Transitive cycle factorizations and prime parking functions*

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Abstract

Minimal transitive cycle factorizations and parking functions are related very closely. Using the correspondence between them, we find a bijection between minimal transitive factorizations of permutations of type $(1, n-1)$ and prime parking functions of length n.

Keywords: Minimal transitive factorization, parking function, noncrossing partition

1 Introduction

For any natural number n, let $\mathcal{S}_n$ denote the group of all permutations of $[n] = \{1, 2, \ldots, n\}$. A cycle of length n can be expressed as a product of $n-1$ transpositions. The total number of different such expressions is n^{n-2}. In general, we may count different ways to form a given permutation as a product of transpositions with some interesting conditions. For example, we may require that the transpositions in the product generate $\mathcal{S}_n$ and the minimal number of transpositions are used. This case has been considered in various literature [1, 5, 8, 16]. Clearly the count depends only on the cycle type of the permutation.

An integer partition of n is a weakly increasing sequence of positive integers, which sums to n. Given an integer partition $\lambda = (\lambda_1, \ldots, \lambda_l)$ of n, let λ° denote the permutation $(1\,2 \cdots \lambda_1)(\lambda_1 + 1 \cdots \lambda_1 + \lambda_2) \cdots (n - \lambda_l + 1 \cdots n)$ of $[n]$ in the cycle notation. The cycle type of λ° is λ. Let $\mathcal{F}_\lambda$ be the set of all m-tuples of transpositions $(\sigma_1, \ldots, \sigma_m)$ such that

(f1) $\sigma_1 \cdots \sigma_m = \lambda^\circ$,

(f2) $\{\sigma_1, \ldots, \sigma_m\}$ generates $\mathcal{S}_n$,

*Running head: Transitive cycle factorizations
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(f3) $m = n + l - 2$, i.e., m is minimal subject to (f1), (f2).

Elements of $\mathcal{F}_\lambda$ are called *minimal transitive factorizations* of λ° (or simply λ). These factorizations are connected with branched covers of the sphere, that is treated in [1, 3, 7, 11].

Goulden and Jackson [8] proved that

$$|\mathcal{F}_\lambda| = (n + l - 2)!\, n^{l-3} \prod_{i=1}^{l} \frac{\lambda_i^{\lambda_i}}{(\lambda_i - 1)!}\,, \tag{1}$$

which was originally suggested by Hurwitz [11]. They proved it for arbitrary λ, but the proof was not combinatorial. Recently, Bousquet-Mélou and Schaeffer [3] enumerated the transitive factorizations by interpreting them as planar constellations and Eulerian trees, and obtained the formula (1) by using the principle of Inclusion–Exclusion. In case $\lambda = (n)$, there have been several combinatorial proofs [2, 6, 9, 10]. For any other λ, there has been no simple combinatorial proof yet. When $\lambda = (1, n-1)$, the expression (1) reduces to

$$|\mathcal{F}_{(1,n-1)}| = (n-1)^n. \tag{2}$$

The aim of this paper is to give a combinatorial proof of (2), using prime parking functions.

2 Preliminaries

Let π be a permutation and $\sigma = (w\,z)$, with $w < z$, a transposition in $\mathcal{S}_n$. If w and z appear in the same cycle in π, then this cycle is cut into two cycles in the product $\sigma\pi$; otherwise, the cycles containing w and z are joined into one cycle in $\sigma\pi$. We call σ a *cut* of π in the former case and a *join* in the latter. For an element $(\sigma_1, \ldots, \sigma_m)$ in $\mathcal{F}_\lambda$, where $\lambda = (\lambda_1, \ldots, \lambda_l)$ is an integer partition of n, σ_i is called a cut (resp. a join), if σ_i is a cut (resp. join) of $\sigma_{i+1} \cdots \sigma_m$. Note that, from Goulden and Jackson [8, Proposition 2.1], there are exactly $l-1$ cuts and $n-1$ joins in these factorizations.

When $\lambda = (n)$, a minimal transitive factorization has $n-1$ transpositions which generate $\mathcal{S}_n$.

A *circle chord diagram*, first introduced in [10], is a circle drawn on the plane with n points on it, labelled $1, \ldots, n$ clockwise, and with $n-1$ chords on these n points, numbered $1, \ldots, n-1$ distinctly. There is a natural injection Γ from factorizations to circle chord diagrams: Γ maps a factorization $(\sigma_1, \ldots, \sigma_{n-1}) \in \mathcal{F}_{(n)}$, where $\sigma_i = (w_i\, z_i)$, to the circle chord diagram in which the chords joining points w_i and z_i are numbered i, for $i = 1, \ldots, n-1$. The circle chord diagrams corresponding to the factorizations are characterized below.

Proposition 1 *The following three properties (c1), (c2), (c3) completely characterize the image set of* Γ, $\Gamma(\mathcal{F}_{(n)})$.

