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On a continued fraction formula of Wall

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Abstract. We study the combinatorics of a continued fraction formula due to Wall. We also derive the orthogonality of little q-Jacobi polynomials from this formula, as Wall did for little q-Laguerre polynomials.

Keywords: Dyck path, continued fraction formula, little q-Jacobi polynomials

1. Introduction

Wall [7] found the following continued fraction formula:

$$\cfrac{1}{1-\cfrac{(1-g_1)t}{1-\cfrac{(1-g_2)g_1t}{1-\cfrac{(1-g_3)g_2t}{1-\cfrac{(1-g_4)g_3t}{1-\ddots}}}}} \times \cfrac{1}{1-\cfrac{g_1t}{1-\cfrac{(1-g_1)g_2t}{1-\cfrac{(1-g_2)g_3t}{1-\cfrac{(1-g_3)g_4t}{1-\ddots}}}}} = \frac{1}{1-t}. \qquad (1)$$

From the modern combinatorial point of view [3, 6], the above identity can be seen as an equation between generating functions of certain weighted Dyck paths. So it is natural to seek for a bijective proof of the identity in this context. We give such a proof in sections 2 and 3.

One of the consequences of (1) is the discovery of the little q-Laguerre polynomials, which were the precursor of the little q-Jacobi polynomials studied by Andrews and Askey [1]. Recall that the monic little q-Jacobi polynomials [4, p.92] are defined by the recurrence relation:

$$xp_n(x) = p_{n+1}(x) + (A_n + C_n)p_n(x) + A_{n-1}C_np_{n-1}(x), \qquad n \geq 1,$$

where $p_{-1}(x) = 0$, $p_0(x) = 1$ and

$$A_n = \frac{q^n(1-aq^{n+1})(1-abq^{n+1})}{(1-abq^{2n+1})(1-abq^{2n+2})}, \quad C_n = \frac{aq^n(1-q^n)(1-bq^n)}{(1-abq^{2n})(1-abq^{2n+1})}. \qquad (2)$$

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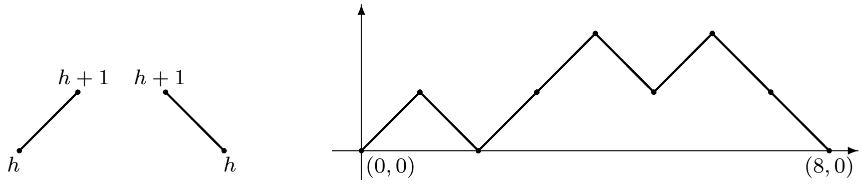

Figure 1. Up and down steps, and a Dyck path of length 8.

The explicit formula of $p_n(x)$ is

$$p_n(x) = \frac{(aq;q)_n}{(abq^{n+1};q)_n}(-1)^n q^{\binom{n}{2}}{}_2\phi_1(q^{-n}, abq^{n+1}; aq \,|\, q; qx).$$

As shown recently by Ismail and Stanton [5] the little q-Jacobi polynomials are orthogonal with respect to the moment sequence $\{\mu_n\}_{n\geq 0}$, where

$$\mu_n = \frac{(aq;q)_n}{(abq^2;q)_n}.$$

In view of (2), this orthogonality is equivalent to the following continued fraction expansion:

$$\sum_{n\geq 0}\mu_n t^n = \cfrac{1}{1-\cfrac{\lambda_1 t}{1-\cfrac{\lambda_2 t}{1-\cfrac{\lambda_3 t}{1-\ddots}}}}, \tag{3}$$

where $\lambda_{2n-1} = A_{n-1}$ and $\lambda_{2n} = C_n$ for $n \geq 1$.

In section 4 we prove that (3), which is equivalent to the orthogonality of the little q-Jacobi polynomials, follows from (1), as Wall [8] did for the special $b = 0$ case.

2. Combinatorial model

A Dyck path [3, 6] of length $2n$ can be described as a sequence $(y_0, y_1, \ldots, y_{2n})$ of non-negative integers such that $y_0 = y_{2n} = 0$, and $y_i = y_{i-1} \pm 1$, $1 \leq i \leq 2n$. It can also be characterized by its step sequence $s = (s_1, s_2, \ldots, s_{2n})$, where $s_i = u$, if $y_i = y_{i-1} + 1$, and $s_i = d$, if $y_i = y_{i-1} - 1$, $1 \leq i \leq 2n$. Call s_i an up step if $s_i = u$ and a down step if $s_i = d$. The level of the step s_i is defined to be y_{i-1}. A Dyck path can be visualized as a connected path in the plane $\mathbf{R}^2$ from $(0, 0)$ to $(2n, 0)$, by drawing a line segment between consecutive points (i, y_i), $0 \leq i \leq 2n$. (See Figure 1.)

