A combinatorial formula for the linearization coefficients of general Sheffer polynomials

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Abstract

We prove a formula for the linearization coefficients of the general Sheffer polynomials, which unifies all the special known results for Hermite, Charlier, Laguerre, Meixner and Meixner-Pollaczek polynomials. Furthermore, we give a new and explicit real version of the corresponding formula for Meixner-Pollaczek polynomials. Our proof is based on some explicit bijections and sign-reversing weight-preserving involutions.

Running head: *Combinatorics of Sheffer polynomials*

1 Introduction

Given a sequence of formal orthogonal polynomials $p_n(x)$ $(n \geq 0)$ with respect to linear functional $\mathcal{L}$, the values $\mathcal{L}(\prod_{i=1}^{m} p_{n_i}(x))$ are usually called the linearization coefficients of the polynomials $p_n(x)$ (see [1]). The problem of finding interesting formulas for the linearization coefficients of the classical polynomials have attracted much attention in the last three decades (see [3, 8, 9, 11, 12, 15, 17] and the references cited there). Once the moment sequence $\mathcal{L}(x^n)$ is determined, this problem can be considered as a formal algebraic calculus. From this point of view, Viennot [15, Ch. 1] combinatorially proved the Favard theorem, using the combinatorial interpretations of the moments and polynomials.

One important class of polynomials is the *Sheffer polynomials*, which includes the classical Hermite, Charlier, Laguerre, Meixner and Meixner-Pollaczek polynomials as special cases. Recall that the monic Sheffer polynomials $P_n(x)$ $(n \geq 0)$ are defined by the three-term recurrence relations [4, p. 164]:

$$P_{n+1}(x) = \big(x - (\alpha\beta + nu_3 + nu_4)\big)P_n(x) - n(\beta + n - 1)u_1u_2P_{n-1}(x), \quad n \geq 0, \tag{1}$$

with $P_{-1}(x) = 0$ and $P_0(x) = 1$.

There are two known combinatorial interpretations of the linear functional corresponding to the above Sheffer polynomials. The first interpretation using *linear permutation statistics* is due to Viennot [15, Ch. 2], who proved the orthogonality by a bijection of Françon and Viennot. Another interpretation using *permutations statistics* related to cycle structures was given by Zeng [19]. As mentioned in [19] the two interpretations are transformed from one to another through Foata's first fundamental transformation [10]. For our purpose we need only the second interpretation related to the cycle structures of permutations, that we recall below.

For any positive integer n, let $\mathcal{S}_n$ denote the set of permutations of $[n] := \{1, \ldots, n\}$. Given a permutation $\sigma \in \mathcal{S}_n$, a value $i \in [n]$ is called a *peak or ascent-descent*, if $\sigma^{-1}(i) < i > \sigma(i)$; *valley or descent-ascent*, if $\sigma^{-1}(i) > i < \sigma(i)$; *double-ascent or ascent-ascent*, if $\sigma^{-1}(i) < i < \sigma(i)$; *double-descent or descent-descent*, if $\sigma^{-1}(i) > i > \sigma(i)$; *fixed point*, if $\sigma(i) = i$. Let $\operatorname{ad}\sigma$ (resp. $\operatorname{da}\sigma$, $\operatorname{aa}\sigma$, $\operatorname{dd}\sigma$ and $\operatorname{fix}\sigma$) be the number of peaks (resp. valleys, double-ascents, double-descents and fixed points) of σ. Note that $\operatorname{ad}\sigma = \operatorname{da}\sigma$. Furthermore, a value i is called *excedance* (resp. *decedance*) if $\sigma(i) > i$ (resp. $\sigma(i) < i$). Let $\operatorname{dec}\sigma$ (resp. $\operatorname{exc}\sigma$) be the number of decedances (resp. excedences) of σ. It is easy to see that $\operatorname{dec}\sigma = \operatorname{ad}\sigma + \operatorname{dd}\sigma$ and $\operatorname{exc}\sigma = \operatorname{aa}\sigma + \operatorname{da}\sigma$. Finally each permutation can be decomposed into disjoint *cycles*. Let $\operatorname{cyc}\sigma$ denote the number of cycles of σ. Consider the linear functional $\mathcal{L} : K[x] \to K$, where K is a commutative ring of characteristic 0, defined by its values on the monomials x^n :

$$\mathcal{L}(x^n) = \sum_{\sigma \in \mathcal{S}_n} u_1^{\operatorname{ad}\sigma} u_2^{\operatorname{da}\sigma} u_3^{\operatorname{aa}\sigma} u_4^{\operatorname{dd}\sigma} \alpha^{\operatorname{fix}\sigma} \beta^{\operatorname{cyc}\sigma}. \tag{2}$$

For instance, the first three values of $\mathcal{L}(x^n)$ are $\mathcal{L}(x) = \alpha\beta$, $\mathcal{L}(x^2) = \alpha^2\beta^2 + u_1u_2\beta$ and $\mathcal{L}(x^3) = \alpha^3\beta^3 + 3u_1u_2\alpha\beta^2 + u_1u_2(u_3 + u_4)\beta$. It was then proved in [19] that the general Sheffer polynomials defined by (1) are orthogonal with respect to the linear functional $\mathcal{L}$:

$$\mathcal{L}(P_{n_1}(x)P_{n_2}(x)) = n_1!(\beta)_{n_1}(u_1u_2)^{n_1}\delta_{n_1\,n_2}. \tag{3}$$

The following important definitions follow Foata and Zeilberger [11]. Given m nonnegative integers $\mathbf{n} = (n_1, \ldots, n_m)$, let $A_1, \ldots, A_{m-1}$ and A_m be m *disjoint totally ordered* sets such that $|A_i| = n_i$ for $i \in [m]$ and set$A_{\mathbf{n}} = A_1 \cup \cdots \cup A_m$. For $a \in A_{\mathbf{n}}$, an integer $i \in [m]$ is called *color* of a if $a \in A_i$, written as $c(a) = i$. Clearly the set $A_{\mathbf{n}}$ is totally ordered with respect to the lexicographical order, i.e., for two elements a and b in $A_{\mathbf{n}}$ we say that $a < b$ iff $c(a) < c(b)$ or $c(a) = c(b)$ and $a < b$. So the above classical weight function on $\mathcal{S}_n$ is still valid for any permutation of $A_{\mathbf{n}}$ considered as a totally ordered set. A permutation σ of $A_{\mathbf{n}}$ is called a *generalized derangement* if $c(a) \neq c(\sigma(a))$ for all $a \in A_{\mathbf{n}}$. Let $\mathcal{D}_{\mathbf{n}}$ be the set of generalized derangements of $A_{\mathbf{n}}$. Note that a generalized derangement π of $A_{\mathbf{n}}$ must also be an *ordinary derangement*, i.e. $\pi(a) \neq a$ for all $a \in A_{\mathbf{n}}$ and that $a > \pi(a)$ if and only if $c(a) > c(\pi(a))$.

Foata and Zeilberger [11] then proved that the linearization coefficients of Laguerre polynomials are the generating functions of $\mathcal{D}_{\mathbf{n}}$ with respect to the number of cycles. In [18] the second author proved that for each classical class of Sheffer polynomials the linearization coefficients are the generating functions of derangements with respect to the corresponding weight functions of their moment sequences. In this paper, we will refine the above results and give a formula for the values $\mathcal{L}(\prod_{i=1}^m P_{n_i}(x))$ for $m \geq 1$ in its full generality. To this end, we need to introduce a new statistic on the set of generalized derangements.

Consider a generalized derangement π of $A_{\mathbf{n}}$, factor it into disjoint cycles and identify each cycle of π with a path $P : s_1 \to s_2 \to \ldots \to s_l \to s_1$ if $\sigma(s_i) = s_{i+1}$ for $1 \leq i \leq l-1$, where s_1 is the smallest element in the cycle. Clearly the valleys (local minima) and peaks (local maxima) appear alternatively in the path P and their numbers are equal. Suppose there is a double-ascent s_p and a double-descent s_q $(p < q)$ in the path P such that $c(s_p) = c(s_q)$. Then the pair $(s_p,\ s_q)$ is called a

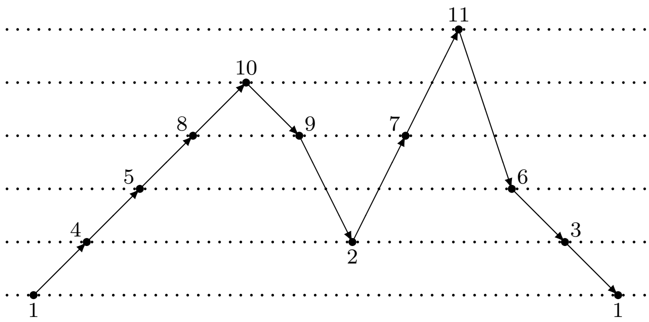


Figure 1: The lattice path representation of a cyclic permutation $\pi = (1\,4\,5\,8\,10\,9\,2\,7\,11\,6\,3)$ of $A_{\mathbf{n}}$ with $\mathbf{n} = (1, 3, 2, 3, 1, 1)$, where the elements of same color are placed on the same level.

color match of π if $c(s_j) \geq c(s_p)$ $(= c(s_q))$ for all $j \in \{p, \dots, q\}$. Let $\operatorname{mat}\sigma$ denote the number of *color matches* of π.

