

## THE COMBINATORICS OF THE AL-SALAM-CHIHARA $q$ -CHARLIER POLYNOMIALS

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*Dedicated to Xavier Viennot on the occasion of his sixtieth birthday*

ABSTRACT. We describe various aspects of the Al-Salam-Chihara  $q$ -Charlier polynomials. These include combinatorial descriptions of the polynomials, the moments, the orthogonality relation and a combinatorial proof of Anshelevich’s recent result on the linearization coefficients.

### 1. INTRODUCTION

The classical Charlier polynomials  $C_n^a(x)$  have been studied combinatorially by several authors [5, 8, 14]. Recall [4] that these polynomials are defined by

$$C_n^a(x) = \sum_{k=0}^n \binom{n}{k} (-a)^{n-k} x(x-1) \cdots (x-(k-1)) \tag{1.1}$$

and satisfy the three term-recurrence relation

$$C_{n+1}^a(x) = (x-a-n) C_n^a(x) - an C_{n-1}^a(x), \quad n \geq 0, \tag{1.2}$$

where  $C_0^a(x) = 1$ ,  $C_{-1}^a(x) = 0$ .

A  $q$ -version  $C_n^a(x; q)$  of these polynomials was studied in [6]. The three-term recurrence relation was

$$C_{n+1}^a(x; q) = (x - aq^n - [n]_q) C_n^a(x; q) - aq^{n-1} [n]_q C_{n-1}^a(x; q),$$

where  $[n]_q = 1 + q + \cdots + q^{n-1}$ ,  $C_0^a(x; q) = 1$ ,  $C_{-1}^a(x; q) = 0$ . The explicit formula analogous to (1.1) is given by

$$C_n^a(x; q) = \sum_{k=0}^n \begin{bmatrix} n \\ k \end{bmatrix}_q (-a)^{n-k} q^{\binom{n-k}{2}} \prod_{i=0}^{k-1} (x - [i]_q).$$

The linearization coefficients for the Charlier polynomials are given by quotients of factorials (see (1.5)). The combinatorial study of the  $q$ -analogues  $C_n^a(x; q)$  in [6] included finding their linearization coefficients, which were given by a double sum, not quotients of factorials, and as a polynomial in  $q$  and  $a$  did not have positive coefficients (see [6, Theorem 3]).

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Anshelevich [2] has recently considered a  $q$ -analogue of the three-term recurrence (1.2) for  $C_n^a(x+a, a)$  and proved that the linearization coefficients of the corresponding polynomials are polynomials of  $a$  and  $q$  with positive integer coefficients (see Theorem 6).

The aim of this paper is to study the combinatorial aspects of a new  $q$ -analogue of Charlier polynomials, which is a re-scaled version of Anshelevich's  $q$ -polynomials and turns out to be a special re-scaled version of the Al-Salam-Chihara polynomials. We shall give a combinatorial proof of Anshelevich's result by using the combinatorial interpretations of the polynomials and their moments. It is inspired by the beautiful proofs for other classical orthogonal polynomials in [7, 10].

This paper is organized as follows: in the next two sections we give the definitions and combinatorial interpretations of the new  $q$ -Charlier polynomials and their moments, Corollary 3 and Theorem 4. The explicit linearization coefficients are given in §4 in Corollary 8. We then give the killing involution in §5. We present a variation  $\hat{C}_n(x|q)$  of the polynomials  $C_n(x, a; q)$  in §6, which has the advantage to include the  $q$ -Hermite polynomials in [10] as a special case.

We collect here some well-known facts about Charlier polynomials.

The generating function is

$$\sum_{n=0}^{\infty} C_n^a(x) \frac{t^n}{n!} = e^{-at}(1+at)^x. \quad (1.3)$$

The moment sequence  $\mu_n$  is given by the following formula:

$$\mu_n = \mathcal{L}(x^n) = \sum_{x=0}^{\infty} x^n e^{-a} \frac{a^x}{x!} = \sum_{k=1}^n S(n, k) a^k, \quad (1.4)$$

where  $S(n, k)$  denotes the Stirling number of the second kind. The orthogonality reads:

$$\mathcal{L}(C_m^a(x)C_n^a(x)) = \sum_{k=0}^{\infty} C_m^a(k)C_n^a(k) \frac{e^{-a}a^k}{k!} = n!a^n \delta_{mn}.$$

The linearization coefficient  $c_{n_1 n_2}^{n_3}$  is defined by:

$$C_{n_1}^a(x)C_{n_2}^a(x) = \sum_{n_3} c_{n_1 n_2}^{n_3} C_{n_3}^a(x).$$

By orthogonality we have  $c_{n_1 n_2}^{n_3} = \mathcal{L}(C_{n_1}^a(x)C_{n_2}^a(x)C_{n_3}^a(x))/\mathcal{L}(C_{n_3}^a(x)C_{n_3}^a(x))$ .

For Charlier polynomials it is easy to derive from (1.3) and (1.4) that

$$\sum_{n_1, \dots, n_k=0}^{\infty} \mathcal{L}(C_{n_1}^a(x) \dots C_{n_k}^a(x)) \frac{t_1^{n_1}}{n_1!} \dots \frac{t_k^{n_k}}{n_k!} = e^{a(e_2(t_1, \dots, t_k) + \dots + e_k(t_1, \dots, t_k))},$$

where  $e_i$  is the elementary symmetric function of degree  $i$ . It follows that

$$\mathcal{L}(C_{n_1}^a(x)C_{n_2}^a(x)C_{n_3}^a(x)) = \sum_l \frac{n_1!n_2!n_3!a^{l+n_3}}{l!(n_3-n_1+l)!(n_3-n_2+l)!(n_1+n_2-n_3-2l)!}. \quad (1.5)$$

In the general case the above generating function implies that  $\mathcal{L}(C_{n_1}^a(x) \dots C_{n_k}^a(x))$  ( $k \geq 1$ ) is a polynomial in  $a$  with positive integer coefficients; a combinatorial interpretation of this coefficient has been given [8, 14].

