Fifty Ways to Leave Your Lover—
er, Find the Volume of a Torus

Dan Drake – University of Minnesota
http://www.math.umn.edu/∼drake
drake@math.umn.edu

1 Introduction

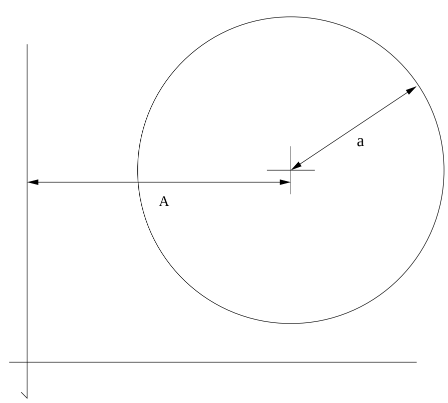
This is almost certainly the first time in this history of human civilization that Paul Simon lyrics have been used in a multivariable calculus class.

In this class we've learned a lot of different kinds of integrals. We'll use almost all of them in this article/document/whatever to find the volume of a torus.

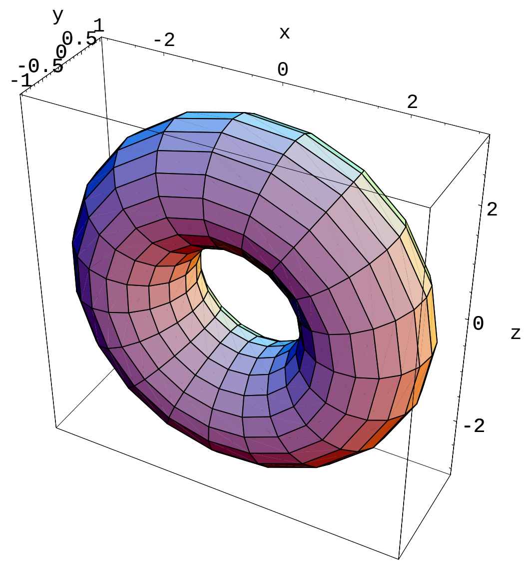
2 Why Homer Simpson is a topologist

Let's begin by making sure we understand what a torus is and what it looks like. The quick answer is "a doughnut", which is why this section has the title it does. However, this author is a particular fan of Bavarian creme-filled doughnuts, which are not torus-shaped. Let's leave the baked goods behind and make our definition a bit more rigorous:

A torus is obtained by rotating a circle of radius a whose center is A units away from a line around that line. A picture of this is shown below:



In three dimensions, a torus looks like this:



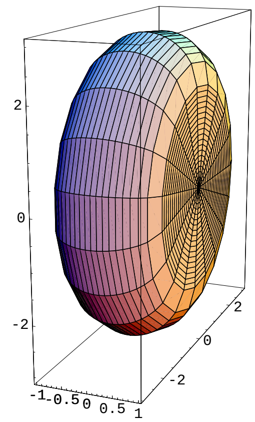
Through this article, we'll assume that $A > a > 0$. You can relax that restriction a bit and get goofy looking torii, but we won't worry about that.

3 Slip out the back, Jack

We'll start out using Calculus I methods to figure out the volume. We can think of the torus as a surface of revolution with another surface of revolution taken out of it. The outside is like the Bavarian creme-filled doughnut I spoke of earlier, and can be described by rotating the semicircle

$$y = f(x) = A + \sqrt{a^2 - x^2} \tag{1}$$

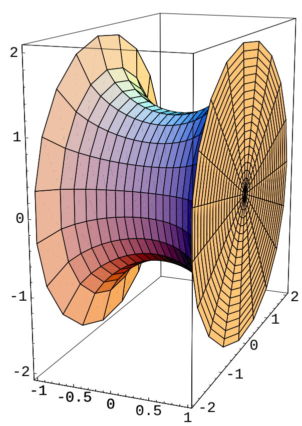
around the x-axis. The resulting object looks this:



As you probably learned in first semester calculus, we find the volume by a single integral. The value of $f(x)$ is the radius of our surface; if we square $f(x)$, multiply it by π and by dx, we can think of the integral as finding the total volume of many very thin disks. The integral is

$$\int_{-a}^{a} \pi[f(x)]^2 \, \mathrm{d}x = \int_{-a}^{a} \pi \left(A + \sqrt{a^2 - x^2}\right)^2 \, \mathrm{d}x, \tag{2}$$

but we've counted too much volume. We need to account for the hole in the middle, which we do with a very similar integral: we use the semicircle $y = g(x) = A - \sqrt{a^2 - x^2}$, which looks like this when revolved around the x-axis:



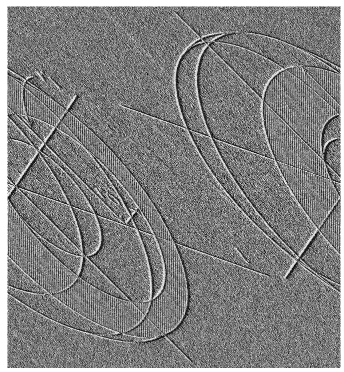
We do a very similar integral to find the volume of the hole, then subtract that. This gives us our first way of finding the volume of a torus:

$$\boxed{\int_{-a}^{a} \pi \left(A + \sqrt{a^2 - x^2}\right)^2 - \pi \left(A - \sqrt{a^2 - x^2}\right)^2 \, \mathrm{d}x.} \tag{3}$$

4 Make a new plan, Stan

The new plan is actually just another way of thinking of the above method. Instead of doing two integrals with surfaces of revolution, we'll think of the torus as a stack of very thin annuli. An annulus (plural *annuli*) is a disk with another disk cut out of its center. A compact disc is a good example of an annulus.

Think of a torus as a bagel, and slice it as you normally would slice a bagel. That gives you two pieces. Now make the same kind of slice, but slice the torus/bagel into three pieces, or four…or millions of slices. As you make more and more slices, the resulting pieces become more like perfect annuli. Here's a picture of the situation, taken from [2], (page 262):



To find the volume, we can just find the area of one of those thin annuli and integrate that from $-a$ to a. The area of an annulus with outer radius R and inner radius r is just the area of a disk of radius R minus the area of a disk with radius r, which is $\pi(R^2 - r^2)$.

So, what's the outer and inner radius for these annuli? The outer radius is along the "outer" semicircle described in equation (1), so $R = A + \sqrt{a^2 - x^2}$. In a very similar way, you can figure out that $r = A - \sqrt{a^2 - x^2}$. That means that at $x = a$, the area of an annulus is

$$\pi\left(\left(A + \sqrt{a^2 - x^2}\right)^2 - \left(A - \sqrt{a^2 - x^2}\right)^2\right).$$

That's the area of a thin annulus; to get the volume of the entire torus, we integrate that over $-a \le x \le a$ to get

$$\int_{-a}^{a} \pi\left(\left(A + \sqrt{a^2 - x^2}\right)^2 - \left(A - \sqrt{a^2 - x^2}\right)^2\right)\, \mathrm{d}x.$$

However, you can see that this is exactly the same as (3) above. This is just another way to think about the surface-of-revolution method.

5 No need to be coy, Roy

Next we'll use double integrals to find the volume. Think of a torus as a sliced bagel and put the top half on the xy plane. A double integral will find the volume underneath the surface of the torus, and we can then multiply by two. In cartesian coordinates, it's difficult to express this surface, but in polar coordinates, it's very easy:

$$f(r,\theta) = \begin{cases} \sqrt{a^2-(r-A)^2} & \text{if } A-a \le r \le A+a \\ 0 & \text{otherwise.} \end{cases}$$

All we have to do is integrate that function over appropriate bounds in cylindrical coordinates. It's clear that θ will range from 0 to 2π; you can see from the above definition that r will range from $A-a$ to $A+a$. The integral is *almost*

$$2\int_0^{2\pi}\int_{A-a}^{A+a} \sqrt{a^2-(r-A)^2}\ \mathrm{d}r\ \mathrm{d}\theta.$$

That's not quite right, because in cylindrical coordinates, we have to multiply by the determinant of the Jacobian of the transformation that takes us from cartesian to cylindrical coordinates. That's a long-winded way to say "multiply the integrand by r". Once we do that, we have the second way of finding the volume of a torus:

$$\boxed{2\int_0^{2\pi}\int_{A-a}^{A+a} r\sqrt{a^2-(r-A)^2}\ \mathrm{d}r\ \mathrm{d}\theta.} \tag{4}$$

6 Hop on the bus, Gus

By now you should know the basic following properties of single, double, and triple integrals: when you integrate the function "1", you get length, area, and volume, respectively. That is,

$$\int_I 1\ \mathrm{d}x = \text{length of interval } I$$
$$\iint_R 1\ \mathrm{d}A = \text{area of the planar region } R$$
$$\iiint_S 1\ \mathrm{d}V = \text{volume of the 3-dimensional solid } S.$$

Therefore, if asked to find the volume of a torus, the easiest answer is that the volume is

$$\iiint_T 1\ \mathrm{d}V$$

where T is the torus[1]. That answer is perfectly legitimate, but of course it isn't very satisfying. Let's use the Divergence Theorem to turn this into something more interesting. Recall that the Divergence Theorem (also called Gauss' Theorem, but everything is Gauss' Theorem) states that

The Divergence Theorem. *If S is a solid region in $\mathbb{R}^3$ and ∂S is the boundary of S, then*

$$\oiint_{\partial S} \boldsymbol{F} \cdot \boldsymbol{n} \, \mathrm{d}\sigma = \iiint_{S} \operatorname{div} \boldsymbol{F} \, \mathrm{d}V$$

for "nice" vector fields $\boldsymbol{F}$.