To combinatorially interpret (1) we need the notion of weighted Dyck paths. There are two types of weight functions $w^{(1)}$ and $w^{(2)}$ on the steps s_i, starting from level h:

$$w^{(1)}(s_i) = \begin{cases} g_h, & \text{if } s_i \text{ is up,} \\ 1 \text{ or } \; -g_h, & \text{if } s_i \text{ is down,} \end{cases}$$

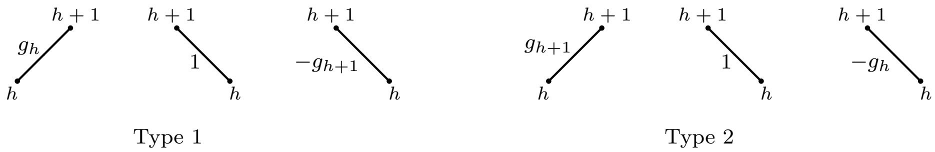

Figure 2. $h \geq 0$, $g_0 = 1$ in type 1, and $g_0 = -1$ in type 2.

where $g_0 = 1$, and

$$w^{(2)}(s_i) = \begin{cases} g_{h+1}, & \text{if } s_i \text{ is up,} \\ 1 \text{ or } -g_{h-1}, & \text{if } s_i \text{ is down,} \end{cases}$$

where $g_0 = -1$. (See Figure 2.)

A *weighted Dyck path* is then defined as a pair $P = (s, w)$, where s is a sequence of steps forming a Dyck path and w is a sequence of weights of type 1 or 2 of the entries in s. The sequence s is called the *step sequence* of P and the sequence w the *weight sequence* of P. The weight of $P = (s, w)$ is the product of entries in w. Let $\mathcal{P}_n^{(i)}$ be the set of weighted Dyck paths of length $2n$ with weight of type i, $i = 1$ or 2. Set

$$\mathcal{P}_n = \bigcup_{k=0}^{n} \left(\mathcal{P}_k^{(1)} \times \mathcal{P}_{n-k}^{(2)} \right),$$

where $\mathcal{P}_0^{(1)} = \mathcal{P}_0^{(2)} = \{(\emptyset, \emptyset)\}$, the set of the empty weighted Dyck path of length 0 whose weight is 1. We now put a weight w on $\mathcal{P}_n$ by $w(x) = w^{(1)}(P)w^{(2)}(Q)$, if $x = (P, Q) \in \mathcal{P}_n$. Then formula (1) is equivalent to the assertion

$$w(\mathcal{P}_n) = \sum_{x \in \mathcal{P}_n} w(x) = 1, \quad \text{for } n \geq 0. \tag{4}$$

We will construct a *weight-preserving sign-reversing* (wpsr) involution ϕ on the set $\mathcal{P}_n$, i.e. ϕ is an involution and $w(\phi(x)) = -w(x)$, if $\phi(x) \neq x$, with one fixed point of weight 1, which proves (4) since

$$\sum_{x \in \mathcal{P}_n} w(x) = \sum_{x:\, \phi(x) = x} w(x) = 1. \tag{5}$$

3. Involution for Wall's formula

For clarity we first define two simple involutions ϕ_1 and ϕ_2 and build the involution ϕ as their composition.

Let $\mathcal{F}_n$ denote the set of paths in $\mathcal{P}_n^{(1)}$ such that there is no step with weight $\pm g_h$ ($h \geq 1$) followed by h steps with weight 1. We define an involution ϕ_1 on $\mathcal{P}_n^{(1)}$ with fixed set $\mathcal{F}_n$, i.e. the set of fixed points of ϕ_1. Let $P = (s, w)$ be a weighted path in $\mathcal{P}_n^{(1)} \setminus \mathcal{F}_n$. Let k be the smallest integer such that $w_k = \pm g_h$ for some $h (\geq 1)$ and $w_{k+1} = w_{k+2} = \cdots = w_{k+h} = 1$. Let $\phi_1(P)$ be the weighted path in $\mathcal{P}_n^{(1)} \setminus \mathcal{F}_n$