Example. For $\mathbf{n} = (1, 3, 2, 3, 1, 1)$, let $A_1 = \{1\}$, $A_2 = \{2, 3, 4\}$, $A_3 = \{5, 6\}$, $A_4 = \{7, 8, 9\}$, $A_5 = \{10\}$ and $A_6 = \{11\}$. Consider the cyclic permutation $\pi : 1 \to 4 \to 5 \to 8 \to 10 \to 9 \to 2 \to 7 \to 11 \to 6 \to 3 \to 1$ of $A_{\mathbf{n}}$. In order to illustrate the different statistics on π, it is instructive to draw a *lattice path* in the plan $\mathbf{N} \times \mathbf{N}$ by placing s_i at (i, j) if $s_i \in A_j$. We give such a representation in Figure 1, called *lattice path representation* of π. It follows that $\operatorname{aa}\pi = 4$, $\operatorname{dd}\pi = 3$, $\operatorname{da}\pi = 2$, $\operatorname{ad}\pi = 2$ and $\operatorname{mat}\pi = 2$, for there are two color matches: (4, 3) of color '2' and (8, 9) of color '4'. Note that (5, 6) is not a color match.

We are now in a position to state our main result.

Theorem 1.1 *The linearization coefficient of Sheffer polynomials is the generating functions of generalized derangements, i.e.*

$$\mathcal{L}\left(\prod_{i=1}^{m} P_{n_i}(x)\right) = \sum_{\sigma \in \mathcal{D}_{\mathbf{n}}} u_1^{\operatorname{ad}\sigma + \operatorname{mat}\sigma} u_2^{\operatorname{da}\sigma + \operatorname{mat}\sigma} u_3^{\operatorname{aa}\sigma - \operatorname{mat}\sigma} u_4^{\operatorname{dd}\sigma - \operatorname{mat}\sigma} \beta^{\operatorname{cyc}\sigma}. \tag{4}$$

Remark. The number of color matches is at most the number of double-ascents or double-descents, i.e. $\operatorname{aa}\sigma, \operatorname{dd}\sigma \geq \operatorname{mat}\sigma$, so each term in the above sum is a monomial with nonnegative exponents.

The proof of theorem 1.1 is of combinatorial nature, this is in the same vein as [8, 5, 12], where the authors were looking for a unified and more transparent approach to the linearization problems. More precisely, we first set up a combinatorial model for the Sheffer polynomials, combining with the combinatorial interpretation for the moment $\mathcal{L}$ in (2) we then interpret the linearization coefficients as the generating functions of some finite structures. Theorem 1.1 is finally proved by constructing some explicit weight-preserving sign-reversing involutions on these structures (See lemma 3.6 and theorem 4.5). Recall that a *weight-preserving sign-reversing* (wpsr) involution ϕ on a set S is an involution such that for all $x \in S$, $\phi(x) \neq x$ implies $w(\phi(x)) = -w(x)$. The set $\operatorname{Fix}\Phi = \{x \in S : \Phi(x) = x\}$ is called the *fixed set of* Φ.

For $\mathbf{n} = (n_1, \dots, n_m)$, set $|\mathbf{n}| = n_1 + \cdots + n_m$. For convenience, we will often identify $A_{\mathbf{n}}$ with $[|\mathbf{n}|]$, that is $A_i = \{n_1 + \cdots + n_{i-1} + 1, \dots, n_1 + \cdots + n_i\}$ for $1 \leq i \leq m$.

This paper is organized as follows. In section 2 we derive all the known results about the linearization coefficients of the classical Hermite, Charlier, Laguerre, Meixner and Meixner-Pollaczek polynomials from theorem 1.1. We also prove a new formula for the linearization coefficients of Meixner-Pollaczek polynomials. In section 3 we set up a preliminary combinatorial model, called L-graphs, for the linearization coefficients of Sheffer polynomials. In section 4 we construct a weight-preserving and sign-reversing involution on the L-graphs in order to prove theorem 1.1. Finally we conclude this paper with some remarks on further extensions in section 5.

2 Applications

2.1 Orthogonality of Sheffer polynomials

First of all the special $m = 2$ case of theorem 1.1 corresponds to the orthogonality relation of Sheffer polynomials [15, 19]. Indeed, if $n_1 \neq n_2$, then $\mathcal{D}_{\mathbf{n}} = \emptyset$; if $n_1 = n_2 = n$, each derangement π in $\mathcal{D}_{\mathbf{n}}$ can be identified with two permutations π_1 and π_2 in S_n, determined by $\pi(1, i) = (2, \pi_1(i))$ and $\pi(2, i) = (1, \pi_2(i))$ for each $i \in [n]$. It is easy to check that $\operatorname{cyc}\pi = \operatorname{cyc}\pi_2$, $\operatorname{ad}\pi = \operatorname{da}\pi = n$ and $\operatorname{mat}\pi = \operatorname{aa}\pi = \operatorname{dd}\pi = 0$. It follows from (4) that

$$\mathcal{L}(P_n(x)P_n(x)) = (u_1u_2)^n \left(\sum_{\pi_1 \in \mathcal{S}_n} 1\right)\left(\sum_{\pi_2 \in \mathcal{S}_n} \beta^{\operatorname{cyc}\pi}\right) = n!(\beta)_n (u_1u_2)^n,$$

where $(\beta)_n = \beta(\beta+1)\cdots(\beta+n-1)$. By combining the above two cases, we obtain (3).

2.2 Hermite polynomials

Set $u_1 = u_2 = \sqrt{u}$, where u is a positive real number, $u_3 = u_4 = 0$, $\alpha = 0$ and $\beta = u^{-1}$, then the moment sequence (2) becomes

$$\mathcal{L}(x^n) = \sum_{\sigma \in \mathcal{D}_n^*} u^{\operatorname{ad}\sigma - \operatorname{cyc}\sigma},$$

where $\mathcal{D}_n^*$ is the set of derangements of $[n]$ without double-descents or double-ascents. Now, letting $u \to 0$, the above sum reduces to 0, if n is odd, and is the number of involutions without fixed points, if n is even, i.e.

$$\mathcal{L}(x^n) = \begin{cases} 0 & \text{if } n \text{ odd}, \\ 1 \cdot 3 \cdot \cdots \cdot (n-1) & \text{if } n \text{ even}. \end{cases}$$

Besides, the three-term recurrence (1) reduces to that of Hermite polynomials:

$$H_{n+1}(x) = xH_n(x) - nH_{n-1}(x).$$

We recover then from theorem 1.1 the following result of Azor et al. [2]:

$$\mathcal{L}\left(\prod_{i=1}^{m} H_{n_i}(x)\right) = \sum_{\pi \in \mathcal{I}_{\mathbf{n}}} 1,$$

where $\mathcal{I}_{\mathbf{n}}$ is the set of all involutions in $\mathcal{D}_{\mathbf{n}}$.

2.3 Charlier polynomials

Set $\alpha = u_1 = u_4 = u$, $u_2 = u_3 = 1$ and $\beta = a/u$, then the moment sequence (2) becomes

$$\mathcal{L}(x^n) = \sum_{\sigma\in\mathcal{S}_n} u^{\mathrm{fix}\,\sigma+\mathrm{dec}\,\sigma-\mathrm{cyc}\,\sigma} a^{\mathrm{cyc}\,\sigma}.$$

Setting $u = 0$ the above sum reduces to the generating function of permutations of which each cycle contains only one decedance or is a singleton. It is easy to see that such permutations are in bijection with the set of partitions of $[n]$ by identifying each cycle with its underlying set, called *block*. Let $\mathrm{bloc}\,\pi$ denote the number of blocks in a partition π. For a positive integer n, let Π_n denote the set of partitions of $[n]$. Hence

$$\mathcal{L}(x^n) = \sum_{\pi\in\Pi_n} a^{\mathrm{bloc}\,\pi},$$

and the three-term recurrence becomes that of Charlier polynomials:

$$C_{n+1}(x) = (x - (a+n))C_n(x) - anC_{n-1}(x).$$

We derive then from theorem 1.1 the following result of Zeng [17]:

$$\mathcal{L}\left(\prod_{i=1}^{m} C_{n_i}(x)\right) = \sum_{\pi\in\Pi_{\mathbf{n}}} a^{\mathrm{bloc}\,\pi},$$

where $\Pi_{\mathbf{n}}$ denotes the set of partitions π of $A_{\mathbf{n}}$ such that the elements of each block of π have distinct colors.

2.4 Laguerre polynomials

Set $u_1 = u_2 = u_3 = u_4 = \alpha = 1$, then the three-term recurrence (1) becomes

$$L_{n+1}(x) = (x - (\beta+2n))L_n(x) - n(\beta+n-1)L_{n-1}(x),$$

and the moment sequence (2) reduces to

$$\mathcal{L}(x^n) = \sum_{\sigma\in\mathcal{S}_n} \beta^{\mathrm{cyc}\,\sigma}.$$

We recover then from theorem 1.1 the following result of Foata-Zeilberger [11]:

$$\mathcal{L}\left(\prod_{i=1}^{m} L_{n_i}(x)\right) = \sum_{\pi\in\mathcal{D}_{\mathbf{n}}} \beta^{\mathrm{cyc}\,\sigma}.$$

2.5 Meixner polynomials

Set $\alpha = \frac{c}{1-c}$, $u_1 = u_4 = \frac{c}{1-c}$ and $u_2 = u_3 = \frac{1}{1-c}$, then the three-term recurrence (1) becomes that of Meixner polynomials:

$$M_{n+1}(x) = (x - (c\beta + (1+c)n)/(1-c))M_n(x) - ((cn(\beta+n-1))/(1-c)^2)M_{n-1}(x),$$

and the moment sequence (2) reduces to

$$\mathcal{L}(x^n) = \left(\frac{c}{1-c}\right)^n \sum_{\sigma\in\mathcal{S}_n} \beta^{\operatorname{cyc}\sigma} c^{-\operatorname{exc}\sigma}. \tag{5}$$

We recover then from theorem 1.1 the following result of Zeng [17]:

$$\mathcal{L}\left(\prod_{i=1}^m M_{n_i}(x)\right) = \left(\frac{c}{1-c}\right)^n \sum_{\sigma\in D_{\mathbf{n}}} \beta^{\operatorname{cyc}\sigma} c^{-\operatorname{exc}\sigma}. \tag{6}$$

Another proof of this result was given by de Médicis [5].

