ON TRANSFORMATION GROUPS WHICH ACT ON TORUS MANIFOLDS

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ABSTRACT. We study the question of what $2n$-dimensional torus manifolds $(M,\ T)$ can have an extension $(M,\ G)$ with codimension zero (transitive) or codimension one principal orbits and classify such $(M,\ G)$, where $\mathrm{rank}\, G = \dim T = n$.

In this article we focus on the case M is the (quasi-)toric manifold, when $(M,\ G)$ has codimension one principal orbits (this paper is written as an article for the "Symposium on Transformation Groups (2006)" in Yokohama).

1. INTRODUCTION

The pair $(M,\ T)$ of the $2n$-dimensional, closed, connected, oriented, smooth manifold M and n-dimensional torus T, is said to be a *torus manifold* [HM03] if the following conditions are satisfies;

(1) M has an effective smooth action of an n-dimensional toral group T.
(2) the fixed point set M^T is the non-empty set (automatically M^T is a finite set).
(3) M is omnioriented.

Here M is called *omnioriented* if an orientation is specified for M and for every *characteristic submanifold* M_i (a T-invariant $(2n-2)$-manifold). Let G be a compact, connected, semisimple, Lie group which has T as a maximal tori (such G is called a *rank* n Lie group). In this paper we study the question of what $2n$-dimensional torus manifolds M can have an extended G-action with codimension zero (transitive) or codimension one principal orbits, and get the following classifications;

Theorem A . *The torus manifold $(M,\ T)$ extends to the transitive action $(M,\ G)$, where G is a compact, connected, semisimple, rank n, Lie group. Then M is diffeomorphic to a product of complex projective spaces $\mathbb{C}P(m)$ or even dimensional spheres S^{2m} and G is locally isomorphic to a product of special unitary groups $SU(m+1)$ or special orthogonal groups $SO(2m+1)$, that is,*

$$M \cong \prod_{i=1}^{a} \mathbb{C}P(l_i) \times \prod_{j=1}^{b} S^{2m_j}, \quad G \approx \prod_{i=1}^{a} SU(l_i+1) \times \prod_{j=1}^{b} SO(2m_j+1),$$

where $\sum_{i=1}^{a} l_i + \sum_{j=1}^{b} m_j = n = \dim T.$

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Theorem B . *The torus manifold* (M, T) *extends to* (M, G) *which has codimension one orbits and* $H^1(M; \mathbb{Z}_2) = 0$*. Then* M *is diffeomorphic to an* S^{2m} *or a* $\mathbb{C}P(m)$ *bundle over a product of complex projective spaces or even dimensional spheres, and* G *is locally isomorphic to a product of* $SU(l+1)$ *or* $SO(2l+1)$ *or* T^1 *, that is,*

$$M \cong G \times_H P(\mathbb{C}^{k_1} \oplus \mathbb{C}^{k_2}),$$
$$M \cong G \times_H S(\mathbb{C}^{k} \oplus \mathbb{R}) \text{ or}$$
$$M \cong G \times_H S(\mathbb{R}^{2k_2+1} \oplus \mathbb{C}^{k_1})$$

where $G/H \cong \prod_{i=1}^{a} \mathbb{C}P(l_i) \times \prod_{j=1}^{b} S^{2m_j}$*, and* H *acts on the fibre by the representation of* H *which is determined by the slice representation of the singular orbit. The above last case is the case one singular orbit is not a torus manifold.*

Automatically we have the follwoing corollary for the (quasi-)toric manifold.

