A new decomposition of derangements

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Abstract

We give a new decomposition of derangements, which gives a direct interpretation of a formula for their generating function. This decomposition also works for counting derangements by number of excedances.

1 Introduction

A permutation π of $[n]=\{1, 2, \ldots, n\}$ is a *derangement*, if $\pi(i) \neq i$, for all $i \in [n]$. A value $i \in [n]$ is an *excedance* of π if $i < \pi(i)$. The number of excedances in π is denoted $\operatorname{exc} \pi$. Let $\mathcal{D}_n$ be the set of derangements of $[n]$, and $d_n(x)$ the polynomial

$$d_n(x) = \sum_{\pi \in \mathcal{D}_n} x^{\operatorname{exc} \pi}. \tag{1}$$

For example, $d_0(x) = 1$, $d_1(x) = 0$, $d_2(x) = x$, $d_3(x) = x + x^2$, $d_4(x) = x + 7x^2 + x^3$. The generating function of $d_n(x)$ can be written as follows [2, 5] :

$$\sum_{n \geq 0} d_n(x) \frac{t^n}{n!} = \frac{1}{1 - \sum_{n \geq 2}(x + x^2 + \cdots + x^{n-1})\, t^n/n!}\,. \tag{2}$$

Of course (2) can be proved by various methods, but, as pointed out by Gessel [4], it seems difficult to directly interpret (2) (even in the $x = 1$ case!) in terms of derangements. In [4] Gessel gave a direct proof of (2) in a different model with $x = 1$. His proof is actually based on a factorization of some *D-permutations*, and cannot be generalized in a straightforward way to prove (2). Our purpose is to give a decomposition of derangements which interprets (2) directly.

A cycle $\sigma = (s_1 s_2 \cdots s_k)$ of length k is a cyclic permutation of a k-set $\{s_1, \ldots, s_k\}$ such that $\sigma(s_i) = s_{i+1}$ for $i \in [k]$, with $s_{k+1} = s_1$, and $s_1 = \min(\sigma)$. In what follows a cycle is always of length greater than or equal to 2. We identify each derangement with the sequence of its cycles sorted in the decreasing order of their minima. The cycle σ is called *unimodal* (resp. *prime*), if there exists i, $2 \le i \le k$, such that $s_1 < \cdots < s_{i-1} < s_i > s_{i+1} > \cdots > s_k$ (resp., in addition, $s_{i-1} < s_k$). Considering that s_1 is the smallest in our case, this definition is consistent with the usual definition of 'unimodal'. According to our definition, any cycle $(s_1\, s_2)$ of length 2 is unimodal and prime.

A sequence of prime cycles $\tau = (\tau_1, \tau_2, \ldots, \tau_m)$ is called a *P-decomposition* of type $(l_1, \ldots, l_m)$, if τ_i is of length l_i and the underlying sets of τ_i, $i \in [m]$, form a partition of $[l_1 + \cdots + l_m]$. Define the *excedance of* τ as the sum of the excedances of its prime cycles, i.e. $\operatorname{exc} \tau = \operatorname{exc} \tau_1 + \cdots + \operatorname{exc} \tau_m$, and weight τ by $x^{\operatorname{exc} \tau}$. It turns out that the right-hand side of (2) is the corresponding generating function of P-decompositions. Indeed, the weight of prime cycles on a given l-set is $x + x^2 + \cdots + x^{l-1}$, and so the generating function of P-decompositions of type $(l_1, \ldots, l_m)$ is given by

$$\binom{l_1 + \cdots + l_m}{l_1\,,\, \ldots\,,\, l_m} \prod_{i=1}^{m} (x + \cdots + x^{l_i - 1}) \frac{t^{l_1 + \cdots + l_m}}{(l_1 + \cdots + l_m)!}.$$

Summing on $l_1, \ldots, l_m \ge 2$ and $m \ge 0$, we obtain the right hand side of (2).

In the next section we will give an algorithm, which permits us to decompose each derangement into a P-decomposition with the same number of excedances and thus to prove (2). We conclude this paper with some further remarks.

2 Unimodal and prime decompositions

The following algorithm decomposes each cycle $\sigma = (s_1 s_2 \cdots s_k)$ into a sequence of unimodal cycles.

U-Algorithm

1. *If σ is unimodal then $U(\sigma) = \sigma$.*
2. *Otherwise, let i be the largest integer such that $s_{i-1} > s_i < s_{i+1}$, and let j be the unique integer greater than i such that $s_j > s_i > s_{j+1}$, recalling that $s_{k+1} = s_1$. Decompose it into two cycles $\sigma_1 = (s_1 \cdots s_{i-1} s_{j+1} \cdots s_k)$ and $\sigma_2 = (s_i s_{i+1} \cdots s_j)$. Note that σ_2 is unimodal. Then set $U(\sigma) = (U(\sigma_1), \sigma_2)$.*

There is an abuse of notation in the description of the U-algorithm (also of the P-algorithm later). Namely, the second σ in $U(\sigma) = \sigma$ actually means (σ), to be consistent with the case of length greater 1, a sequence of length 1 whose only term is σ, and $U(\sigma_1)$ in $(U(\sigma_1), \sigma_2)$ actually means the sequence of cycles in $U(\sigma_1)$ without outside parentheses.