(c1) The chords form a tree on $[n]$.

(c2) The chords meet only at the end points.

(c3) The numbers on the chords of the boundary of any region increase clockwise, starting immediately after the unique arc.

Proof. See Theorem 2.2 in [10]. □

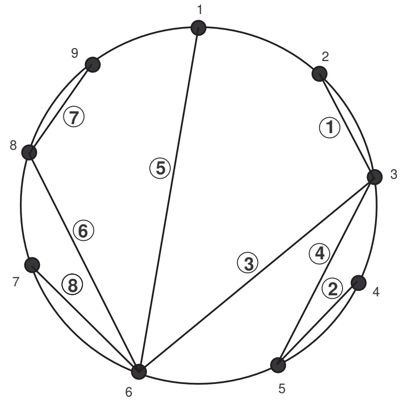

Figure 1: A circle chord diagram with $n = 9$

Let $\mathcal{C}_n$ be the set of all circle chord diagrams which satisfy the above conditions (c1), (c2) and (c3). Clearly, Γ is a bijection from $\mathcal{F}_{(n)}$ onto $\mathcal{C}_n$. For example, the circle chord diagram illustrated in Figure 1 is $\Gamma(F)$, with

$$F = (\,(2\,3), (4\,5), (3\,6), (3\,5), (1\,6), (6\,8), (8\,9), (6\,7)\,).$$

Note that there are 9 regions and each region satisfies (c3) in the proposition.

Using the planarity of circle chord diagrams, Goulden and Yong [10] found a bijection between $\mathcal{C}_n$ and the set $\mathcal{T}_n$ of all labelled trees on $[n]$.

A partition of $[n]$ is a family of nonempty, pairwise disjoint sets $B_1, \ldots, B_k$, called blocks, whose union is $[n]$. The collection of all partitions of $[n]$ will be denoted by Π_n. A partition of $[n]$ is called *noncrossing* when it has the property that for any four elements $a < b < c < d$, if a and c are in the same block and b is in another, then b and d are *not* in the same block. The collection of all noncrossing partitions of $[n]$ will be denoted by NC_n. The collection Π_n can be viewed as a partially ordered set (poset) with the following order (called refinement order): for $\mu, \nu \in \Pi_n$, we have $\mu \leq \nu$ if each block of μ is contained in a block of ν. The set NC_n is also regarded as a poset with the refinement order. The Hasse diagram of NC_4 is shown in Figure 2. In fact, the poset NC_n has many nice properties. See [13] for extensive studies of noncrossing partitions.

A *parking function* of length n is a sequence $(x_1, \ldots, x_n)$ of positive integers such that the increasing rearrangement $y_1 \leq \cdots \leq y_n$ of $x_1, \ldots, x_n$ satisfies $y_i \leq i$ for all i. Let $\mathcal{P}_n$ denote the set of all parking functions of length n. It is well known that $|\mathcal{P}_{n-1}| = n^{n-2}$. For example, the set $\mathcal{P}_3$ is as follows: (An array abc means (a, b, c).)

$$\mathcal{P}_3 = \{111, 112, 113, 121, 122, 123, 131, 132, 211, 212, 213, 221, 231, 311, 312, 321\}$$

Parking functions are related to labelled trees, lattice paths, noncrossing partitions and hyperplane arrangements. For further information see [13] and [14, pp. 94]. Combining the results of Goulden and Yong [10] and Stanley [15, Theorem 3.1], we can define a combinatorial bijection $\Phi : \mathcal{F}_{(n)} \to \mathcal{P}_{n-1}$ by

$$\Phi(\,(w_1\, z_1), \ldots, (w_{n-1}\, z_{n-1})\,) = (w_1, \ldots, w_{n-1}). \tag{3}$$

We will use Φ to establish our main result later. Since the mapping Φ is not shown explicitly in [10, 15], we present an explicit description of Φ in the next section.