Corollary B . *Under the hypothesis in Theorem B, assume* (M, T) *is a (quasi-)toric manifold. Then* (M, G) *is as follows;*

$$M \cong G \times_H P\left(\mathbb{C}^{k_1} \oplus \mathbb{C}^{k_2}\right), \; G \simeq \prod_{i=1}^{a} SU(l_i+1) \times (SU(k_1) \times SU(k_2)) \times T^1,$$

where the subgroup $H \simeq \prod_{i=1}^{a} S(U(1) \times U(l_i)) \times (SU(k_1) \times SU(k_2)) \times T^1$ *acts on* $P(\mathbb{C}^{k_1} \oplus \mathbb{C}^{k_2})$ *by the following representation;*

$$H \ni \left(\begin{pmatrix} t_1 & 0 \\ 0 & A_1 \end{pmatrix}, \cdots, \begin{pmatrix} t_a & 0 \\ 0 & A_a \end{pmatrix}, B_1, B_2, t \right) \mapsto (t_1^{r_1} \cdots t_a^{r_a} t^r B_1, B_2) \in U(k_1) \times SU(k_2).$$

Here G *acts on* M *canonically and* $G/H \cong \prod_{i=1}^{a} \mathbb{C}P(l_i)$, $\sum_{i=1}^{a} l_i + k_1 + k_2 - 1 = n = \dim T$, $(r_1, \cdots, r_a, r) \in \mathbb{Z}^a \oplus (\mathbb{Z} - 0)$.

The manifold in the above corollary is a generalized object for the *Hirzebruch surface* ($\mathbb{C}P(1)$ bundle over $\mathbb{C}P(1)$). In fact the case $k_1 = k_2 = a = 1 = l_1$ is the Hirzebruch surface.

Only Theorem A and Corollary B (as Theorem B′) will be shown in this paper.

In this paper the symbol $A \cong B$ means A is diffeomorphic to B, $A \approx B$ means A is locally isomorphic to B, i.e., Lie algebras of A and B are isomorphic, and $A \simeq B$ means A is isomorphic to B as a Lie group. Moreover we identify (M, G) and (M, G') if their induced effective actions are equivariant diffeomorphic. We call such equivalence relation an *essential isomorphism*.

The outline of proofs for Theorem A, B′ and the organization of this paper are as follows. In the next Section 2 we recall some basic notations and results from the classical Lie theorey [MT91] and the transformation group theory [Bre72]. Then we go on and in Section 3 give a proof for Theorem A. To prove Theorem A, we use the classical classification theorem of the simple Lie group and its maximal rank subgroup. In Section 4 then we prepare from the transformation group theory to prove Theorem B′, we know the local and global structure of (M, G) which has codimension one singular orbits and, especially we know there are two singular orbits in (M, G). In Section 5 we also prepare from the assumption that M is the torus manifold. In particular we know if M is a (quasi-)toric manifold, two singular orbits are (quasi-)toric manifold. In Section 6 we prove Theorem B′. To prove Theorem B′, we use the method of Uchida [Uch77]. The sketch of this method is the following. First, find two singular orbits (by using Theorem A). Second, compute slice representations of two singular isotropy groups. Then we get a closed tubular neighborhood of a singular orbit. Third, consider attaching maps between boundaries of two tubular neighborhoods. Then we know how many (M, G) exist. The uniqueness of each (M, G) follows form Lemma 4.7. Finally we construct such examples of (M, G) directly.

Uchida classified (M, G) such that M is a rational cohomology complex projective space, i.e., a rational cohomology ring $H^*(M; \mathbb{Q})$ is isomorphic to $H^*(\mathbb{C}P(m); \mathbb{Q})$. In his case the cohomology ring of M is fixed, but in our case the cohomology ring of M is not fixed (M is a family of torus manifolds). Hence this research says not only the research of the torus manifold (the toric topological aspect), but also what his method is effective for wider situation (the transformation group aspect). [1]

2. Preparation I (from the Lie theory)

We begin by noting some definitions and facts which will be of use in the codimension zero (transitive) case.

The symbol (W, H) means a manifold W with H-action. We call (W, T) is a *GKM manifold* if it satisfies the following conditions [GHZ06]:

1. the set of zero dimensional orbits (the fixed point set W^T) in the orbit space W/T is zero dimensional.
2. the set of one dimensional orbits in the orbit space W/T is one dimensional.

Under these hypotheses, the union of zero and one dimensional orbits has a structure of a graph. As is easily seen that the torus manifold (M, T) is a GKM manifold. So the torus manifold is a GKM manifold in this sence.