Example 2.1 *Let* $\sigma = (1\ 8\ 4\ 7\ 12\ 14\ 11\ 9\ 13\ 10\ 6\ 3\ 5\ 2)$*. The* U*-algorithm runs as follows:*

$$\begin{aligned} \sigma &\rightarrow (U(1\ 8\ 4\ 7\ 12\ 14\ 11\ 9\ 13\ 10\ 6\ 2), (3\ 5)) \\ &\rightarrow (U(1\ 8\ 4\ 7\ 12\ 14\ 11\ 6\ 2), (9\ 13\ 10), (3\ 5)) \\ &\rightarrow (U(1\ 8\ 2), (4\ 7\ 12\ 14\ 11\ 6), (9\ 13\ 10), (3\ 5)) \\ &\rightarrow ((1\ 8\ 2), (4\ 7\ 12\ 14\ 11\ 6), (9\ 13\ 10), (3\ 5)). \end{aligned}$$

We extend U to any derangement by applying U to each of its cycles; the resulting sequence is called the *unimodal decomposition* of the given derangement. Note that if $U(\pi) = (u_1, \ldots, u_m)$ is the resulting sequence, then the first cycle of π corresponds to the segment $(u_1, \ldots, u_i)$, where i is the smallest integer satisfying $\min(u_1) > \min(u_{i+1})$, and the second to a segment of $(u_{i+1}, \ldots, u_m)$ in the same manner, etc., so that the underlying set of each cycle can be read off from $U(\pi)$. The following result characterizes all the sequences of unimodal cycles obtained by the U-algorithm.

Lemma 2.2 *A sequence of disjoint unimodal cycles,* $u = (u_1, \ldots, u_m)$*, is a unimodal decomposition of a derangement in* $\mathcal{D}_n$ *if and only if the underlying sets of* u_i*,* $i \in [m]$*, form a partition of* $[n]$ *and for each* $i = 2, \ldots, m$*,* $\max(u_{i-1}) > \min(u_i)$*.*

Proof. Clearly it suffices to show the 'if' part. Without loss of generality we may assume that $\min(u_1) < \min(u_i)$, for each $i = 2, \ldots, m$. We build π step by step. Let $\pi^{(1)} = u_1$. For $i > 1$, assume that $\pi^{(i-1)}$ has been built and that $\pi^{(i-1)} = (s_1 s_2 \cdots s_l)$, where $s_1, \ldots, s_l$ is a rearrangement of elements in $u_1, u_2, \ldots, u_{i-1}$. Let $u_i = (r_1 r_2 \cdots r_a)$. Since $\max(u_{i-1}) > \min(u_i)$, there is an integer j such that $s_j > \min(u_i)$, let j_0 be the largest such integer and set $\pi^{(i)} = (s_1 s_2 \cdots s_{j_0} r_1 r_2 \cdots r_a s_{j_0+1} \cdots s_l)$. Let $\pi = \pi^{(m)}$. Clearly $U(\pi) = u$. $\square$

The following algorithm decomposes each unimodal cycle $\sigma = (s_1 s_2 \cdots s_k)$ into a sequence of prime cycles.

P-Algorithm

1. *If* σ *is prime then* $P(\sigma) = \sigma$*.*
2. *Otherwise, let* j *be the smallest integer such that* $s_j > s_i > s_{j+1} > s_{i-1}$ *for some integer* i *greater than 1. Decompose it into two cycles* $\sigma_1 = (s_1 \cdots s_{i-1} s_{j+1} \cdots s_k)$ *and* $\sigma_2 = (s_i s_{i+1} \cdots s_j)$*. Note that* σ_2 *is prime. Then set* $P(\sigma) = (P(\sigma_1), \sigma_2)$*.*

We extend P to any sequence of unimodal cycles by applying P to each of its components. Since each derangement is decomposed into cycles, sorted in the decreasing order of their minima, combining with the U-algorithm, we can factor any derangement π into a sequence of prime cycles, $P \circ U(\pi)$, called the *prime decomposition* of π. The structure of the unimodal decomposition of π can be easily obtained from its prime decomposition. If $P \circ U(\pi) = (\tau_1, \ldots, \tau_m)$, then the first unimodal cycle in $U(\pi)$ corresponds to the segment $(\tau_1, \ldots, \tau_i)$, where i is the smallest integer satisfying $\max(\tau_i) > \min(\tau_{i+1})$, and the second to a segment of $(\tau_{i+1}, \ldots, \tau_m)$ in the same manner, etc.