We have a triple integral whose integrand is 1; let's try to work backwards and convert that integral into a surface integral. We need a vector field whose divergence is exactly 1. Here are a couple possibilities:

$$\begin{aligned} \boldsymbol{F}(x,y,z) &= (x,0,0) \\ &= (0,y,0) \\ &= (0,0,z) \\ &= (x/3, y/3, z/3) \\ &= (x, \sin(\cos(z^9)), \log(\pi + |y| + \sqrt{1+y^4})) \end{aligned}$$

No reasonable person would ever want to work with that last vector field, but its divergence is certainly 1. I came up with the last vector field by just writing down a hairy-looking function of z in the y coordinate, and another hairy-looking function of y in the z coordinate. The partial derivatives will clearly be zero.

Certainly there are infinitely many functions of y and z (there are $\aleph_1$ of them, if you happen to know anything about transfinite cardinals), which means that the awe-inspiring power of calculus enables us to find the volume of a torus using any one of infinitely many vector fields—there are INFINITELY MANY WAYS to find the volume of a torus! Stick that in your pipe and smoke it, Paul Simon.

Let's record one of those ways, using the vector field $\boldsymbol{F}(x,y,z) = \frac{1}{3}(x,y,z)$:

$$\boxed{\oiint_{\partial T} \frac{1}{3}(x,y,z) \cdot \boldsymbol{n} \, \mathrm{d}\sigma.} \tag{5}$$

7 Drop off the key, Lee

The other major theorem we learned this semester is Stokes' Theorem, which relates surface integrals and line integrals. At this point, you might tempted to think of something like this ("SF"

[1] This isn't quite right; a torus is actually the 2-dimensional surface, and not the solid region it encloses.

means a scalar-valued function, "VF" means a vector field):

$$\text{triple integral of SF} = \text{surface integral of VF} \qquad \text{(Divergence Thm)}$$
$$\text{surface integral of VF} = \text{line integral of VF} \qquad \text{(Stokes' Thm)}$$
$$\Downarrow$$
$$\text{triple integral of SF} = \text{line integral of VF}$$

But alas, our reach has exceeded our grasp. This will never happen unless the scalar function is identically zero and the vector field is conservative, and in that case, the integrals will all be zero. There's a couple ways to explain why this is so. First, let's recall Stoke's Theorem:

Stokes' Theorem. *Let S be a smooth surface in $\mathbb{R}^3$. If $\boldsymbol{F}$ is a differentiable vector field, then*

$$\iint_S \operatorname{curl} \boldsymbol{F} \, \mathrm{d}\sigma = \oint_{\partial S} \boldsymbol{F} \, \mathrm{d}\boldsymbol{x},$$

provided some conditions on orientation are met.

The first reason can be summarized by saying "div curl $\boldsymbol{F} = 0$". If we apply the Divergence Theorem to the situation described above in Stokes' Theorem, we would have

$$\iiint_T \operatorname{div}(\operatorname{curl} \boldsymbol{F}) \, \mathrm{d}V = \iint_S \operatorname{curl} F \, \mathrm{d}\sigma.$$

However, if you take any vector field, find its curl, and then find the divergence of the resulting vector field, you'll always get zero. So the triple integral on the left will be zero, making the double integral and in turn the single integral all zero. So there's no chance we'll be able to measure the volume of something using this method (unless its volume is zero, which isn't very interesting).

The second and more serious problem with this is the fact that the Divergence Theorem required a closed surface—one that encloses a three-dimensional solid region. But Stokes' Theorem requires the surface to have some sort of boundary, and no surface that encloses a solid region can have a boundary: compare the surface of a sphere, which has no boundary and encloses a solid region, and a disk, which has a boundary but doesn't enclose any sort of solid region. So we simply can't relate the single integral in Stokes' to the triple integral in the Divergence Theorem.

Despite that unfortunate reality of calculus, we still have found a number of ways to find the volume of a torus. We didn't find 50 ways, but Paul Simon's song didn't list 50 ways to leave your lover, either!

Below are some possibly useful references for the types of integrals and theorems mentioned here.

References

[1] David M. Bressoud, *Second year calculus*, Springer-Verlag, 1991.

[2] Richard E. Williamson and Hale F. Trotter, *Multivariable mathematics*, Prentice-Hall, 1996.