## 2. THE NEW $q$ -CHARLIER POLYNOMIALS

We define the new  $q$ -Charlier polynomials  $C_n(x, a; q)$  by

$$C_{n+1}(x, a; q) = (x - a - [n]_q) C_n(x, a; q) - a[n]_q C_{n-1}(x, a; q). \quad (2.1)$$

The first values of these polynomials are

$$\begin{aligned} C_1(x, a; q) &= x - a, \\ C_2(x, a; q) &= x^2 - (2a + 1)x + a^2, \\ C_3(x, a; q) &= x^3 - (q + 3a + 2)x^2 + (aq + 3a^2 + 2a + q + 1)x - a^3. \end{aligned}$$

The explicit formula which is analogous to (1.1) is

$$C_n(x, a; q) = \sum_{k=0}^n \begin{bmatrix} n \\ k \end{bmatrix}_q q^{k(k-n)} (-a)^{n-k} \prod_{i=0}^{k-1} (x - [i]_q + a(q^{-i} - 1)). \quad (2.2)$$

This is a re-scaled version of the Al-Salam-Chihara polynomials  $Q_n(x, \alpha, \beta; q)$  [12, p. 80–81]:

$$C_n(x, a; q) = \left( \frac{a}{1-q} \right)^{n/2} Q_n \left( \frac{1}{2} \sqrt{\frac{1-q}{a}} \left( x - a - \frac{1}{1-q} \right), \frac{-1}{\sqrt{a(1-q)}}, 0; q \right).$$

Since the generating function of the Al-Salam-Chihara polynomials is known, we derive that

$$\sum_{n=0}^{\infty} C_n(x, a; q) \frac{t^n}{n!_q} = \frac{(-t; q)_{\infty}}{(\sqrt{a(1-q)}te^{i\theta}, \sqrt{a(1-q)}te^{-i\theta}; q)_{\infty}},$$

where  $n!_q = [n]_q [n-1]_q \dots [2]_q [1]_q$  and

$$\cos \theta = \frac{1}{2} \sqrt{\frac{1-q}{a}} \left( x - a - \frac{1}{1-q} \right).$$

We can give a combinatorial interpretation for the  $q$ -Charlier polynomials from a result due to Simion and Stanton [13].

Consider a subset  $B$  of  $[n]$  and a permutation  $\sigma$  on  $[n] \setminus B$ . Then  $\sigma$  consists of fixed points and cycles of length  $> 1$ :

$$C = (k_0, k_1, k_2, \dots, k_s), \quad \text{where } k_s > \max\{k_0, k_1, \dots, k_{s-1}\}.$$

For any  $k \in [n] \setminus B$ , let  $w(k) = 0$  if  $k$  is the maximum of its cycle, otherwise  $k = k_j$  is on a cycle  $C$  as above, then

$$w(k) = k - 1 - |\{i : j < i < s, k_i < k_j\}| - \sum_{\text{cycles } Q, \max(Q) > k_s} (\# \text{ of points on } Q \text{ less than } k).$$

Let  $w(B, \sigma) = \sum_{k \in [n] \setminus B} w(k)$  and let  $\text{cyc}(\sigma)$  be the number of cycles of  $\sigma$ .

**Example 1.** Let  $n = 9$ ,  $B = \{2, 9\}$  and  $\sigma = (6)(47)(3518)$  (in cycle notation with maximum at last). Then we have  $\text{cyc}(\sigma) = 3$  and

$$w(B, \sigma) = (3 - 1 - 1) + (5 - 1 - 1) + (1 - 1) + (4 - 1 - 2) = 5.$$

**Theorem 1.** We have

$$C_n(x, a; q) = \sum_{(B, \sigma)} (-1)^{n - \text{cyc}(\sigma)} a^{|B|} x^{\text{cyc}(\sigma)} q^{w(B, \sigma)}.$$

where  $B \subset [n]$  and  $\sigma$  is a permutation on  $[n] \setminus B$ .

*Proof.* This is the  $r = 1$ ,  $s = 0$ ,  $t = q$ ,  $u = 1$  special case of the quadrabasic Laguerre polynomials [13, p.313].  $\square$

We now assume that each permutation  $\pi$  of  $[n]$  is represented as a product of disjoint cycles,  $\pi = \sigma_1 \sigma_2 \cdots \sigma_k$ , where the cycles are written in the descending order of their maxima and each  $\sigma_i$  has its maximum at the first position. A pair  $(i, j)$ ,  $i > j$ , is called a *Charlier-inversion* in  $\pi = \sigma_1 \sigma_2 \cdots \sigma_k$  if  $i$  is not a maxima of any cycles of  $\pi$  and  $i$  appears to the left of  $j$  in  $\pi$ . For instance,  $(6, 2)$ ,  $(6, 4)$ ,  $(6, 5)$ ,  $(6, 1)$ ,  $(6, 3)$ ,  $(2, 1)$ ,  $(4, 1)$  and  $(4, 3)$  are all Charlier-inversions in  $\pi = (862)(74)(513)$ . Let  $\text{Cinv}(\pi)$  denote the number of Charlier-inversions in  $\pi$ .

**Definition 2.** (*Charlier-labeling of permutations*) A *Charlier-labeling* of a permutation  $\pi = \sigma_1 \sigma_2 \cdots \sigma_k$  is a labeling of integers and cycles in  $\pi$  satisfying the following rules:

- Each integer in  $\pi$  is labeled  $-1$ .
- Each cycle of length 1 is labeled either  $-x$  or  $a$ .
- Each cycle of length  $> 1$  is labeled  $-x$ .

A permutation with a Charlier-labeling is called a Charlier-permutation.

Let  $\tau$  denote a Charlier-permutation with underlying permutation  $\pi$ . Identify  $\text{Cinv}(\tau)$  with  $\text{Cinv}(\pi)$ . Define the weight of  $\tau$ ,  $w(\tau)$ , to be the product of  $q^{\text{Cinv}(\tau)}$  and all the labels of integers and of cycles in  $\tau$ . Since only 1-cycles are allowed two different choices for a label, if  $\pi$  has  $f$  fixed points, there are  $2^f$  distinct Charlier-permutations with  $\pi$  as an underlying permutation. We represent each cycle in a permutation as a sequence starting with the maximum, enclosed with a pair of parentheses. The cycles in a Charlier-permutation are represented in the same way except that 1-cycles with label  $a$  are enclosed with a pair of brackets.