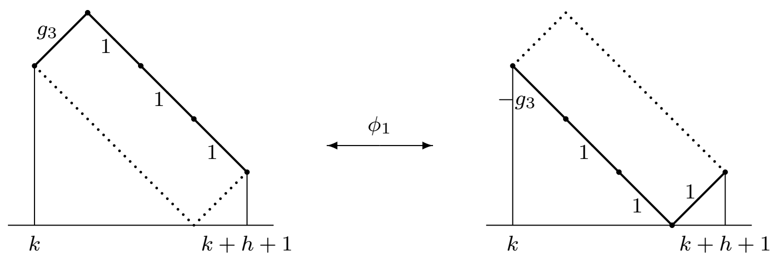

Figure 3. Involution ϕ_1 on $\mathcal{P}_n^{(1)}$.

with weight sequence $w' = (w_1, \ldots, w_{k-1}, \mp g_h, w_{k+1}, \ldots, w_{2n})$. The step sequence of $\phi_1(P)$ is determined by the new weight sequence w'. The map ϕ_1 is illustrated in Figure 3. Clearly we have the following result.

LEMMA 1 *The mapping ϕ_1 is a* wpsr *involution on $\mathcal{P}_n^{(1)}$ whose fixed set is $\mathcal{F}_n$.*

Let $P = (s, w)$ be a weighted Dyck path in $\mathcal{F}_n$. If $s_{2i+1} = u$ is the rightmost up step starting from level 0, then $s_{2j-1} = u$, $s_{2j} = d$ and $w_{2j-1} = w_{2j} = 1$ for $1 \le j \le i$. The path P is called *reduced*, if $i = 0$. In particular,

$$P_{re} = ((s_{2i+1}, s_{2i+2}, \ldots, s_{2n}), (w_{2i+1}, w_{2i+2}, \ldots, w_{2n}))$$

is a reduced path in $\mathcal{F}_{n-i}$, called the reduced component of P.

Given a reduced path $P = (s, w)$ in $\mathcal{F}_n$, let (i_1, j_1), (i_2, j_2), ..., (i_l, j_l) be all the pairs of integers such that $(w_{i_k}, w_{i_k+1}, \ldots, w_{j_k}) = (g_{h_k}, 1, \ldots, 1)$, for $h_k > 0$, and $w_{j_k+1} \neq 1$. We call the segment $(w_{i_k}, w_{i_k+1}, \ldots, w_{j_k})$ an *up-down hook*, since an up step is followed by down steps. Note that $i_1 < j_1 < i_2 < j_2 < \cdots < i_l < j_l$. We define a weighted path $P^{\Delta} = ((s_1^*, s_2^*, \ldots, s_{2n}^*), (w_1^*, w_2^*, \ldots, w_{2n}^*))$ in $\mathcal{P}_n^{(2)}$, called the *shadow* of P, as follows:

$$s_i^* = \begin{cases} d, & \text{if } i \in \{i_1, i_2, \ldots, i_k\}, \\ u, & \text{if } i \in \{j_1, j_2, \ldots, j_k\}, \\ s_i, & \text{otherwise,} \end{cases} \qquad w_i^* = \begin{cases} w_{i+1}, & \text{if } s_i^* = s_{i+1}, \\ -w_{i+1}, & \text{if } s_i^* \neq s_{i+1}, \\ 1, & \text{if } i = 2n. \end{cases}$$

Note that $P \in \mathcal{F}_n$ is reduced if and only if P^{Δ} doesn't touch the level 0 except at the two end points, and that if $w_{i+1} = 1$ then $w_i^* = 1$.

A Dyck path is said to be *prime* if it doesn't touch the level 0 except for the two end points and a weighted path is said to be *prime* if its underlying path is prime.

Given a prime weighted path $Q = (s, w)$ in $\mathcal{P}_n^{(2)}$, let (i_1, j_1), (i_2, j_2), ..., (i_l, j_l) be all the pairs such that $(w_{i_k}, w_{i_k+1}, \ldots, w_{j_k}) = (1, \ldots, 1, g_{h_k})$, for $h_k > 0$, and $w_{i_k-1} \neq 1$. We call the segment $(w_{i_k}, w_{i_k+1}, \ldots, w_{j_k})$ a *down-up hook*, since a down step is followed by up steps. Note again that $i_1 < j_1 < i_2 < j_2 < \cdots < i_l < j_l$. We define a weighted path $Q^{\nabla} = ((s_1^\circ, s_2^\circ, \ldots, s_{2n}^\circ), (w_1^\circ, w_2^\circ, \ldots, w_{2n}^\circ))$ in $\mathcal{F}_n$, called the