2.6 Meixner-Pollaczek polynomials

Let i be a complex number such that $i^2 = -1$. Set $\alpha = \delta$, $u_1 = u_4 = \delta + i$, $u_2 = u_3 = \delta - i$ and $\beta = \eta$, then the three-term recurrence (1) becomes that of Meixner-Pollaczek polynomials:

$$P_{n+1}(x) = (x - (\delta\eta + 2\delta n))P_n(x) - n(\eta + n - 1)(1 + \delta^2)P_{n-1}(x),$$

and the moment sequence (2) reduces to

$$\mathcal{L}(x^n) = \sum_{\sigma\in\mathcal{S}_n} \eta^{\operatorname{cyc}\sigma}\delta^{\operatorname{fix}\sigma}(\delta+i)^{\operatorname{dec}\sigma}(\delta-i)^{\operatorname{exc}\sigma}. \tag{7}$$

We recover then from theorem 1.1 the following result of Zeng [18]:

$$\mathcal{L}\left(\prod_{k=1}^m P_{n_k}(x)\right) = \sum_{\sigma\in\mathcal{D}_{\mathbf{n}}} (\delta+i)^{\operatorname{dec}\sigma}(\delta-i)^{\operatorname{exc}\sigma}\eta^{\operatorname{cyc}\sigma}. \tag{8}$$

It is interesting to note that we naturally obtain a *complex* expression instead of *real* one. Although it was proved in [18] that the right-hand side of (8) is a polynomial of δ and η with nonnegative integers, an explicit real expression was missing. We now proceed to give such an explicit *real* formula.

Recall that we identify a cyclic permutation σ with the path $P: s_1 \to s_2 \to \ldots \to s_k \to s_1$ if $\sigma(s_i) = s_{i+1}$ for $1 \le i \le k-1$, where s_1 is the smallest element. Suppose that $k \ge 2$. Clearly the local minima and maxima appear alternatively in the above path and their numbers are equal. The sequence of its extreme elements $(s_{j_1}, s_{i_1}, \ldots, s_{j_l}, s_{i_l})$ is called *extreme sequence*, i.e., $1 = j_1 < i_1 < j_2 < i_2 < j_3 < \cdots < j_l < i_l \le k$ and i_r (resp. j_r) for $1 \le r \le p$ are the positions of local maxima (resp. minima). Let $\mathcal{C}_{\mathbf{n}}(\alpha)$ be the set of $|\mathbf{n}|$-cycles in $\mathcal{D}_{\mathbf{n}}$ with fixed extreme sequence $\alpha = (\alpha_1, \ldots, \alpha_{2l})$.

Theorem 2.1 *There holds*

$$\sum_{\sigma\in\mathcal{C}_{\mathbf{n}}(\alpha)} (1+ui)^{\operatorname{dec}\sigma}(1-ui)^{\operatorname{exc}\sigma} = \sum_{\sigma\in\mathcal{C}_{\mathbf{n}}(\alpha)} (1+u^2)^{\operatorname{ad}\sigma+\operatorname{mat}\sigma}. \tag{9}$$

Proof. Consider a cyclic permutation σ in $\mathcal{C}_{\mathbf{n}}(\alpha)$ identified with the path $p: s_1 \to s_2 \to \cdots \to s_n \to s_1$, where s_1 is the smallest element. The vertex s_j is called an *excedance* if $c(s_j) < c(s_{j+1})$, a *decedance* if $c(s_j) > c(s_{j+1})$. Weight each vertex s_i by $w_i = 1$ or $-ui$ (resp. ui) if it is an excedance (resp. decedance). Let $w = (w_1, \ldots, w_n)$ be the weight sequence and define the weight of the pair (p, w) by $v(p, w) = w_1 \cdots w_n$. Clearly the left-hand side of (9) is the generating function of all such pairs.

A pair (p, w) is said to be a *bad guy* if there are two vertices s_j, $s_{j'}$ with $j' > j$ satisfying one of the following three conditions:

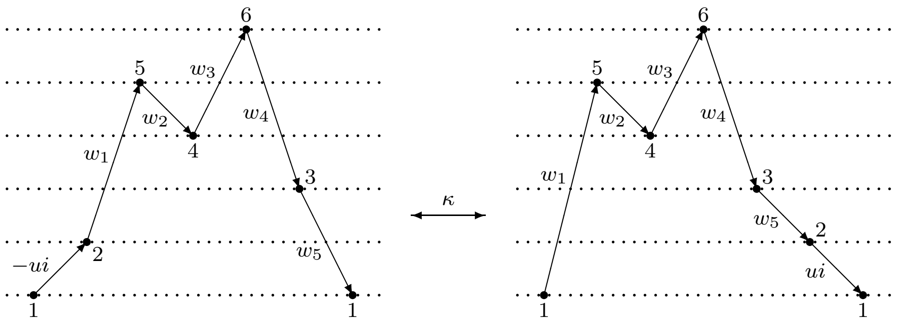


Figure 2: Involution $\kappa : \sigma = (1\,2\,5\,4\,6\,3) \mapsto \sigma^* = (1\,5\,4\,6\,3\,2)$

(i) s_j is a double-ascent with weight $-ui$ and j' is the smallest integer such that $c(s_{j'-1}) > c(s_j) > c(s_{j'})$,

(ii) $s_{j'}$ is a double-descent with weight ui and j is the largest integer such that $c(s_{j-1}) < c(s_{j'}) < c(s_j)$,

(iii) s_j is a valley and j' is the smallest integer such that $s_{j'}$ is a peak, or s_j and $s_{j'}$ form a color match and $(w_j, w_{j'}) = (1, ui)$ or $(-ui, 1)$.

We now set up a killing involution κ on the set of all bad guys. Let (p, w) be a bad guy and (j, j') be the pair such that s_j and $s_{j'}$ satisfy one of the above three conditions in (p, w) and $\min\{s_i, s_j\}$ is the smallest among all such pairs. We form a new pair (p^*, w^*) as follows:

(i) If s_j is a double-ascent with weight $-ui$, then we define the path

$$p^* : s_1 \to \cdots \to s_{j-1} \to s_{j+1} \to \cdots \to s_{j'-1} \to s_j \to s_{j'} \to \cdots \to s_n \to s_1.$$

The weights of vertices are directly inherited from (p, w) except the weight of s_j in p^*, which is ui.

(ii) If $s_{j'}$ is a double-descent in p with weight ui, then we define p^* by moving $s_{j'}$ to the left of s_j and changing the weight to $-ui$.

(iii) If s_j is a valley and $s_{j'}$ a peak or s_j and $s_{j'}$ form a color match, then we define $p^* = p$ and switch $(w_j, w_{j'}) = (1, ui)$ to $(w_j^*, w_{j'}^*) = (-ui, 1)$ or $(w_j, w_{j'}) = (-ui, 1)$ to $(w_j^*, w_{j'}^*) = (1, ui)$.

Clearly this is an involution such that $v(p, w) = -v(p^*, w^*)$ on the set of bad guys. In the remaining pairs (p, w), each double-ascent (resp. valley) s_j can be associated to a unique double-descent (resp. peak) $s_{j'}$ $(j > j')$ such that $(s_j, s_{j'})$ is a color match (resp. (valley, peak)) and $w_j w_{j'} = 1$ or u^2. □

Remark. By definition we see immediately that

$$\sum_{\sigma \in \mathcal{C}_{\mathbf{n}}(\alpha)} (1+ui)^{\operatorname{dec}\sigma}(1-ui)^{\operatorname{exc}\sigma} = (1+u^2)^{\operatorname{ad}\sigma} \sum_{\sigma \in \mathcal{C}_{\mathbf{n}}(\alpha)} (1+ui)^{\operatorname{dd}\sigma}(1-ui)^{\operatorname{aa}\sigma}.$$

Hence identity (9) is equivalent to

$$\sum_{\sigma \in \mathcal{C}_{\mathbf{n}}(\alpha)} (1+ui)^{\operatorname{dd}\sigma}(1-ui)^{\operatorname{aa}\sigma} = \sum_{\sigma \in \mathcal{C}_{\mathbf{n}}(\alpha)} (1+u^2)^{\operatorname{mat}\sigma}.$$

If $\mathbf{n}=(1,1,\ldots,1)$, then $\operatorname{mat}\sigma=0$ for any $\sigma\in\mathcal{C}_{\mathbf{n}}(\alpha)$ and we have the following result:

$$\sum_{\sigma\in\mathcal{C}_n(\alpha)}(1+\mu i)^{\operatorname{dd}\sigma}(1-\mu i)^{\operatorname{aa}\sigma}=|\mathcal{C}_n(\alpha)|.$$

Recall that the cycle index z_σ of a permutation σ is defined by $z_\sigma=\prod_{i\geq 1}z_i^{c_i(\sigma)}$, where $c_i(\sigma)$ is the number of cycles of length $i\geq 1$ of σ.

