1 Determine whether the following statements are true or false. Mark each statement CLEARLY: **T** (true) or **F** (false).

40 Points

false: Every planar graph G has chromatic number $\chi(G) = 4$.

false: There exists integers $n \geq r-1 > 2$ and a graph G with $|G| = n$, $\|G\| > \frac{r-2}{2r-2}n^2$, and $\chi(G) < r$.

false: If G has chromatic number $\chi(G) = k$, then G has a vertex of degree at most k^2.

true: Every hamiltonian cubic graph is 3-edge colorable.

false: There exists a function $f : \mathbb{N} \to \mathbb{N}$ such that every graph G with girth k has connectivity $\kappa(G) \leq f(k)$.

true: There exists a k-regular graph with $k \geq 2$ which is not hamiltonian.

true: Every planar graph has an independent set of size at least $\lceil \frac{n}{4} \rceil$.

true: For all positive integers a and b, if a graph has $ab+1$ vertices and chromatic number at most a, then it has a vertex of degree at most $(a-1)b$.

false: Every k-connected graph G with $|G| \geq 3$ and $\chi(G) \geq |G|/k$ has a hamiltonian cycle.

true: If (A, B) is an ϵ-regular pair in a graph G, then (A, B) is an ϵ-regular pair in the complement $\overline{G}$.

true: For every integer $n > 1$ the set of pairs of an n-element set can be paritioned into $n+1$ parts such that the sets in each part are pairwise disjoint.

true: Every graph G with m edges satisfies $\chi(G) \leq \frac{1}{2} + \sqrt{2m + \frac{1}{4}}$

true: There exists a function $f : \mathbb{N} \to \mathbb{N}$ such that any graph G with independence number $\alpha(G) \leq k$ and clique number $\omega(G) \leq k$ implies that $|G| \leq f(k)$.

true: The Turán graphs $T_{r-1}(n)$ are hamiltonian for all $n \geq r-1 \geq 3$.

false: If a graph has a vertex of degree k, then its chromatic number is at most $k+1$.

true: The sequence $(1, 1, 1, 1, 1, 1, n-1, n-1, n-1, n-1)$ is not hamiltonian.

false: If a graph is hamiltonian, then its degree sequence is hamiltonian.

true: The list chromatic number (or choice number) of a graph can be arbitrarily larger than its chromatic number.

false: There exists a cubic graph with more than 1000 vertices and no independent set greater than 333.

true: For constant $p \in (0, 1)$, almost every graph in $\mathcal{G}(n, p)$ contains an induced cycle of length 10^6.

2 20 points If a graph G has chromatic number $\chi(G) = k > 1$, show that its vertex set can be partitioned into two non-empty parts V_1 and V_2 such that the induced subgraphs $G[V_1]$ and $G[V_2]$ satisfy

$$\chi(G[V_1]) + \chi(G[V_2]) = k$$

Solution. Since $\chi(G) = k$ there exists a partition of the vertex set $V = C_1 \cup C_2 \cup \cdots \cup C_k$ where each $C_i \neq \emptyset$. Let $V_1 = C_1$ and $V_2 = C_2 \cup \cdots \cup C_k$. Clearly $G[V_1]$ is a set of isolated vertices, therefore $\chi(G[V_1]) = 1$. The original coloring of G implies that $\chi(G[v_2]) \leq k - 1$. Moreover, if $\chi(G[V_2]) < k - 1$, then we could color G by $k - 1$ colors, since the vertices from V_1 can be colored by a single color distinct from the ones used to color $G[V_2]$. Therefore $\chi(G[V_2]) = k - 1$.

3 20 Points Show that for every constant $p \in (0, 1)$, almost no graph in $\mathcal{G}(n, p)$ has a separating complete subgraph.

Solution. Consider a graph G with property $\mathcal{P}_{2,1}$. (This was defined in class and in section 11.3 in the textbook.) We claim that a graph with property $\mathcal{P}_{2,1}$ has the following property: *For any pair of vertices u and v in G there exists a pair of vertices w_1 and w_2 such that w_1 is neighbor to u and v, w_2 is neighbor to u and v, w_1 and w_2 are not neighbors.* To see this consider vertices u and v and an arbitrary vertex x. By property $\mathcal{P}_{2,1}$ there exists a vertex w_1 which is neighbor to u and v, but not to x. Using property $\mathcal{P}_{2,1}$ again it follows that there exists a vertex w_2 which is neighbor to u and v, but not to w_1. Now it is easily seen that a graph G with property $\mathcal{P}_{2,1}$ has no complete separating subgraph: Consider a complete subgraph $H \subset G$ and two arbitrary vertices u and v in $G - V(H)$. By the property above, there are two non-adhacent vertices w_1 and w_2 in G which are both neighbors of u and v. Since H is complete it follows that w_1 and w_2 cannot both belong to H, therefore H does not separate G. The statement now follows since almost all graphs in $\mathcal{G}(n, p)$ have property $\mathcal{P}_{2,1}$ for any constant $p \in (0, 1)$.

4 20 Points Show that for every integer r there exists an integer $n = n(r)$ such that every connected graph on n vertices contains and induced subgraph H where H is either K_r, $K_{1,r}$, or a path on $r + 1$ vertices.

Solution. Let $m = R(r)$ the symmetric Ramsey number for graphs. Suppose that G has a vertex v of degree at least m. Then in the neighborhood $N(v)$ there exists an r-clique or r independent vertices. In the first case this implies that there is a K_{r+1} subgraph, and in the second case this implies there is an induced $K_{1,r}$ subgraph.

Otherwise, the maximum degree $\Delta(G) < m$. This implies that for any vertex v, the number of vertices in the neighborhood $N(v)$ cannot exceed m. Thus if G has sufficiently many vertices, then there must exist vertices at distance 2 from v. Again, by using the maximum degree, the number of vertices at distance ≤ 2 from v cannot exceed $m + m^2$. Continuing in this way we see that the number of vertices at distance $\leq r + 1$ from v cannot exceed $m + m^2 + \cdots + m^{r+1}$. Therefore, the more vertices that G has the larger the diameter of G must be (and since G is connected, the diameter is finite). So if G has sufficiently many vertices there must exists a pair of vertices whose distance is at least $r + 1$, and the shortest path connecting them is an induced path.