Contents

Chapter 1

Functions and Limits

1.4 Limit of Functions

Limit of function values

Example 1.4.1. ff

Theorem 1.4.2. *If $f(x)$, $g(x)$ has limit at $x = a$ and values are L, M Then the following hold.*

(1) $\lim_{x\to a}(f(x)+g(x)) = L + M$ *(sum rule)*

(2) $\lim_{x\to a}(f(x)-g(x)) = L - M$ *(Difference rule)*

(3) $\lim_{x\to a}(kf(x)) = kL$ *(Constant multiple)*

(4) $\lim_{x\to a} f(x)\cdot g(x) = LM$ *(Product rule)*

(5) $\lim_{x\to a}\frac{f(x)}{g(x)} = \frac{L}{M},\ M \neq 0$ *(Quotient rule).*

Poly and rational function

Theorem 1.4.3. *For any polynomial $f(x) = a_0x^n + a_1x^{n-1} + \cdots + a_n$,*

$$\lim_{x\to a} f(x) = a_0a^n + a_1a^{n-1} + \cdots + a_n = f(a).$$

Example 1.4.4. Use Theorem 1.4.2 (3), (4) we see

$$\lim_{x\to a} 2x^2 = 2\lim_{x\to a} x^2 = 2\lim_{x\to a} x \lim_{x\to a} x = 2a\cdot a = 2a^2$$

Theorem 1.4.5. *If $f(x)$, $g(x)$ are polynomials and $g(a) \neq 0$ then*

$$\lim_{x\to a}\frac{f(x)}{g(x)} = \frac{f(a)}{g(a)}.$$

Example 1.4.6. Find

$$\lim_{x\to 2} \frac{x^3+x+2}{x+1}.$$

sol.

$$\lim_{x\to 2} \frac{x^3+x+2}{x+1} = \frac{2^3+2+2}{2+1} = 4$$

□

Eliminating zero denominator

Example 1.4.7.

$$\lim_{x\to 1} \frac{x^3-1}{x^2-1}.$$

sol. Now denominator is zero. But as long as $x \neq 1$ it holds that

$$\frac{x^3-1}{x^2-1} = \frac{(x-1)(x^2+x+1)}{(x-1)(x+1)} = \frac{x^2+x+1}{x+1}$$

$$\lim_{x\to 1} \frac{x^3-1}{x^2-1} = \lim_{x\to 1} \frac{x^2+x+1}{x+1} = \frac{3}{2}. \qquad \square$$

□

Example 1.4.8. $\lim_{x\to 0} \frac{\sqrt{x^2+1}-1}{x^2}$

sol.

$$\lim_{x\to 0} \frac{\sqrt{x^2+1}-1}{x^2} = \frac{x^2+1-1}{x^2(\sqrt{x^2+1}+1)} = \frac{1}{\sqrt{x^2+1}+1}$$

□

Example 1.4.9.

$$\lim_{x\to\infty} \frac{x^3+x}{2x^3+3} = \frac{1}{2}.$$

sol. Factor out x^3

$$\frac{x^3+x}{2x^3+3} = \frac{1+x^2}{2+3/x^3}$$

As $x \to \infty$ Theorem 1.4.2 (5) limit is 1/2.

□

Sandwich Theorem

Theorem 1.4.10 (Sandwich Theorem)**.** *If, for an interval* (α, β) *containing* a*, it holds* $g(x) \leq f(x) \leq h(x)$

$$\lim_{x \to a} g(x) = \lim_{x \to a} h(x) = L$$

then

$$\lim_{x \to a} f(x) = L$$

This hold for left or right limit also.

Example 1.4.11. Suppose that

$$v(x) - x^2 \leq u(x) \leq v(x) + 3x^2$$

holds for all $x \neq a$. Then

$$\lim_{x \to a} u(x) = \lim_{x \to a} v(x)$$

provided the limit of the r.h.s term exists.

Example 1.4.12. We will later see that

$$-|\theta| \leq \sin\theta \leq |\theta|$$

for all θ. Hence $\lim_{x \to 0} \sin\theta = 0$ by Sandwich theorem.

Example 1.4.13. Similarly, from the inequality

$$0 \leq 1 - \cos\theta \leq |\theta|$$

for all θ. Hence $\lim_{x \to 0} \cos\theta = 1$.

Theorem 1.4.14 (Sandwich Theorem, inequality)**.** *If* $f(x) \leq g(x)$ *holds for all* x *in an interval* (α, β) *containing* c *except possibly* c*, then*

$$\lim_{x \to c} f(x) \leq \lim_{x \to c} g(x)$$

This hold for left- right limit also.

When limit do not exist ?

Three cases: Jump, infinity, oscillation:

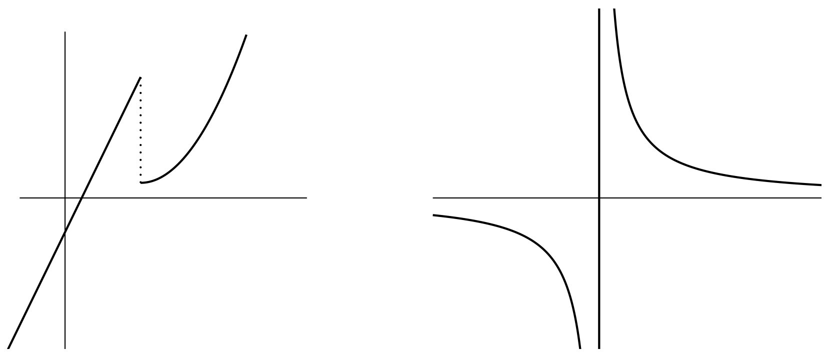


Figure 1.1: Broken graph or undefined(infinity)

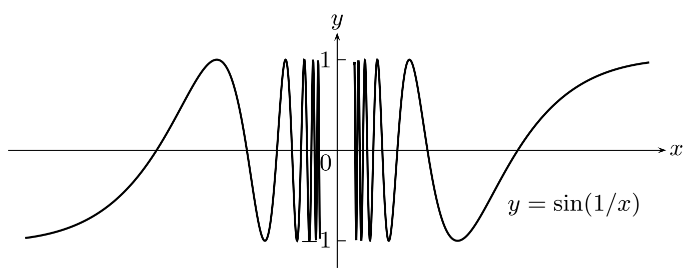


Figure 1.2: $y = \sin \frac{1}{x}$

1.5 Precise definition

Definition 1.5.1. Let $f(x)$ be defined on an open interval containing a except possibly a. If there is a number L such that for any positive ε, there exists $\delta > 0$ such that

$$0 < |x - a| < \delta \Rightarrow |f(x) - L| < \varepsilon$$

we say $f(x)$ has **limit** L at $x = a$ and write

$$\lim_{x \to a} f(x) = L$$

Remark 1.5.2. In general, δ depends on x, f, ε. Also, note that we do not check at $x = a$. $0 < |x - a|$.

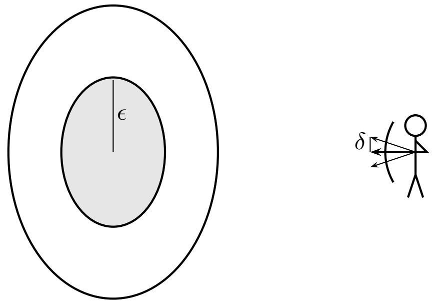


Figure 1.3: How to control arrow well to hit bull's eye within ϵ tolerance?

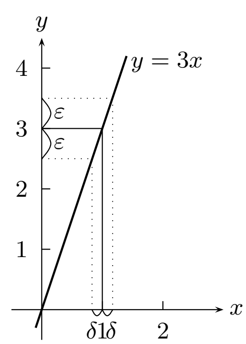


Figure 1.4: Linear case, $y = 3x$

Remark 1.5.3. L is unique

Example 1.5.4 (Linear case)**.** Show $y = 3x + 1$ has limit 7 at $x = 2$.

sol. Suppose the following holds.

$$|y - 7| = |3x + 1 - 7| < \varepsilon$$

Then we take

$$3|x - 2| < \varepsilon, \ |x - 2| < \varepsilon/3$$

So $\delta = \dfrac{\varepsilon}{3}$.

□

Finding δ algebraically when ϵ is given

Example 1.5.5. For the limit $\lim x \to 2\sqrt{x+7} = 3$, find a $\delta > 0$ that works for $\epsilon = 1$. Repeat with $\epsilon = 0.1, \ 0.001$, etc.

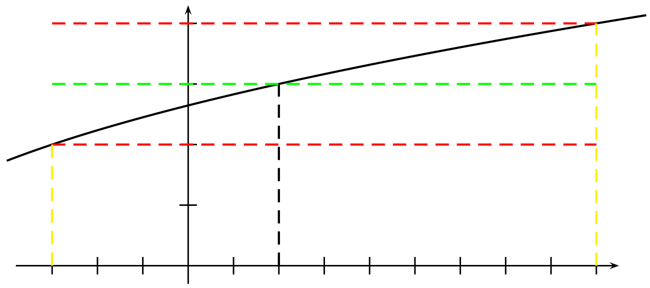


Figure 1.5: Choosing δ

sol. Step 1) Solve the inequality $|\sqrt{x+7}-3|<2$:

$$\begin{aligned}-1 < \sqrt{x+7}-3 &< 1\\ 2 < \sqrt{x+7} &< 4\\ 4 < x+7 &< 16\\ -3 < x &< 9\end{aligned}$$

Step 2). Find a value $\delta > 0$ to place the centered interval $(2-\delta, 2+\delta)$ inside the interval $-3 < x < 9$. A choice of delta is $\delta = 5$. Any value smaller than that works.

□

Example 1.5.6. Limit of $f(x) = (x^2-1)/(x-1)$ is 2 at $x = 1$.

sol.

$$|x+1-2| < \varepsilon$$

$0 < |x-1| < \varepsilon$. Hence we may $\delta = \varepsilon$.

□

Example 1.5.7. Show that $y = x^2$ has limit 4 at $x = 2$.

sol. (Method 1) We try to solve the exact values of x where

$$|x^2-4| = |(x-2)(x+2)| < \varepsilon \tag{1.1}$$

holds. Thus we see

$$-\varepsilon < x^2 - 4 < \varepsilon$$

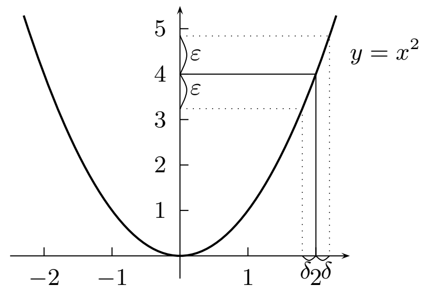


Figure 1.6: The graph of $y = x^2$

from which we get

$$\sqrt{4-\varepsilon} < x < \sqrt{4+\varepsilon}$$

Now for any x in the interval $(2-\delta_1, 2+\delta_2)$, (1.1) holds. But the interval is not an symmetric interval.Thus the idea is to choose $\delta := \min\{\delta_1, \delta_2\}$ so that

$$|x-2| < \delta \Rightarrow x \in (2-\delta_1, 2+\delta_2) \Rightarrow |x^2-4| < \varepsilon$$

(Method 2) Choose certain interval near $x = 2$ where the following holds:

$$|x^2-4| = |(x-2)(x+2)| < \varepsilon \tag{1.2}$$

Suppose $\delta < 1$. So $0 < |x-2| < 1$ and Since $|x+2|$ between 3 and 5 we have

$$|x^2-4| = |(x-2)(x+2)| \le |x-2| \cdot \max_{1\le x\le 3} |x+2| \le 5|x-2|$$

To satisfy (1.2) we take $|x-2| < \varepsilon/5$. Since δ was chosen to satisfy $\delta < 1$, we may choose $\delta = \min\{1, \varepsilon/5\}$.

□

Example 1.5.8. Show $y = \sqrt{x-1}$ has limit 2 at $x = 5$.

sol. We need to find an interval near $x = 5$ where the inequality is true:

$$|\sqrt{x-1} - 2| < \varepsilon$$

(Method 1) Just solve the inequality exactly.

$$\begin{aligned}
-\varepsilon &< \sqrt{x-1} - 2 < \varepsilon \\
2-\varepsilon &< \sqrt{x-1} < 2+\varepsilon \\
(2-\varepsilon)^2 &< x-1 < (2+\varepsilon)^2 \\
-4\varepsilon + \varepsilon^2 &< x-5 < 4\varepsilon + \varepsilon^2
\end{aligned}$$

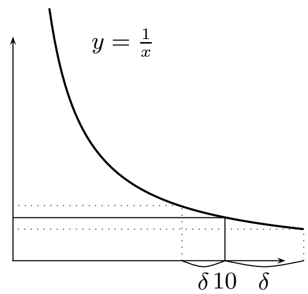


Figure 1.7: Choose smaller δ

We may assume $0 < \varepsilon < 1$. Hence we can choose

$$\delta = \min\{4\varepsilon - \varepsilon^2, 4\varepsilon + \varepsilon^2\} = 4\varepsilon - \varepsilon^2.$$

(Method 2)

$$\begin{aligned}\sqrt{x-1} - 2 &= \frac{(\sqrt{x-1}-2)(\sqrt{x-1}+2)}{\sqrt{x-1}+2} \\ &= \frac{x-5}{\sqrt{x-1}+2}\end{aligned}$$

Taking absolute values

$$\begin{aligned}\left|\sqrt{x-1} - 2\right| &= \left|\frac{(\sqrt{x-1}-2)(\sqrt{x-1}+2)}{\sqrt{x-1}+2}\right| \\ &= \left|\frac{x-5}{\sqrt{x-1}+2}\right| \\ &\leq \left|\frac{x-5}{2}\right|\end{aligned}$$

This will be less than ϵ if $|x-5| < 2\epsilon$. Hence $\delta = 2\varepsilon$.

□

Proving theorems with definitions

Example 1.5.9. Assume $\lim_{x\to a} f(x) = L, \quad \lim_{x\to a} g(x) = M$. Prove $\lim_{x\to a} (f(x) + g(x)) = L + M$.

sol. Let ϵ be given.

$|f(x) + g(x) - (L + M)| = |f(x) - L + g(x) - M| \leq |f(x) - L| + |g(x) - M|$.

We want to show this term is less than ϵ when x lies within certain interval. From the definitions we see there exists two deltas $\delta_1, \delta_2 > 0$ such that

$$0 < |x-a| < \delta_1 \Rightarrow |f(x) - L| < \epsilon/2$$
$$0 < |x-a| < \delta_2 \Rightarrow |g(x) - L| < \epsilon/2.$$

Thus for those x with $0 < |x-a| < \min(\delta_1, \delta_2)$, we see the desired inequality holds.

□

1.6 One sided limit

Limit as x approaches ∞

$f(x) = 1/x \to 0$ as x grows. ε-δ

Definition 1.6.1. We say $f(x)$ has limit L as x approaches ∞, if for every positive ε, there is M such that for all

$$x > M \Rightarrow |f(x) - L| < \varepsilon.$$

We write

$$\lim_{x\to\infty} f(x) = L$$

Example 1.6.2. Explain

$$\lim_{x\to\infty} \frac{x+2}{x+1} = 1$$

sol. $f(x) = 1 + 1/(x+1)$ We want to know when the following holds:

$$|f(x) - 1| = \frac{1}{|x+1|}$$

Thus we solve $|x+1| > 1/\varepsilon$ for x. It will holds when $x > M$ where

$$M = \frac{1}{\varepsilon} - 1$$

□

Useful limits:

Theorem 1.6.3. *(1)* $\lim_{\theta\to 0} \sin\theta = 0$

(2) $\lim_{\theta\to 0} \cos\theta = 1$

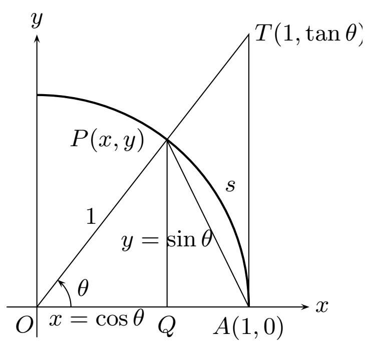


Figure 1.8:

(3) $\lim_{\theta \to 0} \frac{\sin\theta}{\theta} = 1$

Proof. (1) Referring to Fig 1.8, let P be the point on the unit circle so that θ is the angle between the x-axis and line OP. Then the length of segment $\overline{PQ}$ is $\sin\theta$. The arc from $(1,0)$ to P has length $s = \theta$. Hence

$$0 < \overline{PQ} < \overline{PA} < \theta$$

and it holds that

$$0 < \sin\theta < \theta.$$

As θ approaches 0, we see

$$\lim_{\theta \to 0^+} \sin\theta = 0$$

Since $\sin(-\theta) = -\sin\theta$

$$-\theta < \sin(-\theta) < 0$$

Hence

$$\lim_{\theta \to 0^-} \sin\theta = 0$$

(2) Comparing $\overline{QA}$ and $\overline{PA}$, we see

$$0 < \overline{QA} < \overline{PA} < \theta$$

$$0 < 1 - \cos\theta < \theta$$

Also, as θ approaches to 0 $1 - \cos\theta$ approaches to 0.

