrix} \frac{\partial f_1}{\partial x}, & \frac{\partial f_1}{\partial y}, & \frac{\partial f_1}{\partial z} \\ \frac{\partial f_2}{\partial x}, & \frac{\partial f_2}{\partial y}, & \frac{\partial f_2}{\partial z} \\ \frac{\partial f_3}{\partial x}, & \frac{\partial f_3}{\partial y}, & \frac{\partial f_3}{\partial z} \end{bmatrix} \text{ while } D\mathbf{g} = \begin{bmatrix} x'(t) \\ y'(t) \\ z'(t) \end{bmatrix}$$

$$\text{Hence } D\mathbf{f} \circ D\mathbf{g} = \begin{bmatrix} \frac{\partial f_1}{\partial x}x'(t) + \frac{\partial f_1}{\partial y}y'(t) + \frac{\partial f_1}{\partial z}z'(t) \\ \frac{\partial f_2}{\partial x}x'(t) + \frac{\partial f_2}{\partial y}y'(t) + \frac{\partial f_2}{\partial z}z'(t) \\ \frac{\partial f_3}{\partial x}x'(t) + \frac{\partial f_3}{\partial y}y'(t) + \frac{\partial f_3}{\partial z}z'(t) \end{bmatrix}$$

Example 5.8. Let $\mathbf{f} : U \subset \mathbb{R}^n \to \mathbb{R}^m$ be given by $\mathbf{f} = (f_1, \cdots, f_m)$ and $g(\mathbf{x}) = \sin[\mathbf{f}(\mathbf{x}) \cdot \mathbf{f}(\mathbf{x})]$. Compute $Dg(\mathbf{x})$.

sol.

$$Dg(\mathbf{x}) = \cos[\mathbf{f}(\mathbf{x})\cdot\mathbf{f}(\mathbf{x})]D[\mathbf{f}(\mathbf{x})\cdot\mathbf{f}(\mathbf{x})]$$

We compute $D[\mathbf{f}(\mathbf{x})\cdot\mathbf{f}(\mathbf{x})]$ which is

$$\begin{aligned} Dh &= \left[2f_1\tfrac{\partial f_1}{\partial x_1} + \cdots + 2f_m\tfrac{\partial f_m}{\partial x_1}, \cdots, 2f_1\tfrac{\partial f_1}{\partial x_n} + \cdots + 2f_m\tfrac{\partial f_m}{\partial x_n}\right] \\ &= 2\mathbf{f}(\mathbf{x})D\mathbf{f}(\mathbf{x}) \end{aligned}$$

where $D\mathbf{f}(\mathbf{x})$ is the derivative of $\mathbf{f}$, Finally, we see $Dg(\mathbf{x}) = 2[\cos[\mathbf{f}(\mathbf{x})\cdot\mathbf{f}(\mathbf{x})]\mathbf{f}(\mathbf{x})D\mathbf{f}(\mathbf{x})$

Geometric meaning of derivative

For a curve $\mathbf{c} = (x(t), y(t), z(t)) \colon \mathbb{R} \to \mathbb{R}^3$ and function $\mathbf{f} \colon \mathbb{R}^3 \to \mathbb{R}^3$, the composite function $\mathbf{p}(t) := \mathbf{f}\circ\mathbf{c}(t) = \mathbf{f}(x(t), y(t), z(t)) = (h_1(t), h_2(t), h_2(t))$ is another curve D in $\mathbb{R}^3$ mapped by f, i.e, $f(C) = D$. Then $\mathbf{c}'(t) = (x'(t), y'(t), z'(t))$ is a velocity vector(tangent vector) of C and $\mathbf{p}'(t)$ is a velocity vector(tangent vector) at D. Here the chain rule

$$\mathbf{p}'(t) = \mathbf{D}f(\mathbf{c}(t))\mathbf{c}'(t)$$

shows the derivative $\mathbf{D}f(\mathbf{c}(t))$ maps the tangent vector $\mathbf{c}'(t)$ at C to the tangent vector $\mathbf{p}'(t)$ at D. (Figure 2.9)

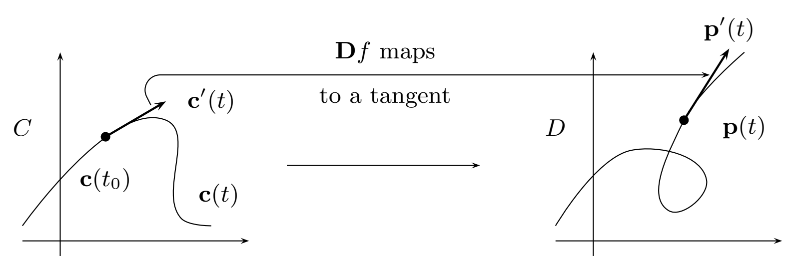


그림 2.9: a tangent vector is mapped to another by derivative

Example 5.9. Given $f(x,y) = \sqrt{1-x^2-y^2}$.