For each pair  $(B, \sigma)$  in Theorem 1, where  $B \subset [n]$  and  $\sigma$  a permutation of  $[n] \setminus B$ , one can associate a Charlier-permutation  $\tau$  of  $[n]$  as follows: each element of  $B$  gives rise a 1-cycle with brackets, each cycle  $(a_1 a_2 \dots a_l)$  of  $\sigma$  gives rise a cycle  $(a_l a_{l-1} \dots a_1)$  of  $\tau$  with reverse order and the maximal element at the first position. It is not hard to see that  $w(B, \sigma) = \text{Cinv}(\tau)$ . For instance, the Charlier-permutation corresponding to the pair  $(B, \sigma)$  in the above example is  $\tau = [9](8153)(74)(6)[2]$  with weight

$$(-1)^9 (-x)^3 a^2 q^{0+3+1+1} = a^2 q^5 x^3,$$

because there are nine integers of label  $-1$ , three cycles of label  $-x$ , two cycles of label  $a$ , five Charlier-inversions, i.e.  $(5, 3), (5, 4), (5, 2), (3, 2), (4, 2)$ .

One can restate Theorem 1 as follows.

**Corollary 3.** *The  $q$ -Charlier polynomial  $C_n(x, a; q)$  is the generating function of Charlier-permutations of  $[n]$ :*

$$C_n(x, a; q) = \sum_{\tau} w(\tau),$$

where  $\tau$  runs through all permutations of  $[n]$ .

### 3. THE MOMENTS

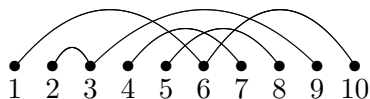
For the Charlier polynomials  $C_n^a(x)$ , the  $n^{\text{th}}$  moment  $\mu_n$  is the generating function for set partitions of  $\{1, 2, \dots, n\}$  according to the number of blocks (see (1.4)). There is a natural  $q$ -analogue for the polynomials  $C_n^a(x; q)$  [6, Eq. (3.1)], whose  $n^{\text{th}}$  moment is

$$\mu_n = \sum_{k=1}^n S_q(n, k) a^k.$$

Note that  $S_q(n, k)$  is the most natural  $q$ -analogue of the Stirling numbers of the second kind, and may also be interpreted as a generating function for set partitions with  $k$  blocks according to a  $q$ -statistic. It has a remarkably simple expression as a single sum [6, Eq. (3.3)]. In this section we identify an appropriate  $q$ -statistic on set partitions which yields the  $n^{\text{th}}$  moment  $\mu_n$  for  $C_n(x, a; q)$ , and give an explicit formula for it.

Recall that if  $\pi$  is a partition of  $M = \{1, \dots, m\}$ , then a *crossing* of  $\pi$  is a quadruple  $(a, b, c, d)$  of elements of  $M$  such that  $a < b < c < d$ , the elements  $a, c$  are in some block of the partition and  $b, d$  are in another block. For two elements  $e$  and  $f$  of  $M$ , with  $e < f$ , we say that  $f$  *follows*  $e$  in  $\pi$  if  $e$  and  $f$  belong to the same block of  $\pi$ , and there is no element  $g$  of this block with  $e < g < f$ . We define a *restricted crossing* to be a crossing  $(a, b, c, d)$  such that  $c$  follows  $a$  and  $d$  follows  $b$ . Similarly a *nesting* is a quadruple  $(a, b, c, d)$  of elements of  $M$  such that  $a < b < c < d$ , the elements  $a, d$  are in some block of the partition and  $b, c$  are in another block. We define a *restricted nesting* to be a nesting  $(a, b, c, d)$  such that  $d$  follows  $a$  and  $c$  follows  $b$ . The restricted crossings and nestings have a natural interpretation in the graphic line representation of partitions. This representation consists in drawing the  $m$  points on the  $x$ -axis in the plane and, if  $f$  follows  $e$ , joining the point  $e$  and  $f$  by an arc above the  $x$ -axis.

For instance, the graph of  $\pi = \{1, 6, 10\} - \{2, 3, 9\} - \{4, 7\} - \{5, 8\}$  is the following:



Let  $\text{rc}(\pi)$  (resp.  $\text{rn}(\pi)$ ) be the number of restricted crossings (resp. restricted nestings) of a partition  $\pi$ . The number of blocks of  $\pi$  is denoted by  $\text{block}(\pi)$ . For the above  $\pi$

we have  $\text{block}(\pi) = 4$ ,  $\text{rc}(\pi) = 7$  and there are  $\text{rn}(\pi) = 3$  restricted nestings, namely  $(1, 2, 3, 6)$ ,  $(3, 4, 7, 9)$ ,  $(3, 5, 8, 9)$ .

We can derive the combinatorial interpretation of the moments from the continued fraction expansion of the ordinary generating functions of partitions with respect to the corresponding statistics (see [3, 11]) and the three-term recurrence relation (2.1).

**Theorem 4.** *The  $n^{\text{th}}$ -moment of the  $q$ -Charlier polynomials  $C_n(x, a; q)$  is*

$$\mu_n(a) := \mathcal{L}_q(x^n) = \sum_{\pi \in \Pi_n} a^{\text{block}(\pi)} q^{\text{rc}(\pi)} = \sum_{\pi \in \Pi_n} a^{\text{block}(\pi)} q^{\text{rn}(\pi)},$$

where  $\Pi_n$  denotes the set of partitions of  $[n] := \{1, \dots, n\}$ .

The first values of  $\mu_n(a)$  are as follows:

$$\mu_1(a) = a, \quad \mu_2(a) = a + a^2, \quad \mu_3(a) = a + 3a + a^3, \quad \mu_4(a) = a + (6+q)a^2 + 6a^3 + a^4.$$

It is possible to derive an explicit formula for the moments from the known measure for the Al-Salam-Chihara polynomials and facts about  $q$ -Hermite polynomials. We do not give the details of this calculation.

Let  $\theta_0 = 1$ , and for odd values of  $m \geq 1$  let

$$\theta_m = \sum_{k=0}^{\lfloor m/2 \rfloor} \binom{m}{k} \sum_{l=0}^{\lfloor m/2 \rfloor - k} \frac{(-1)^{m-l} (a(1-q))^{k+l}}{(2\sqrt{a(1-q)})^m} \frac{1 - q^{m-2k}}{1 - q^{m-2k-l}} \begin{bmatrix} m - 2k - l \\ l \end{bmatrix}_q q^{\binom{l}{2}},$$

while for even values of  $m \geq 1$  let

$$\theta_m = \sum_{k=0}^{\lfloor m/2 \rfloor - 1} \binom{m}{k} \sum_{l=0}^{\lfloor m/2 \rfloor - k} \frac{(-1)^{m-l} (a(1-q))^{k+l}}{(2\sqrt{a(1-q)})^m} \frac{1 - q^{m-2k}}{1 - q^{m-2k-l}} \begin{bmatrix} m - 2k - l \\ l \end{bmatrix}_q q^{\binom{l}{2}} + \frac{1}{2^m} \binom{m}{m/2}.$$