(3) We note that

$$\begin{array}{ccccc} \overline{PQ} & \leq & \theta & \leq & \overline{TA} \\ \sin\theta & \leq & \theta & \leq & \tan\theta \\ 1 & \leq & \theta/\sin\theta & \leq & 1/\cos\theta \end{array}$$

As θ approaches 0, $1/\cos\theta$ approaches 1 by (2). Hence $\lim_{\theta\to 0} \frac{\sin\theta}{\theta} = 1$. ☐

Example 1.6.4. $\lim_{\theta\to 0}\frac{\sin 2\theta}{\theta} = \lim_{\theta\to 0} 2\frac{\sin 2\theta}{2\theta} = 2\lim_{x\to 0}\frac{\sin x}{x} = 2.$

Example 1.6.5.

$$\lim_{\theta\to 0}\frac{\tan\theta\sin\theta}{\theta^2} = 1.$$

sol. Thm 1.4.2 (4)

$$\begin{aligned}\lim_{\theta\to 0}\frac{\tan\theta\sin\theta}{\theta^2} &= \lim_{\theta\to 0}\frac{\sin^2\theta}{\theta^2}\cdot\frac{1}{\cos\theta}\\ &= \lim_{\theta\to 0}\left(\frac{\sin\theta}{\theta}\right)^2\cdot\lim_{\theta\to 0}\frac{1}{\cos\theta}\\ &= \lim_{\theta\to 0}\left(\frac{\sin\theta}{\theta}\right)\cdot\lim_{\theta\to 0}\left(\frac{\sin\theta}{\theta}\right)\\ &= 1.\end{aligned}$$

□

1.7 Continuity

Definition 1.7.1. If a function $f(x)$ is **continuous** at an interior point $x = c$ if f

$$\lim_{x\to c} f(x) = f(c)$$

A function $f(x)$ is **continuous at a left end point** $x = a$ (resp. right end point $x = b$) if

$$\lim_{x\to a^+} f(x) = f(a), \quad (resp. \lim_{x\to b^-} f(x) = f(b))$$

If f is continuous at all points of its domain, we say f is a **continuous function**.

Consider f on (c, d). For a in c, d $y = f(x)$ is continuous at $x = a$ iff

(1) $f(a)$ exists

(2) $\lim_{x\to a} f(x)$ exists.

(3) $\lim_{x\to a} f(x) = f(a)$ holds.

Theorem 1.7.2. *If f, g are continuous at $x = a$ then*

(1) $f \pm g$ is continuous at $x = a$.

(2) *For any constant k, the function kf is continuous at $x = a$*

(3) *fg is continuous at $x = a$*

(4) *When $g(a) \neq 0$, f/g is continuous at $x = a$*

Corollary 1.7.3. *Polynomials $f(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$ are continuous at all points.*

Corollary 1.7.4. *For two polynomials $f(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$ and $g(x) = b_m x^m + b_{m-1}x^{m-1} + \cdots + b_1 x + b_0$, the rational function $f(x)/g(x)$ is continuous where $g(a)$ is nonzero 0.*

Composite function

Theorem 1.7.5. *Suppose f is continuous at a and g continuous at $f(a)$ then $g \circ f$ is continuous at a.*

Proof. Since g is conti at $f(a)$ and $\lim_{x\to a} f(x) = f(a)$

$$\lim_{x\to a} g(f(x)) = g(\lim_{x\to a} f(x)) = g(f(a))$$

□

Theorem 1.7.6 (Intermediate Value theorem)**.** *Suppose f is continuous on $[a, b]$. Then for any value y_0 between $f(a)$ and $f(b)$ there is a point $c \in [a, b]$ such that $f(c) = y_0$.*

When does this breaks down?

1.8 Limit Involving Infinite and vertical asymptote

Example 1.8.1.

$$\lim_{x\to 0} \frac{1}{x}$$

$$\lim_{x\to\infty} \left(x + \frac{1}{x}\right)$$

Sandwich theorem revisited

Sandwich theorem holds when $x \to \infty$.

Vertical Asymptotes

Oblique asymptote?

Chapter 2

Differentiation

2.1 Tangents and Derivatives at a point

Finding tangent to the graph of a function

Definition 2.1.1. The slope of the curve $y = f(x)$ at a point $P = (x_0, f(x_0))$ is the number

$$\lim_{h \to 0} \frac{f(x_0 + h) - f(x_0)}{h}$$

provided it exists. The **tangent line** to the curve at the point P is the line through P with this slope.

The rate of change of $y = f(x)$ between $P(x_0, y_0)$ and $Q(x_1, y_1)$ is

$$m_{sec} = \frac{f(x_1) - f(x_0)}{x_1 - x_0}$$

In the limit, it is the slope of tangent line.

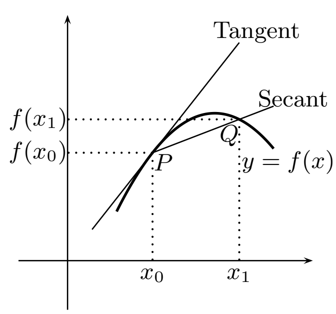


Figure 2.1: Tangent and secant

Hence the slope of tangent line at P is

$$m_{tan} = \lim_{x_1 \to x_0} \frac{f(x_1) - f(x_0)}{x_1 - x_0} = \lim_{h \to 0} \frac{f(x_0 + h) - f(x_0)}{h}$$

Rate of change:Derivative at a point

Definition 2.1.2. The derivative of a function $f(x)$ at a point is given by

$$f'(x_0) = \lim_{h \to 0} \frac{f(x_0 + h) - f(x_0)}{h}$$

exists, it is called **derivative** at $x = x_0$.

Example 2.1.3. dd

2.2 Derivative as a function

Definition 2.2.1. If the derivative of a function $f(x)$ at a

$$f'(x) = \lim_{h \to 0} \frac{f(x + h) - f(x)}{h}$$

exists, it is called the **derivative** (function) of $f(x)$. (Treated as a function) f is said to be **differentiable** at x. If f is differentiable at all points of domain we say f is **differentiable.** We also use the notation df/dx, $(d/dx)f$ for f'.

One sided derivative

Definition 2.2.2. Suppose f is defined on $[a, b]$. Then at each end point the one sided derivative is defined by

$$f'(a^+) = \lim_{h \to 0^+} \frac{f(a + h) - f(a)}{h},$$
$$f'(b^-) = \lim_{h \to 0^+} \frac{f(b) - f(b - h)}{h}.$$

Example 2.2.3. Using definition, find $(d/dx)x^3$.

sol. Set $f(x) = x^3$

$$f(x + h) - f(x) = h\{(x + h)^2 + x(x + h) + x^2\}$$

$$\lim_{h \to 0} \frac{1}{h}\{f(x + h) - f(x)\} = \lim_{h \to 0}\{(x + h)^2 + x(x + h) + x^2\} = 3x^2$$

So $f'(x) = 3x^2$

□

Example 2.2.4. The one sided derivatives of $f(x) = |x|$ at $x = 0$ are $f'(0^-) = -1$ and $f'(0^+) = 1$. Hence f is not differentiable at $x = 0$. (Figure 2.2)

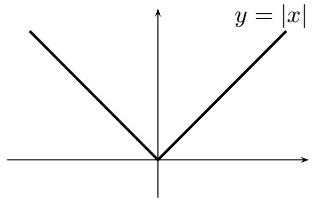


Figure 2.2: $y = |x|$

Example 2.2.5. The function defined by

$$f(x) = \begin{cases} x\sin(1/x), & x \neq 0 \\ 0, & x = 0 \end{cases}$$

is conti at $x = 0$ but not differentiable.

sol. Since $-|x| \leq f(x) \leq$ and $\lim_{x\to 0} |x| = \lim_{x\to 0} = 0$ by theorem f is conti. at $x = 0$. But the limit

$$\lim_{h\to 0} \frac{1}{h} \left\{ h \sin \frac{1}{h} - 0 \right\} = \lim_{h\to 0} \sin \frac{1}{h}$$

does not exist, f is not differentiable $x = 0$.

□

Differentiable functions are continuous

Theorem 2.2.6. *If f is differentiable at $x = a$, then f is conti. at $x = a$.*

Proof. By definition of derivative we have

$$\lim_{h\to 0} \{f(a+h) - f(a)\} = \left(\lim_{h\to 0} \frac{1}{h} \{f(a+h) - f(a)\} \right) \cdot \left(\lim_{h\to 0} h \right) = 0.$$

Hence $f(x)$ is conti. at $x = a$. □

Intermediate Value property of derivatives(Darboux's theorem)

Theorem 2.2.7. *If a, b are any two points in an interval where f is differentiable, then f' takes any value between $f'(a)$ and $f'(b)$.*

2.3 Differentiation Rules

Proposition 2.3.1. *Suppose f, g are differentiable functions. Then*

(1) *For any constant C, $\frac{dC}{dx} = 0$ for any constant C.*

(2) *When n is positive integer , $\frac{d}{dx}x^n = nx^{n-1}$.*

(3) *For any constant C, $\frac{d(Cu)}{dx} = C\frac{du}{dx}$.*

(4) $\frac{d}{dx}(u \pm v) = \frac{du}{dx} \pm \frac{dv}{dx}$.

(5) $\frac{d(uv)}{dx} = v\frac{du}{dx} + u\frac{dv}{dx}$.

(6) $\frac{d}{dx}\left(\frac{u}{v}\right) = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$.

(7) *When n is negative integer, $\frac{d}{dx}x^n = nx^{n-1}$.*

Proof. (4)

$$\begin{aligned}\frac{d}{dx}(uv) &= \lim_{h\to 0}\frac{u(x+h)v(x+h) - u(x)v(x)}{h}\\ &= \lim_{h\to 0}\left\{\frac{u(x+h) - u(x)}{h}v(x+h) + u(x)\frac{v(x+h) - v(x)}{h}\right\}\\ &= u'(x)v(x) + u(x)v'(x).\end{aligned}$$

Figure for product rule. □

Higher order derivative

$$f''(x) = \frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right)$$

$$y^{(n)} = \frac{dy^{(n-1)}}{dx} = \frac{d^ny}{dx^n}$$

2.4 The Derivative as a Rate of change

Definition 2.4.1. The **instantaneous rate of change** of f at x_0 is

$$f'(x_0) = \lim_{h\to 0}\frac{f(x_0+h) - f(x_0)}{h}$$

provided the limit exists.

Definition 2.4.2. (Velocity) is the derivative of a position function w.r.t time. If a moving object position is given by $s = f(t)$, then the **velocity** at t is

$$v(t) = \lim_{h\to 0} \frac{f(t+h) - f(t)}{h}$$

provided the limit exists.

Definition 2.4.3. (Speed) **Speed** is

$$|v(t)|$$

Acceleration is

$$a(t) = \frac{dv}{dt} = \frac{d^2 s}{dt^2}$$

Third derivative is called a Jerk

2.5 Derivative of Trig functions

Use definition to find the derivative of $f(x) = \sin x$.

$$\begin{aligned}
&= \lim_{h\to 0} \frac{\sin(x+h) - \sin x}{h} \\
&= \lim_{h\to 0} \frac{\sin x \cos h + \cos x \sin h - \sin x}{h} \\
&= \sin x \lim_{h\to 0} \frac{\cos h - 1}{h} + \cos x \lim_{h\to 0} \frac{\sin h}{h} \\
&= \sin x \cdot 0 + \cos x \cdot 1 = \cos x.
\end{aligned}$$

Hence $\frac{d}{dx} \sin x = \cos x$. Similarly, we have

$$\frac{d}{dx} \cos x = -\sin x$$

Other trigonometric functions are defined by the following relation and their derivatives can be found using differentiation rules:

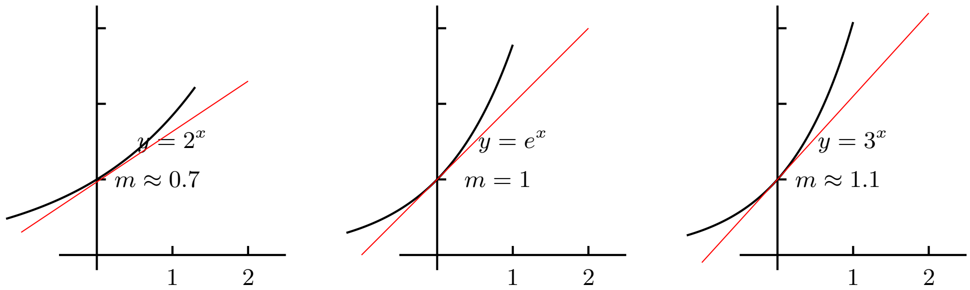
$$\tan x = \frac{\sin x}{\cos x}, \quad \sec x = \frac{1}{\cos x}, \quad \csc x = \frac{1}{\sin x}, \quad \cot x = \frac{\cos x}{\sin x}$$

The derivative of $\tan x$ is

$$\frac{d}{dx} \tan x = \frac{(\sin x)' \cos x - (\cos x)' \sin x}{\cos^2 x} = \frac{1}{\cos^2 x} = \sec^2 x$$

Summarizing, we have

Proposition 2.5.1. *(1)* $\frac{d}{dx} \sin x = \cos x$



(2) $\dfrac{d}{dx}\cos x = -\sin x$

(3) $\dfrac{d}{dx}\tan x = \sec^2 x$

(4) $\dfrac{d}{dx}\sec x = \sec x \tan x$

(5) $\dfrac{d}{dx}\csc x = -\csc x \cot x$

(6) $\dfrac{d}{dx}\cot x = -\csc^2 x$

2.6 Exponential functions

We define for any $a > 0$ and any real x

$$a^x = \lim_{r \to a} a^r, \qquad \text{rational}$$

Rules for exponentiation. For $a, b > 0$ we have

(1) $a^x \cdot a^y = a^{x+y}$

(2) $\frac{a^x}{a^y} = a^{x-y}$

(3) $(a^x)^y = (a^y)^x = a^{xy}$

(4) $a^x b^x = (ab)^x$

(5) $\frac{a^x}{b^x} = \left(\frac{a}{b}\right)^x$

The natural exponential function e^x

We define e to be the number such that the slope of tangent line to $f(x) = e^x$ at 0 is 1. e is a irrational number $e = 2.78182\cdots$. Why do we use this strange number? This simplifies computation in later sections(diff. integration)

Differentiation of exponential function

Let us compute the derivative of $f(x) = a^x$.

$$\begin{aligned} \lim_{h\to 0} \frac{a^{x+h} - a^x}{h} &= \lim_{h\to 0} \frac{a^x a^h - a^x}{h} \\ &= \lim_{h\to 0} a^x \left(\frac{a^h - 1}{h}\right) \\ &= a^x \lim_{h\to 0} \frac{a^h - 1}{h} \end{aligned}$$

Note that when $x = 0$ this limit $\lim_{h\to 0} \frac{1}{h}(a^h - 1)$ is the slope of tangent line to a^x at $x = 0$. As a special case, if $a = e$, then we know

$$f'(0) = \lim_{h\to 0} \frac{e^h - 1}{h} = 1$$

by definition of the number e! So we have

$$\frac{d}{dx} e^x = e^x.$$

In Chapter 3 we will see this number is obtained from $(1 + \frac{1}{x})^x$ as x approaches unboundedly.

Exponential growth

Example 2.6.1 (exponential growth)**.** It is used to model the amount of money to grow when the interest is compounded. Say P dollar is initially invested in the bank with interest rate of r per year, t is the time in years, then the total amount of money after t years is

$$y = Pe^{rt}$$

Example 2.6.2 (exponential decay)**.** It is used to model the amount of radioactive material. e.g, Carbon 14 case, If A is initial original amount of carbon 14.

$$y = Ae^{-1.2\times 10^{-4} t}$$

Carbon 14 is used to detect how old is the dead organisms such as shells, wooden artifacts.