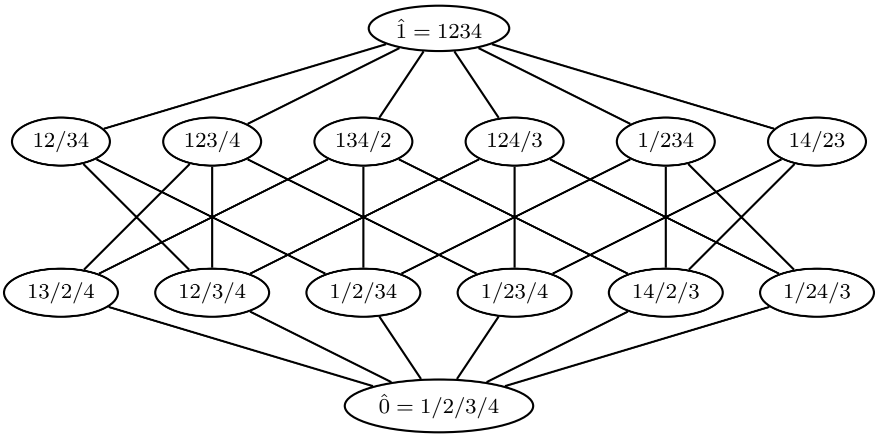

Figure 2: The Hasse diagram of NC_4

3 Bijection Φ

Let $\mathcal{M}_n$ be the set of all maximal chains in NC_n. We define a mapping $\Sigma : \mathcal{C}_n \to \mathcal{M}_n$ through the following process. Let C be a circle chord diagram in $\mathcal{C}_n$. Then $C = \Gamma(\,(w_1\,z_1), \dots, (w_{n-1}\,z_{n-1})\,)$ for some factorization $(\,(w_1\,z_1), \dots, (w_{n-1}\,z_{n-1})\,) \in \mathcal{F}_{(n)}$. Let $\mathfrak{m} = [\mu_0 < \mu_1 < \cdots < \mu_{n-1}]$ be a maximal chain in $\mathcal{M}_n$ such that $\mu_0 = \hat{0}$, $\mu_{n-1} = \hat{1}$, and while i changes from $n-1$ to 1, μ_i is refined to μ_{i-1} by 'splitting' the blocks in μ_i with the chord $(w_i\,z_i)$: To be more specific, B is a block in μ_{i-1} if and only if for some block $\widetilde{B}$ in μ_i,

$$B = \widetilde{B} \cap [w_i + 1, z_i] \quad \text{or} \quad B = \widetilde{B} \cap (\,[n] \setminus [w_i + 1, z_i]\,),$$

where $[a, b]$ denotes the set $\{a, a+1, \dots, b\}$. Since C satisfies (c2), the chords $(w_1\,z_1), \dots, (w_{n-1}\,z_{n-1})$ do not cross in C, and so each μ_i is an element of NC_n with $\mathrm{bk}(\mu_{i-1}) \le \mathrm{bk}(\mu_i) + 1$, where $\mathrm{bk}(\mu)$ is the number of blocks in μ. Moreover, w_i and z_i are in the same block in μ_i, which implies $\mathrm{bk}(\mu_{i-1}) \ge \mathrm{bk}(\mu_i) + 1$; if they are not in the same block, then for some $j > i$, the chord $(w_i\,z_i)$ crosses or coincides with $(w_j\,z_j)$, contradicting to the condition (c2). Thus μ_i covers μ_{i-1} for each i and $\mathfrak{m}$ is really in $\mathcal{M}_n$. Set $\Sigma(C) = \mathfrak{m}$.

Now let us find the inverse of Σ. Suppose that $\mu, \nu \in \mathrm{NC}_n$, where ν covers μ. Then there exist $A \in \nu$ and $A', A'' \in \mu$ satisfying $A = A' \cup A''$, with $\min A' > \min A''$. We define the base, middle, left, right and left-right sets of (μ, ν), denoted by B, M, L, R and LR respectively, as follows (Figure 3):

$$\begin{aligned} B(\mu,\nu) = A, \qquad M(\mu,\nu) &= A', \\ L(\mu,\nu) &= \{i \in A : i < \min A'\}, \\ R(\mu,\nu) &= \{i \in A : i > \max A'\}, \\ LR(\mu,\nu) = L(\mu,\nu) &\cup R(\mu,\nu) = A''. \end{aligned}$$

Note that $L(\mu, \nu)$ and $M(\mu, \nu)$ are *not* empty.

For $\mu, \nu \in \mathrm{NC}_n$ such that ν covers μ, define $\Delta_1(\mu, \nu)$ and $\Delta_2(\mu, \nu)$ by

$$\Delta_1(\mu,\nu) = \max L(\mu,\nu), \qquad \Delta_2(\mu,\nu) = \max M(\mu,\nu),$$

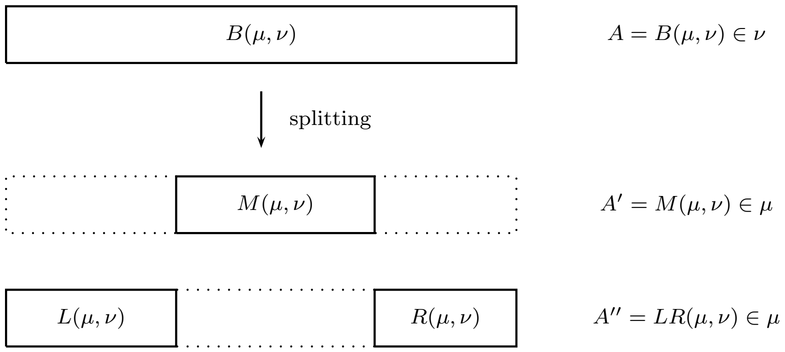

Figure 3: Base, middle, left, right and left-right set of (μ,ν).

and define $\Delta(\mu,\nu)$ to be the transposition $(\Delta_1(\mu,\nu)\,\Delta_2(\mu,\nu)\,)$ in $\mathcal{S}_n$.