**Proposition 5.** *The  $n^{\text{th}}$ -moment of the  $q$ -Charlier polynomials  $C_n(x, a; q)$  is given by*

$$\mu_n(a) = u^{-n} \sum_{m=0}^n \binom{n}{m} (-v)^{n-m} \theta_m,$$

where

$$u = \frac{1}{2} \sqrt{\frac{1-q}{a}} \quad \text{and} \quad v = -\frac{a(1-q) + 1}{2\sqrt{a(1-q)}}.$$

#### 4. THE ORTHOGONALITY RELATION AND THE LINEARIZATION OF PRODUCTS

The orthogonality of the  $q$ -Charlier polynomials reads as follows:

$$\mathcal{L}_q(C_n(x, a; q) C_m(x, a; q)) = n!_q a^n \delta_{mn}.$$

In this section we state Anshelevich's linearization result, which generalizes the orthogonality relation, in Theorem 6, and explicitly evaluate the coefficients in Corollary 8.

Set  $n = n_1 + n_2 + \dots + n_k$ . Denote by

$$\pi_{n_1, n_2, \dots, n_k} \in \Pi_n$$

the partition whose blocks are intervals of consecutive integers of lengths  $n_1, n_2, \dots, n_k$ . Denote

$$\Pi(n_1, n_2, \dots, n_k) = \{\pi \in \Pi_n : \pi \text{ has no singleton and } \pi \wedge \pi_{n_1, n_2, \dots, n_k} = \hat{0}\},$$

the partitions *without singleton* and *inhomogeneous* with respect to  $\pi_{n_1, n_2, \dots, n_k}$ , that is, the collection of all partitions which do not put together elements of the  $k$  distinguished subsets in the same block. Thus  $\pi = \{1, 6, 10\} - \{2, 3, 9\} - \{4, 7\} - \{5, 8\} \in \Pi(2, 4, 4)$ .

In 2005 Anshelevich [2] considered the re-scaled version  $C_n(x + a, a; q)$  and proved the following

**Theorem 6** (Anshelevich). *The linearization coefficients of  $q$ -Charlier polynomials are the generating functions of the inhomogeneous partitions:*

$$\mathcal{L}_q(C_{n_1}(x, a; q) \cdots C_{n_k}(x, a; q)) = \sum_{\pi \in \Pi(n_1, n_2, \dots, n_k)} q^{\text{rc}(\pi)} a^{\text{block}(\pi)}. \quad (4.1)$$

For example, if  $k = 3$  and  $n_1 = n_2 = 2$  and  $n_3 = 1$ , then

$$\Pi(2, 2, 1) = \{\{(1, 3, 5)(2, 4)\}, \{(1, 4, 5)(2, 3)\}, \{(2, 3, 5)(1, 4)\}, \{(2, 4, 5)(1, 3)\}\}.$$

It is easy to see that the corresponding generating function in (4.1) is

$$a^2 q^2 + a^2 + a^2 q + a^2 q = a^2(1 + q)^2.$$

If  $k = 2$ , equation (4.1) gives the orthogonality relation. When  $k = 3$ , there is an explicit formula for the generating function in (4.1).

**Theorem 7.** *We have*

$$\sum_{\pi \in \Pi(n_1, n_2, n_3)} q^{\text{rc}(\pi)} t^{\text{block}(\pi)} = \sum_{l \geq 0} \frac{n_1!_q n_2!_q n_3!_q t^{l+n_3} q^{\binom{n_1+n_2-n_3-2l}{2}}}{l!_q (n_3 - n_1 + l)!_q (n_3 - n_2 + l)!_q (n_1 + n_2 - n_3 - 2l)!_q}.$$

*Proof.* First we verify the  $q = 1$  case, and then give an argument for the  $q$  case.

Let  $N_1 = [n_1]$ ,  $N_2 = [n_1 + n_2] \setminus [n_1]$  and  $N_3 = [n_1 + n_2 + n_3] \setminus [n_1 + n_2]$ . The type of a subset  $S$  of  $[n_1 + n_2 + n_3]$  is defined to be the triple  $(|S \cap N_1|, |S \cap N_2|, |S \cap N_3|)$ .

Consider the inhomogeneous partitions of the colored set  $[n_1 + n_2 + n_3]$  without singleton. Let  $a, b, c$  and  $d$  be respectively the numbers of blocks of type :  $A = (1, 1, 1)$ ,  $B = (1, 1, 0)$ ,  $C = (1, 0, 1)$  and  $D = (0, 1, 1)$ . Then

$$a + b + c = n_1, \quad a + b + d = n_2, \quad a + c + d = n_3.$$

Solving the equations and setting  $b = l$  we get

$$a = n_1 + n_2 - n_3 - 2l, \quad c = n_3 - n_2 + l, \quad d = n_3 - n_1 + l.$$

The total number of blocks is equal to  $a + b + c + d = n_3 + l$ , the power of  $t$  in Theorem 7.

Given an inhomogeneous partition  $\pi$  with  $a$  blocks of type  $A$ ,  $b$  blocks of type  $B$ ,  $c$  blocks of type  $C$ , and  $d$  blocks of type  $D$ , the types of elements of  $[n_1]$  form a multiset permutation  $w_1$  of  $A^a B^b C^c$ . Similarly we may define words  $w_2$  and  $w_3$  of lengths  $n_2$  and  $n_3$  as multiset permutations of  $A^a B^b D^d$  and  $A^a C^c D^d$ . The number of such words  $(w_1, w_2, w_3)$

is given by a product of three trinomial coefficients. The number of ways to choose edges to connect like types of letters is a factorial, so there is a total of

$$\begin{aligned} & \binom{n_1}{a, b, c} \binom{n_2}{a, b, d} \binom{n_3}{a, c, d} (a!)^2 b! c! d! \\ &= \frac{n_1! n_2! n_3!}{l!(n_3 - n_1 + l)! (n_3 - n_2 + l)! (n_1 + n_2 - n_3 - 2l)!}, \end{aligned}$$

inhomogeneous partitions.

We include  $q$  in the above argument by keeping track of the possible restricted crossings of  $\pi$ .

If  $\pi$  has words  $(w_1, w_2, w_3)$ , then some crossings are guaranteed from the  $w_i$ , independent of how the edges are attached to the letters.