2.7 Chain rule

Chain Rule

Theorem 2.7.1. *(1) If $f(u)$ is differentiable at $u = g(x)$ and g is differentiable at x, then the composite function $f \circ g$ is differentiable at x and*

$$(f \circ g)'(x) = f'(g(x))g'(x)$$

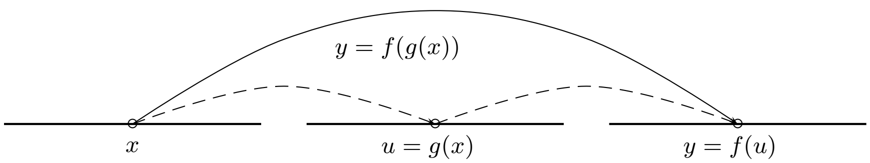


Figure 2.3: Chain rule

If $y = f(u)$, $u = g(x)$ *then*

$$\left.\frac{dy}{dx}\right|_{x} = \left.\frac{dy}{du}\right|_{u=f(x)} \cdot \left.\frac{du}{dx}\right|_{x}$$

Proof. (Intuitive) Let $\Delta u = g(x+\Delta x) - g(x)$ be the change of u corresponding to the change of Δx. (We assume $g(x+\Delta x) \neq g(x)$) We might consider

$$\frac{\Delta y}{\Delta x} = \frac{\Delta y}{\Delta u}\frac{\Delta u}{\Delta x}, (\Delta u \neq 0)$$

$$\begin{aligned} \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} &= \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta u}\frac{\Delta u}{\Delta x} \\ &= \lim_{\Delta u \to 0} \frac{\Delta y}{\Delta u} \lim_{\Delta x \to 0} \frac{\Delta u}{\Delta x} \\ &= \frac{dy}{du}\frac{du}{dx} \end{aligned}$$

□

Repeated Use

Example 2.7.2. $y = \cos(1+x^4)^5$

sol. Given function is the composite of $y = \cos u$ and $u = (1+x^4)^5$. Hence

$$\frac{dy}{du} = -\sin u \frac{du}{dx}.$$

On the other hand, since $(1+x^4)^5$ is a composite function of $u = v^5$ and $v = 1 + x^4$, use Chain rule again

$$\frac{du}{dx} = 5v^4 \cdot 4x^3.$$

$$\frac{dy}{dx} = -\sin(1+x^4)^5 \cdot 20(1+x^4)^4 x^3.$$ □

□

Outside-Inside Rule

Example 2.7.3. The derivative of $|x|$ can be computed as follows: Notice that $|x| = \sqrt{x^2}$. Hence

$$\frac{d}{dx}\sqrt{x^2} = \frac{1}{2\sqrt{x^2}} \cdot 2x = \frac{x}{|x|}, \quad x \neq 0.$$

Example 2.7.4. Find slope of tangent line to $y = (1-2x)^3$ at $x = 1$.

Derivatives of Power function

Find derivative of $f(x) = u^r(x)$ for any real r and $x > 0$.

Theorem 2.7.5. *Since* $u^r = e^{r \ln u}$

$$\frac{d}{dx}u^r = u^r \frac{d \ln u}{dx} = u^r \frac{1}{u}\frac{du}{dx} = u^{r-1}\frac{du}{dx}.$$

2.8 Implicit differentiation

Suppose x, y satisfy $y^5 + \sin xy = x^3 y$. Assuming y is a differentiable function of x, take derivative w.r.t x. This procedure is called an **Implicit differentiation**.

$$\frac{d}{dx}\left(y^5\right) + \frac{d}{dx}(\sin xy) = \frac{d}{dx}\left(x^3 y\right)$$
$$5y^4\frac{dy}{dx} + (\cos xy)\left(y + x\frac{dy}{dx}\right) = 3x^2 y + x^3\frac{dy}{dx}.$$

Hence

$$\frac{dy}{dx} = \frac{3x^2 y - y\cos xy}{5y^4 + x\cos xy - x^3}$$

Example 2.8.1. (1) $x^3 + y^3 = 3xy$. Find dy/dx and d^2y/dx^2

(2) Find equation of tangent line to $x^3 + y^3 = 3xy$ at $(3/2, 3/2)$

sol.

(1) Taking derivative

$$(3y^2 - 3x)\frac{dy}{dx} = 3y - 3x^2$$

Chain rule

$$\left(6y\frac{dy}{dx} - 3\right)\frac{dy}{dx} + (3y^2 - 3x)\frac{d^2y}{dx^2} = 3\frac{dy}{dx} - 6x.$$

Hence

$$\frac{dy}{dx} = \frac{y - x^2}{y^2 - x},$$

$$\frac{d^2y}{dx^2} = \frac{-2\left((y-x^2)/(y^2-x)\right)^2 y + 2(y-x^2)/(y^2-x) - 2x}{y^2 - x}.$$

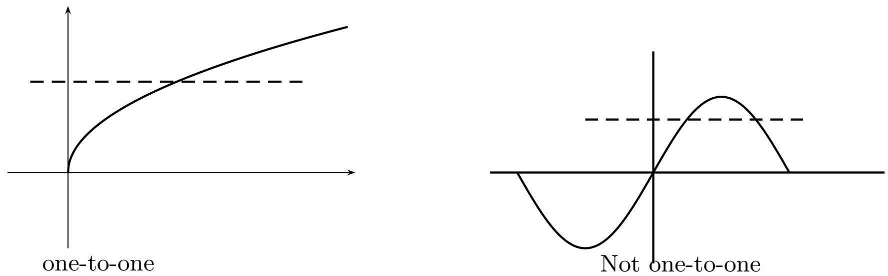


Figure 2.4: Horizontal line test

(2) At $(3/2, 3/2)$, $dy/dx = -1$ The tangent line is

$$y = -\left(x - \frac{3}{2}\right) + \frac{3}{2} = -x + 3. \qquad \square$$

2.9 Inverse functions and Their Derivatives

Definition 2.9.1. A function f is one-to-one on a domain D if $f(x_1) \neq f(x_2)$ whenever $x_1 \neq x_2$.

Definition 2.9.2. Suppose a function f is one-to-one on a domain D with range R. The **inverse function** f^{-1} exists and is defined by

$$f^{-1}(b) = a \text{ if } f(a) = b.$$

The domain of f^{-1} is R and range is D.

$$(f^{-1} \circ f)(x) = x, \quad x \in D$$
$$(f \circ f^{-1})(y) = y, \quad y \in R$$

Horizontal line test

Derivatives of inverse function

Theorem 2.9.3. *Suppose f is one-to-one and differentiable in I. If $f'(x)$ exists and is never zero, then f^{-1} exists, differentiable. Furthermore for $a \in I$, $f(a) = b$, then*

$$(f^{-1})'(b) = \frac{1}{f'(a)}.$$

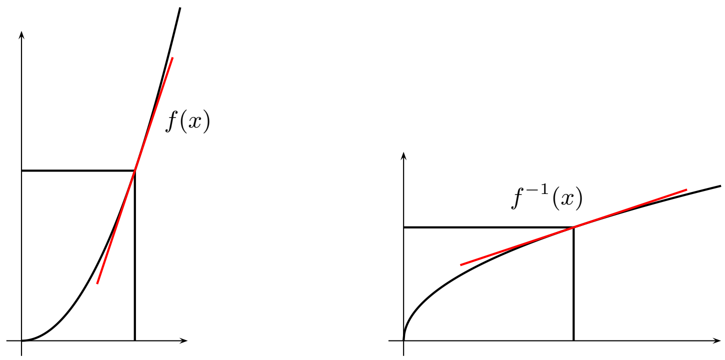


Figure 2.5: Slope of inverse function

Set $y = f(x)$. Then the inverse function is $x = f^{-1}(y)$, and its derivative is

$$\left.\frac{dx}{dy}\right|_{y=f(a)} = \frac{1}{dy/dx|_{x=a}}, \quad a \in I$$

Proof. Differentiate $x = (f^{-1} \circ f)(x)$ w.r.t x using the Chain rule

$$1 = (f^{-1})'(f(a))f'(a)$$

or

$$(f^{-1})'(b) = 1/f'(a)$$

□

Usually, we use the notation $y = f^{-1}(x)$. The graph of $y = f(x)$ and that of $y = f^{-1}(x)$ are symmetric w.r.t the line $y = x$.

Example 2.9.4. (1) $f(x) = x^3 - 2$. Find $(f^{-1})'$ at $f(2) = 6$

(2) $f(x) = x^7 + 8x^3 + 4x - 2$. Find $(f^{-1})'(-2)$.

sol. (1) Since $f' = 7x^6 + 24x^2 + 4 \geq 4$ inverse f^{-1} exists. Since $f(0) = -2$ we have

$$(f^{-1})'(-2) = (f^{-1})'(f(0)) = \frac{1}{f'(0)} = \frac{1}{4}.$$

(2) $y = \sin^{-1} x$, $x = \sin y$. Hence

$$\begin{aligned}\frac{d}{dx}\sin^{-1} x = \frac{dy}{dx} &= \frac{1}{dx/dy} = \frac{1}{(d/dy)\sin y} \\ &= \frac{1}{\cos y} = \frac{1}{\sqrt{1-\sin^2 y}} = \frac{1}{\sqrt{1-x^2}}.\end{aligned}$$

□

□

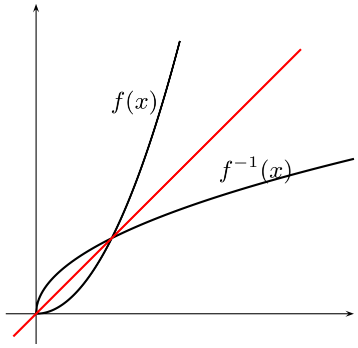


Figure 2.6: Graph of inverse function is symmetric about $y = x$

2.10 Logarithmic functions

Definition 2.10.1. The logarithmic function with base a, $y = \log_a x$ is the inverse function of $y = a^x (a > 0, a \neq 1)$

$$\log_e x \text{ is written as } \ln x \text{ and called } \textit{natural logarithmic function}$$

$$\log_{10} x \text{ is written as } \log x \text{ and called } \textit{common logarithmic function}$$

$$\ln x = y \Leftrightarrow e^y = x$$

Properties

(1) Product rule: $\log_a xy = \log_a x + \log_a y$

(2) Quotient rule: $\log_a \frac{x}{y} = \log_a x - \log_a y$

(3) Product rule: $\log_a \frac{1}{y} = -\log_a y$

(4) Power rule: $\log_a x^y = y \log_a x$

Example 2.10.2. f

Inverse properties

(1) Base a: $a^{\log_a x} = x$, $\log_a(a^x) = x (a > 0, a \neq 1, x > 0)$

(2) Base e: $e^{\ln x} = x$, $\ln(e^x) = x (x > 0)$

Hence (by substituting a^x for x)

$$\begin{aligned} a^x &= e^{\ln a^x} \\ &= e^{x \ln a} \\ &= e^{(\ln a) x} \end{aligned}$$

Every exponential function can be written as

$$a^x = e^{x \ln a}$$

Exponential growth

Example 2.10.3 (exponential growth)**.** It is used to model the amount of money to grow when the interest is compounded. Say P dollar is initially invested in the bank with interest rate of r per year, t is the time in years, then the total amount of money after t years is

$$y = Pe^{rt}$$

Example 2.10.4 (exponential decay)**.** It is used to model the amount of radioactive material. e.g, Carbon 14 case, If A is initial original amount of carbon 14.

$$y = Ae^{-1.2 \times 10^{-4} t}$$

Carbon 14 is used to detect how old is the dead organisms such as shells, wooden artifacts.

Change of base

$$\log_a x = \frac{\ln x}{\ln a} \quad (a > 0, a \neq 1, x > 0)$$

Derivatives of log function

Recall $f(x) = e^x$ iff $f^{-1}(x) = \ln x$.

$$\begin{aligned} (f^{-1})'(x) &= \frac{1}{f'(f^{-1}(x))} \\ &= \frac{1}{e^{f^{-1}(x)}} \quad (f'(u) = u) \\ &= \frac{1}{e^{\ln x}} \\ &= \frac{1}{x} \end{aligned}$$

Alternative way: $y = \ln x$ is written as $e^y = x$. Hence

$$\begin{aligned} \frac{d}{dx} e^y &= 1 \\ e^y \frac{dy}{dx} &= 1 \\ \frac{dy}{dx} &= \frac{1}{e^y} = \frac{1}{x} \end{aligned}$$

Example 2.10.5.

$$\frac{d}{dx}(\ln u) = \frac{1}{u}\frac{du}{dx}$$

$$\frac{d}{dx}\ln(x^4+2) = \frac{1}{x^4+2}\cdot 4x^3$$

Derivatives of $\ln|x|$

$\ln|x| = 1/x$

Derivatives of a^u

$a^x = e^{x\ln a}$. So

$$(a^x)' = e^{x\ln a}\ln a = a^x \ln a$$

In general,

$$\frac{d}{dx}a^u = a^u \ln a\frac{du}{dx}$$

Derivatives of $\log_a u$

$$\log_a x = \frac{\ln x}{\ln a}$$

So

$$\frac{d}{dx}\log_a x = \frac{1}{x\ln a}$$

and

$$\frac{d}{dx}\log_a u = \frac{1}{u\ln a}\frac{du}{dx}$$

Logarithmic Differentiation

Find dy/dx if $y = \frac{(x^2+1)^{1/3}(x-3)^{1/2}}{x+5}$

Derivatives of Power function

We prove the following theorem which was stated earlier.

Theorem 2.10.6. *Since* $u^r = e^{r\ln u}$

$$\frac{d}{dx}u^r = u^r\frac{d\ln u}{dx} = u^r\frac{1}{u}\frac{du}{dx} = u^{r-1}\frac{du}{dx}.$$

Example 2.10.7. Differentiate $f(x) = x^x, x > 0$

sol. Write $f(x) = x^x = e^{x\ln x}$. So

$$\begin{aligned} f'(x) &= \frac{d}{dx}(e^{x\ln x}) \\ &= (e^{x\ln x})\frac{d}{dx}(x\ln x) \\ &= e^{x\ln x}(\ln x + x\cdot\frac{1}{x}) \\ &= x^x(\ln x + 1) \end{aligned}$$

□

The number e

In section 2.6 we saw e was defined so that it satisfies

$$\lim_{h\to 0}\frac{e^h - 1}{h} = \ln e = 1.$$

An important property is the following.

Theorem 2.10.8. *The number e satisfies*

$$e = \lim_{x\to 0}(1+x)^{1/x}.$$

Proof. If $f(x) = \ln x$. Then $f'(1) = \frac{1}{x}|_{x=1} = 1$. By definition,

$$1 = f'(1) = \lim_{x\to 0^+}\frac{\ln(1+x) - \ln 1}{x} = \ln[\lim_{x\to 0^+}(1+x)^{\frac{1}{x}}].$$

Now exponentiate. □

2.11 Inverse trig functions

Example 2.11.1. Half life of Polonium 210. The time for radioactive substance required to decay by half is independent of the initial quantity. $y = y_0e^{-kt}$. So $t = \ln 2/k$. For Polonium 210, $k = 5\cdot 10^{-3}$.

Inverse sine

Restrict the function $\sin x$ on $[-\pi/2, \pi/2]$. Then $\sin x\colon [-\pi/2, \pi/2] \to [-1, 1]$ is one-to - one function. So the inverse exists. Define

$$\sin^{-1} x\colon [-1, 1] \longrightarrow [-\pi/2, \pi/2].$$

whenever $x = \sin y$ for $x \in [-\pi/2, \pi/2]$. Graph is as in figure 2.7. $\sin^{-1} x$ is sometimes written as $\arcsin x$.

Example 2.11.2. (1) $\sin^{-1}(1/2) = \pi/6$

(2) $\sin^{-1} 1 = \pi/2$

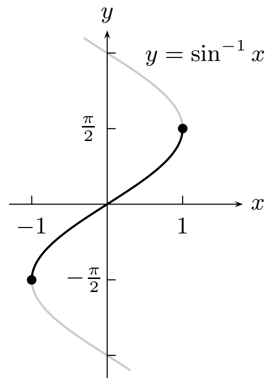


Figure 2.7: $y = \sin^{-1} x$

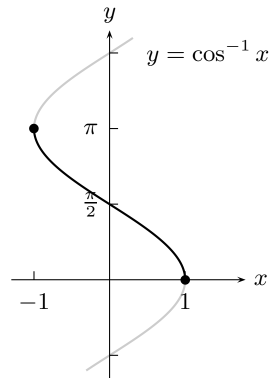


Figure 2.8: $y = \cos^{-1} x$

Inverse cosine

Restrict $\cos x$ to $[0, \pi]$, we obtain $\cos^{-1} x$.

$$\cos^{-1} x \colon [-1, 1] \longrightarrow [0, \pi]$$

If $\cos x = y$ for any $x \in [0, \pi]$ then $\cos^{-1} y = x$ is defined and figure is in 2.8 written as $\cos^{-1} x$ or $\arccos x$.