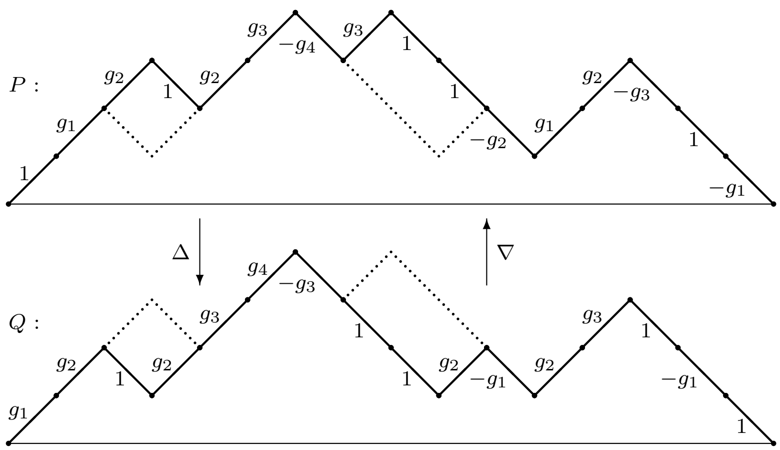

Figure 4. $P^{\Delta} = Q$ and $Q^{\nabla} = P$.

shade of Q, as follows:

$$s_i^{\circ} = \begin{cases} u, & \text{if } i \in \{i_1, i_2, \ldots, i_k\}, \\ d, & \text{if } i \in \{j_1, j_2, \ldots, j_k\}, \\ s_i, & \text{otherwise,} \end{cases} \qquad w_i^{\circ} = \begin{cases} 1, & \text{if } i = 1, \\ w_{i-1}, & \text{if } s_i^{\circ} = s_{i-1}, \\ -w_{i-1}, & \text{if } s_i^{\circ} \neq s_{i-1}. \end{cases}$$

Shadow and shade are illustrated in Figure 4. The line segments which are different from those in the original path are represented as dotted lines.

Note that each up-down hook in P becomes a down-up hook in P^{Δ} and each down-up hook in Q becomes a up-down hook in Q^{∇}.

Lemma 2 *The shadow and shade are well-defined and the mapping $\Delta : P \mapsto P^{\Delta}$ is an injection from the set of reduced paths in $\mathcal{F}_n$ to $\mathcal{P}_n^{(2)}$ such that $w^{(2)}(P^{\Delta}) = -w^{(1)}(P)$.*

Proof: We need to show that P^{Δ} and Q^{∇} are legitimate paths in $\mathcal{P}_n^{(2)}$ and $\mathcal{F}_n$, respectively. Comparing the weights of type 1 and type 2, it is easy to see that if $s_i^* = s_i$ or $i \in \{i_1, i_2, \ldots, i_k\}$, then the step s_i^* can be weighted by w_i^* in type 2. If $i \in \{j_1, j_2, \ldots, j_k\}$, then $s_i^* = u$ and $w_i \neq 1$, which guarantee that s_i^* can be weighted by w_i^*. Hence $P^{\Delta} \in \mathcal{P}_n^{(2)}$ is well-defined. This is illustrated in Figure 4. The case of Q^{∇} is similar and so we omit.

Since w^* is a rearrangement of the entries in w with some changes in sign, we first note that $|w^{(2)}(P^{\Delta})| = |w^{(1)}(P)|$. So it suffices to show that the signs of $w^{(2)}(P^{\Delta})$, $w^{(1)}(P)$ differ, which follows from the fact that the number of down steps with weight 1 in P is one less than the number of those in P^{Δ}, because $w_1 = 1$ is an up step in P while $w_{2n}^* = w_1$ is a down step in P^{Δ}.