- any occurrence in  $w_1$  of  $B$  preceding  $C$  or  $A$  preceding  $C$  gives a crossing,
- any occurrence in  $w_2$  of  $D$  preceding  $B$ ,  $D$  preceding  $A$ , or  $A$  preceding  $B$  gives a crossing,
- any occurrence in  $w_3$  of  $C$  preceding  $A$  or  $C$  preceding  $D$  gives a crossing.

The remaining crossings are

- crossings of edges of types  $ABAB$  and  $BABA$ , where the first two letters are in  $w_1$  and the last two letters are in  $w_2$ ,
- crossings of edges of types  $ADAD$  and  $DADA$ , where the first two letters are in  $w_1$  and the last two letters are in  $w_3$ ,
- crossings amongst edges of the same type.

Construct  $\pi$  in the following manner. Fix a word  $w_2$ , the guaranteed crossings in  $w_2$  are exactly the inversions in  $w_2$  if the letters are ordered  $BAD$ , thus the crossing generating function for  $w_2$  is [1, p. 41]

$$\left[ \begin{array}{c} n_2 \\ a, b, d \end{array} \right]_q.$$

Choose  $c$  of the positions in  $[n_1]$  for the locations of  $C$  in  $w_1$ , the  $C$ -inversions in  $w_1$  give the crossing generating function

$$\left[ \begin{array}{c} n_1 \\ c \end{array} \right]_q.$$

Also choose the  $c$  positions in  $w_3$  for  $C$ , the  $C$ -inversions in  $w_3$  contribute

$$\left[ \begin{array}{c} n_3 \\ c \end{array} \right]_q.$$

Match these  $2c$  positions with  $c$  inhomogeneous edges, the crossing generating function is

$$c!_q.$$



Connect the  $a + b$  letters of  $w_2$  of type  $A$  or  $B$  to the remaining  $a + b$  positions of  $w_1$  in  $(a + b)!$  ways. The crossings here have type  $ABAB$ ,  $BABA$ , and the same type  $AA$ ,  $BB$ . The generating function is

$$(a + b)!_q.$$

Connect the  $a + d$  letters of  $w_2$  of type  $A$  or  $D$  to the remaining  $a + d$  positions of  $w_3$  in  $(a + d)!$  ways. The crossings here have type  $ADAD$ ,  $DADA$ , and the same type  $AA$ ,  $DD$ . The generating function is

$$(a + b)!_q.$$

Any pair of edges, each of type  $A$ , always has one remaining crossing which is not accounted for, this is

$$q^{\binom{a}{2}}.$$

Multiplying the above corresponding generating functions yields the formula. □

**Corollary 8.** *We have the following linearization formula:*

$$C_{n_1}(x, a; q)C_{n_2}(x, a; q) = \sum_{n_3} K_{n_1 n_2 n_3} C_{n_3}(x, a; q), \tag{4.2}$$

where

$$K_{n_1 n_2 n_3} = \sum_{l \geq 0} \frac{n_1!_q n_2!_q a^l q^{\binom{n_1+n_2-n_3-2l}{2}}}{l!_q (n_3 - n_1 + l)!_q (n_3 - n_2 + l)!_q (n_1 + n_2 - n_3 - 2l)!_q}.$$

Corollary 8 may also be proven using the Askey-Wilson integral, see [9, p. 422].

### 5. A COMBINATORIAL PROOF OF THEOREM 6

In this section we prove Theorem 6, using the combinatorial interpretation of the polynomials given in Corollary 3 and the moments given in Theorem 4.

**5.1. Generalized Charlier-permutations.** We fix  $\mathbf{n} = (n_1, \dots, n_k)$ , where  $n_i$ 's are positive integers. Let  $n$  denote  $n_1 + n_2 + \dots + n_k$ . For  $1 \leq i \leq k$ , let  $N_i$  denote the set of all integers  $j$  such that  $n_1 + \dots + n_{i-1} < j \leq n_1 + \dots + n_i$ ,  $n_0 = 0$ . Then  $[n] = N_1 \cup \dots \cup N_k$ . A *generalized Charlier-permutation*  $\tau$  of type  $\mathbf{n}$  is a sequence  $(\tau_k, \tau_{k-1}, \dots, \tau_1)$  where  $\tau_i$  is a Charlier-permutation of  $N_i$ . The weight of a generalized Charlier-permutation is the product of the weights of its Charlier-permutations.

The following are examples of generalized Charlier-permutations of type  $\mathbf{n} = (2, 4, 3)$ :

$$\begin{aligned} &(9\ 7)(8) \mid (6\ 5)(4\ 3) \mid (2)(1), \quad (9\ 7)(8) \mid (6\ 5)(4\ 3) \mid [2](1), \quad (9\ 7)(8) \mid (6\ 4)(5\ 3) \mid (2\ 1), \\ &(9\ 7)[8] \mid (6\ 4)(5\ 3) \mid (2\ 1), \quad (9\ 8)[7] \mid (6\ 4)(5\ 3) \mid (2\ 1), \quad (9\ 8)[7] \mid (6)(5\ 4)(3) \mid (2\ 1). \end{aligned}$$

**5.2. Charlier-partitions.** Combining generalized Charlier-permutations and moments discussed in the previous sections, we want to interpret

$$\mathcal{L}_q(C_{n_1}(x, a; q) \cdots C_{n_k}(x, a; q))$$

as the weight generating function of some objects. The weight of any generalized Charlier-permutation can be regarded as a monomial in  $x$  of degree the number of cycles labeled  $-x$ . Applying  $\mathcal{L}_q$  to the monomial is equivalent to considering all possible partitions of such cycles, where cycles are ordered as they appear in the generalized Charlier-permutation. We call the resulting objects *Charlier-partitions* of  $\mathbf{n}$ . A Charlier-partition is represented as  $(\tau, \nu)$ , where  $\tau = (\tau_k, \tau_{k-1}, \dots, \tau_1)$  is a generalized Charlier-permutation and  $\nu$  is a partition of cycles labeled  $-x$  in  $\tau$ . We regard each 1-cycle with label  $a$  as a block by itself in  $\nu$ . The weight of  $(\tau, \nu)$  is defined by

$$w(\tau, \nu) = q^{\text{rc}(\nu)} a^{\text{block}(\nu)} w(\tau)|_{x=1}.$$

Then clearly we have the following identity:

$$\mathcal{L}_q(C_{n_1}(x, a; q) \cdots C_{n_k}(x, a; q)) = \sum_{(\tau, \nu)} w(\tau, \nu). \quad (5.1)$$