Example 2.11.3. (1) $\cos^{-1}(1/2) = \pi/3$

(2) $\cos^{-1} 0 = \pi/2$

Example 2.11.4.

$$\sin^{-1} x + \cos^{-1} x = \frac{\pi}{2}, \quad \cos^{-1} x + \cos^{-1}(-x) = \pi$$

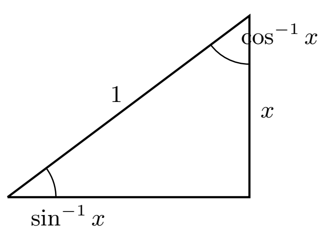


Figure 2.9: $\sin^{-1} x$

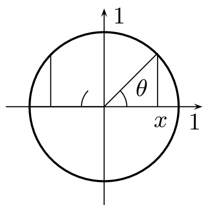


Figure 2.10: $\theta = \cos^{-1} x$

Inverse of $\tan x$

On $(-\pi/2, \pi/2)$ $\tan x$ has inverse $\tan^{-1} x$

$$\tan^{-1} x \colon \mathbb{R} \longrightarrow (-\pi/2, \pi/2)$$

for any $x \in \mathbb{R}$, $\tan x = \alpha$ iff $\tan^{-1} \alpha = x$. See figure 2.11. It is written as $\tan^{-1} x$ or $\arctan x$.

$\tan^{-1} 1 = \pi/4$ $\tan^{-1} 0 = 0$

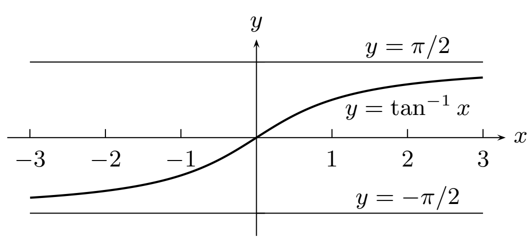


Figure 2.11: $y = \tan^{-1} x$

Example 2.11.5. Find the derivative of $\tan^{-1} x$.

From $y = f(x) = \tan x$, we see

$$\begin{aligned}(f^{-1})'(y) &= \frac{1}{f'(x)} \\ &= \frac{1}{1 + \tan^2 x} \\ &= \frac{1}{1 + y^2}\end{aligned}$$

Thus $(f^{-1})'(x) = \frac{1}{1+x^2}$.

Example 2.11.6. Find derivatives

(1) $y = \sin^{-1} x, \quad (|x| \le 1)$.

(2) $y = \sec^{-1} x, \quad (|x| \ge 1)$.

sol. (3) Let $y = \sec^{-1} x$. Then $x = \sec y$. Taking derivative w.r.t x we get $1 = \sec y \tan y (dy/dx)$. Thus

$$\frac{dy}{dx} = \frac{1}{\sec y \tan y}.$$

We need to change it to expression in x.

For $x > 1$, $\tan y = \sqrt{x^2 - 1}$. Hence, we have

$$\frac{dy}{dx} = \frac{1}{x\sqrt{x^2-1}}, \quad x > 1.$$

For $x < -1$, use $(x \to -x)$ to get

$$\frac{dy}{dx} = \frac{1}{-x\sqrt{x^2-1}}, \quad x < -1$$

Hence

$$\frac{d}{dx} \sec^{-1} x = \frac{1}{|x|\sqrt{x^2-1}}, \quad |x| > 1$$

□

Proposition 2.11.7. *Similarly, we get the derivatives of inverse trig. function*

(1) $\displaystyle \frac{d}{dx} \sin^{-1} x = \frac{1}{\sqrt{1-x^2}}$

(2) $\displaystyle \frac{d}{dx} \cos^{-1} x = -\frac{1}{\sqrt{1-x^2}}$

(3) $\displaystyle \frac{d}{dx} \tan^{-1} x = \frac{1}{1+x^2}$

(4) $\displaystyle \frac{d}{dx} \csc^{-1} x = -\frac{1}{|x|\sqrt{x^2-1}}, \ |x| > 1$

(5) $\displaystyle \frac{d}{dx} \sec^{-1} x = \frac{1}{|x|\sqrt{x^2-1}}, \ |x| > 1$

(6) $\displaystyle \frac{d}{dx} \cot^{-1} x = -\frac{1}{1+x^2}$

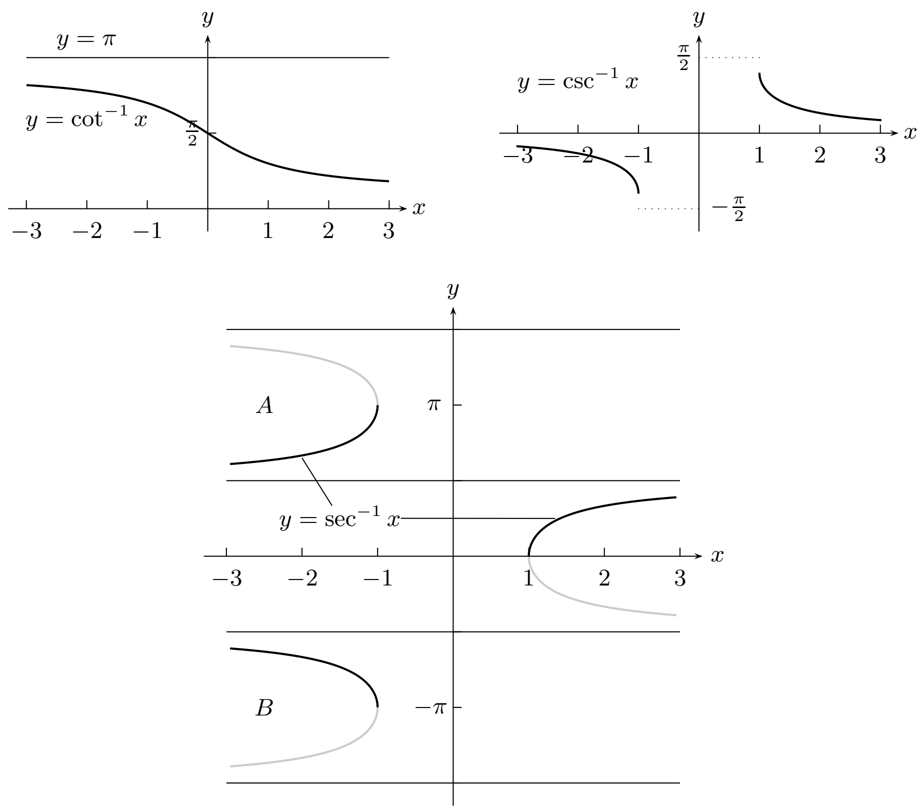


Figure 2.12:

Other inverse trig function

Inverses of $\csc x$, $\sec x$, $\cot x$

$\csc^{-1} x : \mathbb{R} - (-1, 1) \to [-\pi/2, \pi/2] - \{0\}$

$\sec^{-1} x : \mathbb{R} - (-1, 1) \to [0, \pi] - \{\pi/2\}$

$\cot^{-1} x : \mathbb{R} \to (0, \pi)$

We see the following relation hold

Proposition 2.11.8. *(1)* $\cot^{-1} x = (\pi/2) - \tan^{-1} x$

(2) $\sec^{-1} x = \cos^{-1}(1/x)$

(3) $\csc^{-1} x = \sin^{-1}(1/x)$

Example 2.11.9. (1) Find $\sin(\cos^{-1}(3/5))$

(2) Simplify $\tan(\sin^{-1} a)$

sol. (1) Let $\theta = \cos^{-1}(3/5)$. Then $\cos\theta = 3/5$ and $0 \le \theta \le \pi$

$$\sin\theta = \sqrt{1 - \frac{9}{25}} = \frac{4}{5}.$$

(2) Let $\theta = \sin^{-1} a$. Then $\sin\theta = a$ and $-\pi/2 \le \theta \le \pi/2$

$$\cos\theta = \sqrt{1-a^2}.$$

Hence

$$\tan\theta = \sin\theta/\cos\theta = a/\sqrt{1-a^2}.$$ □

2.12 Related rates

Related rates

y is a function of x and $x(x = x(t))$ is a function of time t the rate of change dy/dx and dy/dt satisfies $dy/dt = (dy/dx)(dx/dt)$.

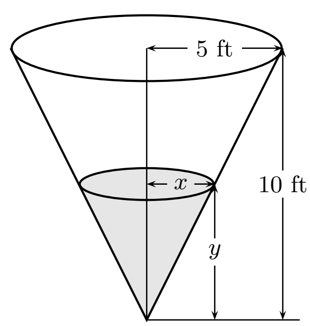


Figure 2.13: Conic Tank

Example 2.12.1. Figure 2.13. Water is being poured into a conical tank at the rate of 9ft^3/min. Find rate of rising water level when depth of water is 6ft.

sol.

(1) V : Volume of water at t

(2) x: Radius of water surface at t

(3) y: Depth of water at t

We have $x = y/2$

$$V = \frac{1}{3}\pi x^2 y = \frac{\pi}{12}y^3,$$
$$\frac{dV}{dt} = \frac{\pi}{4}y^2\frac{dy}{dt}$$

Substitute $dV/dt = 9$, $y = 6$. Then from $9 = \pi/436 \cdot dx/dt|_{y=6}$, we get $dx/dt|_{y=6} = 1\pi$.

□

Example 2.12.2. Rocket is rising at the speed of 300m/sec vertically. At 1000m above the ground, a camera man is watch 1000m away. Find the rate of change of this camera angle θ.

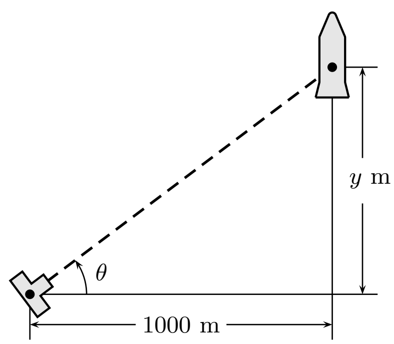


Figure 2.14: A camera following the tip of a rising rocket

sol. From figure 2.14 we see the angle of elevation θ and the height y is related by

$$\tan\theta = \frac{y}{1000}$$

Differentiating w.r.t t

$$\sec^2\theta\frac{d\theta}{dt} = \frac{1}{1000}\frac{dy}{dt}$$

When

$$y = 1000, \quad \theta = 45^\circ, \quad \frac{dy}{dt} = 300$$
$$\left.\frac{d\theta}{dt}\right|_{y=1000} = \frac{300}{1000}\cdot\frac{1}{2}\cdot\frac{180}{\pi} \approx 8.59(\text{degree/sec})$$

□

2.13 Linearization and differential

Definition 2.13.1.

$$L(x) = f(a) + f'(a)(x - a)$$

is called the **linearization** of f at a.

Example 2.13.2. (1) Find linearization of $\cos x$ at $\pi/2$

(2) Find appro value of $\sqrt{1.003}$ using linearization of $\sqrt{1+x}$ $x = 0$.

(3) Find linearization of $\frac{1}{\sqrt[3]{x^4+1}}$ at $x = 0$

(4) Find appro value of $\sqrt{4.8}$

(5) Find linearization of $\cos x$ at $\pi/2$. Ans $-x + \pi/2$.

(6) Find linearization of $(1+x)^k$. $1 + kx$

Differential

Definition 2.13.3. Let $y = f(x)$ be differentiable. The differential dx is an independent variable. The quantity dy defined by

$$dy := f'(x)dx$$

is called the **differential** of f.

The geometric meaning of differential is given in Figure 3.8.

$$\Delta y = f(a + dx) - f(a), \quad f(a + dx) = f(a) + \Delta y \approx f(a) + dy$$

We see that dy is precisely the change of the tangent line as x changes by an amount of $dx = \Delta x$. In other words, dy is an approximation of exact change Δy.

Example 2.13.4. Find differential of

(1) $y = x^3 - \sin x$

(2) $y = \sin u(x)$

(3) $\tan(3x)$

(4) $d(\frac{x}{1+x})$.

Estimating with differentials

Radius of a circle is enlarged from 10 to 10.1. Use dA to estimate the increase in area. Compare with exact increase.

$A = \pi r^2$,

$$dA = 2\pi r dr = 2\pi(10)(0.1) = 2\pi m^2$$

Actual increase is $A(10.1) - A(10) = 2\pi((10.1)^2 - 100) = 2.01\pi$.

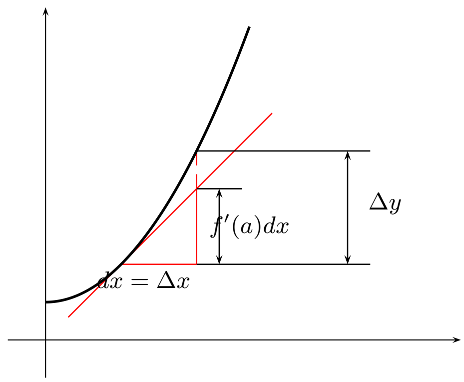


Figure 2.15: Differential $dy = f'(a)dx$ and Δy

Error in differential approximation

We estimate the change in y in more detail.

Theorem 2.13.5. *We have*

$$\Delta f = f'(a)\Delta x + \epsilon \Delta x$$

where $\epsilon \to 0$ as $\Delta x \to 0$.

Proof.

$$\begin{aligned}
\text{approximation error} &= \Delta f - df \\
&= \Delta f - f'(a)\Delta x \\
&= f(a+\Delta x) - f(a) - f'(a)\Delta x \\
&= \left(\frac{f(a+\Delta x) - f(a)}{\Delta x} - f'(a)\right)\Delta x \\
&= \epsilon \Delta x
\end{aligned}$$

Since f is differentiable, we know $\epsilon := \left(\frac{f(a+\Delta x)-f(a)}{\Delta x} - f'(a)\right)$ approaches 0 as Δx approaches 0. Thus

$$\underset{\substack{\text{true} \\ \text{change}}}{\Delta f} = \underset{\substack{\text{estimated} \\ \text{change}}}{f'(a)\Delta x} + \underset{\text{error}}{\epsilon \Delta x}$$

□

Proof of Chain rule

Assume $y = f(u)$ is a diff'ble function of u and $u = g(x)$ is a diff'ble function of x. Then the composite function $y = f(g(x))$ is diff'ble and by theorem there exist ϵ_1, ϵ_2 which approaches 0 as $\Delta u, \Delta x$ approaches 0 in such a way that

$$\begin{aligned} \Delta y &= f'(u_0)\Delta u + \epsilon_2 \Delta u \\ \Delta u &= g'(x_0)\Delta x + \epsilon_1 \Delta x \end{aligned}$$

Hence

$$\Delta y = (f'(u_0) + \epsilon_2)(g'(x_0) + \epsilon_1)\Delta x$$

$$\frac{\Delta y}{\Delta x} = (f'(u_0) + \epsilon_2)(g'(x_0) + \epsilon_1)$$

Let $\Delta x \to 0$. Then we obtain the Chain rule.

Example 2.13.6. Converting mass to energy: The Newton's law

$$F = m\frac{dv}{dt} = ma$$

is not exactly true when an object is moving at very high speed, because the mass increases with velocity. In Einstein's correction, the mass is

$$m = \frac{m_0}{\sqrt{1 - v^2/c^2}} \approx m_0(1 + \frac{v^2}{2c^2})$$

So the new mass is

$$m \approx m_0 + \frac{mv^2}{2c^2}$$

By multiplying c^2

$$(m - m_0)c^2 \approx \frac{1}{2}mv^2 - \frac{1}{2}m0^2 = \Delta(KE)$$

Thus the change in the mass corresponds to the change in the Kinetic Energy.

Power function

Derivative of a power function$y = u^r(x)$ for rational number r.

Theorem 2.13.7. *For any rational number r*

$$\frac{d}{dx}u^r = ru^{r-1}\frac{du}{dx}.$$

Proof. Assume $r > 0$. We have $r = p/q$ for some positive integer p and q. Hence $y = u^r = r^{p/q}$ can be written as $y^q = u^p$. Taking derivative w.r.t. x we obtain

$$qy^{q-1}\frac{dy}{dx} = pu^{p-1}\frac{du}{dx}$$

$$\frac{dy}{dx} = \frac{pu^{p-1}}{qy^{q-1}}\frac{du}{dx} = \frac{p}{q}\frac{y}{u}\frac{du}{dx} = ru^{r-1}\frac{du}{dx}$$

□

Example 2.13.8. Find the equation of tangent and normal line to the curve $y = (1 + x^{1/3})^{2/3}$ at $(-8, 1)$.