Corollary 2.2 *For any sequence $\mathbf{n}=(n_1,\ldots,n_m)$ of nonnegative integers, we have*

$$\sum_{\sigma\in\mathcal{D}_{\mathbf{n}}}(\delta+i)^{\operatorname{dec}\sigma}(\delta-i)^{\operatorname{exc}\sigma}z_\sigma=\sum_{\sigma\in\mathcal{D}_{\mathbf{n}}}\delta^{|\mathbf{n}|-2\operatorname{ad}\sigma-2\operatorname{mat}\sigma}(\delta^2+1)^{\operatorname{ad}\sigma+\operatorname{mat}\sigma}z_\sigma, \tag{10}$$

and for any positive integer n, we have

$$\sum_{\sigma\in\mathcal{S}_n}(\delta+i)^{\operatorname{dec}\sigma}(\delta-i)^{\operatorname{exc}\sigma}\delta^{\operatorname{fix}\sigma}z_\sigma=\sum_{\sigma\in\mathcal{S}_n}\delta^{n-2\operatorname{ad}\sigma}(\delta^2+1)^{\operatorname{ad}\sigma}z_\sigma. \tag{11}$$

Proof. Since each permutation in $\mathcal{D}_{\mathbf{n}}$ can be decomposed as a set of disjoint cycles of length at least 2 and the weight functions dec, exc, ad and mat are additive with respect to the cycle-decomposition, we easily derive (10) from identity (9). Since there is no color match in any permutation of $\mathcal{S}_n$, it follows from (10) that for any subset T of $[n]$,

$$\sum_{\sigma\in\mathcal{D}_T}(\delta+i)^{\operatorname{dec}\sigma}(\delta-i)^{\operatorname{exc}\sigma}z_\sigma=\sum_{\sigma\in D_T}\delta^{|T|-2\operatorname{ad}\sigma}(\delta^2+1)^{\operatorname{ad}\sigma}z_\sigma.$$

Multiplying by $(\delta z_1)^{n-|T|}$ and summing over all subsets T of $[n]$ we obtain (11). □

Setting $z_i=\eta$ for all $i\in[n]$ in (11), we recover from (7) the classical interpretation [15] for the moment sequence of Meixner-Pollaczek polynomials:

$$\mathcal{L}(x^n)=\sum_{\sigma\in\mathcal{S}_n}\delta^{n-2\operatorname{ad}\sigma}(\delta^2+1)^{\operatorname{ad}\sigma}\eta^{\operatorname{cyc}\sigma}.$$

Finally, we derive from (8) and (10) a new interpretation for the linearization coefficient of Meixner-Pollaczek polynomials.

Theorem 2.3 *We have*

$$\mathcal{L}\left(\prod_{k=1}^m P_{n_k}(x)\right)=\sum_{\sigma\in\mathcal{D}_{\mathbf{n}}}\delta^{|\mathbf{n}|-2\operatorname{ad}\sigma-2\operatorname{mat}\sigma}(\delta^2+1)^{\operatorname{ad}\sigma+\operatorname{mat}\sigma}\eta^{\operatorname{cyc}\sigma}.$$

Notice that, comparing with (8), there is no occurrence of $\sqrt{-1}$ in the above expression.

3 Combinatorial models

3.1 Sheffer Polynomials and P-graphs

In what follows a *tree* is a directed rooted tree, whose edges are all directed toward the root. Let S be a finite set of positive integers. A *P-tree* on vertex set S is an edge labeled tree such that each edge $i\overset{w}{\Longrightarrow}j$ satisfies the following:

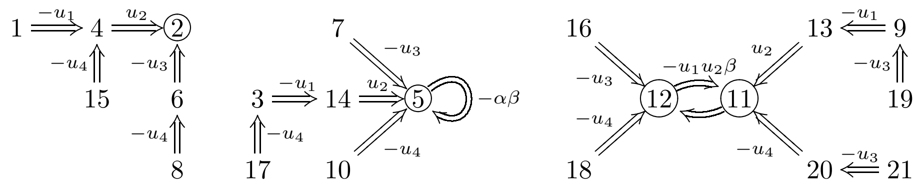


Figure 3: A P-graph with vertex set $[21]$, where the roots 2, 5, 11 and 12 are circled.

1. If $i > j$, then i is the greatest vertex in the path from i to the root and $w = -u_3$, $-u_4$ or u_2 and if $w = u_2$, then $k \overset{-u_1}{\Longrightarrow} i$ appears in the tree, for a unique k less than i.
2. If $i < j$, then $w = -u_1$ and $j \overset{u_2}{\Longrightarrow} k$ appears in the tree, for a unique k less than j.

Remark. There are only four types of edges in a P-tree: $i \overset{-u_1}{\Longrightarrow} j$, $i \overset{u_2}{\Longrightarrow} j$, $i \overset{-u_3}{\Longrightarrow} j$, $i \overset{-u_4}{\Longrightarrow} j$. In addition, edges $i \overset{-u_1}{\Longrightarrow} j$ and $j \overset{u_2}{\Longrightarrow} k$ can appear only as the path $i \overset{-u_1}{\Longrightarrow} j \overset{u_2}{\Longrightarrow} k$.

Definition 3.1 *A* P-graph *on a finite set S is a pair (T, V), where T is a set of P-trees whose underlying vertices form a partition of S and V is a family of subsets of T of size 1 or of size 2 satisfying the following condition: If two P-trees g_1 and g_2 constitute a 2-subset in V then their roots are smaller than vertices adjacent to each root. A 1-subset in V is called a* P-singleton*, and a 2-subset in V is called a* P-doubleton*.*

A P-graph can be visualized by drawing a *loop* at the root of a P-tree in a *P-singleton*, and a *2-cycle* formed with the roots of two P-trees in a P-doubleton. A P-singleton or loop has weight $-\alpha\beta$ and a P-doubleton or 2-cycle has weight $-u_1u_2\beta$. The *weight of a P-graph g*, denoted by $w(g)$, is the product of weights of its edges, loops and 2-cycles. For instance, the weight of P-graph in Figure 3 is $u_1^4u_2^4u_3^5u_4^6\alpha\beta^2$. The number of isolated P-trees of g (i.e. the P-trees not belonging to any P-singleton or P-doubleton) is denoted by $\operatorname{iso} g$. Let $\mathcal{P}_n$ be the set of P-graphs on $[n]$.

Proposition 3.2 *The Sheffer polynomial $P_n(x)$ is the generating function of P-graphs on $[n]$, i.e.*

$$P_n(x) = \sum_{g \in \mathcal{P}_n} w(g)\, x^{\operatorname{iso} g}.$$

Proof. For $n = 1$ and 2, we note first that $P_1(x) = x - \alpha\beta$ and $P_2(x) = x^2 - (2\alpha\beta + u_3 + u_4)x + \alpha^2\beta^2 + u_3\alpha\beta + u_4\alpha\beta - \beta u_1u_2$. On the other hand, it is easy to see that there are exactly two and nine P-graphs on $\{1\}$ and $\{1, 2\}$ respectively, whose weights correspond to the monomials in the above polynomials. For the general case we proceed by induction on $n \geq 3$. By definition the greatest vertex in a P-tree must be of degree 0, 1 or 2. In the last case, it should be adjacent to a vertex of degree 1. In order to compute the generating function of the P-graphs on $[n+1]$, we classify them in five cases and, by inductive hypothesis, write the corresponding generating function next to each case:

- $n+1$ forms an isolated P-tree: $xP_n(x)$,
- $n+1$ forms a separate P-singleton: $-\alpha\beta P_n(x)$,
- $n+1$ is attached to a P-graph on $[n]$ at some $i \in [n]$, creating an edge $n+1 \overset{-u_3}{\Longrightarrow} i$ or $n+1 \overset{-u_4}{\Longrightarrow} i$: $-nu_3P_n(x) - nu_4P_n(x)$,
- $n+1$ forms a 2-cycle with some $i \in [n]$: $-nu_1u_2\beta P_{n-1}(x)$,

- for some $i \in [n]$, $n+1$ is attached to a P-graph on $[n] \setminus \{i\}$ at some $j \in [n] \setminus \{i\}$, creating two edges $i \overset{-u_1}{\Longrightarrow} n+1 \overset{u_2}{\Longrightarrow} j$: $-n(n-1)u_1u_2P_{n-1}(x)$.

Summing all the cases, we get the recurrence relation (1). □

Remark. Viennot [15] has given a general model for general orthogonal polynomials based on the three term recurrence relations. The P-graphs are more structured ad hoc models, which reflect literally the three-term recurrence relation (1) and can be seen somehow as an enriched version of Viennot's model.

3.2 Linearization coefficients and L-graphs

Given a linearly ordered set E of cardinality n, we can identify each permutation $\sigma \in \mathcal{S}_n$ with its functional digraph on E: for each x and y in E, there is an edge $x \to y$ iff $y = \sigma(x)$. We put weight on each edge $x \to \sigma(x)$ as follows:

- α, if x is a fixed point, i.e. $\sigma(x) = x$,
- u_1, if $\sigma(x)$ is a peak, i.e. $x < \sigma(x) > \sigma^2(x)$,
- u_2, if $\sigma(x)$ is a valley, i.e. $x > \sigma(x) < \sigma^2(x)$,
- u_3, if $\sigma(x)$ is a double-ascent, i.e. $x < \sigma(x) < \sigma^2(x)$,
- u_4, if $\sigma(x)$ is a double-descent, i.e. $x > \sigma(x) > \sigma^2(x)$.

The weight of a cycle is β times the product of the weight of its edges. The weight of a permutation σ, denoted by $w(\sigma)$, is the product of weights of all cycles of σ. Weighting the functional digraph of a permutation this way, the resulting generating function of $\mathcal{S}_n$ is equal to the moment $\mathcal{L}(x^n)$ in (2).