(1) Find the tangent vector at

$$\alpha(t) = \left(\cos t, \frac{1}{2}\sin t, \sqrt{1 - \cos^2 t - \frac{1}{4}\sin^2 t}\right)$$

at $t = \pi/2$ and show this vector lies on the tangent plane at $\alpha(\pi/2) = (0, 1/2, \sqrt{3}/2)$ of the graph G.

Continuation $z = f(x, y) = \sqrt{1 - x^2 - y^2}$. Let $S = \Phi(G)$ be the image of G under the mapping $\Phi(x, y, z) = (x, y, z/2)$. Then it is the graph of $g(x, y, z) = (1/2)\sqrt{1 - x^2 - y^2}$. Now do

(a) Find a tangent to S along the curve $\beta(t) = \Phi \circ \alpha(t)$ at $t = \pi/2$.

(b) Show this vector lies in a tangent plane of S at $\beta(\pi/2)$.

sol. (1) tangent vector to the curve α is

$$\alpha'(t)|_{\pi/2} = \left(-\sin t, (1/2)\cos t, \frac{(3/4)\cos t \sin t}{\sqrt{1-\cos^2 t - (1/4)\sin^2 t}} \right)\Bigg|_{\pi/2}$$
$$= (-1, 0, 0)$$

The tangent plane of G at $\alpha(\pi/2) = (0, 1/2, \sqrt{3}/2)$ is

$$z = \frac{\sqrt{3}}{2} - \frac{\sqrt{3}}{3}(y - \frac{1}{2})$$

This plane is perpendicular to the vector $(0, \sqrt{3}/3, 1)$ and the vector $(0, \sqrt{3}/3, 1)$ is perpendicular to $(-1, 0, 0)$. Hence the vector $(-1, 0, 0)$ is parallel to tangent plane.

(2) The tangent vector to the curve β is

$$\beta'(\pi/2) = \mathbf{D}\Phi|_{(0,1/2,\sqrt{3}/2)}\alpha'(\pi/2)$$
$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \frac{1}{2} \end{bmatrix} \begin{bmatrix} -1 \\ 0 \\ 0 \end{bmatrix}$$

Hence $\beta'(\pi/2) = (-1, 0, 0)$. The tangent plane of $g(x, y, z)$ at $\beta(\pi/2) = (0, 1/2, \sqrt{3}/4)$ is

$$z = \frac{\sqrt{3}}{4} - \frac{\sqrt{3}}{6}(y - \frac{1}{2})$$

Hence by similar argument as before $\beta'(\pi/2) = (-1, 0, 0)$ lies in the tangent plane.

Example 5.10. $g(x, y) = (x^2 + 1, y^2), \quad f(u, v) = (u + v, u, v^2)$. Find $f \circ g$ at $(1, 1)$.

sol.

$$Df = \begin{bmatrix} 1 & 1 \\ 1 & 0 \\ 1 & 2v \end{bmatrix}, \quad Dg = \begin{bmatrix} 2x & 0 \\ 0 & 2y \end{bmatrix}$$

Example 5.11. $f(x, y) = (x^2 + 1, y^2), \quad x = r\cos\theta, y = r\sin\theta$. Find $\frac{\partial f}{\partial \theta}$.

sol.

$$\frac{\partial f}{\partial \theta} = -r\frac{\partial f}{\partial x}\sin\theta + r\frac{\partial f}{\partial y}\cos\theta$$

Example 5.12. $f(x,y) = (\cos y + x^2, e^{x+y})$, $g(u,v) = (e^{u^2}, u - \sin v)$. Find $f \circ g$ at $(0,0)$.

sol. $f \circ g(u,v) = (\cos(u - \sin v) + e^{2u^2}, e^{(e^{u^2}+u-\sin v)})$.

$$Df = \begin{bmatrix} 2x & -\sin y \\ e^{x+y} & e^{x+y} \end{bmatrix}, \quad Dg = \begin{bmatrix} 2ue^{u^2}, & 0 \\ 1 & -\cos y \end{bmatrix}$$

$$g(0,0) = (1,0), \quad Df = \begin{bmatrix} 2 & 0 \\ e & e \end{bmatrix}, \quad Dg = \begin{bmatrix} 0, & 0 \\ 1 & -1 \end{bmatrix}$$