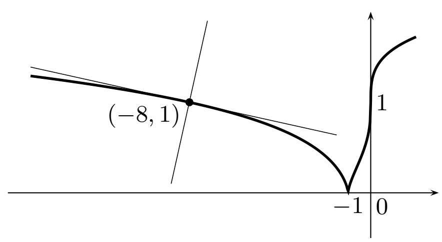


Figure 2.16: Tangent and normal

sol.

$$y'\Big|_{x=-8} = \frac{2}{3}(1 + x^{1/3})^{-1/3} \cdot \frac{1}{3}x^{-2/3}\Big|_{x=-8} = -\frac{1}{18}$$

Tangent line is $y = -1/18(x + 8) + 1 = -x/18 + 5/9$ normal line is $y = 18(x + 8) + 1 = 18x + 145$.

□

Example 2.13.9 (Slope of tangent in Polar coordinate)**.** Express dy/dx for the equation given in polar coordinate $r = f(\theta)$ in θ.

sol. Using the relation $x = r\cos\theta, y = r\sin\theta$, we see $x = f(\theta)\cos\theta, y = f(\theta)\sin\theta$. Hence the slope of tangent is

$$\frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta} = \frac{f'(\theta)\sin\theta + f(\theta)\sin\theta}{f'(\theta)\cos\theta - f(\theta)\sin\theta}$$

□

Chapter 3

Applications of Derivatives

3.1 Absolute Maximum, Minimum

Definition 3.1.1. Let f be defined on a domain D. Then f has **absolute(global) maximum** at c if

$$f(x) \leq f(c), \quad \text{for all } x \in D.$$

It has **absolute(global) minimum** at c if

$$f(x) \geq f(c), \quad \text{for all } x \in D.$$

Theorem 3.1.2. *[Extreme Value Theorem] If f is continuous on a closed interval $[a, b]$. Then f assumes both absolute maximum M and absolute minimum m in $[a, b]$. In other words, there are numbers x_1, x_2 in $[a, b]$ such that $f(x_1) = m$ and $f(x_2) = M$, and $m \leq f(x) \leq M$ for all $x \in [a, b]$.*

Maximum or minimum are called **extreme values.**

Definition 3.1.3. Let f be defined on a domain D. Then f has **relative(local) maximum** at c if

$$f(x) \leq f(c), \quad \text{for all } x \text{ in some interval containing } c.$$

It has **relative(local) minimum** at c if

$$f(x) \geq f(c), \quad \text{for all } x \text{ in some interval containing } c.$$

Theorem 3.1.4 (First derivative theorem)**.** *Suppose f is differentiable and if f has local max(min) at an interior point c then $f'(c) = 0$.*

Proof. Suppose $f'(c) > 0$ there is an interval near c such that for all x in $I = (c - \delta, c + \delta)$

$$\frac{f(x) - f(c)}{x - c} > 0$$

hold. If $x \in I$, $x > c$

$$f(x) - f(c) = (x - c)\frac{f(x) - f(c)}{x - c} > 0$$

$f(x) > f(c)$ Hence f cannot have maximum at c. If $x < c$

$$f(x) - f(c) = (x - c)\frac{f(x) - f(c)}{x - c} < 0$$

f cannot have minimum at c. So f cannot have local extreme at c The case $f'(c) < 0$ is similar. Hence $f'(c) = 0$. □

Remark 3.1.5. This is not a necessary condition for a function to have a local extreme. Often, a function has an extreme value where f is not differentiable! So the points where f is not differentiable is also point of interest. Thus, we define

Definition 3.1.6. If $f'(c) = 0$ or $f'(c)$ does not exists, we say c is **critical point** of f.

Remark 3.1.7. How to find Absolute max(min) on I

(1) Evaluate f at all critical points

(2) Check all end points and compare

Example 3.1.8. Find max(min) of $f(x) = \left|4 - x^2\right|$ on $[-3, 3]$.

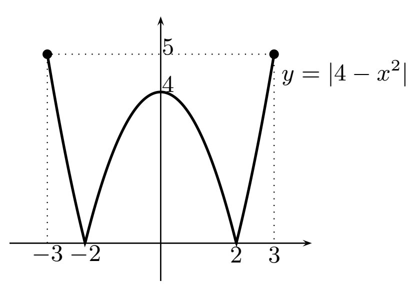


Figure 3.1: $y = |4 - x^2|$

sol. By Theorem 3.1.2 f has absolute minimum and maximum. Its graph is as in Figure 3.1The critical points of f are -2, 0, 2. $f(-2) = f(2) = 0$, $f(0) = 4$ while at end points f assumes $f(-3) = f(3) = 5$. Hence maximum is 5 (at $x = -3$ or $x = 3$) and minimum is 0 (at $x = -2$ or $x = 2$).

□

Example 3.1.9. Find absolute extrema of $f(x) = 10x(2 - \ln x)$ on $[1, e^2]$.

sol. $f'(x) = 10(1 - \ln x)$. So critical point is e. Fig 3.1 Check end points $1, e^2$. We see maximum is

□

Example 3.1.10. Find absolute extrema of $f(x) = x^{2/3}$ on $[-2, 3]$.

sol. Derivative at 0 does not exist. By drawing graph we see $x = 0$ is cusp but local and absolute minimum.

□

Example 3.1.11. Find exrtema of $f(x) = x^{2/3}(1 - x)^{3/2}$ $(x \le 1)$.

sol. Since

$$\begin{aligned} f'(x) &= \frac{2}{3}x^{-1/3}(1-x)^{3/2} - \frac{2}{3}x^{2/3}(1-x)^{1/2} \\ &= x^{-1/3}(1-x)^{1/2}\left(\frac{2}{3}(1-x) - \frac{3}{2}x\right) \\ &= \frac{\sqrt{1-x}(4-13x)}{6x^{1/3}} \end{aligned}$$

Extreme points are $x = 0$, $x = 4/13$. Here $f(0) = 0$ is local min and $f(4/13) = (4/13)^{2/3}(9/13)^{3/2}$ is local max. (Refer to Fig 3.2)

□

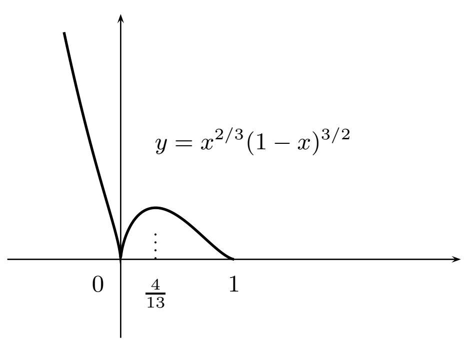


Figure 3.2: $y = x^{2/3}(1 - x)^{3/2}$

Remark 3.1.12. In above example the point $(0,0)$ satisfies

$$\lim_{x\to 0^+} f'(x) = +\infty, \quad \lim_{x\to 0^-} f'(x) = -\infty$$

This kind of point is called a **cusp** .

Example 3.1.13. Piping from Oil hole from the see to the refinery on the shore.

Underwater pipe cost 500,000 per mile while land pipe cost 300,000 per mile

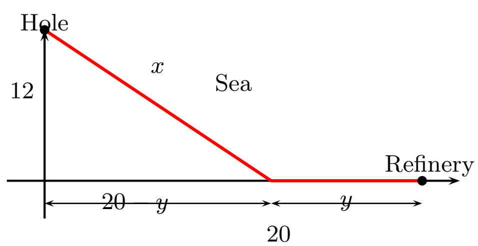


Figure 3.3: Refinery and hole in the sea

sol. cost is $c = 500,000x + 300,000y$. But $x^2 = 12^2 + (20-y)^2$. Thus $R(y)\cdot 10^{-5} = 5\sqrt{144+(20-y)^2}$

$$R'(y) = \frac{-5(20-y)}{\sqrt{144+(20-y)^2}} + 3$$

$y = 11$.

□

3.2 Mean Value Theorem

Theorem 3.2.1 (Rolle's Theorem)**.** *If f is continuous on $[a,b]$ and differentiable on (a,b). Suppose $f(a) = f(b)$ holds then there exists a $c \in (a,b)$ such that $f'(c) = 0$.*

Proof. By theorem 3.1.2, f must attain maximum and minimum at some point c. Then there are two possibilities:

(1) max or min occurs at interior point.

(2) both max or min occur at end points.

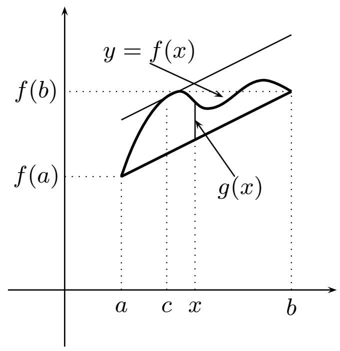


Figure 3.4:

In the first case, by theorem 3.1.4 there is a point where $f'(c) = 0$. In the second case, the maximum is equal to minimum since $f(a) = f(b)$. Hence f is constant and $f'(c) = 0$ for any $c \in (a, b)$. □

Theorem 3.2.2 (Mean Value Theorem)**.** *If f is continuous on $[a, b]$ and differentiable on (a, b). Then there exists a $c \in (a, b)$ such that*

$$f'(c) = \frac{f(b) - f(a)}{b - a}$$

Proof. Consider the line given by the equation

$$g(x) = \left(f(a) + \frac{f(b) - f(a)}{b - a}(x - a) \right)$$

Then (3.4 .)

$$h(x) = f(x) - g(x) = f(x) - \left(f(a) + \frac{f(b) - f(a)}{b - a}(x - a) \right)$$

h continuous on $[a, b]$ diff'ble in (a, b) and $h(a) = h(b)$. Hence by Thm 3.3.9 (Rolle's) there exists $c \in (a, b)$ such that $h'(c) = 0$. That is

$$h'(c) = f'(c) - \frac{f(b) - f(a)}{b - a}$$

□

Application of MVT

Theorem 3.2.3. *(1) If f satisfies $f'(x) = 0$ on an interval I, then f is constant on I.*

(2) If $f'(x) = g'(x)$ on I, then $f(x) = g(x) + C$, $x \in I$

Proof. (1) By Mean Value Theorem 3.2.2, we have for any $x, z \in I$,

$$\frac{f(z) - f(x)}{z - x} = f'(c)$$

for some $c \in (x, z) \subset I$. Since $f'(c) = 0$, we have $f(z) = f(x)$. This holds for any x, z, hence f is constant.

□

Definition 3.2.4. Suppose F defined on I. satisfies $F'(x) = f(x)$, F is and **anti-derivative** of f.

Example 3.2.5. Show that $\pi/4 + 3/25 < \tan^{-1} 4/3 < \pi/4 + 1/6$ holds.

sol. Apply Thm 3.2.2(MVT) to

$$f(x) = \tan^{-1} \text{ on } [1, 4/3]$$

Since $f'(x) = 1/(1 + x^2)$, there is a point $c \in (1, 4/3)$ such that

$$\frac{\tan^{-1}(4/3) - \pi/4}{4/3 - 1} = \frac{1}{1 + c^2}$$

holds. Hence

$$\frac{1}{1 + (4/3)^2} < 3\left(\tan^{-1}\frac{4}{3} - \frac{\pi}{4}\right) < \frac{1}{1 + 1^2}$$

$$\frac{3}{25} < \tan^{-1}\frac{4}{3} - \frac{\pi}{4} < \frac{1}{6}.$$

□

□

Proof of Log rule $\ln bx = \ln b + \ln x$

Consider

$$\frac{d}{dx}\ln(bx) = \frac{1}{x} = \frac{d}{dx}\ln x$$

So by above result,

$$\ln(bx) = \frac{1}{x} = \ln x + C$$

Place $x = 1$ to see $C = \ln b$.

Proof of Log rule $\ln^r x = r \ln x$

Consider
$$\frac{d}{dx}\ln^r x = \frac{1}{x^r}\frac{d}{dx}(x^r) = \frac{1}{x^r} r x^{r-1} = \frac{r}{x} = \frac{d}{dx}(r \ln x)$$
Thus $\ln^r x$ and $r \ln x$ have same derivative.

3.3 Monotonic function and derivative

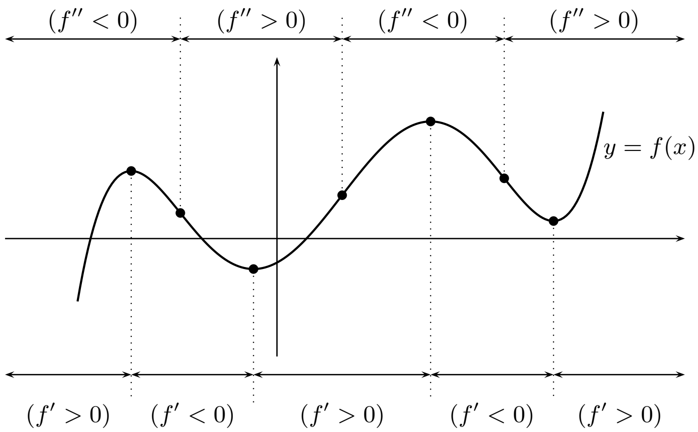


Figure 3.5: · ·

Definition 3.3.1. f is said to be a **increasing function** if $f(x) \le f(z)$ holds for all all x, z $(x < z)$.

First derivative test

Theorem 3.3.2. *(1) Suppose $f'(x) > 0$ for all x, then f is increasing on I.*

(2) Suppose $f'(x) < 0$ for all x, then f is decreasing.

Proof.
$$f'(c) = \frac{f(z) - f(x)}{z - x}$$
for some $c \in (x, z)$. Since $f'(c) \ge 0$ and $z > x$ we see $f(z) \ge f(x)$. If f is not one-to-one, there exists two point $v, w \in I$, $v < w$ such that $f(v) = f(w)$. Then $f'(x) = 0$ on (v, w). This contradicts to $f'(x) = 0$ at finite points. □

Example 3.3.3. Investigate the increase and decrease of $f(x) = x^5 - 5x^4 + 5x^3 + 1$.

sol. $f'(x) = 5x^4 - 20x^3 + 15x^2 = 5x^2(x-1)(x-3)$ We obtain the following table.

x	$\cdots$	1	$\cdots$	0	$\cdots$	3	$\cdots$
f'	+	0	−	0	−	0	+
f	$\nearrow$	2	$\searrow$	1	$\searrow$	−26	$\nearrow$

Hence f is mon. inc. on $x \leq 1$ and mon. dec on $1 \leq x \leq 3$, mon. inc. on $3 \leq x$.

□

First derivative test for local extrema

Theorem 3.3.4. *(1) If f' changes from negative to positive at c then f has local minimum at c*

(2) If f' changes from positive to negative at c then f has local maximum at c

Proof. f is decreasing on $[c-\delta, c]$ and increasing on $[c, c+\delta]$. Hence, $f(x)$ has local minimum at c. □

Example 3.3.5. (1) $f(x) = (x^2-3)e^x$

(2) $x^{1/3}(x-4)$

Second derivative test for extreme values

Theorem 3.3.6. *Suppose $f'(c) = 0$. Then*

(1) $f(c)$ is local maximum if $f''(c) < 0$.

(2) $f(c)$ is local minimum if $f''(c) > 0$.

Proof. (1) Since

$$f''(c) = \lim_{x \to c} \frac{f'(x)}{x-c} < 0$$

there is a δ such that

$$\frac{f'(x)}{x-c} < 0$$

for all $x \in (c-\delta, c+\delta)$ $(x \neq c)$. Hence if $c - \delta < x < c$ then $f'(x) > 0$, and if $c < x < c+\delta$ then $f'(x) < 0$. Hence f' change from positive to negative at c. By theorem 3.3.4, $f(c)$ is a local maximum. □

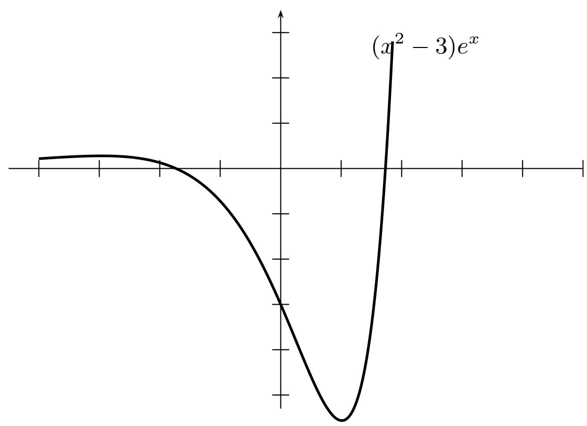


Figure 3.6: $y = (x^2 - 3)e^x$

3.4 Concavity and sketching

Definition 3.4.1. Graph of $y = f(x)$ is

(1) **concave up** if f' is increasing on I,

(2) **concave down** if f' is decreasing on I,

(3) a **inflection point** if the concavity of f changes across a point c.