Definition 3.3 *An L-graph on $A_{\mathbf{n}}$ is an $(m+1)$-tuple $(g_1, \ldots, g_m, \sigma)$, where g_i is a P-graph on A_i for each $i \in [m]$ and σ a permutation of isolated trees in g_1, ..., g_{m-1} and g_m, linearly ordered according to their roots.*

We naturally identify σ with its functional digraph on the set of roots of isolated trees in g_1, ..., g_m. The weight of an L-graph $(g_1, \ldots, g_m, \sigma)$ is $w(g_1)\cdots w(g_m)w(\sigma)$. We now describe an L-graph in graph terminology. A component of a graph on vertex set $A_{\mathbf{n}}$ is called *monochromatic*, if all its vertices are of the same color. An L-graph on $A_{\mathbf{n}}$ is then a digraph whose connected components are monochromatic P-singletons, monochromatic P-doubletons or cycles of monochromatic P-trees, i.e., (r+1)-tuples $(t_1, \ldots, t_r, \sigma)$ for some integer $r \geq 1$, where t_i $(1 \leq i \leq r)$ are monochromatic P-trees and σ is a cyclic permutation of S_r. A vertex adjacent to the root in a P-tree is called a *pendant* of the root. Any pendant of a root of a P-tree in an L-graph g is called a *pendant* of g. A cycle of P-trees is called an *L-singleton*, if it is a 1-cycle of a monochromatic P-tree; and an *L-doubleton*, if it is a monochromatic 2-cycle whose pendants, if any, are greater than both of its roots. For example, the functional digraph of a permutation in $\mathcal{S}_{\mathbf{n}}$ is an L-graph without pendants.

The *weight of an L-graph* is then equal to the product of the weight inherited from P-graphs and the weight of the permutation of roots of its isolated P-trees. Each edge in a cycle of P-trees is weighted as in the above. Let $\mathcal{L}_{\mathbf{n}}$ denote the set of all L-graphs on vertex set $A_{\mathbf{n}}$.

Example. Figure 4 illustrates an L-graph, which has a 6-cycle, two 2-cycles, an L-singleton, a P-singleton, an L-doubleton, a P-doubleton, with weight $u_1^8u_2^8u_3^9u_4^6\alpha^2\beta^7$.

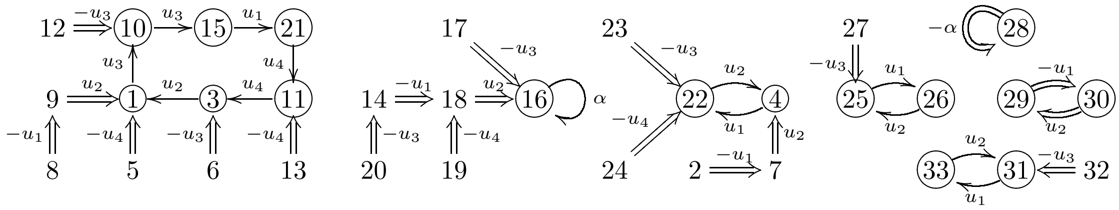


Figure 4: An L-graph on $A_{\mathbf{n}}$ with $\mathbf{n} = (9,4,7,7,6)$.

Proposition 3.4 *The linearization coefficient of the Sheffer polynomials is the generating function of L-graphs, i.e.*

$$\mathcal{L}\left(\prod_{i=1}^{m} P_{n_i}(x)\right) = \sum_{g\in\mathcal{L}_{\mathbf{n}}} w(g). \tag{12}$$

Proof. Applying (2) and using proposition 3.2, we obtain

$$\begin{aligned}\mathcal{L}\left(\prod_{i=1}^{m} P_{n_i}(x)\right) &= \sum_{(g_1,\ldots,g_m)} w(g_1)\cdots w(g_m)\,\mathcal{L}(x^{\mathrm{iso}\,g_1+\cdots+\mathrm{iso}\,g_m})\\ &= \sum_{(g_1,\ldots,g_m,\sigma)} w(g_1)\cdots w(g_m)w(\sigma),\end{aligned}$$

where the sum is over all L-graphs $(g_1,\ldots,g_m,\sigma)$ on $A_{\mathbf{n}}$. □

Definition 3.5 *A skew-derangement is a permutation π of $A_{\mathbf{n}}$ without color matches such that if there is any $a\in A_{\mathbf{n}}$ such that $c(a)=c(\pi(a))$, then $c(\pi^{-1}(a))\neq c(a)$ and $c(\pi(a))\neq c(\pi^2(a))$, and the smaller in $\{a,\pi(a)\}$ is a valley and the larger a peak.*

Note that any generalized derangement without color matches is a skew-derangement and any skew-derangement is a derangement in the classical sense. Let $\mathcal{D}'_{\mathbf{n}}$ denote the set of skew-derangements of $A_{\mathbf{n}}$. We now give a useful variant version of main theorem 1.1.

Theorem 3.6 *There is a one-to-one correspondence $\pi\mapsto\pi'$ from $\mathcal{D}_{\mathbf{n}}$ onto $\mathcal{D}'_{\mathbf{n}}$ such that*

$$(u_1u_2/u_3u_4)^{\mathrm{mat}\,\pi} w(\pi) = w(\pi'),$$

where $w(\pi)=u_1^{\mathrm{ad}\,\pi}u_2^{\mathrm{da}\,\pi}u_3^{\mathrm{aa}\,\pi}u_4^{\mathrm{dd}\,\pi}\beta^{\mathrm{cyc}\,\pi}$. Hence

$$\sum_{\pi\in\mathcal{D}_{\mathbf{n}}} (u_1u_2/u_3u_4)^{\mathrm{mat}\,\pi} w(\pi) = \sum_{\pi\in\mathcal{D}'_{\mathbf{n}}} w(\pi). \tag{13}$$

Proof. It suffices to describe the correspondence for cyclic permutations. Recall that we identify a cyclic permutation π with the path

$$\pi:\quad s_1 \overset{<}{\longrightarrow} s_2 \longrightarrow \cdots \longrightarrow s_l \overset{>}{\longrightarrow} s_1,$$

where $\pi(s_i)=s_{i+1}$ for $1\le i\le l-1$, and s_1 is the smallest element. If π has no color match then set $\pi'=\pi$, otherwise suppose it has only one color match (s_p,s_q), $p<q$, where s_p is a double-ascent and

s_q a double-descent. If $s_p < s_q$, then define π' by $\pi'(s_{p-1}) = s_q$, $\pi'(s_{q-1}) = s_{q+1}$, $\pi'(s_q) = s_p$, and $\pi'(s_i) = \pi(s_i)$, for $i \neq p-1,\, q-1,\, q$, which corresponds to the path

$$\pi':\quad s_1 \xrightarrow{<} \cdots \longrightarrow s_{p-1} \xrightarrow{u_1} s_q \xrightarrow{u_2} s_p \xrightarrow{<} \cdots \longrightarrow s_{q-1} \xrightarrow{>} s_{q+1} \longrightarrow \cdots \longrightarrow s_1;$$

if $s_p > s_q$ by $\pi'(s_{p-1}) = s_{p+1}$, $\pi'(s_p) = s_{q+1}$, $\pi'(s_q) = s_p$, and $\pi'(s_i) = \pi(s_i)$, for $i \neq p-1,\, p,\, q$, which corresponds to the path

$$\pi':\quad s_1 \xrightarrow{<} \cdots \longrightarrow s_{p-1} \xrightarrow{<} s_{p+1} \longrightarrow \cdots \xrightarrow{u_2} s_q \xrightarrow{u_1} s_p \xrightarrow{>} s_{q+1} \longrightarrow \cdots \longrightarrow s_1.$$

If π has more than one color match, then we apply the above process to each color match. It remains to show that the resulting skew-derangement π' is independent of the order of color matches chosen. Indeed, suppose (s_{p_1}, s_{q_1}) and (s_{p_2}, s_{q_2}) are two color matches such that $p_1 < p_2$. Then either $p_1 < q_1 < p_2 < q_2$ or $p_1 < p_2 < q_2 < q_1$. In both cases it is easy to see that we get the same π'.

Since the above application changes each color match of π into a generalized fixed point of π', which induces a weight change by the factor u_1u_2/u_3u_4 for each color match, the weight change amounts to $(u_1u_2/u_3u_4)^{\operatorname{mat}\pi}$. □

Therefore to prove theorem 1.1 it suffices to prove the right hand side of (12) is equal to that of (13).

4 Involutions on L-graphs

4.1 Involution Φ on $\mathcal{L}_{(1,k,1)}$

For clarity, we first consider the special case $\mathbf{n} = (1, k, 1)$, where k is some positive integer, and identify the underlying set $A_{\mathbf{n}}$ with $\{a, 1, \ldots, k, b\}$, where $c(a) = 1$, $c(1) = \cdots = c(k) = 2$ and $c(b) = 3$. If there is no risk of confusion, we indicate only the root of a P-tree, omitting possible pendants attached to each root. For example, the following cycle

$$1 \twoheadrightarrow 10 \twoheadrightarrow 15 \twoheadrightarrow 21 \twoheadrightarrow 11 \twoheadrightarrow 3 \longrightarrow 1$$

represents the left-most component of the L-graph in Figure 4. The following facts will be useful.

Fact 4.1 *The connected components of an L-graph in $\mathcal{L}_{(1,k,1)} \setminus \mathcal{D}'_{(1,k,1)}$ are of the following types: monochromatic P-singletons, L-singletons, monochromatic P-doubletons, L-doubletons, monochromatic 2-cycles which are not L-doubletons, two color 2-cycles and r-cycles ($r \geq 3$).*

We proceed to define a *wpsr* involution Φ on $\mathcal{L}_{(1,k,1)}$. We first present in Figure 5 rules, i.e. pairs of patterns, involved in the involution Φ for connected components. For example, rule c_1 associates each cycle which has a double-ascent "2" of color 2 with a preimage "1" of color 2 to a cycle by moving "2" as a pendant of "1" with weight $-u_3$.

Each rule c_i, $1 \leq i \leq 9$, induces another rule, c_i', called the *dual* of c_i, by reversing the arrows in the cycles, with appropriate weights. For example, the dual of rule c_1 associates each cycle which has a double-descent "2" of color 2 with a image "1" of color 2 to a cycle by moving "2" as pendant of "1" with weight $-u_4$.

We include only the dual of c_1 in Figure 5 and omit the duals of other rules. In e_3, for simplicity, we break the convention of writing a cycle with the smallest element in the first position.