We now define $\Omega : \mathcal{M}_n \to \mathcal{F}_{(n)}$ as follows: Given a maximal chain $\mathfrak{m} = [\mu_0 < \mu_1 < \cdots < \mu_{n-1}]$,

$$\Omega(\mathfrak{m}) = (\Delta(\mu_0,\mu_1),\ldots,\Delta(\mu_{n-2},\mu_{n-1})\,).$$

It easily follows from construction that Ω is well-defined and injective.

Proposition 2 *The composition map $\Gamma \circ \Omega$ is the inverse of Σ. So the mappings Σ and Ω are all bijections.*

Proof. Consider the maximal chain

$$\Sigma(C) = [\mu_0 < \mu_1 < \cdots < \mu_{n-1}]\,,$$

where $C = \Gamma(\,(w_1\,z_1),\ldots,(w_{n-1}\,z_{n-1})\,)$. Recall that w_i and z_i belong to the same block A in μ_i. The block A is split into two blocks A' and A'' in μ_{i-1}, where

$$A' = A \cap [w_i+1, z_i], \qquad A'' = A \cap (\,[1,w_i] \cup [z_i+1,n]\,).$$

On the other hand,

$$M(\mu_{i-1},\mu_i) = A', \qquad L(\mu_{i-1},\mu_i) = A \cap [1,w_i].$$

Since w_i, z_i are in A,

$$\max L(\mu_{i-1},\mu_i) = w_i\,, \qquad \max M(\mu_{i-1},\mu_i) = z_i.$$

Thus $\Delta(\mu_{i-1},\mu_i) = (w_i\,z_i)$ and

$$\Omega(\Sigma(C)\,) = (\,(w_1\,z_1),\ldots,(w_{n-1}\,z_{n-1})\,)\,.$$

Applying the both sides of the above with Γ yields that $(\Gamma \circ \Omega) \circ \Sigma$ is the identity map on $\mathcal{C}_n$. Thus $\Gamma \circ \Omega$ is surjective. On the other hand, since Ω is injective, $\Gamma \circ \Omega$ is the inverse of Σ. $\square$

Given a maximal chain $\mathfrak{m} = [\,\mu_0 < \cdots < \mu_{n-1}\,] \in \mathcal{M}_n$, define $\Omega_1(\mathfrak{m})$ by

$$\Omega_1(\mathfrak{m}) = (\Delta_1(\mu_0,\mu_1),\ldots,\Delta_1(\mu_{n-2},\mu_{n-1})\,).$$

Stanley [15, Theorem 3.1] showed that the map Ω_1 (in his notation, Λ) is a bijection between $\mathcal{M}_n$ and $\mathcal{P}_{n-1}$. Thus we have the following consequence.

Theorem 1 *The map $\Phi : \mathcal{F}_{(n)} \to \mathcal{P}_{n-1}$ defined by*

$$\Phi(\,(w_1\,z_1),\ldots,(w_{n-1}\,z_{n-1})\,) = (w_1,\ldots,w_{n-1})$$

is a bijection.

Proof. It is easy to show that

$$\Phi = \Omega_1 \circ \Sigma \circ \Gamma\,.$$

So Φ is a bijection. □

Remark. It is well known that $|\mathcal{P}_{n-1}| = n^{n-2}$. So Theorem 1 gives a bijective proof of (1) for the case $\lambda = (n)$. In fact, the resulting bijection Φ is the same as given by Biane [2], who proved it without Stanley's result.

In Figure 4, we give an example which summarizes all the consequences in this section. Note that all maps are bijective and all diagrams commute.

4 Main Result

We now proceed to construct a combinatorial proof of (2). Let $(\tau_1,\ldots,\tau_m)$ be an element of $\mathcal{F}_{(1,n-1)}$. Since λ has two parts, i.e. $l = 2$, we have $m = n$ and there is only one cut among $\tau_1,\ldots,\tau_{n-1}$ (Note that τ_n cannot be a cut). Moreover, since $\tau_1\cdots\tau_n = (1)(2\cdots n)$, the unique cut has the form $(1\,r)$ for some r, $2 \le r \le n$. Let $\mathcal{G}^r_{n,k}$ be the set of all minimal factorizations $(\tau_1,\ldots,\tau_n) \in \mathcal{F}_{(1,n-1)}$ which have a cut $(1\,r)$ at the k-th position, i.e. $\tau_k = (1\,r)$. Set

$$\mathcal{G}^r_n = \bigcup_{k=1}^{n-1} \mathcal{G}^r_{n,k}.$$

Clearly,

$$\mathcal{F}_{(1,n-1)} = \bigcup_{r=2}^{n} \mathcal{G}^r_n.$$

We first observe that the cardinality of $\mathcal{G}^r_{n,k}$ is independent of r.