We say (M, T) *exteds to* (M, G) or (M, G) is an *extension* of (M, T), if the restricted action of $\varphi : G \times M \to M$ to T ($\varphi|_T : T \times M \to M$) coincides with the action (M, T).

The following theorem becomes a trigger to start this research.

Theorem 2.1 ([GHZ06]). *Suppose W is a H-homogeneous manifold. Then the following are equivalent;*

1. (W, T) *is a GKM manifold.*
2. *The Euiler characteristic of W is non-zero* ($\chi(W) \neq 0$).

[1] While the condition, that (M, G) is an extension of the torus manifold (M, T), seems to be a strong condition.

(3) *W is of the form* $W = H/K$, *where* K *is a closed connected subgroup of* H *containig* T ($T \subset K \subset H$).

This theorem says if the torus manifold $(M,\ T)$ extends to the transitive action $(M,\ G)$ and denote such M as G/H, i.e. $M \cong G/H$, then rank G = rank H^o holds (we can also have rank G = rank H^o directly, because of $\chi(G/H) = \chi(M) = \chi(M^T) \neq 0$ and the classical result in [MT91]), where H^o is a connected component of H. So we need to consider the maximal rank subgroup of G.

Before to consider the maximal rank subgroup, we recall the results in the Lie theory (see [MT91] Chapter V). In general, a compact, connected, Lie group has the following structure;

$$G \simeq (G_1 \times \cdots \times G_k)/N$$

where G_i is a simple, simply connected, compact Lie group or toral Lie group ($i = 1,\ \cdots, k$) and N is some finite central normal subgroup in $G_1 \times \cdots \times G_k$, that is, we can take a finite covering map

$$p : \widetilde{G} = G_1 \times \cdots \times G_k \to G.$$

In this paper we assume $(M,\ G)$ is identified by the essential isomorphism, so we may assume all our Lie groups as $G_1 \times \cdots \times G_k$ because the action $\varphi : G \times M \to M$ and $\widetilde{\varphi} : \widetilde{G} \times M \overset{p\times 1}{\to} G \times M \overset{\varphi}{\to} M$ are essential isomorphic.

Let G/H be a quotient space of a maximal rank subgroup H in G. Then $\widetilde{H} = p^{-1}(H) \subset \widetilde{G}$ is a maximal rank subgroup in $\widetilde{G}$. So we can put $\widetilde{H}^o = H_1 \times \cdots \times H_k$ where $H_i \subset G_i$ is a connected, maximal rank subgroup, because of the following lemma.

Lemma 2.2. *Let* G_i *be a compact, connected, Lie group and* G *be a product of such Lie group, i.e.,* $G = G_1 \times \cdots \times G_k$. *Assume* H *is a closed maximal rank subgroup in* G. *Then* $H = H_1 \times \cdots \times H_k$ *where* H_i *is a maximal rank subgroup in* G_i.

Proof. It is sufficient to prove the case $k = 2$.

Assume $G = G_1 \times G_2$. Let H be a maximal rank closed subgroup of G. Then we can put as follows;

$$G_1 \times G_2 \supset H \supset T_1 \times T_2$$

where T_i is a maximal tori in G_i ($i = 1,\ 2$).

Put $H_1 = \{g_1 \mid (g_1,\ g_2) \in H \subset G_1 \times G_2\}$ and $H_2 = \{g_2 \mid (g_1,\ g_2) \in H \subset G_1 \times G_2\}$. Since $T_1 \times T_2 \subset H$, we have $T_i \subset H_i \subset G_i$ for $i = 1,\ 2$. Now $T = T_1 \times T_2$ is a maximal tori in H, hence we have $\cup_{b\in H} bTb^{-1} = H$. Therefore, for all $a = (a_1,\ a_2) \in H$, there exists $b = (b_1,\ b_2) \in H$ such that $b^{-1}(a_1,\ a_2)b \in T = T_1 \times T_2$. Take $a_2 = e \in H_2$ (the identity element in H_2), then $t = (b_1^{-1}a_1b_1,\ e) \in T_1 \times 1$. So we have $btb^{-1} = (a_1,\ e) \in H$. This relation says $H_1 \times 1 \subset H$, and similary we also have $1 \times H_2 \subset H$. Hence $H_1 \times H_2 \subset H$. Consequently $H = H_1 \times H_2$ ($H_1 \times H_2 \supset H$ can be easily shown). ∎