Theorem 3.4.2. *Suppose f'' exists on I. Then*

(1) f is concave up, if $f''(x) > 0$ for all $x \in I$,

(2) f is concave down if $f''(x) < 0$ for all $x \in I$.

Example 3.4.3. Sketch $f(x) = x + \sin x$.

sol. $f'(x) = 1 + \cos x,\ f''(x) = -\sin x$

x	$\cdots$	$-\pi$	$\cdots$	0	$\cdots$	π	$\cdots$	2π	$\cdots$	3π	$\cdots$
f'	$\cdots$	0	$+$	$+$	$+$	0	$+$	$+$	$+$	0	$\cdots$
f''	$\cdots$	0	$+$	0	$-$	0	$+$	0	$-$	0	$\cdots$
f	$\cdots$		$\nearrow$		$\nearrow$		$\nearrow$		$\nearrow$		$\cdots$

Hence f is increasing for all x and for integer n concave down on $(2n\pi, 2n\pi+\pi)$ concave up on $(2n\pi + \pi, 2n\pi + 2\pi)$. Points of inflection are $(n\pi, n\pi)$. (n is integer)

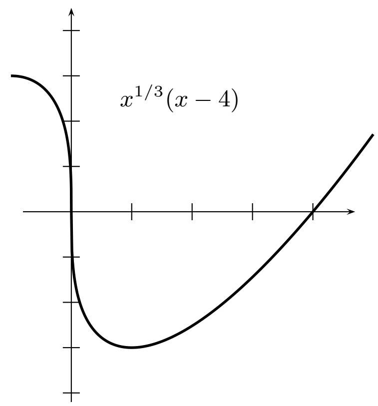


Figure 3.7: $y = x^{1/3}(x-4)$

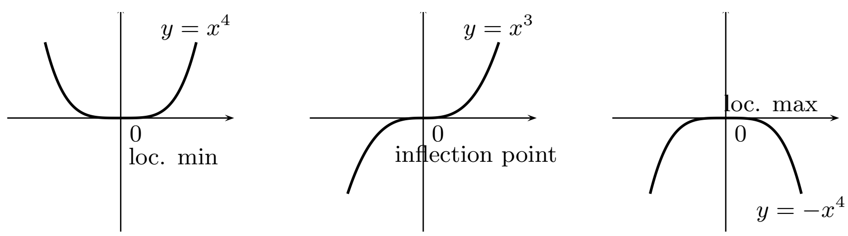


Figure 3.8: $y = x^4,\ y = x^3,\ y = -x^4$

□

Example 3.4.4. Show $f(\pi/4)$ is a local maximum when $f(x) = \sin^4 x \sin 4x + \cos^4 x \cos 4x$.

sol. Compute f' and f'' at $x = \pi/4$.

$$f'(x) = 4(\sin^3 x - \cos^3 x)\sin 5x,$$
$$f''(x) = 12 \sin x \cos x(\sin x + \cos x)\sin 5x + 20 \cos 5x(\sin^3 x - \cos^3 x)$$

Since $f'(\pi/4) = 0$ and $f''(\pi/4) = -6 < 0$. Hence $f(\pi/4)$ is local maximum.

□

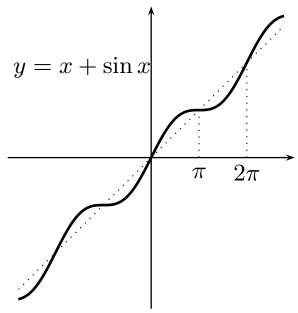


Figure 3.9: $y = x + \sin x$

Example 3.4.5. $y = x^4$ An inflection point may not exists when $y'' = 0$.

Example 3.4.6. For $y = x^{1/3}$ an inflection point may exist even if y'' does not exist.

Example 3.4.7. Sketch $f(x) = \frac{(x+1)^2}{1+x^2}$. Check inflection point and horizontal asymptote.

Example 3.4.8. Sketch $f(x) = e^{1/2x}$. Check inflection point and horizontal asymptote.

sol. Since f is not defined at $x = 0$, we investigate the behavior as $x \to^{\pm}$. Next we compute the derivative of $f = e^{\frac{1}{2x}}$. Since $f' = -\frac{1}{2x^2}e^{\frac{1}{2x}}$ we see f is decreasing function where it is defined. Now check second derivative:

$$f'' = \frac{1}{2x^3}\frac{4x+1}{2x}e^{\frac{1}{2x}}$$

Thus $x = -1/4$ is a point of inflection.

□

Theorem 3.4.9. *Suppose f'' continuous on I.*

(1) If $f'(c) = 0$ and $f''(c) < 0$ then f has local maximum

(2) If $f'(c) = 0$ and $f''(c) > 0$ then f has local minimum

(3) If $f'(c) = 0$ and $f''(c) = 0$ then the test fails. We need more information.

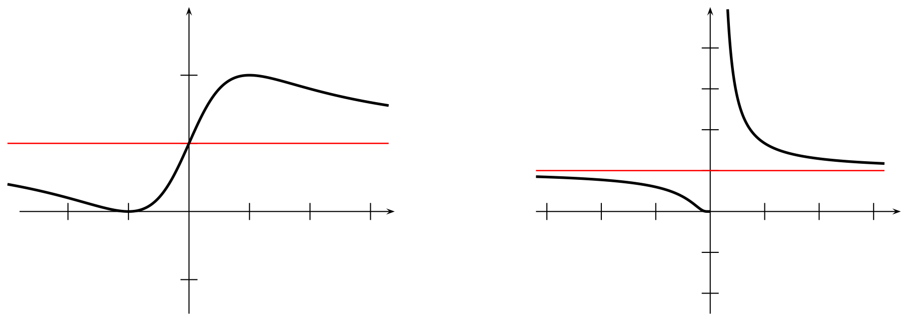


Figure 3.10: $y = \frac{(x+1)^2}{1+x^2}$ and $y = e^{\frac{1}{2x}}$

3.5 Parametrization

parametric equation

Suppose x, y are given by parametric equations $x = f(t)$, $y = g(t)$ for $t \in I$. Furthermore, f and g are differentiable and $f' \neq 0$. Then $t = f^{-1}(x)$ exists and $y(x) = (g \circ f^{-1})(x)$ is well defined. By implicit function theorem, the derivative of y w.r.t x is given by

$$\frac{dy}{dx} = \frac{dg}{dt} \cdot \frac{dt}{dx} = \frac{dg/dt}{df/dt}$$

Example 3.5.1. Sketch the path traced by the point $P(x, y)$ where
$x = t + \frac{1}{t}$, $y = t - \frac{1}{t}$.

Second derivative

If $x = f(t)$, $y = g(t)$ define y as a twice differentiable function of x at the point where $dy/dx \neq 0$, then

$$\begin{aligned} \frac{d^2y}{dx^2} &= \frac{d}{dt}\left(\frac{dy}{dx}\right) / \frac{dt}{dx} \\ &= \frac{dy'/dt}{dx/dt} \end{aligned}$$

Example 3.5.2. Compute dy/dx and d^2y/dx^2 when $x = t - t^2$, $y = t - t^3$.

Remark: Do not attempt to find $\frac{d}{dx}(\frac{dt}{dx})$.

sol. Chain rule and implicit differentiation

$$
\begin{aligned}
\frac{dy}{dx} &= \frac{dy/dt}{dx/dt} = \frac{1-3t^2}{1-2t} \\
\frac{d^2y}{dx^2} &= \frac{d}{dt}\left(\frac{dy}{dx}\right)\frac{dt}{dx} \\
&= \frac{d}{dt}\left(\frac{1-3t^2}{1-2t}\right)\cdot(1-2t) \\
&= \frac{2-6t+6t^2}{(1-2t)^3}.
\end{aligned}
$$

□

3.6 Applied Optimziation

Skip

3.7 Intermediate form aand L'Hopital's Rule

L'Hopital's Rule

When $f(a) = g(a) = 0$ or $f(a) = g(a) = \infty$, tshe limit

$$\lim_{x\to a}\frac{f(x)}{g(x)}$$

cannot be found by substituting a

Theorem 3.7.1 (L'Hopital's Rule: First form)**.** *Suppose that $f(a) = g(a) = 0$, that $f'(a)$, $g'(a)$ exist, then*

$$\lim_{x\to a}\frac{f(x)}{g(x)} = \frac{f'(a)}{g'(a)}.$$

Proof. $\displaystyle\lim_{x\to a}\frac{f(x)}{g(x)} = \frac{\lim_{x\to a}(f(x)-f(a))/(x-a)}{\lim_{x\to a}(g(x)-g(a))/(x-a)} = \frac{f'(a)}{g'(a)}.$ □

Example 3.7.2. (1) $\displaystyle\lim_{x\to 0}\frac{\sqrt{1+x}-1}{x}\left(\frac{0}{0}\right) = \frac{1/2\sqrt{1+x}}{1}\bigg|_{x=0} = \frac{1}{2}.$

(2) $\displaystyle\lim_{x\to 1}\frac{x^2-1}{x-1}\left(\frac{0}{0}\right) = \frac{2x}{1}\bigg|_{x=1} = 2.$

Example 3.7.3. (1) $\lim_{x\to(\pi/2)^-} \frac{\cos x}{\sin x - 1} \left(\frac{0}{0}\right) = \lim_{x\to(\pi/2)^-} \frac{-\sin x}{\cos x} = -\infty.$

(2) $\lim_{x\to 0}\left(\frac{1}{\sin x} - \frac{1}{x}\right)(\infty - \infty) = \lim_{x\to\infty} \frac{x - \sin x}{x \sin x}\left(\frac{0}{0}\right)$

$$= \lim_{x\to 0} \frac{1-\cos x}{\sin x + x\cos x}\left(\frac{0}{0}\right) = \lim_{x\to 0}\frac{\sin x}{2\cos x - x\sin x} = \frac{0}{2} = 0$$

Theorem 3.7.4 (L'Hopital's Rule: Stronger form). *Suppose that $f(a) = g(a) = 0$ and f, g are differentiable on (a, b). (The case $f'(c) = g'(c) = 0$ is allowed) and that $g'(x) \neq 0$ for $x \neq a$. Then*

$$\lim_{x\to a}\frac{f(x)}{g(x)} = \lim_{x\to a}\frac{f'(x)}{g'(x)}$$

as long as the rhs limit exists.

The proof is based on

Theorem 3.7.5 (Cauchy's Mean value theorem). *Suppose f and g are conti in $[a, b]$, diff'ble in (a, b). If $g' \neq 0$ on (a, b) then $g(b) \neq g(a)$ and there exist $c \in (a, b)$ such that*

$$\frac{f(b) - f(a)}{g(b) - g(a)} = \frac{f'(c)}{g'(c)}$$

Proof. Suppose $g(b) = g(a)$ then by thm 3.2.2

$$g'(c) = \frac{g(b) - g(a)}{b - a} = 0$$

for some $c \in (a, b)$. This contradict to $g' \neq 0$. So, $g(b) \neq g(a)$. Next consider the function F defined by

$$F(x) = f(x) - f(a) - \left(\frac{f(b) - f(a)}{g(b) - g(a)}\right)(g(x) - g(a))$$

We apply Rolle's theorem 3.2.1 to F. F satisfies the condition of Rolle's thm. Hence there exist $c \in (a, b)$ such that $F'(c) = 0$. Since

$$F'(c) = f'(c) - \left(\frac{f(b) - f(a)}{g(b) - g(a)}\right) g'(c) = 0$$

we have

$$\frac{f'(c)}{g'(c)} = \frac{f(b) - f(a)}{g(b) - g(a)}$$

□

Proof. First show

$$\lim_{x\to c^+}\frac{f(x)}{g(x)}=\lim_{x\to c^+}\frac{f'(x)}{g'(x)}$$

When $c<x<b$ use thm 3.7.5(Cauchy' MVT) on $[c,x]$. Then there is $d\in(c,x)$ s.t.

$$\frac{f'(d)}{g'(d)}=\frac{f(x)-f(c)}{g(x)-g(c)}=\frac{f(x)}{g(x)}$$

and $d\to c^+$ as as $x\to c^+$

$$\begin{aligned}\lim_{x\to c^+}\frac{f(x)}{g(x)}&=\lim_{d\to c^+}\frac{f'(d)}{g'(d)}\\&=\lim_{x\to c^+}\frac{f'(x)}{g'(x)}\end{aligned}$$

The following can be shown the same way.

$$\lim_{x\to c^-}\frac{f(x)}{g(x)}=\lim_{x\to c^-}\frac{f'(x)}{g'(x)}$$

□

Intermediate form ∞/∞, $\infty\cdot 0$, $\infty-\infty$

Example 3.7.6.

(1) $\displaystyle\lim_{x\to 0} x\sin\frac{1}{x}$

(2) $\displaystyle\lim_{x\to 0^+}\sqrt{x}\ln x$

(3) $\displaystyle\lim_{x\to(\pi/2)^-}\frac{\tan x}{1+\tan x}\left(\frac{\infty}{\infty}\right)=\lim_{x\to(\pi/2)^-}\frac{\sec^2 x}{\sec^2 x}=1.$

(4) $$\lim_{x\to\infty}\frac{\pi/2-\tan^{-1}x}{1/x}\left(\frac{0}{0}\right)=\lim_{x\to\infty}\frac{-1/(1+x^2)}{-1/x^2}$$
$$=\lim_{x\to\infty}\frac{x^2}{1+x^2}\left(\frac{\infty}{\infty}\right)=\lim_{x\to\infty}\frac{2x}{2x}=1.$$

(5) $$\lim_{x\to\infty}\frac{\sqrt{9x+1}}{\sqrt{4x+1}}\left(\frac{\infty}{\infty}\right)=\lim_{x\to\infty}\frac{9/(2\sqrt{9x+1})}{4/(2\sqrt{4x+1})}\left(\frac{0}{0}\right)\qquad=\lim_{x\to\infty}\frac{-(81/4)(9x+1)^{-3/2}}{-4(4x+1)^{-3/2}}\left(\frac{0}{0}\right).$$

(6) $\displaystyle\lim_{x\to\pi/2}\frac{\sec x}{1+\tan x}$

(7) $\displaystyle\lim_{x\to\infty}\frac{\ln x}{2\sqrt{x}}$

Wrong use of L'hopital's rule

$$\lim_{x\to\infty}\frac{\pi/2-\tan^{-1}x}{1/x}\left(\frac{0}{0}\right)=\lim_{x\to\infty}\frac{-1/(1+x^2)}{-1/x^2}$$
$$=\lim_{x\to\infty}\frac{x^2}{1+x^2}\left(\frac{\infty}{\infty}\right)=\lim_{x\to\infty}\frac{2x}{2x}=1.$$

In this case we can find limit as follows:

$$\lim_{x\to\infty}\frac{\sqrt{9x+1}}{\sqrt{4x+1}}=\sqrt{\lim_{x\to\infty}\frac{9x+1}{4x+1}}=\sqrt{\frac{9}{4}}=\frac{3}{2}.$$

$$\lim_{x\to 0}\frac{1-\cos x}{x+x^2}=\lim_{x\to 0}\frac{\sin x}{1+2x}=0$$

But do not continue.

Intermediate form 0^∞, ∞^0, $\infty-\infty$

Example 3.7.7. Use continuity

If $\lim\ln f(x)=L$ then $f(x)=\lim e^{\ln f(x)}=e^L$. Here a may be either finite or infinite.