Fact 4.2 *Let σ be an r-cycle ($r \geq 3$) without pendants of an L-graph in $\mathcal{L}_{(1,k,1)} \setminus \mathcal{D}'_{(1,k,1)}$. If σ has no movable root under rules c_1 or c'_1, then it is one of the following types:*

(i) $a \to i_1 \to \cdots \to i_p \to b \to j_1 \to \cdots \to j_q \to a$, $p, q \geq 1$, *and if* $p > 1$ ($q > 1$, *resp.*), *then the sequence* $i_1, \ldots, i_p$ ($j_1, \ldots, j_q$, *resp.*) *alternates, with* $i_{p-1} > i_p$ ($j_1 < j_2$, *resp.*).

(ii) $a \to i_1 \to \cdots \to i_p \to b \to a$, $p \geq 3$, *and the sequence* $i_1, \ldots, i_p$ *alternates, with* $i_{p-1} > i_p$.

(iii) $a \to b \to j_1 \to \cdots \to j_q \to a$, $q \geq 3$, *and the sequence* $j_1, \ldots, j_q$ *alternates, with* $j_1 < j_2$.

(iv) $a \to i_1 \to \cdots \to i_p \to a$, $p \geq 2$, *and the sequence* $i_2, \ldots, i_{p-1}$ *alternates.*

(v) $b \to i_1 \to \cdots \to i_p \to b$, $p \geq 3$, *p is odd, and the sequence* $i_1, \ldots, i_p$ *alternates, with* $i_{p-1} > i_p$.

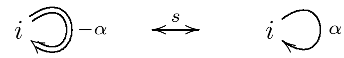
(vi) $i_1 \to \cdots \to i_p \to i_1$, $p \geq 4$, *p is even, and the sequence* $i_1, \ldots, i_p$ *alternates, and i_1 is the greatest.*

In each rule, the unique root (pendant, resp.) which appears as a pendant (root, resp.) in the other is called the *movable root* (pendant, resp.). By a *movable element* we mean a movable root, a movable pendant, the root of a P- or L-singleton or the larger of the roots of a P- or L-doubleton.

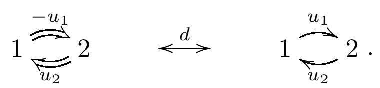
Lemma 4.3 *Any L-graph in $\mathcal{L}_{(1,k,1)} \setminus \mathcal{D}'_{(1,k,1)}$ has a movable element.*

Proof. Any L-graph in $\mathcal{L}_{(1,k,1)} \setminus \mathcal{D}'_{(1,k,1)}$ contains a connected component σ, whose restriction on its underlying set is not a skew-derangement.

1) If σ is a P- or L-singleton, the rule s is:



2) If σ is a P- or L-doubleton, the rule d is :



3) If σ is a monochromatic 2-cycle whose larger root is greater than a pendant of the smaller root, or a monochromatic 3-cycle, or a monochromatic 4-cycle with one peak and one valley, then σ has a movable element under one of rules c_1, c'_1.

4) If σ is a two color 2-cycle (with pendants), or two color 3-cycle then σ has a movable element under one of rules c_1, c'_1, c_9 and c'_9,

In the remaining cases σ is an r-cycle ($r \geq 4$). We will show that at least one of the rules or their duals in Figure 5 is applicable to σ. We distinguish two types: I) σ has no pendants and II) σ has pendants.

I) σ has no pendants. Suppose that σ has no movable element under rules c_1 and c'_1. In view of Fact 4.2 we indicate for each case applicable rules, which induce the corresponding movable elements. Case (i): Assume that $p \geq q$.

- $(p, q) = (1, 1)$: rule c_3 or c'_3
- $(p, q) = (2, 1)$: rule c_3, c_4 or c'_7
- $(p, q) = (2, 2)$: rule c_8 or c'_8
- (p, q) with $p = 3$: rule c_2 or c_5

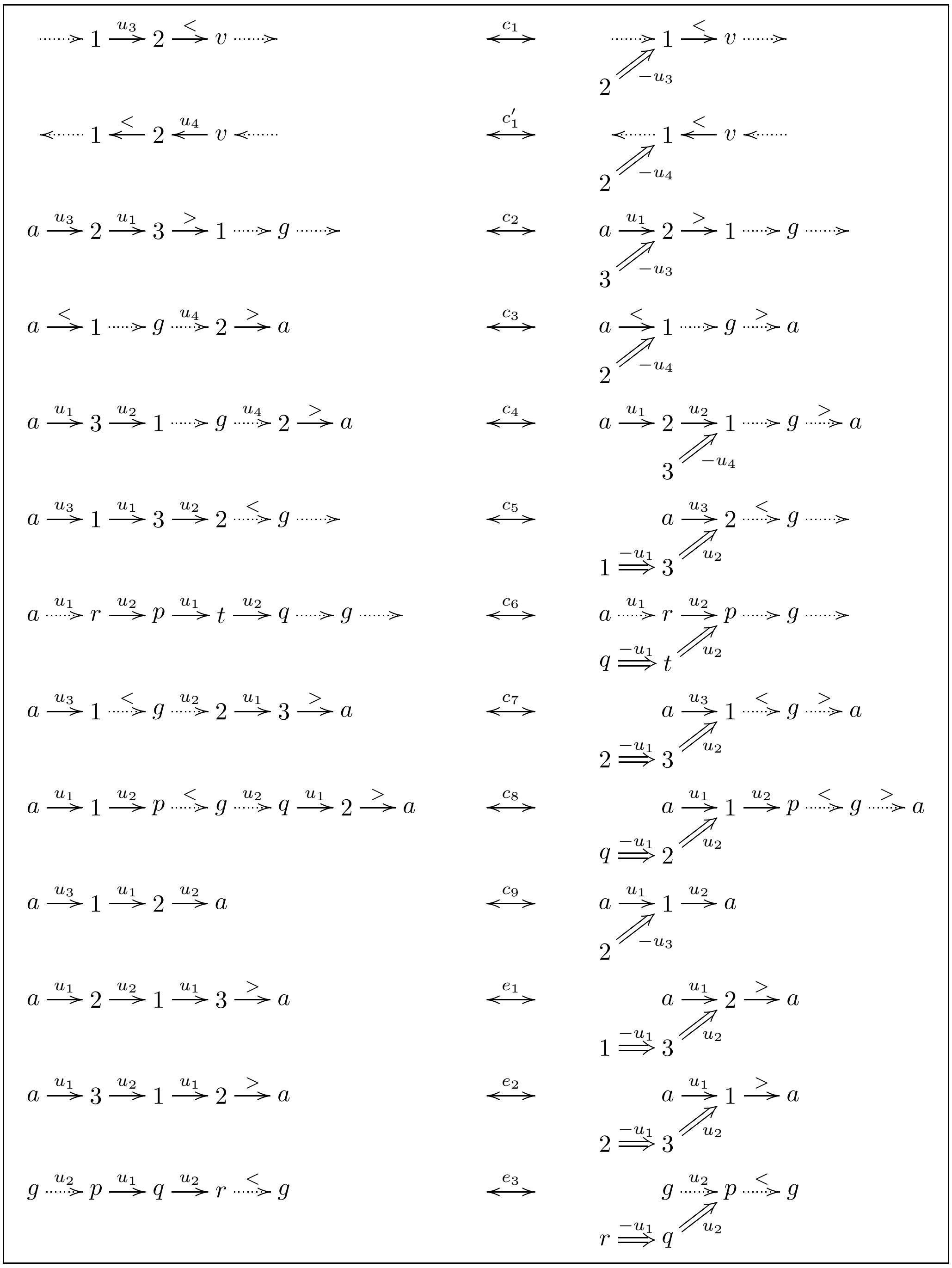


Figure 5: Integers $1, 2, 3$ and 4 denote labels of the same color in that order, p, q, r, t denote integers in $[k]$, $v > 2$, and g with color 2 or 3 is the greatest on the cycle. There can be more elements around dotted arrows. Each integer is the root of a P-tree.

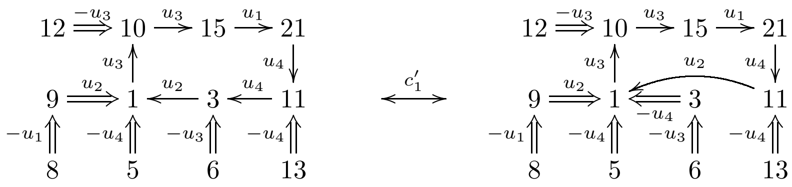


Figure 6: The smallest movable element of the L-graph in Figure 4 is root 3.

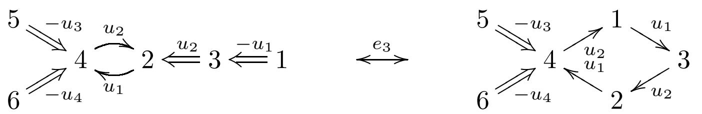


Figure 7: The correspondence Φ for two L-graphs on vertex set $[6]$ with $\mathbf{n} = (3, 3)$.

- (p, q) with $p \geq 4$: rule c_6

Case (ii): If $p = 3$, then either c_2 or c_5 is applicable; otherwise, c_6 is applicable.
Case (iii): If $p = 3$, then either c_2' or c_5' is applicable; otherwise, c_6' is applicable.
Case (iv): If $p = 2$, then either c_9 or c_9' is applicable; if $p = 3$, then c_3, c_3', e_1 or e_2 is applicable; otherwise, we pretend that the greatest element has a greater color and apply case (i).
Case (v): Rule e_3 is applicable.
Case (vi): Rule e_3 is applicable.