If $(G/H,\ T)$ is a torus manifold, then this action is effective. So $G_i \neq H_i$ for each factor of $G = G_1 \times \cdots \times G_k$ and $H = H_1 \times \cdots \times H_k$. In particular G_i is not a toral group, but a simple Lie group.

In the next section we will mention about codimension zero (transitive) case.

3. Codimension zero case (transitive case)

Let $(M,\ T)$ be a torus manifold and G be a compact, connected, semisimple, Lie group which has T as a maximal tori, i.e., rank $G = \dim T = n$. Assume $(M,\ T)$ extends to the transitive action $(M,\ G)$. Then the following theorem holds.

Theorem A . *The torus manifold* $(M,\ T)$ *extends to the transitive action* $(M,\ G)$*, where* G *is a compact, connected, semisimple, rank* n*, Lie group. Then* M *is diffeomorphic to a product of complex projective spaces* $\mathbb{C}P(m)$ *or even dimensional spheres* S^{2m} *and* G *is locally isomorphic to a product of* $SU(m+1)$ *or* $SO(2m+1)$ *, that is,*

$$M \cong \prod_{i=1}^{a} \mathbb{C}P(l_i) \times \prod_{j=1}^{b} S^{2m_j}, \quad G \approx \prod_{i=1}^{a} SU(l_i+1) \times \prod_{j=1}^{b} SO(2m_j+1),$$

where $\sum_{i=1}^{a} l_i + \sum_{j=1}^{b} m_j = n = \dim T.$

In this section this theorem is proved.

We would like to classify $(M,\ G)$ up to the essential isomorphism, so we can put $G = G_1 \times \cdots \times G_k$ from the argument Section 2. Moreover we can put

$$M \cong G/H = G_1/H_1 \times \cdots \times G_k/H_k = M_1 \times \cdots \times M_k.$$

Here G_i is a simply connected, simple, Lie group such that $\sum_{i=1}^{k} \text{rank } G_i = n = \dim T$ and H_i is a maximal rank subgroup of G_i for all $i = 1, \cdots, k$.

The maximal tori T in H^o devides into

$$T = T_1 \times \cdots \times T_k \subset H_1^o \times \cdots \times H_k^o = H^o \subset H = H_1 \times \cdots \times H_k \subset G_1 \times \cdots \times G_k = G$$

where $T_i \subset H_i$ is a maximal tori in the simple Lie group G_i $(i = 1, \cdots, k)$. Then the following lemma holds.

Lemma 3.1. *Each pair* $(M_i,\ T_i) = (G_i/H_i,\ T_i)$ $(i = 1, \cdots, k)$ *satisfies* $2\dim T_i = \dim M_i$.

Proof. Since $(M,\ T) = (G/H,\ T)$ is a torus manifold, we have the following equations;

$$2\dim T = 2n = \dim(G/H) = \sum_{i=1}^{k} \dim(G_i/H_i) = 2\sum_{i=1}^{k} \dim T_i. \tag{3.1}$$

The tangential representation on $p = (p_1, \cdots, p_k) \in M^T = M_1^{T_1} \times \cdots \times M_k^{T_k}$ is devided into

$$T_p(M) = T_{p_1}(M_1) \oplus \cdots \oplus T_{p_k}(M_k).$$

Moreover each factor $T_{p_i}(M_i)$ is devided into

$$T_{p_i}(M_i) = V(\alpha_1) \oplus \cdots \oplus V(\alpha_l)$$

by the tangential representation of T_i. Here the weight representations $\alpha_1, \cdots, \alpha_l \in \mathfrak{t}_i^*$ are linearly independent because (M, T) is a torus manifold.