Given a Charlier-partition  $(\tau, \nu)$  of  $\mathbf{n}$  with  $\tau = (\tau_k, \tau_{k-1}, \dots, \tau_1)$ , we draw the corresponding diagram on the plane as follows:

- The  $n$  integers in  $\tau$  are arranged on the horizontal axis in the order they appear in  $\tau$ , one step apart.
- The 1-cycles with label  $a$  are framed with a box.
- The maximum in each cycle, except that in a box, is circled, so that we can recover the cycle structure and labels.
- If a cycle  $\sigma$  follows a cycle  $\sigma'$  in a block of  $\nu$ , then we draw an arc above the horizontal line between the last element of  $\sigma'$  and the first element, that is also the maximum of  $\sigma$ , making as few crossings as possible. The smallest number of crossings agrees with the restricted crossings in  $\nu$ ,  $\text{rc}(\nu)$ .
- Draw a straight edge between two adjacent elements if and only if they are in the same cycle.

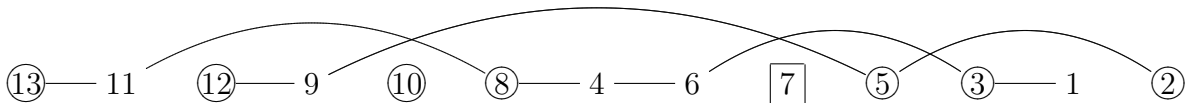
**Example 2.** Let  $\mathbf{n} = (3, 5, 5)$ . Then  $(\tau, \nu)$  is a Charlier-partition of  $\mathbf{n}$ , where

$$\tau = ((13 \ 11)(12 \ 9)(10), (8 \ 4 \ 6)[7](5), (3 \ 1)(2))$$

is a generalized Charlier-permutation of type  $\mathbf{n}$  and

$$\nu = \{(13 \ 11), (8 \ 4 \ 6), (3 \ 1)\}, \{(12 \ 9), (5), (2)\}, \{(10)\}$$

is a partition of cycles of  $\tau$  with weight  $-x$ . The corresponding diagram can be illustrated as follows:



**5.3. Involution.** The weight function in Equation (5.1) has many cancelations. We will give a combinatorial weight-preserving sign-reversing involution  $\phi$  with fixed set  $\Pi(n_1, n_2, \dots, n_k)$  defined on the set of all Charlier-partitions of type  $\mathbf{n}$ .

Let  $(\tau, \nu)$  be a Charlier-partition of  $\mathbf{n}$ . The involution  $\phi$  will be defined depending on three different cases of  $(\tau, \nu)$ .

CASE 1. If  $(\tau, \nu)$  has a circled 1-cycle in a block by itself or a boxed 1-cycle, then define  $\phi(\tau, \nu)$  by picking up the smallest 1-cycle and switching its box to circle or vice versa. Since a boxed 1-cycle contributes  $-a$  and a circled 1-cycle  $a$ ,  $\phi$  is weight-preserving sign-reversing in this case.

CASE 2. We now assume that  $(\tau, \nu)$  has no 1-cycles, boxed or circled in a block by itself. Find the rightmost integer  $\alpha$ , if it exists, in  $\tau$ , say in  $\tau_i$ , such that it has a neighbor  $\beta$  in  $\tau_i$ , along the straight edge or an arc, to its right.

CASE 2.1. Assume that  $\alpha$  and  $\beta$  are in the same cycle  $\sigma$  ending with  $\alpha\beta$ , i.e.  $\sigma = (\dots\alpha\beta)$ . Since  $\alpha$  is the penultimate entry in  $\sigma$ ,  $\beta$  is not the maximum in  $\sigma$ . Suppose the contribution of  $\beta$  to  $\text{Cinv}(\tau_i)$  is  $j$ . Then  $\tau_i$  is of the form

$$\tau_i = (\dots) \cdots (\dots\alpha\beta)(t_m)(t_{m-1}) \cdots (t_{j+2})(t_j) \cdots (t_1)$$

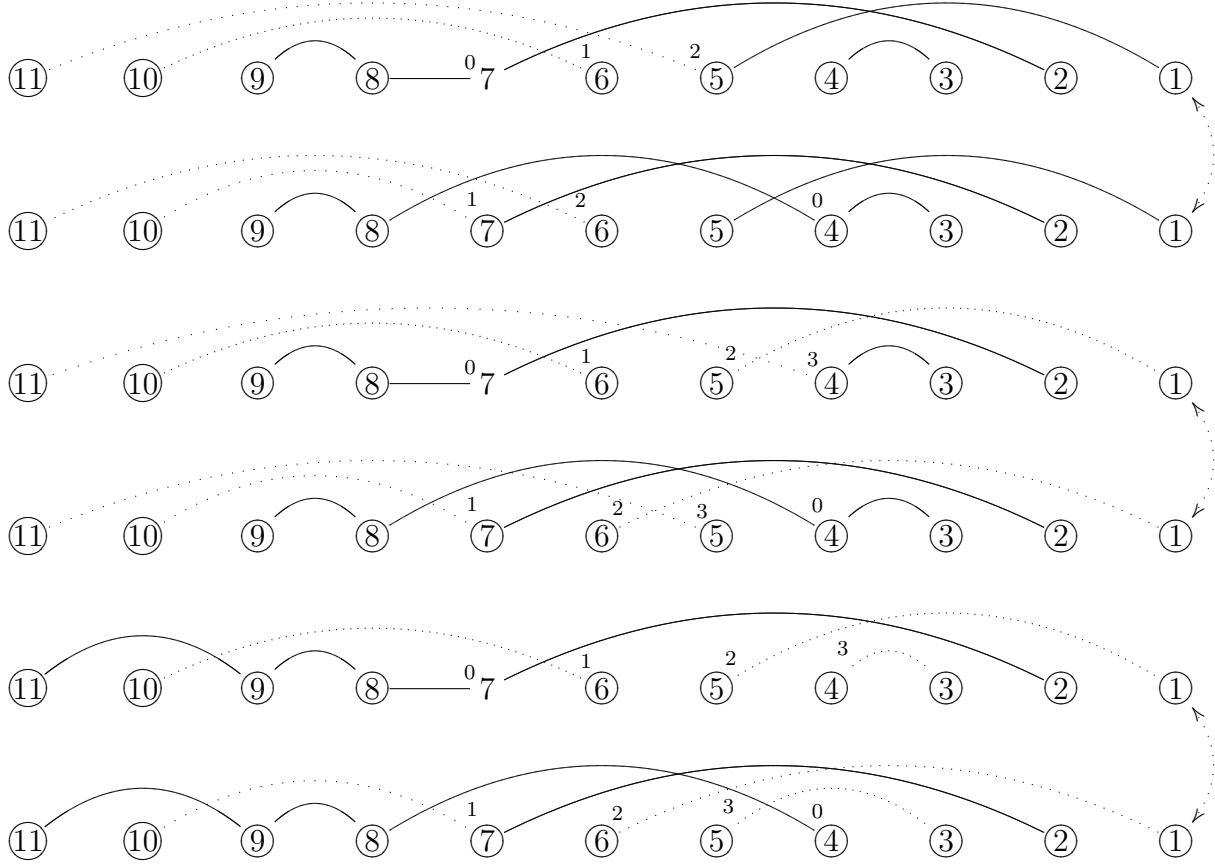
with  $t_1 < t_2 < \dots < t_m$  and  $t_{j+1} = \beta$ . Let  $\tau'_i = (\dots) \cdots (\dots\alpha)(t_m)(t_{m-1}) \cdots (t_1)$ . Integers  $t_m, t_{m-1}, \dots, t_{j+2}$  are moved to the left by one step and  $\beta$  occupies the position of  $t_{j+2}$ . We make some changes on the diagram of  $(\tau, \nu)$  as follows, to obtain the diagram of the Charlier-partition  $(\tau', \nu')$ :