Example 2.3 *Let* $\sigma = (1\ 8\ 4\ 7\ 12\ 14\ 11\ 9\ 13\ 10\ 6\ 3\ 5\ 2)$. *Its unimodal decomposition is* $U(\sigma) = ((1\ 8\ 2), (4\ 7\ 12\ 14\ 11\ 6), (9\ 13\ 10), (3\ 5))$. *Since only the second cycle in* $U(\sigma)$ *is not prime, it suffices to decompose the second cycle. The algorithm runs as follows:*

$$\begin{aligned} (4\ 7\ 12\ 14\ 11\ 6) &\rightarrow (P(4\ 7\ 11\ 6), (12, 14)) \\ &\rightarrow ((4\ 6), (7\ 11), (12\ 14)). \end{aligned}$$

Therefore $P \circ U(\sigma) = ((1\ 8\ 2), (4\ 6), (7\ 11), (12\ 14), (9\ 13\ 10), (3\ 5))$.

Theorem 2.4 *Any* P*-decomposition of* $[n]$ *is the prime decomposition of a derangement in* $\mathcal{D}_n$.

Proof. Let $\tau = (\tau_1, \tau_2, \ldots, \tau_m)$ be a P-decomposition of $[n]$. We first construct a sequence of unimodal cycles as follows: starting from the right, if there is any pair of adjacent τ_i and τ_{i+1} such that $\max(\tau_i) < \min(\tau_{i+1})$, then we insert the elements of τ_{i+1} in τ_i just before the maximum of τ_i and obtain a new cycle $\tau_i * \tau_{i+1}$. Repeat this process with $(\tau_1, \ldots, \tau_i * \tau_{i+1}, \ldots, \tau_m)$, until there are no more such pairs. By lemma 2.2, the resulting sequence σ is a unimodal decomposition of some $\pi \in \mathcal{D}_n$, i.e. $U(\pi) = \sigma$. It follows that $P \circ U(\pi) = P(\sigma) = \tau$. □

From the U-algorithm it is clear that the number of excedances in a cycle is the same as the sum of excedances in each unimodal component. Also the prime decomposition has the same property. Thus we have proved (2).

3 Remarks

If instead of derangements we let $A_n(x)$ denote the sum of $x^{\operatorname{exc} \pi}$ for all permutations π of $[n]$, then

$$\sum_{n \geq 0} A_n(x) \frac{t^n}{n!} = e^t \sum_{n \geq 0} d_n(x) \frac{t^n}{n!} = \frac{1}{1 - \sum_{n \geq 1} (x-1)^{n-1} t^n / n!}\,. \qquad (3)$$

The last expression can be interpreted as the generating function of sequences of unimodal segments (the minimal element is allowed to be at the end) with appropriate weights. A similar proof can be given for (3), but in this case a weight-preserving sign-reversing involution is needed. The polynomials $xA_n(x)$ are actually the well-known

Eulerian polynomials and have several other combinatorial interpretations in addition to counting permutations by number of excedances [6].

Our decompositions work also for permutations of a multiset $\{1^{n_1}, 2^{n_2}, \ldots, m^{n_m}\}$. More precisely, let $w = w_1 w_2 \ldots w_n$ be such a permutation and $\delta(w) = p_1 p_2 \ldots p_n$ the nondecreasing rearrangement of the letters in w, where $n = n_1 + \cdots + n_m$. Then w is a *multiderangement* if $p_i \neq w_i$ for each $i = 1, \ldots, n$, while the statistic of *excedance* of w is defined by $\operatorname{exc} w = \#\{i : w_i > p_i\}$. Let $\mathcal{R}(\mathbf{n})$ be the set of all such permutations and define

$$d_{\mathbf{n}}(x) = \sum_{w \in \mathcal{R}(\mathbf{n})} x^{\operatorname{exc} w}.$$

Using Foata's factorization of multipermutations (see [3]) we can factorize each multiderangement as a product of cycles of length at least 2, combining with our two decompositions we get the following result:

$$\sum_{n_1, \ldots, n_m \geq 0} d_{\mathbf{n}}(x) x_1^{n_1} \ldots x_m^{n_m} = \frac{1}{1 - x\, e_2 - (x + x^2)\, e_3 - \cdots - (x + x^2 + \cdots + x^{m-1})\, e_m},$$

where e_i $(2 \leq i \leq m)$ is the i-th elementary symmetric function of $x_1, \ldots, x_m$. The above result was first proved by Askey and Ismail [1] using MacMahon's Master Theorem.

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