II) σ has pendants. If σ has any movable roots under above rules, then we are done; otherwise, the roots of σ form a skew-derangement. If a pendant p is attached to v with an edge of weight $-u_3$ or $-u_4$, then it is movable under one of rules c_1, c_1', …, c_4, c_4', c_9 and c_9'; otherwise, σ has a pattern, $t \overset{-u_1}{\Longrightarrow} p \overset{u_2}{\Longrightarrow} v$, and s is movable under one of rules c_5, c_5',…, c_8, c_8', e_1, e_2 and e_3. □

For each $\sigma \in \mathcal{L}_{(1,k,1)}$ we define $\Phi(\sigma)$ as follows: if it is a skew-derangement then $\Phi(\sigma) = \sigma$; otherwise $\Phi(\sigma)$ is obtained by applying the appropriate rule to the smallest movable element in σ.

Theorem 4.4 *The map Φ defined as above is a weight-preserving sign-reversing involution on $\mathcal{L}_{(1,k,1)}$ such that* $\mathrm{Fix}\,\Phi = D'_{(1,k,1)}$.

Proof. Since Φ is clearly weight-preserving sign-reversing, it suffices to show that an element i is the smallest movable element in σ if and only if it is the smallest movable element in σ'. It can be shown by examining each rule. We omit the details. □

Example. Figure 6 illustrates a correspondence under rule c_1' and Figure 7 a correspondence under rule e_3.

4.2 Involution Φ for the general case

We now describe how to apply the rules for a cycle in $\mathcal{L}_{(1,k,1)}$ to a cycle σ in $\mathcal{L}_{\mathbf{n}}$,where $\mathbf{n} = (n_1, \ldots, n_m)$. Since $m = 1$ case has been dealt with in the proof of lemma 4.3, we may assume that $m > 1$.

To determine movable elements of color $c(q)$, we proceed as follows:

- Replace each vertex v with $c(v) < c(q)$ (resp., $c(q) < c(v)$) by the letter a (resp., b) and contract the cycle by replacing each segment containing repeated a's (resp., b's) by a single a (resp., b);

- Assume that the resulting cycle has k ($k \geq 1$) letters a's, which divide the cycle into k intervals. Clearly, by identifying the two a's at its ends, each interval defines a cycle, called *subcycle*.
- Furthermore, if any subcycle has more than one letter b, then using these b's instead of a's as before, divide it into cycles, again called subcycles. Each of the resulting subcycles has then at most one a and at most one b. Determine movable elements of each subcycle by the rules in Figure 5.
- Repeat this procedure, color by color, starting from the smallest color, until we find a movable element, which should exist by lemma 4.3, and apply to σ the appropriate rule involving the smallest movable element, and define $\Phi(\sigma)$ to be the resulting cycle.

Example. Suppose σ is the following cycle:

$$o_1 \to p_1 \to q_1 \to q_2 \to r_1 \to q_3 \to r_2 \to q_4 \to q_5 \to s_1 \to s_2 \to q_6 \to o_2 \to o_3 \to q_7 \to q_8 \to p_2 \to o_1 ,$$

where o_1, o_2, o_3 have color 1; p_1, p_2 have color 2; $q_1, \ldots, q_8$ have color 3; r_1, r_2 have color 4; s_1, s_2 have color 5. To determine movable elements of color 3, replace first o_1, o_2, o_3, p_1, p_2 with the letter a, and r_1, r_2, s_1, s_2 with the letter b. The result is

$$a \longrightarrow a \to q_1 \to q_2 \longrightarrow b \to q_3 \longrightarrow b \to q_4 \to q_5 \longrightarrow b \longrightarrow b \to q_6 \longrightarrow a \longrightarrow a \to q_7 \to q_8 \longrightarrow a \longrightarrow a .$$

Contracting the cycle yields

$$a \to q_1 \to q_2 \longrightarrow b \to q_3 \longrightarrow b \to q_4 \to q_5 \longrightarrow b \to q_6 \longrightarrow a \to q_7 \to q_8 \longrightarrow a .$$

This cycle is further divided into two subcycles by the two a's:

$$a \to q_1 \to q_2 \longrightarrow b \to q_3 \longrightarrow b \to q_4 \to q_5 \longrightarrow b \to q_6 \longrightarrow a , \qquad a \to q_7 \to q_8 \longrightarrow a .$$

Now, the first subcycle has three b's, which produce three subcycles, and finally σ is split into four subcycles:

$$a \to q_1 \to q_2 \longrightarrow b \to q_6 \longrightarrow a , \quad b \to q_3 \longrightarrow b , \quad b \to q_4 \to q_5 \longrightarrow b , \quad a \to q_7 \to q_8 \longrightarrow a .$$

Determine movable elements of each subcycle by the rules in Figure 5. An application of a rule in a subcycle induces an application of the rule to the original σ, by tracing the procedure backward.

In summary we obtain the following result.

Theorem 4.5 *The map Φ, defined as above, is a weight-preserving sign-reversing involution on $\mathcal{L}_{\mathbf{n}}$ with fixed set $\mathcal{D}'_{\mathbf{n}}$.*

Now our main theorem 1.1 follows from theorems 3.6 and 4.5.

5 Further extensions

We provide some partial results on two possible generalizations of the linearization coefficients for the Sheffer polynomials and make some comments.

5.1 Permutation enumerations

For any permutation $\pi \in \mathcal{S}_{\mathbf{n}}$, set

$$\mathrm{Fix}\,\pi = \{a \in A_{\mathbf{n}} : c(\pi(a)) = c(a)\}, \quad \mathrm{Fix}_i\pi = \mathrm{Fix}\,\pi \cap A_i.$$

For the Laguerre polynomials $L_n(x)$, Foata and Zeilberger [11] proved that

$$\mathcal{L}\left(\prod_{i=1}^{m} L_{n_i}(\lambda_i x)\right) = \sum_{\pi\in\mathcal{S}_{\mathbf{n}}} \beta^{\mathrm{cyc}\,\pi} \prod_{i=1}^{m} (\lambda_i - 1)^{|\,\mathrm{Fix}_i\,\pi|} \lambda_i^{|A_i \setminus \mathrm{Fix}_i\,\pi|}. \tag{14}$$

So it is natural to ask whether there is any interesting result about $\mathcal{L}\left(\prod_{i=1}^{m} P_{n_i}(\lambda_i x)\right)$. In what follows we will show that a reasonable generalization of (14) does not seem exist in general. We shall denote by e_k the k-th $(0 \le k \le m)$ elementary symmetric polynomial of $x_1, \ldots, x_m$, that is,

$$\sum_{j=0}^{m} e_j x^j = \prod_{k=1}^{m} (1 + x_k x).$$

By convention, we define $e_k = 0$ if $k > m$. For $\mathbf{n} = (n_1, \ldots, n_m)$, set $\mathbf{x}^{\mathbf{n}} = x_1^{n_1} \cdots x_m^{n_m}$ and $\mathbf{n}! = n_1! \cdots n_m!$.

For the Hermite polynomials $H_n(x)$, the generating function and moments [4, p. 145] are

$$\begin{aligned}
\sum_{n=0}^{\infty} H_n(x) \frac{t^n}{n!} &= \exp\,(xt - t^2/2), \\
\mathcal{L}(x^n) &= \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} x^n e^{-x^2} dx = \begin{cases} (2m-1)!! & \text{if } n = 2m; \\ 0 & \text{if } n = 2m+1. \end{cases}
\end{aligned}$$

Proposition 5.1 *There holds*

$$\mathcal{L}\left(\prod_{i=1}^{m} H_{n_i}(\lambda_i x)\right) = \sum_{\pi\in\mathcal{I}_{\mathbf{n}}} \prod_{i=1}^{m} (\lambda_i^2 - 1)^{|\,\mathrm{Fix}_i\,\pi|} \lambda_i^{|A_i \setminus \mathrm{Fix}_i\,\pi|},$$

where $\mathcal{I}_{\mathbf{n}}$ is the set of involutions without singleton of $A_{\mathbf{n}}$.

Proof. Using the generating function and moment of Hermite polynomials we have

$$\begin{aligned}
\sum_{\mathbf{n}} \mathcal{L}\left(\prod_{i=1}^{m} H_{n_i}(\lambda_i x)\right) \frac{\mathbf{x}^{\mathbf{n}}}{\mathbf{n}!} &= \mathcal{L}\left(\prod_{i=1}^{m} \exp(\lambda_i x_i x - x_i^2/2)\right) \\
&= \exp \frac{1}{2}\left[(\lambda_1 x_1 + \cdots + \lambda_m x_m)^2 - (x_1^2 + \cdots x_m^2)\right] \\
&= \exp\left[\sum_{i=1}^{m} (\lambda_i^2 - 1)\frac{x_i^2}{2} + \sum_{i\neq j} \lambda_i \lambda_j x_i x_j\right].
\end{aligned}$$

It is then easy to derive the desired equation from the above formula. □

For the Charlier polynomials $C_n(x)$, the generating function and moments [4, p. 170] are

$$\begin{aligned}
\sum_{n=0}^{\infty} C_n(x) \frac{t^n}{n!} &= e^{-at}(1+t)^x, \\
\mathcal{L}(x^n) &= e^{-a} \sum_{x=0}^{\infty} x^n \frac{a^x}{x!}.
\end{aligned}$$

Proposition 5.2 *There holds*

$$\mathcal{L}\left(\prod_{i=1}^{m} C_{n_i}(\lambda_i x)\right) = \sum_{\pi\in\mathcal{P}_{\mathbf{n}}} a^{\mathrm{bloc}\,\pi} \prod_{i=1}^{m}(\lambda_i-1)^{\mathrm{sing}_i\,\pi} \prod_{B}\prod_{i=1}^{m} \lambda_i(\lambda_i-1)\cdots(\lambda_i-|A_i\cap B|+1),$$

where $\mathrm{sing}_i\,\pi$ *is the number of singletons of color* i *in* π *and* B *is a non-singleton bloc of* π.