**Algorithm: Stretch**

- Initially, start with the diagram of  $(\tau, \nu)$  with all arcs and edges.
- Delete the straight edge between  $\alpha$  and  $\beta$  in the diagram.
- Rearrange  $t_m, t_{m-1}, \dots, t_1$  in descending order, leaving the arcs and edges intact in their present positions.
- For  $l$  from  $m - 1$  down to  $m - j$ , make the arc arriving from left at the position of  $t_l$  to arrive at  $t_{l+1}$ , if it exists; if there is no such arc at position  $t_l$  and there are no arcs at position  $t_{l+1}$ , then make the arc arriving from right at position  $t_l$  to arrive at position  $t_{l+1}$ .
- Add an arc between  $\alpha$  and  $t_{m-j}$ .

Let  $\phi(\tau, \nu) = (\tau', \nu')$ . We need to show that  $\phi$  is a weight-preserving sign-reversing involution. The involution part will be clear after the next subcase is introduced. Clearly we have  $w(\tau', \nu') = -w(\tau, \nu)$  when  $q = 1$ , since  $(\tau', \nu')$  has one more 1-cycle than  $(\tau, \nu)$ , contributing  $-1$  to  $w(\tau', \nu')$ . So it suffices to prove that the exponents of  $q$  in  $w(\tau, \nu)$  and  $w(\tau', \nu')$  are the same. This can be done easily by induction on  $j$ . The loss in Charlier inversions from  $\tau$  to  $\tau'$  exactly matches the gain in restricted crossings from  $\nu$  to  $\nu'$  for all  $j$ .

The following are some examples with  $\mathbf{n} = (3, 6, 2)$ .



Three sets of  $(\tau, \nu)$  and  $(\tau', \nu')$  of type  $\mathbf{n} = (3, 6, 2)$  with  $\alpha = 8$ .

CASE 2.2. We now assume that  $\alpha$  and  $\beta$  are in different cycles. Clearly  $\beta$  forms a 1-cycle and adjoins  $\alpha$  by an arc. Moreover, all integers to the right of  $\alpha$  in  $\tau_i$  form 1-cycles and are in descending order. Suppose there are exactly  $j$  integers between  $\alpha$  and  $\beta$ . Then  $\tau_i$  is of the form  $\tau_i = (\cdots) \cdots (\cdots \alpha)(t_m)(t_{m-1}) \cdots (t_1)$  with  $t_1 < t_2 < \cdots < t_m$  and  $t_{m-j} = \beta$ . Let  $\tau'_i = (\cdots) \cdots (\cdots \alpha t_{j+1})(t_m)(t_{m-1}) \cdots (t_{j+2})(t_j) \cdots (t_1)$ . Integers  $t_m, t_{m-1}, \dots, t_{j+2}$  are moved to the right by one step and  $t_{j+1}$  occupies the position of  $t_m$ . We make some changes on the diagram of  $(\tau, \nu)$  as follows, to obtain the diagram of the Charlier-partition  $(\tau', \nu')$ :

**Algorithm: Compress**

- Initially, start with the diagram of  $(\tau, \nu)$  with all arcs and edges.
- Delete the arc between  $\alpha$  and  $\beta = t_{m-j}$  in the diagram.
- For  $l$  from  $m - j$  to  $m - 1$ , make the arc arriving from left at the position of  $t_{l+1}$  to arrive at  $t_l$ , if it exists; if there is no such arc at position  $t_{l+1}$  and there are no arcs at position  $t_l$ , then make the arc arriving from right at position  $t_{l+1}$  to arrive at position  $t_l$ .
- Rearrange  $t_m, t_{m-1}, \dots, t_{j+1}$  as  $t_{j+1}, t_m, t_{m-1}, \dots, t_{j+2}$ .

- Add a straight edge between  $\alpha$  and  $t_{j+1}$ .

We let  $\phi(\tau, \nu) = (\tau', \nu')$ .

If  $(\tau, \nu)$  falls on CASE 2.1 then  $(\tau', \nu')$  falls on CASE 2.2 and vice versa. The algorithms Stretch and Compress are inverses to each other. If the roles of  $(\tau, \nu)$  and  $(\tau', \nu')$  are exchanged in the examples for algorithm Stretch, then they become examples of algorithm Compress.

CASE 3. Assume that  $(\tau, \nu)$  does not fall in Cases 1 and 2. All the cycles in  $(\tau, \nu)$  are 1-cycles, every block in  $\nu$  has at least two cycles, and each block of  $\nu$  has at most one element in  $N_i$  for each  $i = 1, 2, \dots, k$ . So  $\nu$  is a partition in  $\Pi(n_1, n_2, \dots, n_k)$ . Let  $\phi(\tau, \nu) = (\tau, \nu)$ . The Charlier-partition  $(\tau, \nu)$  becomes a fixed point of  $\phi$ .