Lemma 1 *Let θ be the permutation $(1)(2\cdots n)$. For any r and s, $2 \le r \le s \le n$, the mapping $\Theta_{s-r} : \mathcal{G}^r_{n,k} \to \mathcal{G}^s_{n,k}$ defined by*

$$\Theta_{s-r}(\tau_1,\ldots,\tau_n) = (\theta^{s-r}\tau_1\theta^{r-s},\ldots,\theta^{s-r}\tau_n\theta^{r-s})$$

is a bijection. Moreover, Θ_{s-r} can be extended to the bijection between $\mathcal{G}^r_n$ and $\mathcal{G}^s_n$.

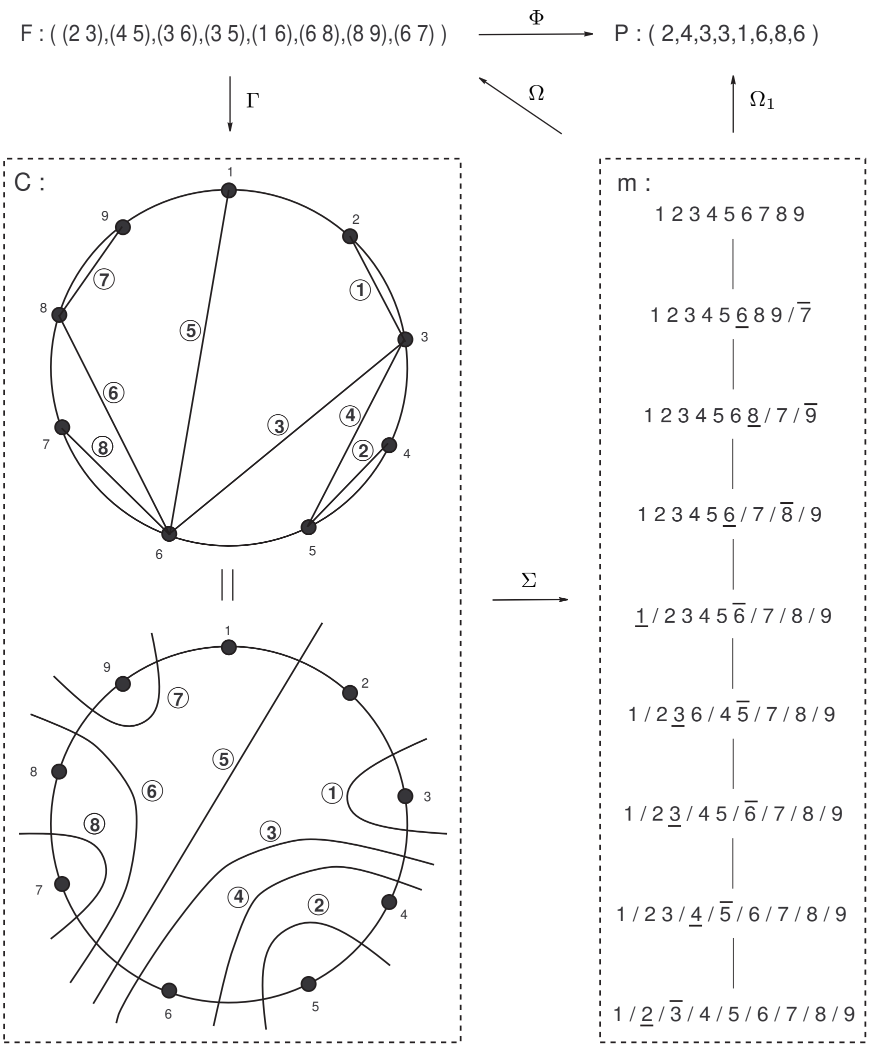

Figure 4: Bijections between $\mathcal{F}_{(n)}$, $\mathcal{C}_n$, $\mathcal{M}_n$ and $\mathcal{P}_{n-1}$.