(1) $\lim_{x\to 0^+}(1+x)^{1/x}$

(2) $\lim_{x\to\infty}x^{1/x}$

(3) $\lim_{x\to 0}\left(\frac{1}{\sin x}-\frac{1}{x}\right)$

3.8 Newton's Methods

3.9 Hyperbolic functions

Definitions

For any function $f(x)$ we have

$$f(x)=\underbrace{\frac{f(x)+f(-x)}{2}}_{\text{Even function}}+\underbrace{\frac{f(x)-f(-x)}{2}}_{\text{Odd function}}$$

In particular, e^x has the form

$$e^x=\frac{e^x+e^{-x}}{2}+\frac{e^x-e^{-x}}{2} \tag{3.1}$$

Definition 3.9.1. A (hyperbolic function) is defined as

$$\textbf{hyperbolic cosine } \cosh x = \frac{e^x + e^{-x}}{2},$$
$$\textbf{hyperbolic sine } \sinh x = \frac{e^x - e^{-x}}{2},$$
$$\textbf{hyperbolic tangent } \tanh x = \frac{\sinh x}{\cosh x} = \frac{e^x - e^{-x}}{e^x + e^{-x}},$$
$$\textbf{hyperbolic cotangent } \coth x = \frac{1}{\tanh x} = \frac{e^x + e^{-x}}{e^x - e^{-x}},$$
$$\textbf{hyperbolic secant } \operatorname{sech} x = \frac{1}{\cosh x} = \frac{2}{e^x + e^{-x}},$$
$$\textbf{hyperbolic cosecant } \operatorname{csch} x = \frac{1}{\sinh x} = \frac{2}{e^x - e^{-x}}.$$

See Fig 3.11.

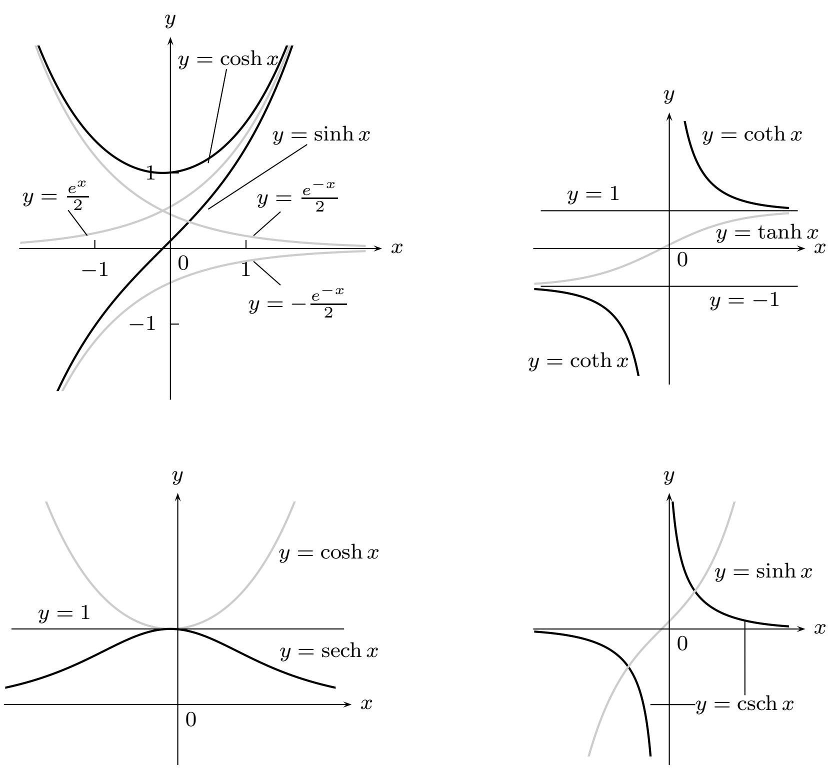


Figure 3.11: hyperbolic functions

Proposition 3.9.2.

(1) $\sinh 2x = 2 \sinh x \cosh x$

(2) $\cosh 2x = \cosh^2 x + \sinh^2 x$

(3) $\sinh^2 x = \dfrac{\cosh 2x - 1}{2}$

(4) $\cosh^2 x = \dfrac{\cosh 2x + 1}{2}$

(5) $\cosh^2 x - \sinh^2 x = 1$

(6) $\tanh^2 x = 1 - \operatorname{sech}^2 x$

(7) $\coth^2 x = 1 + \operatorname{csch}^2 x$

Derivatives of hyperbolic functions

Proposition 3.9.3.

(1) $\dfrac{d}{dx}(\sinh u) = \cosh u \dfrac{du}{dx}$

(2) $\dfrac{d}{dx}(\coth u) = -\operatorname{csch}^2 u \dfrac{du}{dx}$

(3) $\dfrac{d}{dx}(\cosh u) = \sinh u \dfrac{du}{dx}$

(4) $\dfrac{d}{dx}(\operatorname{sech} u) = -\operatorname{sech} u \tanh u \dfrac{du}{dx}$

(5) $\dfrac{d}{dx}(\tanh u) = \operatorname{sech}^2 u \dfrac{du}{dx}$

(6) $\dfrac{d}{dx}(\operatorname{csch} u) = -\operatorname{csch} u \coth u \dfrac{du}{dx}$

Proposition 3.9.4.

(1) $\displaystyle\int \sinh u \, du = \cosh u + C$

(2) $\displaystyle\int \cosh u \, du = \sinh u + C$

(3) $\displaystyle\int \operatorname{sech}^2 u \, du = \tanh u + C$

(4) $\displaystyle\int \operatorname{csch}^2 u \, du = -\coth u + C$

(5) $\displaystyle\int \operatorname{sech} u \tanh u \, du = -\operatorname{sech} u + C$

(6) $\displaystyle\int \operatorname{csch}\ u \coth u du = -\operatorname{csch} u + C$

Example 3.9.5. (1) The integral of $\sinh^2 x$ is obtained in a similar method to that of $\sin^2 x$.

$$\begin{aligned}\int_0^1 \sinh^2 x\,dx &= \int_0^1 \frac{\cosh 2x - 1}{2}\,dx\\ &= \frac{1}{2}\left[\frac{\sinh 2x}{2} - x\right]_0^1\\ &= \frac{\sinh 2}{4} - \frac{1}{2}.\end{aligned}$$

(2) By definition of $\sinh x$

$$\begin{aligned}\int_0^{\ln 2} 4e^x \sinh x\,dx &= \int_0^{\ln 2} 4e^x \frac{e^x - e^{-x}}{2}\,dx = \int_0^{\ln 2} (2e^{2x} - 2)\,dx\\ &= \left[e^{2x} - 2x\right]_0^{\ln 2}\\ &= 3 - 2\ln 2.\end{aligned}$$

Inverse hyperbolic functions

$y = \sinh x$ is a one-to -one function from $(-\infty,\infty)$ onto $(-\infty,\infty)$. Thus its inverse function $y = \sinh^{-1} x$ is well defined on all of $(-\infty,\infty)$. Thus the **inverse hyperbolic sine** is

$$\sinh^{-1} x : (-\infty,\infty) \to (-\infty,\infty).$$

For $y = \cosh x$, we restrict the domain to $x \geq 0$. Then $y = \cosh^{-1} x$ exists on $[1,\infty)$. Thus the **inverse hyperbolic cosine** is

$$y = \cosh^{-1} : [1,\infty) \to [0,\infty).$$

Likewise if we restricted $y = \operatorname{sech} x$ to $x \geq 0$, then inverse function $y = \operatorname{sech}^{-1} x$ exists on $(0, 1]$. $y = \tanh x$, $y = \coth x$, $y = \operatorname{csch} x$ are all one-to-one on $(-\infty,\infty)$. Hence inverse functions

$$\begin{aligned}y &= \tanh^{-1} x : (-1, 1) \to (-\infty,\infty),\\ y &= \coth^{-1} x : |x| > 1 \to (-\infty,\infty)\\ y &= \operatorname{csch}^{-1} x : (-\infty,\infty) \setminus \{0\} \to (-\infty,\infty).\end{aligned}$$

Proposition 3.9.6.

(1) $\sinh^{-1} x = \ln\left(x + \sqrt{x^2 + 1}\right), \quad -\infty < x < \infty$

(2) $\cosh^{-1} x = \ln\left(x + \sqrt{x^2 - 1}\right), \quad x \geq 1$

(3) $\tanh^{-1} x = \frac{1}{2} \ln \frac{1 + x}{1 - x}, \quad |x| < 1$

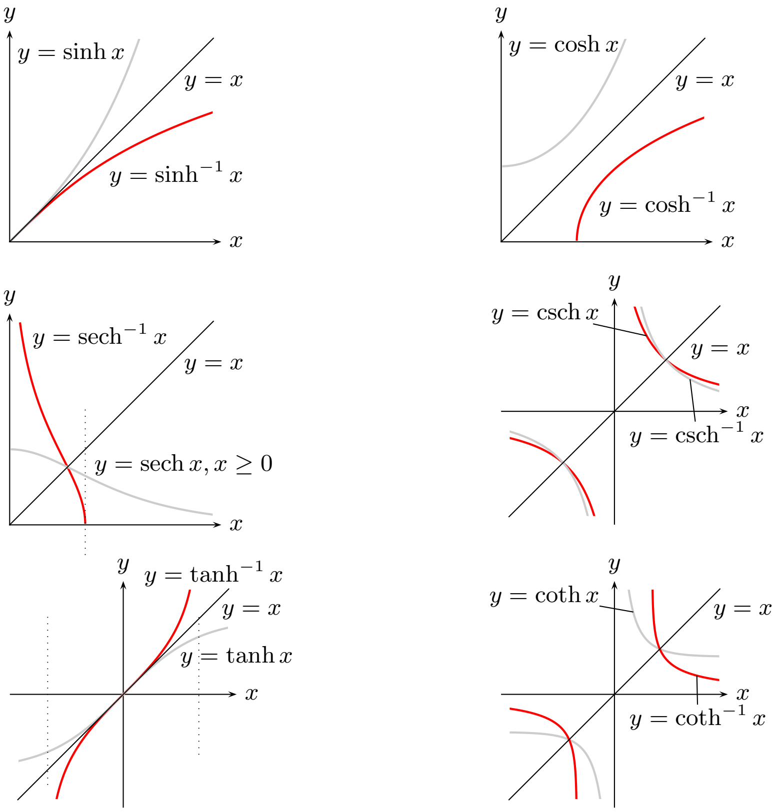


Figure 3.12: Inverse hyperbolic functions

(4) $\operatorname{sech}^{-1} x = \ln\left(\dfrac{1+\sqrt{1-x^2}}{x}\right), \quad 0 < x \le 1$

(5) $\operatorname{csch}^{-1} x = \ln\left(\dfrac{1}{x} + \dfrac{\sqrt{1+x^2}}{|x|}\right), \quad x \neq 0$

(6) $\coth^{-1} x = \dfrac{1}{2}\ln\dfrac{x+1}{x-1}, \quad |x| > 1$

Proof. We prove for $\sinh^{-1} x$ only.

$$\begin{aligned} y = \sinh x &= \frac{e^x - e^{-x}}{2}, \\ e^x - e^{-x} &= 2y, \\ e^{2x} - 2ye^x - 1 &= 0. \end{aligned}$$

Solving this equation for e^x we have

$$e^x = y + \sqrt{y^2+1}.$$

Since $y - \sqrt{y^2+1}$ is negative, we only choose positive sign. So $x = \ln(y + \sqrt{y^2+1})$ hence $y = \ln(x + \sqrt{x^2+1})$ is the inverse of $\sinh^{-1} x$. □

Properties of inverse hyperbolic functions

Proposition 3.9.7.

(1) $\operatorname{sech}^{-1} x = \cosh^{-1} \frac{1}{x}$

(2) $\operatorname{csch}^{-1} x = \sinh^{-1} \frac{1}{x}$

(3) $\coth^{-1} x = \tanh^{-1} \frac{1}{x}$

Proposition 3.9.8.

(1) $\frac{d(\sinh^{-1} u)}{dx} = \frac{1}{\sqrt{1+u^2}} \frac{du}{dx}$

(2) $\frac{d(\cosh^{-1} u)}{dx} = \frac{1}{\sqrt{u^2-1}} \frac{du}{dx}, \quad u > 1$

(3) $\frac{d(\tanh^{-1} u)}{dx} = \frac{1}{1-u^2} \frac{du}{dx}, \quad |u| < 1$

(4) $\frac{d(\coth^{-1} u)}{dx} = \frac{1}{1-u^2} \frac{du}{dx}, \quad |u| > 1$

(5) $\frac{d(\operatorname{sech}^{-1} u)}{dx} = \frac{-du/dx}{u\sqrt{1-u^2}}, \quad 0 < u < 1$

(6) $\frac{d(\operatorname{csch}^{-1} u)}{dx} = \frac{-du/dx}{|u|\sqrt{1+u^2}}, \quad u \neq 0$

Proposition 3.9.9.

(1) $\int \frac{du}{\sqrt{1+u^2}} = \sinh^{-1} u + C$

(2) $\int \frac{du}{\sqrt{u^2-1}} = \cosh^{-1} u + C, \quad u > 1$

(3) $\int \frac{du}{1-u^2} = \begin{cases} \tanh^{-1} u + C, & |u| < 1 \\ \coth^{-1} u + C, & |u| > 1 \end{cases},$

(4) $\int \frac{du}{u\sqrt{1-u^2}} = -\operatorname{sech}^{-1} |u| + C = -\cosh^{-1}\left(\frac{1}{|u|}\right) + C$

(5) $\int \frac{du}{u\sqrt{1+u^2}} = -\operatorname{csch}^{-1} |u| + C = -\sinh^{-1}\left(\frac{1}{|u|}\right) + C$

Exercise 3.9.10. (1) Show

(a) $\sinh(x+y) = \sinh x \cosh y + \cosh x \sinh y$

(b) $\cosh(x+y) = \cosh x \cosh y + \sinh x \sinh y$

(2) Find derivatives

(a) $x - \tanh x^2$

(b) $\ln(\operatorname{sech} x)$

(c) $\frac{1}{2} \ln |\tanh x|$

(d) $\tan^{-1}(\sinh x)$

(e) $(x^2+1) \operatorname{sech}(\ln x)$

(f) $(1-x) \tanh^{-1} x$

(g) $\sinh^{-1}(\tan x)$

(h) $(1-x^2) \coth^{-1} x$

(i) $\tan^{-1}(\sin x)$, $-\pi/2 < x < \pi/2$

(j) $\operatorname{sech}^{-1}(\sin x)$, $0 < x < \pi/2$

(3) Find the following integrals

(a) $\displaystyle\int_{-1}^{0} \cosh(2x+1)\, dx$

(b) $\displaystyle\int_{-\pi}^{\pi} \tanh 2x\, dx$

(c) $\displaystyle\int_{0}^{1/2} 4e^{-x} \sinh x\, dx$

(d) $\displaystyle\int \frac{\cosh(\ln x)}{x}\, dx$

(e) $\displaystyle\int_{0}^{\ln 2} \tanh^2 x\, dx$

(f) $\displaystyle\int_{1}^{4} \frac{\cosh \sqrt{x}}{\sqrt{x}}\, dx$

(g) $\displaystyle\int_{\ln 2}^{\ln 3} \cosh^2 x\, dx$

(h) $\displaystyle\int_{-\ln 2}^{\ln 2} \sqrt{\cosh(2x-1)}\, dx$

(i) $\displaystyle\int \operatorname{sech}^3 5x \tanh 5x\, dx$

(j) $\displaystyle\int \tanh^3 x\, dx$

(4) Prove theorem 3.9.6.

(5) Find the integral.

(a) $\displaystyle\int_0^1 \frac{dx}{\sqrt{1+2x^2}}$

(b) $\displaystyle\int_{4/5}^{12/13} \frac{dx}{x\sqrt{1-x^2}}$

(c) $\displaystyle\int_{5/4}^{5/3} \frac{dx}{\sqrt{x^2-1}}$

(d) $\displaystyle\int_0^{1/2} \frac{dx}{\sqrt{1-x^2}}$

(e) $\displaystyle\int_{5/4}^{2} \frac{dx}{\sqrt{1-x^2}}$

(f) $\displaystyle\int_0^{2\sqrt{3}} \frac{dx}{\sqrt{4+x^2}}$

(g) $\displaystyle\int_1^2 \frac{dx}{x\sqrt{4+x^2}}$

(h) $\displaystyle\int_0^{\pi} \frac{\cos x dx}{\sqrt{1+\sin^2 x}}$

(6) Find the volume of the region when the graph $y = \operatorname{sech} x$ is rotated about x axis between $-\ln\sqrt{3} \le x \le \ln\sqrt{3}$.

(7) Find the centroid of the volume obtained when the region between $y = \tanh x$, $y = 1$, $x = 0$, $x = \ln\sqrt{199}$ is rotated about $y = 1$.