제 6 절 Gradient and directional derivatives

Gradient

Definition 6.1. Let $f\colon U \subset \mathbb{R}^n \to \mathbb{R}$ be differentiable. The **gradient** of at $\mathbf{x}_0$ is

$$\nabla f = \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z}\right)$$

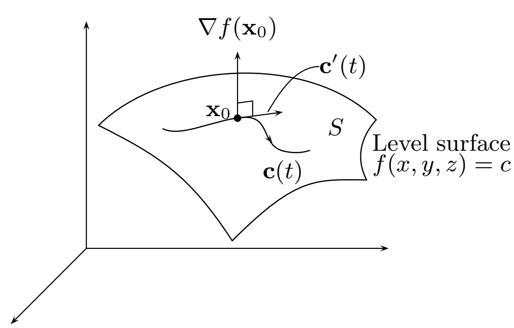


그림 2.10: gradient at $\mathbf{x}_0$ is perpendicular to tangent plane through $\mathbf{x}_0$

directional derivative

Definition 6.2. For a given vector $\mathbf{v} \in \mathbf{R}^n$ and $\mathbf{x}_0 \in \mathbf{R}^n$, the **directional derivative**) of $f\colon \mathbb{R}^n \to \mathbb{R}$ at $\mathbf{x}_0$ along $\mathbf{v}$ is $\mathbf{D_v}f(\mathbf{x})$ defined by (Fig 2.11)

$$\frac{d}{dt}f(\mathbf{x}+t\mathbf{v})\Big|_{t=0}$$

Usually we take unit vector $\mathbf{v} \in \mathbf{R}^n$ ($\|\mathbf{v}\| = 1$)

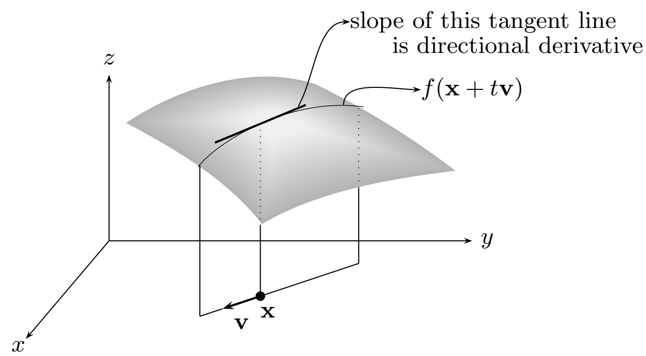


그림 2.11: Directional Derivative

Theorem 6.3. *If $f(\mathbf{x})\colon \mathbb{R}^3 \to \mathbb{R}$, the directional derivative of f at $\mathbf{x}$ along $\mathbf{v}$ is given by*

$$Df(\mathbf{x})\mathbf{v} = \mathrm{grad}\, f(\mathbf{x}) \cdot \mathbf{v} = \nabla f \cdot \mathbf{v}$$

Proof. Let $c(t) = \mathbf{x} + t\mathbf{v}$ so that $f(\mathbf{x}+t\mathbf{v}) = f(\mathbf{c}(t))$. Then by the chain rule $\frac{d}{dt}f(\mathbf{c}(t)) = \nabla f(\mathbf{c}(t)) \cdot \mathbf{c}'(t)$. Hence

$$\frac{d}{dt}f(\mathbf{x}+t\mathbf{v})\Big|_0 = \nabla f \cdot \mathbf{v} = \mathbf{D}(f)\mathbf{c}'(t) = \Big(\frac{\partial f}{\partial x_1}, \frac{\partial f}{\partial x_2}, \dots, \frac{\partial f}{\partial x_n}\Big) \cdot \mathbf{v} \tag{2.2}$$

□

This is the rate of change of f along $\mathbf{v}$. The rate of change of f along a curve is given as

$$\frac{d}{dt}f(\mathbf{c}(t))\Big|_0 = \nabla f \cdot \mathbf{c}'(t)\big|_0 = \nabla f(\mathbf{x}) \cdot \mathbf{v} \tag{2.3}$$

Example 6.4. Compute the rate of change of $f(x,y,z) = xy - z^2$ at $(1,0,1)$ along $(1,1,1)$.

sol. The unit vector to $(1,1,1)$ is $\mathbf{v} = (1/\sqrt{3})(1,1,1)$. The gradient of f at $(1,0,1)$ is

$$\begin{aligned}\nabla f(1,0,1) &= (f_z, f_y, f_z)|_{(1,0,1)} = (y, x, 2z)|_{(1,0,1)} \\ &= (0,1,-2)\cdot\Big(\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\Big) = -\frac{1}{\sqrt{3}}\end{aligned}$$