Proof. Using the generating function and moments of Charlier polynomials we have

$$\begin{aligned}\sum_{\mathbf{n}} \mathcal{L}\left(\prod_{i=1}^{m} C_{n_i}(\lambda_i x)\right)\frac{\mathbf{x}^{\mathbf{n}}}{\mathbf{n}!} &= \mathcal{L}\left(\prod_{i=1}^{m} e^{-a x_i}(1+x_i)^{\lambda_i x}\right)\\ &= \exp\left[-a - a\sum_{i=1}^{m} x_i + a\prod_{i=1}^{m}(1+x_i)^{\lambda_i}\right]\\ &= \exp\left[a\sum_{i=1}^{m}(\lambda_i-1)x_i + a\sum_{|\mathbf{k}|\geq 2}\prod_{i=1}^{m}(-\lambda_i)_{k_i}\frac{(-x_i)^{k_i}}{k_i!}\right].\end{aligned}$$

It is then easy to derive the desired equation from the last result. □

For the Meixner polynomials $M_n(x;\beta,c)$, the generating function and moments [4, p. 176] are

$$\sum_{n\geq 0} M_n(x;\beta,c)\frac{t^n}{n!} = (1-t/c)^x(1-t)^{-x-\beta},$$

$$\mathcal{L}(x^n) = (1-c)^\beta \sum_{x\geq 0} x^n \frac{c^x(\beta)_x}{x!}.$$

Hence

$$\sum_{\mathbf{n}} \mathcal{L}\left(\prod_{i=1}^{m}(-c)^{n_i} M_{n_i}(\lambda_i x;\beta,c)\right)\frac{\mathbf{x}^{\mathbf{n}}}{\mathbf{n}!} = (1-c)^\beta \sum_{x=0}^{\infty}\prod_{i=1}^{m}(1+x_i)^{\lambda_i x}(1+cx_i)^{-\lambda_i x-\beta}\frac{c^x(\beta)_x}{x!},$$

which is equal to

$$\left[(1-c)^{-1}\prod_{i=1}^{m}(1+cx_i) - c(1-c)^{-1}\prod_{i=1}^{m}(1+x_i)\left(1-\frac{(1-c)x_i}{1+x_i}\right)^{1-\lambda_i}\right]^{-\beta}.$$

It turns out that the expression inside the above brackets can be further written as

$$1-\sum_{k=2}^{m}(c+\cdots+c^{k-1})e_k(\mathbf{x}) - c\sum_{|\mathbf{k}|\geq 1}(1-c)^{|\mathbf{k}|-1}\prod_{i=1}^{m}\frac{(\lambda_i-1)_{k_i}x_i^{k_i}}{(1+x_i)^{k_i-1}},$$

which reduces to the case of Laguerre polynomials [11] when $c=1$:

$$1-\sum_{k=2}^{m}(k-1)e_k(\mathbf{x}) - \sum_{i=1}^{m}(\lambda_i-1)x_i\prod_{j\neq i}(1+x_j) = \prod_{i=1}^{m}(1+x_i) - \sum_{i=1}^{m}\lambda_i x_i\prod_{j\neq i}(1+x_j),$$

for $\sum_{i=1}^{m} x_i\prod_{j\neq i}(1+x_j) = \sum_{k=1}^{m} k e_k$. However, if $c\neq 1$, the expression is not a polynomial in $x_1,\ldots,x_m$, so we cannot expect to have a reasonable interpretation as for Hermite, Charlier and Laguerre polynomials.

5.2 q-analogs

Another interesting question is to find a q-analog of theorem 1.1. Till now a satisfying q-analog has been obtained only for Hermite polynomials by Ismail et al. [13]. The problem for q-Charlier polynomials was studied by de Médicis et al. [6]. We define the q-Laguerre polynomials [7, 14] by

$$L_n^{(\alpha)}(x;q) = \sum_{k=0}^{n}(-1)^{n-k}\begin{bmatrix} n \\ k \end{bmatrix}_q q^{\binom{n-k}{2}} \prod_{i=k}^{n-1}(\alpha + [i+1]_q)x^k, \tag{15}$$

where $\begin{bmatrix} n \\ k \end{bmatrix}_q$ is the usual q-binomial coefficient. It is not hard to see that this is a rescaled version of Wall polynomials [4, p.198]:

$$L_n^{(\alpha)}(x;q) = \left(\frac{1+\alpha(1-q)}{1-q}q\right)^n W_n\left(\frac{(1-q)qx}{1+\alpha(1-q)}; \frac{q}{1+\alpha(1-q)}, q\right).$$

It follows that $L_{-1}^{(\alpha)}(x;q) = 0$, $L_0^{(\alpha)}(x;q) = 1$, and

$$L_{n+1}^{(\alpha)}(x;q) = (x - b_n)L_n^{(\alpha)}(x;q) - \lambda_n L_{n-1}^{(\alpha)}(x;q),$$

where $b_n = q^n([n]_q + \alpha + 1 + q[n]_q)$ and $\lambda_n = q^{2n-1}[n]_q(\alpha + 1 + q[n-1]_q)$. The corresponding n-th moment of q-Laguerre polynomials [4, p.198] reads as follows:

$$\mathcal{L}_q(x^n) = \prod_{i=0}^{n-1}(\alpha + 1 + q[i]_q).$$

Although we have the following q-analog of the orthogonality:

$$\mathcal{L}_q(L_m^{(\alpha)}(x;q)L_n^{(\alpha)}(x;q)) = q^{n(n+1)}[n]_q! \prod_{i=0}^{n-1}(\alpha + 1 + q[i]_q)\delta_{m,n}, \tag{16}$$

the usual q-analog of the linearization coefficients doesn't give a combinatorial refinement of the ordinary case ($q = 1$), as the polynomial

$$\mathcal{L}_q((L_2^{(0)}(x;q))^3) = q^{15} + 5\,q^{14} + 14\,q^{13} + 26\,q^{12} + 31\,q^{11} + 20\,q^{10} - q^9 - 12\,q^8 - 6\,q^7 + q^6 + q^5$$

has negative coefficients. On the other hand, applying the linear functional $\mathcal{L}_q$ to the following q-binomial formula:

$$(x-1)(x-q)\cdots(x-q^{n-1}) = \sum_{k=0}^{n}(-1)^k \begin{bmatrix} n \\ k \end{bmatrix}_q q^{\binom{k}{2}} x^{n-k},$$

we obtain

$$\mathcal{L}_q\left(q^{\binom{n}{2}} L_1^{(0)}(x)L_1^{(0)}(x/q)\cdots L_1^{(0)}(x/q^{n-1})\right) = [n]_q! \sum_{k=0}^{n} \frac{(-1)^k q^{\binom{k}{2}}}{[k]_q!}, \tag{17}$$

which is the well-known q-derangement numbers [16]. This suggests to consider the following q-analog:

$$\begin{aligned} \mathcal{L}_q^*(0,\mathbf{n}) &= \mathcal{L}_q\left(q^{\sum_{i=1}^m (i-1)n_i} L_{n_1}^{(0)}(x)L_{n_2}^{(0)}(x/q)\cdots L_{n_m}^{(0)}(x/q^{m-1})\right) \\ &= \prod_{i=1}^{m}[n_i]_q! \sum_{\mathbf{k}} (-1)^{|\mathbf{n}|-|\mathbf{k}|} \begin{bmatrix} k_1 + \cdots + k_m \\ k_1, \ldots, k_m \end{bmatrix}_q \prod_{i=1}^{m} \begin{bmatrix} n_i \\ k_i \end{bmatrix}_q q^{\binom{n_i - k_i}{2} + (i-1)(n_i - k_i)}. \end{aligned}$$

Conjecture 5.3 *The polynomial* $\mathcal{L}_q^*(0,\mathbf{n})$ *has nonnegative integral coefficients if* $n_1 \geq n_2 \geq \cdots \geq n_m$.

Remark. For $\mathbf{n} = (1,2,1)$, we get $\mathcal{L}_q^*(0,\mathbf{n}) = q^6 + q^5 - q^4 + 2q^2 + q$. Further Maple experiments seem to suggest that the quotient $\mathcal{L}_q^*(0,\mathbf{n})/\prod_{i=1}^m [n_i]_q!$ is a polynomial of q with nonnegative integral coefficients if $n_1 = n_2 = \cdots = n_m$. Note that the latter has negative coefficients when $\mathbf{n} = (8,5,4)$.

For $m = 3$ and $n_1 \geq n_2 \geq n_3$, it is easy to see by applying q-Chu-Vandermonde formula that

$$\mathcal{L}_q^*(0,\mathbf{n}) = [n_1]_q![n_2]_q![n_3]_q! \sum_{l\geq 0} (-1)^{n_2+n_3-l} \begin{bmatrix} l \\ n_1 \end{bmatrix}_q \sum_{k\geq 0} \begin{bmatrix} n_2 \\ l-k \end{bmatrix}_q \begin{bmatrix} n_3 \\ k \end{bmatrix}_q \begin{bmatrix} l \\ k \end{bmatrix}_q q^{M(k,l)},$$

where $M(k,l) = n_1^2 + \binom{n_2-l+k+1}{2} + \binom{n_3-k+1}{2} + n_3 - k$. If $n_3 = 0$, it reduces to

$$\mathcal{L}_q^*(0,\mathbf{n}) = [n_1]_q![n_2]_q! \sum_{l\geq 0} (-1)^{n_2-l} \begin{bmatrix} l \\ n_1 \end{bmatrix}_q \begin{bmatrix} n_2 \\ l \end{bmatrix}_q q^{n_1^2+\binom{n_2-l}{2}+n_2-l+k} = [n_1]_q![n_2]_q! q^{n_1^2} \delta_{n_1\, n_2},$$

which is clearly a polynomial of q with nonnegative integral coefficients.

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