Proof. Clearly it is a bijection since $\theta^{s-r}\tau_i\theta^{r-s}$ fixes 1 and changes r into s in τ_i. $\square$

A parking function $(x_1,\ldots,x_n)$ is said to be *prime*, if the increasing rearrangement $y_1 \le \cdots \le y_n$ of $x_1,\ldots,x_n$ satisfies $y_i < i$ for $i = 2,\ldots,n$. Let $\mathcal{Q}_n$ denote the set of *prime parking functions* $Q = (x_1,\ldots,x_n)$ of length n and define $\mathcal{Q}_{n,k}$ to be the set of all $Q \in \mathcal{Q}_n$ in which $x_k = 1$ is the leftmost occurrence of 1. Clearly, we have

$$\mathcal{Q}_n = \bigcup_{k=1}^{n-1} \mathcal{Q}_{n,k}.$$

It is clear that the prime parking functions satisfy the following:

Lemma 2 *$(x_1,\ldots,x_n) \in \mathcal{Q}_{n,k}$ if and only if $(x_1,\ldots,x_{k-1},x_{k+1},\ldots,x_n) \in \mathcal{P}_{n-1}$ with $x_i \ne 1$ for $i = 1, 2, \ldots, k-1$.*

We will eventually show that $\mathcal{G}^2_{n,k}$ and $\mathcal{Q}_{n,k}$ have the same cardinality.

Let $\mathcal{A}_{n,k}$ be the set consisting of all $(\sigma_1,\ldots,\sigma_{n-1}) \in \mathcal{F}_{(n)}$ such that

(a1) 1 occurs in $\sigma_k,\ldots,\sigma_{n-1}$.

(a2) 2 does not occur in $\sigma_1,\ldots,\sigma_{k-1}$,

Lemma 3 *The map $\Psi_1 : \mathcal{G}^2_{n,k} \to \mathcal{A}_{n,k}$ defined by*

$$\Psi_1(\tau_1,\ldots,\tau_n) = ((1\,2)\tau_1(1\,2),\ldots,(1\,2)\tau_{k-1}(1\,2),\tau_{k+1},\ldots,\tau_n),$$

is a bijection.

Proof. Set $\widetilde{\tau}_i = (1\,2)\tau_i(1\,2)$. Assume that $(\tau_1,\ldots,\tau_n) \in \mathcal{G}^2_{n,k}$. Since $(\tau_1,\ldots,\tau_n) \in \mathcal{F}_{(1,n-1)}$, $\tau_1\cdots\tau_n = (1)(2\,\cdots\,n)$ and $\tau_k = (1\,2)$. Multiplying both sides with $(1\,2)$ yields $\widetilde{\tau}_1\cdots\widetilde{\tau}_{k-1}\tau_{k+1}\cdots\tau_n = (1\,\cdots\,n)$. This implies that $(\widetilde{\tau}_1,\ldots,\widetilde{\tau}_{k-1},\tau_{k+1},\ldots,\tau_n) \in \mathcal{F}_{(n)}$. In addition, since $\tau_k = (1\,2)$ is a cut, $\tau_{k+1}\cdots\tau_n$ doesn't fix 1 and so there exists a j greater than k such that τ_j contains 1. Moreover, since τ_k is the unique cut in $(\tau_1,\ldots,\tau_n)$, 1 doesn't appear in any of $\tau_1,\ldots,\tau_{k-1}$ (otherwise, 1 can not be a fixed point in $\tau_1\cdots\tau_n$). Thus $\widetilde{\tau}_1,\ldots,\widetilde{\tau}_{k-1}$ do not have 2.

Hence $(\widetilde{\tau}_1,\ldots,\widetilde{\tau}_{k-1},\tau_{k+1},\ldots,\tau_n)$ is in $\mathcal{A}_{n,k}$.

We now show that Ψ_1 has the inverse. Define the map $\Psi'_1 : \mathcal{A}_{n,k} \to \mathcal{G}^2_{n,k}$ by

$$\Psi'_1(\sigma_1,\ldots,\sigma_{n-1}) = (\widetilde{\sigma}_1,\ldots,\widetilde{\sigma}_{k-1},(1\,2),\sigma_k,\ldots,\sigma_{n-1}).$$

We first prove that Ψ'_1 is well-defined. Since $(\sigma_1,\ldots,\sigma_{n-1}) \in \mathcal{F}_{(n)}$, we have $\sigma_1\cdots\sigma_{n-1} = (1\,\cdots\,n)$. Multiplying both sides with $(1\,2)$ yields $\widetilde{\sigma}_1\cdots\widetilde{\sigma}_{k-1}(1\,2)\sigma_k\cdots\sigma_{n-1} = (1)(2\ldots n)$. This implies that $F = (\widetilde{\sigma}_1,\ldots,\widetilde{\sigma}_{k-1},(1\,2),\sigma_k,\ldots,\sigma_{n-1}) \in \mathcal{F}_{(1,n-1)}$. Since $(\sigma_1,\ldots,\sigma_{n-1}) \in \mathcal{F}_{(n)}$ has no cut, none of σ_k, $\ldots$, σ_{n-1} is a cut in F. We now show that $(1\,2)$ is a cut in F. Suppose that it is a join in F. The unique cut of F is $\widetilde{\sigma}_j$ for some j less than k. So 1 is a fixed point of $\widetilde{\sigma}_j\cdots\widetilde{\sigma}_{k-1}(1\,2)\sigma_k\cdots\sigma_{n-1}$. But from the condition (a1), 1 appears at least twice among $\widetilde{\sigma}_{j+1},\ldots,\widetilde{\sigma}_{k-1},(1\,2),\sigma_k,\ldots,\sigma_{n-1}$ and 1 is contained in a cycle of length at least 3 in $\widetilde{\sigma}_{j+1}\cdots\widetilde{\sigma}_{k-1}(1\,2)\sigma_k\cdots\sigma_{n-1}$. So $\widetilde{\sigma}_j$ is of the form $(1\,m)$ and 2 appears in σ_j, contradicting to the condition (a2). Therefore, the k-th transposition $(1\,2)$ is a cut and we have $(\widetilde{\sigma}_1,\ldots,\widetilde{\sigma}_{k-1},(1\,2),\sigma_k,\ldots,\sigma_{n-1}) \in \mathcal{G}^2_{n,k}$.