(8) Find the solution of differential equation.

$$x\frac{d^2y}{dx^2} = \sqrt{1+\left(\frac{dy}{dx}\right)^2}, \quad y(1) = 0, \quad \frac{dy}{dx}(1) = 0$$

Chapter 4

Integration

4.1 Anti-derivatives

If $F'(x) = f(x)$ then F is an anti-derivative. In general, $F(x) + C$ is an anti-derivative.

D.E. Find the solution of $y' = x^2$

Definition 4.1.1. Indefinite integral. The set of all anti-derivative of f is denoted by

$$\int f(x)\,dx$$

4.2 Estimating with finite sums

4.3 Sigma and limit

4.4 Definite integral

Suppose $f(x)$ is a positive(temporarily) continuous function on $[a, b]$. Let A be the region under the graph of $y = f(x)$, between $x = a$, $x = b$ To find area, we divide $[a, b]$ into small subintervals. (4.1). For example, uniform n-subintervals are $\{x_0, x_1, \ldots, x_n\}$

$$x_i = a + i(b - a)/n, \quad i = 0, 1, \cdots, n$$

The area A is approximated by

$$S_n = \sum_{i=0}^{n-1} f(x_i)(x_{i+1} - x_i)$$

In general, nonuniform intervals allowed except that $\max_i |x_{i+1} - x_i| \to 0$ as $n \to \infty$.

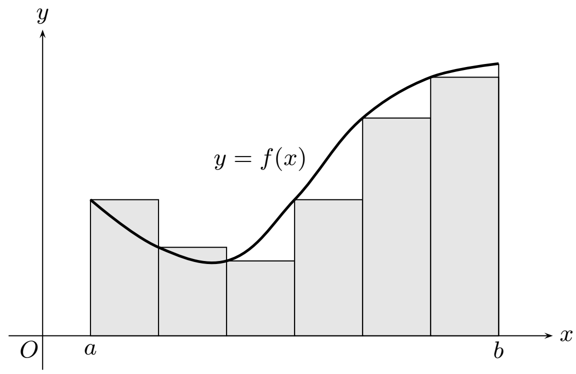


Figure 4.1: S_6

$f(x_i)$ can be replaced by $f(c_i)$. A **partition** of $[a, b]$ is a set of points $x_0, x_1, \ldots, x_n$ such that $a = x_0 < x_1 < \cdots < x_{n-1} < x_n = b$. $P = \{x_0, x_1, \ldots, x_n\}$ is a partition of $[a, b]$ and $f(x)$ is defined on $[a, b]$. For any sequence $\{c_1, c_2, \ldots, c_n\}$ satisfying $x_{i-1} \leq c_i \leq x_i$ The **Riemann sum** $R(f, P)$ of $f(x)$ w.r.t P.(4.2)

$$R(f, P) = \sum_{i=1}^{n} f(c_i)(x_{i+1} - x_i)$$

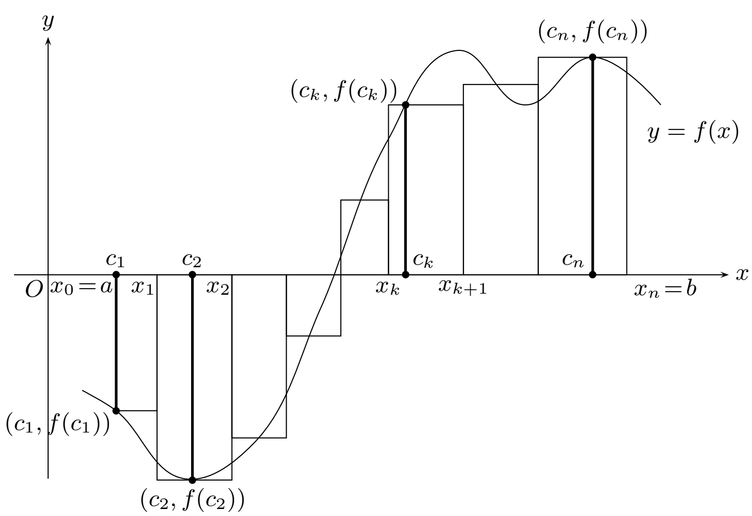


Figure 4.2: $R(f, P)$

Definition 4.4.1. $\|P\|$: **norm** of $P = \{x_0, x_1, \ldots, x_n\}$ is defined by

$$\|P\| = \max_{0 \leq i < n} (x_{i+1} - x_i)$$

Suppose the Riemann sum $R(f, P)$ of $f(x)$ approaches some number I as the norm $\|P\|$ of partition P approaches 0 then this number is defined as the area under the graph.

Definition 4.4.2 (Definite Integral as limit of Riemann Sum)**.** Let $f(x)$ be defined on $[a, b]$. We say a number I is the **definite integral of f over** $[a, b]$ if the following holds:

> For any $\epsilon > 0$ there corresponds a $\delta > 0$ such that for any partition and any choice of points c_k in $x_k \leq c_k \leq x_{k+1}$ such that for any $\|P\| < \delta$ and points in $[a, b]$ and for partition $P = \{x_0, x_1, \ldots, x_n\}$ for any choice of $\{c_0, c_1, \ldots, c_{n-1}\}(x_i \leq c_i \leq x_{i+1})$, we have $\|R(f, P) - I\| < \epsilon$.

This number I is denoted by $\int_a^b f(x)\,dx$ and called the **definite integral** of $f(x)$ on $[a, b]$. We say $f(x)$ is **integrable** on $[a, b]$.

We write it as

$$\int_a^b f(x)\,dx = \lim_{\|P\| \to 0} \sum_{i=0}^{n-1} f(c_i)(x_i - x_{i-1})$$

Non-integrable function

$$f(x) = \begin{cases} 0, & x \text{ rational number} \\ 1, & x \text{ irrational number} \end{cases}$$

$$g(x) = \begin{cases} 0, & x \in [0, 1] \\ 1, & x \in (1, 2] \end{cases}$$

Theorem 4.4.3 (Definite integral)**.** *The following holds:*

(1) $\displaystyle\int_a^b (f(x) + g(x))\,dx = \int_a^b f(x)\,dx + \int_a^b g(x)\,dx$

(2) $\displaystyle\int_a^b kf(x)\,dx = k\int_a^b f(x)\,dx$

(3) $\displaystyle\int_a^c f(x)\,dx = \int_a^b f(x)\,dx + \int_b^c f(x)\,dx,\ (a \leq b \leq c)$

(4) If $f(x) \geq 0$ *on* $[a, b]$*, then* $\displaystyle\int_a^b f(x)\,dx \geq 0$

(5) $\displaystyle\min_{x \in [a,b]} f(x) \cdot (b - a) \leq \int_a^b f(x)\,dx \leq \max_{x \in [a,b]} f(x) \cdot (b - a)$

(6) $\left|\int_a^b f(x)\,dx\right| \le \int_a^b |f(x)|\,dx$

Definition 4.4.4.

$$\int_a^a f(x)\,dx = 0, \qquad \int_b^a f(x)\,dx = -\int_a^b f(x)\,dx.$$

Example 4.4.5. Find $\int_0^a x^2\,dx$ by definition.

sol. Suppose $P = \{x_0, x_1, \ldots, x_n\}$ is a uniform partition of $[0, a]$. $x_i = i\,a/n$

$$\begin{aligned}\sum_{i=1}^n x_i^2(x_i - x_{i-1}) &= \sum_{i=1}^n \left(\frac{i\cdot a}{n}\right)^2 \frac{a}{n} \\ &= \frac{a^3}{n^3}\sum_{i=1}^n i^2 \\ &= \frac{a^3(n+1)(2n+1)}{6n^2}\end{aligned}$$

So

$$\begin{aligned}\int_0^a x^2\,dx &= \lim_{n\to\infty}\sum_{i=1}^n x_i^2(x_i - x_{i-1}) \\ &= \lim_{n\to\infty}\frac{a^3(n_+1)(2n+1)}{6n^2} \\ &= \frac{a^3}{3}\end{aligned}$$

□

□

Theorem 4.4.6 (Mean value theorem for integral)**.** *Let $a < b$. If $f(x)$ is conti. on closed interval $[a, b]$ then there is a c in $[a, b]$*

$$f(c) = \frac{1}{b-a}\int_a^b f(x)\,dx$$

This value is called the average of f on $[a, b]$ denoted by $av(f)$

Proof. Since $f(x)$ is continuous on $[a, b]$ there are min and max;

$$f(x_0) = \min_{x\in[a,b]} f(x), \quad f(x_1) = \max_{x\in[a,b]} f(x)$$

for some x_0, x_1 in $[a, b]$.

First, if $x_0 = x_1$ then $f(x)$ is constant and the equality holds for all $c \in [a, b]$. Suppose $x_0 < x_1$. Then by intermediate value theorem, $f(x)$ assumes all values between $f(x_0)$ and $f(x_1)$ in $[x_0, x_1]$. Since

$$f(x_0) \le \frac{1}{b-a} \int_a^b f(x)\, dx \le f(x_1)$$

So there is a c such that

$$f(c) = \frac{1}{b-a} \int_a^b f(x)\, dx$$

for some $c \in [x_0, x_1]$.

The case $x_0 < x_1$ is the same. □

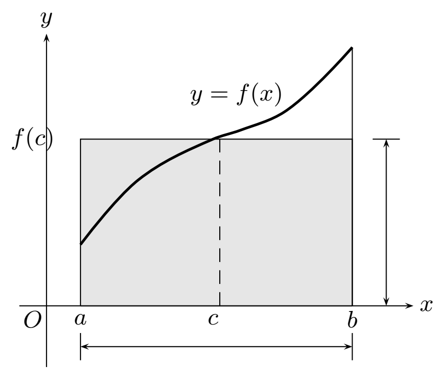


Figure 4.3: MVT for Integral

4.5 Fundamental theorem of Calculus

If f is integrable on I, the integral from a fixed point a to another point x defines a new function $F(x) = \int_a^x f(t)\, dt$. If f is conti, this new function is differentiable, and

Theorem 4.5.1 (Fundamental theorem of Calculus I)**.** *Suppose $f(x)$ is conti on $[a, b]$. Then the function $F(x)$ defined by*

$$F(x) = \int_a^x f(t)\, dt$$

is differentiable on (a, b) and

$$F'(x) = \frac{d}{dx} \int_a^x f(t)\, dt = f(x)$$

Proof. By definition,

$$F'(x) = \lim_{h\to 0} \frac{F(x+h) - F(x)}{h}$$

$$\begin{aligned} \frac{F(x+h) - F(x)}{h} &= \frac{1}{h}\left(\int_a^{x+h} f(t)\,dt - \int_a^x f(t)\,dt\right) \\ &= \frac{1}{h}\int_x^{t+h} f(t)\,dt \end{aligned}$$

By MVT there is c between x and $x+h$ s.t.

$$\frac{1}{h}\int_x^{x+h} f(t)\,dt = f(c).$$

Now $f(c)$ approaches $f(x)$ as $h \to 0$. Hence

$$\begin{aligned} F'(x) &= \lim_{h\to 0} \frac{F(x+h) - F(x)}{h} \\ &= \lim_{h\to 0} \frac{1}{h}\int_x^{t+h} f(t)\,dt \\ &= f(x) \end{aligned}$$

□

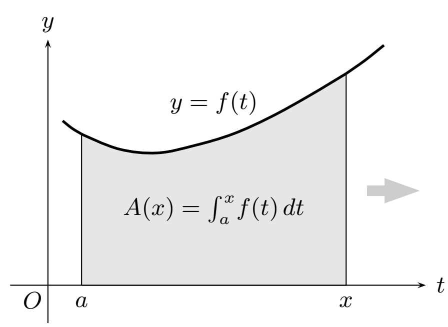


Figure 4.4: $A'(x) = f(x)$

Theorem 4.5.2 (Fundamental theorem of Calculus II)**.** *Suppose $f(x)$ is conti on $[a, b]$. If $F(x)$ is the anti-derivative of $f(x)$, then*

$$\int_a^b f(x)\,dx = F(b) - F(a).$$

Proof. Set $G(x) = \int_a^x f(t)\,dt$. Then by (1), $G(x)$ is anti-derivative of $f(x)$ on $[a, b]$. Since $F(x)$ is also an anti-derivative of $f(x)$, we have

$$G(x) = F(x) + C.$$

But $G(a) = F(a) + C = \int_a^a f(t)\,dt = 0$, hence $C = -F(a)$. Hence $G(b) = \int_a^b f(t)\,dt = F(b) - F(a)$. □

Example 4.5.3. (1) $\dfrac{d}{dx}\displaystyle\int_1^{x^2} \cos t\,dt$.

(2) $\dfrac{d}{dx}\displaystyle\int_{3+x^2}^{9} \frac{1}{1+e^t}\,dt$.

Example 4.5.4. Find derivative of $A(x)$ when $h(t)$ is continuous and $u(x)$, $v(x)$ are differentiable. Find the derivative of

$$A(x) = \int_{u(x)}^{v(x)} h(t)\,dt$$

sol. Let $H(t)$ be an antiderivative of $h(t)$. Then $A(x) = H(v(x)) - H(u(x))$ and $A'(x) = h(v(x))v'(x) - h(u(x))u'(x)$.

□

In other words,

$$\frac{d}{dx}\int_{u(x)}^{v(x)} h(t)\,dt = h(v(x))v'(x) - h(u(x))u'(x).$$

Example 4.5.5. $\displaystyle\int_a^b e^x\,dx = e^b - e^a$.

Total area.

4.6 Indefinite integrals and substitution

Recall

$$\frac{d}{dx}\left(\frac{u^{n+1}}{n+1}\right) = u^n\frac{du}{dx}$$

So we have

$$\int u^n du = \frac{u^{n+1}}{n+1} + C$$

Example 4.6.1. $\int \sqrt{1+x^2}\,2x dx = \int u^{1/2} du =$

chain rule

$$\frac{d}{dx}F(u(x)) = \frac{d}{du}F(u)\frac{d}{dx}u(x)$$

$$\int \frac{d}{du}F(u)\frac{d}{dx}u(x)\,dx = F(u(x)) + C$$

$$\int \frac{d}{du}F(u)\,du = F(u) + C$$

$$\int \frac{d}{du}F(u)\frac{d}{dx}u(x)\,dx = \int \frac{d}{du}F(u)\,du$$

Proposition 4.6.2. $\displaystyle\int f(g(x)) \cdot g'(x)\,dx = \int f(u)\,du.$

Proof. Let $F(u)$ be an anti-derivative of $f(u)$.

$$\frac{d}{dx}F(g(x)) = f(g(x))\,g'(x)$$

$$\int f(g(x))\,g'(x)\,dx = \int f(u)\,du$$

□

Example 4.6.3. $\displaystyle\int_0^{\pi/2} e^{\sin x}\cos x\,dx = \int_0^1 e^u\,du.$

Example 4.6.4. Find

$$\int \sec x\,dx$$

The idea is to multiply $\sec x + \tan x$ both the numerator and denominator:

$$\begin{aligned}
\int \sec x\,dx &= \int \sec x \cdot \frac{\sec x + \tan x}{\sec x + \tan x}\,dx \\
&= \int \frac{\sec^2 x + \sec x \tan x}{\sec x + \tan x}\,dx \\
&= \int \frac{du}{u} \\
&= \ln|\sec x + \tan x| + C
\end{aligned}$$

Similarly, we obtain

$$\int \csc x\,dx = -\ln|\csc x + \cot x| + C$$

Example 4.6.5.

$$\int \sin^2 x\,dx = \int \frac{1 - \cos 2x}{2}dx$$

4.7 Area between curves

If $f(x) > 0$, $\int_a^b f(x)\,dx$ is the area defined by

$$y = f(x), \quad a \le x \le b$$

In general, when $f(x) \le g(x)$ on $[a, b]$, the area defined by

$$\int_a^b (g(x) - f(x))\,dx$$

Use of symmetry

If there is any symmetry it is useful to take advantage of it.

Proposition 4.7.1.

$$\textit{For even function,} \quad \int_{-a}^{a} f(x)\,dx = 2\int_0^a f(x)\,dx$$

$$\textit{For odd function,} \quad \int_{-a}^{a} f(x)\,dx = 0$$

Integration w.r.t y

When the region is determined by functions of y, we need to integrate w.r.t. y.

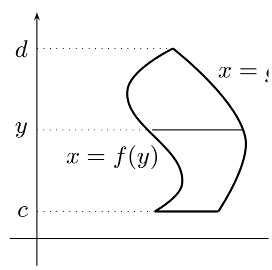


Figure 4.5: Region by between functions of y

$$A = \int_c^d (g(y) - f(y))\,dy$$