Direction of fastest increase

$\nabla f \cdot \mathbf{v}$ is the rate of change of f along the direction $\mathbf{v}$ Then

$$\mathbf{D_v} f = \|\mathbf{v}\|\|\nabla f\| \cos\theta$$

Here θ is the angle between $\mathbf{v}$ and ∇f. Hence if $\theta = 0$ the directional derivative ∇f has maximum value $\|\nabla f\|$, and if $\theta = \pi$ has minimum $-\|\nabla f\|$. Also, if $\theta = \pi/2$ then the directional derivative is 0. Hence

Theorem 6.5. *Suppose $f : \mathbb{R}^n \to \mathbb{R}$ is differentiable at $\mathbf{x}_0$. Then f increases(decreases) fastest at $\mathbf{x}_0$ along $\nabla f(\mathbf{x}_0)(-\nabla f(\mathbf{x}_0))$. Also, f does not change along the perpendicular direction to $\nabla f(\mathbf{x}_0)$*

Example 6.6. In what direction from $(0,1)$ does $f(x,y) = x^2 - y^2$ increases fastest?

Gradient is normal to the level set

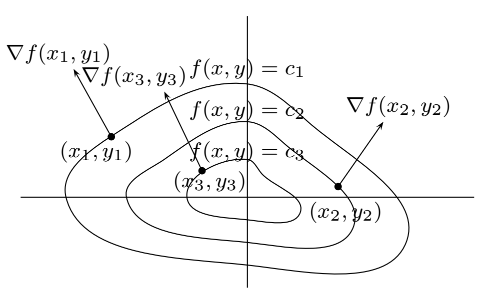


그림 2.12: gradient is perpendicular to level curve.

See the graph (2.10) in 3D case. Also see the book.

Consider the level set(Surface) $S = \{(x, y, z) \in \mathbb{R}^3 \mid f(x, y, z) = k\}$ of $f(x, y, z)$. Suppose a curve $\mathbf{c}$ passes the point $\mathbf{x}_0 = (x_0, y_0, z_0)$ lies on the surface S Then $f(\mathbf{c}(t)) = k$ holds. Then we have by chain rule

$$0 = \frac{d}{dt} f(\mathbf{c}(t)) = \nabla f(\mathbf{c}(t)) \cdot \mathbf{c}'(t)$$

(Fig 2.10) Hence the tangent vector $\mathbf{c}'(t_0)$ at $\mathbf{x}_0$ is normal to the gradient $\nabla f(\mathbf{x}_0)$.

Theorem 6.7. *Suppose $f(x, y, z)$ is differentiable and $\nabla f(\mathbf{x}_0) \neq 0$. Then $\nabla f(\mathbf{x}_0)$ is normal to the level surface $S = \{(x, y, z) \in \mathbb{R}^3 \mid f(x, y, z) = k\}$.*

We can also define

Definition 6.8. The plane S in Theorem 6.7 at $\mathbf{x}_0 = (x_0, y_0, z_0)$ is **tangent plane**. In other words, if S is a level surface $f(x, y, z) = k$ and $\mathbf{x}_0$ is on S, then the tangent plane to S at $\mathbf{x}_0$ is given by

$$\nabla f(\mathbf{x}_0) \cdot (\mathbf{x} - \mathbf{x}_0) = 0, \qquad \text{or}$$
$$\frac{\partial f}{\partial x}(\mathbf{x}_0)(x - x_0) + \frac{\partial f}{\partial y}(\mathbf{x}_0)(y - y_0) + \frac{\partial f}{\partial z}(\mathbf{x}_0)(z - z_0) = 0.$$

Example 6.9. Find equation of tangent plane to $3xy + z^2 + 4$ at $(1, 1, 1)$.

sol. $-\nabla f = (3y, 3x, 2z)$ at $(1, 1, 1)$, it is $(3, 3, 2)$. Thus tangent plane is

$$(3, 3, 2) \cdot (x - 1, y - 1, z - 1) = 0.$$

We often speak of vector field. $\nabla f(\mathbf{x})$. (A vector function is often called a **vector fields** Draw graph of a vector field.

Example 6.10. The gravitational force of a mass m at (x, y, z) produced by a mass M at origin is

$$\mathbf{F} = \frac{GmM}{r^2}\mathbf{n}$$

G is gravitational constant, $\mathbf{r} = (x, y, z)$, $r = \|\mathbf{r}\|$, $\mathbf{n} = \mathbf{r}/r$. Note $\mathbf{F} = -\nabla GmM/r$. Here $V = -GmM/r$ is called potential.

$$\begin{aligned}\nabla V &= \nabla\Big(-\frac{GmM}{\sqrt{x^2 + y^2 + z^2}}\Big) \\ &= \Big(-\frac{GmM}{2\sqrt{x^2 + y^2 + z^2}}(2x, 2y, 2z)\Big) = -\mathbf{F}\end{aligned}$$