Clearly, $\Psi'_1 \circ \Psi_1$ is the identity map and so Ψ_1 is a bijection. $\square$

We now show that $\mathcal{A}_{n,k}$ and $\mathcal{Q}_{n,k}$ are in 1-1 correspondence.

Lemma 4 *The map* $\Psi_2 : \mathcal{A}_{n,k} \to \mathcal{Q}_{n,k}$ *defined by*

$$\Psi_2(\sigma_1, \ldots, \sigma_{n-1}) = (\widetilde{a}_1, \ldots, \widetilde{a}_{k-1}, 1, a_k, \ldots, a_{n-1}),$$

where $\sigma_i = (a_i\, b_i)$ *and* $\widetilde{a}_i$ *is the image of* a_i *under the transposition* $(1\,2)$*, is a bijection.*

Proof. Let $(\sigma_1, \ldots, \sigma_{n-1})$ be an element of $\mathcal{A}_{n,k}$. Then $(a_1, \ldots, a_{n-1}) \in \mathcal{P}_{n-1}$ with $2 \notin \{a_1, \ldots, a_{k-1}\}$ and $1 \in \{a_k, \ldots, a_{n-1}\}$, which imply $(\widetilde{a}_1, \ldots, \widetilde{a}_{k-1}, a_k, \ldots, a_{n-1}) \in \mathcal{P}_{n-1}$ with $1 \notin \{\widetilde{a}_1, \ldots, \widetilde{a}_{k-1}\}$. So by Lemma 2, $(\widetilde{a}_1, \ldots, \widetilde{a}_{k-1}, 1, a_k, \ldots, a_{n-1}) \in \mathcal{Q}_{n,k}$.

The map Ψ_2 has the inverse. It can be shown easily that the map $\Psi_2' : \mathcal{Q}_{n,k} \to \mathcal{A}_{n,k}$ defined by

$$\Psi_2'(x_1, \ldots, x_{k-1}, 1, x_{k+1}, \ldots, x_n) = \Phi^{-1}(\widetilde{x}_1, \ldots, \widetilde{x}_{k-1}, w_{k+1}, \ldots, x_n)$$

is the inverse of Ψ_2.

Hence Ψ_2 is a bijection. □

Theorem 2 *The mapping* $\Psi : \mathcal{G}^2_{n,k} \to \mathcal{Q}_{n,k}$ *defined by*

$$\Psi(\,(a_1\, b_1), \ldots, (a_n\, b_n)\,) = (a_1, \ldots, a_n),$$

is a bijection. Moreover, Ψ *can be extended to a bijection between* $\mathcal{G}^2_n$ *and* $\mathcal{Q}_n$.

Proof. Since $(a_i\, b_i) \neq (1\,2)$ for $i = 1, 2, \ldots, k-1$, we clearly have $\Psi = \Psi_2 \circ \Psi_1$, where Ψ_1 and Ψ_2 are the bijections defined in Lemmas 3 and 4.

Since $\mathcal{G}^2_n = \cup_{k=1}^{n-1} \mathcal{G}^2_{n,k}$ and $\mathcal{Q}_n = \cup_{k=1}^{n-1} \mathcal{Q}_{n,k}$, Ψ can be defined as a mapping from $\mathcal{G}^2_n$ to $\mathcal{Q}_n$. □

Now we are ready to finish a combinatorial proof of (2).

Corollary 1 *The mapping* $\Theta : \mathcal{Q}_n \times [n-1] \to \mathcal{F}_{(1,n-1)}$ *defined by*

$$\Theta(Q, t) = \Theta_{t-1}(\Psi^{-1}(Q)),$$

is a bijection.