Combining Cases 1, 2 and 3, the mapping  $\phi$  is a weight-preserving sign-reversing involution on the set of all Charlier-partitions of type  $\mathbf{n}$  with fixed set  $\Pi(n_1, n_2, \dots, n_k)$ . This constitutes a combinatorial proof of Theorem 6.

### 6. A VARIATION

Consider the polynomials  $\hat{C}_n(x|q)$  defined by

$$\hat{C}_{n+1}(x|q) = (x - b[n]_q) \hat{C}_n(x|q) - a[n]_q \hat{C}_{n-1}(x|q), \quad n \geq 0, \tag{6.1}$$

where  $\hat{C}_0(x|q) = 1$  and  $\hat{C}_{-1}(x|q) = 0$ . Then (see [3, 11]) the polynomials  $\hat{C}_n(x|q)$  are orthogonal with respect to the linear functional  $\hat{\mathcal{L}}_q$  defined by

$$\hat{\mathcal{L}}_q(x^n) = \hat{\mu}_n = \sum_{\pi \in \Pi'_n} q^{\text{rc}(\pi)} a^{\text{block}(\pi)} b^{n-2\text{block}(\pi)},$$

where  $\Pi'_n$  is the set of partitions of  $[n]$  without singleton.

These polynomials may be obtained from  $C_n(x, a; q)$  as follows: let  $p_n(x)$  be the polynomial  $C_n(x + a, a; q)$  with  $a$  replaced by  $a/b^2$ , then  $\hat{C}_n(x|q) = b^n p_n(x/b)$ . It follows from (2.2) that

$$\hat{C}_n(x|q) = \sum_{k=0}^n \begin{bmatrix} n \\ k \end{bmatrix} (-1)^{n-k} q^{k^2 - kn} \left(\frac{a}{b}\right)^{n-k} \prod_{i=0}^{k-1} \left(x + \frac{a}{b} q^{-i} - b[i]_q\right). \tag{6.2}$$

The first values of these polynomials are

$$\begin{aligned} \hat{C}_1(x|q) &= x, \\ \hat{C}_2(x|q) &= x^2 - bx - a, \\ \hat{C}_3(x|q) &= x^3 - b(q+2)x^2 + (b^2(1+q) - 2a - aq)x + ab(1+q). \end{aligned}$$

Since the linearization coefficients are invariant by translation of  $x$ , we have

$$\frac{\hat{\mathcal{L}}_q(\prod_{i=1}^k \hat{C}_{n_i}(x|q))}{\hat{\mathcal{L}}_q((\hat{C}_{n_k}(x|q))^2)} = \frac{\mathcal{L}_q(\prod_{i=1}^k C_{n_i}(x, a; q))}{\mathcal{L}_q((C_{n_k}(x, a; q))^2)} \Bigg|_{a \rightarrow a/b^2} \cdot b^{n_1+n_2+\dots+n_{k-1}-n_k}. \tag{6.3}$$

As  $\hat{\mathcal{L}}_q(\hat{C}_{n_k}(x|q)\hat{C}_{n_k}(x|q)) = \mathcal{L}_q(C_{n_k}(x, a; q)C_{n_k}(x, a; q)) = a^{n_k n_k!_q}$ , we derive immediately from (6.3) and Theorem 6 the following

**Theorem 9** (Anshelevich). *The linearization coefficients of the polynomials  $\hat{C}_n(x|q)$  are the generating functions of the inhomogeneous partitions:*

$$\hat{\mathcal{L}}_q\left(\hat{C}_{n_1}(x|q)\cdots\hat{C}_{n_k}(x|q)\right) = \sum_{\pi \in \Pi(n_1, n_2, \dots, n_k)} q^{\text{rc}(\pi)} a^{\text{block}(\pi)} b^{n_1 + \dots + n_k - 2 \text{block}(\pi)}.$$

Anshelevich [2] presented the above theorem as a generalization of several other previously known results and proved it by the same method for Theorem 6. We have just shown that Theorem 6 and Theorem 9 are actually equivalent.

Now Corollary 8 implies the following

**Corollary 10.** *We have the following linearization formula:*

$$\hat{C}_{n_1}(x|q)\hat{C}_{n_2}(x|q) = \sum_{n_3} \hat{K}_{n_1 n_2 n_3} \hat{C}_{n_3}(x|q), \quad (6.4)$$

where

$$\hat{K}_{n_1 n_2 n_3} = \sum_{l \geq 0} \frac{n_1!_q n_2!_q a^l b^{n_1 + n_2 - n_3 - 2l} q^{\binom{n_1 + n_2 - n_3 - 2l}{2}}}{l!_q (n_3 - n_1 + l)!_q (n_3 - n_2 + l)!_q (n_1 + n_2 - n_3 - 2l)!_q}.$$

When  $a = 1$  and  $b = 0$  the polynomials  $\hat{C}_n(x|q)$  reduce to a family of  $q$ -Hermite polynomials  $\tilde{H}_n(x|q)$  (see [10, (2.11)]) and we get the corresponding combinatorial interpretation for the linearization coefficients of the  $q$ -Hermite polynomials in [10]:

$$\hat{\mathcal{L}}_q\left(\tilde{H}_{n_1}(x|q)\cdots\tilde{H}_{n_k}(x|q)\right) = \sum_{\pi} q^{\text{rc}(\pi)}, \quad (6.5)$$

where the summation is over all inhomogeneous 2-partitions  $\pi$  of  $[n_1 + \dots + n_k]$ , i.e., inhomogeneous partitions of which each block contains only two elements.

In particular, when  $a = 1$  and  $b = 0$ , identity (6.4) reduces to

$$\tilde{H}_{n_1}(x|q)\tilde{H}_{n_2}(x|q) = \sum_{l=0}^{\min(n_1, n_2)} \begin{bmatrix} n_1 \\ l \end{bmatrix}_q \begin{bmatrix} n_2 \\ l \end{bmatrix}_q l!_q \tilde{H}_{n_1 + n_2 - 2l}(x|q). \quad (6.6)$$

## 7. REMARKS

The  $q$ -Charlier polynomials in [6] have a natural  $q$ -Stirling number associated with their moments, a simple explicit formula, but a complicated and non-positive linearization formula. In contrast, Al-Salam-Chihara  $q$ -Charlier polynomials have a complicated  $q$ -Stirling number associated with their moments, a complicated explicit formula, but the most natural linearization formula.

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