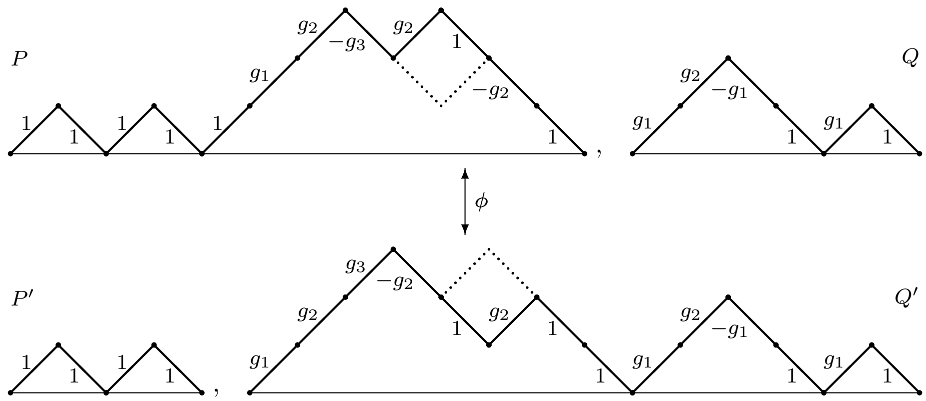

Figure 5. $\phi_2 : (P, Q) \mapsto (P', Q')$

It remains to show that the correspondence is injective. Let Q be a path in $\mathcal{P}_n^{(2)}$ such that $Q = P^{\Delta}$ for some reduced path P in $\mathcal{F}_n$. Then we can show that $Q^{\nabla} = P$, which implies that Δ is one-to-one. Details are omitted. ■

Let $X_n = \cup_{k=0}^{n}(\mathcal{F}_k \times \mathcal{P}_{n-k}^{(2)})$. We define the involution ϕ_2 on X_n as follows. For any pair (P, Q) in X_n, if the reduced component P_{re} of P is not empty, then cut off P_{re} from P and attach P_{re}^{Δ} to Q from the left; if P_{re} is empty but Q is not empty, then cut off the leftmost prime component Q_{pr} of Q, which is the initial segment of Q that comes back to the level 0 for the first time, and attach Q_{pr}^{∇} to P from the right; if both P_{re} and Q are empty, i.e. $(P, Q) = (((u, d; \ldots ; u, d), (1, 1; \ldots ; 1, 1)), (\emptyset, \emptyset))$, then do nothing. Let $\phi_2(P, Q)$ be the resulting pair and extend the domain of ϕ_2 to $\mathcal{P}_n$ by setting $\phi_2(P, Q) = (P, Q)$ for $(P, Q) \in \mathcal{P}_n \setminus X_n$. Then ϕ_2 is a wpsr involution on P_n, which is illustrated in Figure 5.

Finally, combining the involutions ϕ_1 and ϕ_2, we define the mapping ϕ on P_n by $\phi(P, Q) = \phi_2(\phi_1(P), Q)$ and have the following result.

THEOREM 1 *For any integer $n \geq 0$, the mapping ϕ is a wpsr involution on $\mathcal{P}_n$, which has the unique fixed point $(((u, d; \ldots ; u, d), (1, 1; \ldots ; 1, 1)), (\emptyset, \emptyset))$ of weight* 1.

Obviously (4) and then Wall's formula (1) follow from the above theorem.

4. Orthogonality of little *q*-Jacobi polynomials

Note first that the coefficients λ_n in (3) can be parameterized as follows :

$$\lambda_n = (1 - g_n)g_{n-1}, \qquad n \geq 1,$$

where $g_0 = 1$ and for $n \geq 1$

$$g_{2n-1} = \frac{aq^n(1 - bq^n)}{1 - abq^{2n}}, \quad g_{2n} = \frac{q^n(1 - abq^{n+1})}{1 - abq^{2n+1}}.$$

Consider the formal power series

$$\phi(b,t)=\sum_{n\geq 0}a_n(b)t^n=\cfrac{1}{1-\cfrac{(1-g_1)t}{1-\cfrac{(1-g_2)g_1t}{1-\cfrac{(1-g_3)g_2t}{1-\ddots}}}}. \tag{6}$$

Wall's formula (1) implies then

$$(1-t)\phi(b,t)=1-\cfrac{g_1t}{1-\cfrac{(1-g_1)g_2t}{1-\cfrac{(1-g_2)g_3t}{1-\ddots}}}=1-\frac{aq(1-bq)t}{1-abq^2}\phi(bq,qt).$$

Equating the coefficients of t^n yields that $a_0(b)=1$ and

$$a_n(b)=a_{n-1}(b)-\frac{aq^n(1-bq)}{1-abq^2}a_{n-1}(bq). \tag{7}$$

From this identity we derive readily that

$$a_n(b)=\frac{(aq;q)_n}{(abq^2;q)_n}=\mu_n.$$

Since the continued fractions in (3) and (6) are identical, this proves (3).

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