Proof. Θ_{t-1} and Ψ^{-1} are bijections and $\Theta_{t-1}(\mathcal{G}^2_n) = \mathcal{G}^{t+1}_n$ for $t = 1, \ldots, n-1$. Since

$$\mathcal{F}_{(1,n-1)} = \bigcup_{r=2}^{n} \mathcal{G}^r_n,$$

Θ is a bijection. (See Figure 5.) □

Since $|\mathcal{Q}_n| = (n-1)^{n-1}$ ([14, pp. 95]), Corollary 1 is a combinatorial proof of $|\mathcal{F}_{(1,n-1)}| = (n-1)^n$.

5 Remark

With a variation of the *depth-first-search*, we are able to find a bijection between $\mathcal{Q}_{n,k}$ and the set $\mathcal{R}_{n,k}$ of rooted labelled trees on $[n]$ such that the root k is a local minimum of the tree. Chauve, Dulucq and Guibert [4] enumerated the cardinality of $\mathcal{R}_{n,k}$ with a bijective proof. Their result is:

$$|\mathcal{R}_{n,k}| = (n-k)\, n^{n-k-1}(n-1)^{k-2}.$$

So we are able to count not only $\mathcal{F}_{(1,n-1)}$ but also $\mathcal{G}^2_{n,k}$, $k = 1, 2, \ldots, n-1$, which are refinements of $\mathcal{F}_{(1,n-1)}$. Moreover, by this refinement, our approach can be extended to handle the case $\lambda = (2, n-2)$.

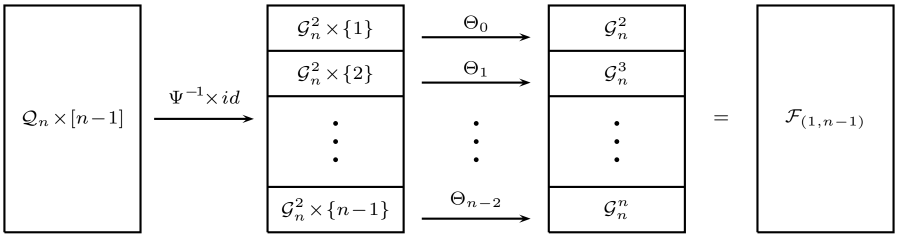

Figure 5: The map $\Theta : \mathcal{Q}_n \times [n-1] \to \mathcal{F}_{(1,n-1)}$

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References

[1] V.I. Arnold, *Topological classification of trigonometric polynomials and combinatorics of graphs with an equal number of vertices and edges*, (English. Russian original) Funct. Anal. Appl. **30** no.1 (1996), 1–14; translation from Funkts. Anal. Prilozh. *30* no. 1 (1996), 1–17.

[2] P. Biane, *Parking functions of types A and B*, Electronic J. Combinatorics **9** no. 1 (2002), N7.

[3] M. Bousquet-Mélou and G. Schaeffer, *Enumeration of planar constellations*, Adv. in Appl. Math. **24** no. 4 (2000), 337–368 .

[4] C. Chauve, S. Dulucq and O. Guibert, *Enumeration of some labelled trees*, Technical Report RR-1226-99, LaBRI, Université Bordeaux I (1999).

[5] M. Crescimanno and W. Taylor, *Large N phases of chiral QCD_2*, Neuclear Phys. B, **437** (1995), 3–24.

[6] J. Dénes, *The representation of a permutation as the product of a minimal number of transpositions and its connection with the theory of graphs*, Publ. Math. Institute Hung. Acad. Sci. **4** (1959), 63–70.

[7] C.L. Ezell, *Branch point structure of covering maps onto nonorientable surfaces*, Trans. Amer. Math. Soc. **243** (1978), 123–133.

[8] I. P. Goulden and D. M. Jackson, *Transitive factorizations into transpositions and holomorphic mappings on the sphere*, Proc. Amer. Math. Soc. **125** (1997), 51–60.

[9] I. P. Goulden and S. Pepper, *Labelled trees and factorizations of a cycle into transpositions*, Discrete Math. **113** (1993), 263–268.

[10] I. P. Goulden and A. Yong, *Tree-like properties of cycle factorizations*, Journal of Combinatorial Theory, Series A **98** (2002), 106–117.

[11] A. Hurwitz, *Über Riemann'sche Flächen mit gegebenen Verzweigungspunkten*, Mathematische Annalen, **39** (1891), 1–60.

[12] P. Moszkowski, *A solution to a problem of Dénes: a bijection between trees and factorizations of cycle permutations*, European J. Combinatorics **10** (1989), 13–16.

[13] R. Simion, *Noncrossing partitions*, Discrete Math. **217** (2000), 367–409.

[14] R. P. Stanley, *Enumerative Combinatorics vol.* **2**, Combridge University Press (1999).

[15] R. P. Stanley, *Parking functions and noncrossing partitions*, Electronic J. Combinatorics **4** no. 2 (1997), R20.

[16] V. Strehl, *Minimal transitive products of transpositions—the reconstruction of a proof of A. Hurwitz*, Semin. Lothar. Comb. 37, B37c, 12 p. (1996)