If there exist a factor such that $2 \dim T_j > \dim G_j/H_j$, then there is a factor such that $2 \dim T_i < \dim G_i/H_i$ by the equation (3.1). However this contradicts to the tangential representation $\alpha_1, \cdots, \alpha_l \in \mathfrak{t}_i^*$ (where $2l = \dim G_i/H_i > \dim \mathfrak{t}_i^*$) are linearly independent. Hence $\dim G_i/H_i = 2 \dim T_i$ for each $i = 1, \cdots, k$. $\square$

From this lemma, to prove Theorem A, it is sufficient to consider each factor G_i/H_i constructed by a simple Lie group G_i and its maximal rank subgroup H_i such that $\dim G_i/H_i = \dim G_i - \dim H_i = 2 \text{ rank } G_i$.

Put S be a simple, compact Lie group and S' be a closed, maximal rank, maximal subgroup of S. Then the following beautiful classification for such (S, S') are known by the classical Lie theory ([MT91] Chapter V).

S	A_l	B_l			C_l	
S'	$A_{i-1} \times A_{l-i} \times T^1$	$B_{l-1} \times T^1$	$B_{i-1} \times D_{l-i+1}$	D_l	$C_{i-1} \times C_{l-i+1}$	$A_{l-1} \times T^1$

S	D_l $(l \geq 3)$			E_6		
S'	$D_{l-1} \times T^1$	$D_{i-1} \times D_{l-i+1}$	$A_{l-1} \times T^1$	$D_5 \times T^1$	$A_1 \times A_5$	$A_2 \times A_2 \times A_2$

S	E_7				E_8				
S'	$D_6 \times A_1$	A_7	$A_2 \times A_5$	$E_6 \times T^1$	D_8	A_8	$A_4 \times A_4$	$E_6 \times A_2$	$E_7 \times A_1$

S	F_4			G_2	
S'	$C_3 \times A_1$	$A_2 \times A_2$	B_4	A_2	$A_1 \times A_1$

Here $A_l \approx SU(l+1)$, $B_l \approx SO(2l+1)$, $C_l \approx Sp(l)$, $D_l \approx SO(2l)$ are classical Lie groups and E_6, E_7, E_8, F_4, G_2 are exceptional Lie groups. Remark each dimension of S is as follows;

$$\dim A_l = l^2 + 2l, \quad \dim B_l = \dim C_l = (2l+1)l, \quad \dim D_l = l(2l-1)$$
$$\dim E_6 = 78,\ \dim E_7 = 133,\ \dim E_8 = 248,\ \dim F_4 = 52,\ \dim G_2 = 14.$$

Therefore we have $\dim S/S' = \dim S - \dim S'$ as follows;

$\dim A_l/(A_{i-1} \times A_{l-i} \times T^1) = -2i^2 + 2li + 2i \ (1 \leq i < l)$,
$\dim B_l/(B_{l-1} \times T^1) = 4l - 2$,
$\dim B_l/(B_{i-1} \times D_{l-i+1}) = -2l - 4i^2 + 6i - 2 + 4li \ (1 < i < l)$, $\quad \dim B_l/D_l = 2l$,
$\dim C_l/(C_{i-1} \times C_{l-i+1}) = -4i^2 + 8i - 4 + 4li - 4l \ (1 \leq i < l)$,
$\dim C_l/(A_{l-1} \times T^1) = l^2 + l$, $\quad \dim D_l/(D_{l-1} \times T^1) = 4l - 4$,
$\dim D_l/(D_{i-1} \times D_{l-i+1}) = 8i - 4i^2 + 4li - 4 - 4l$, $\quad \dim D_l/(A_{l-1} \times T^1) = l^2 - l$,
$\dim E_6/(D_5 \times T^1) = 32$, $\quad \dim E_6/(A_1 \times A_5) = 40$, $\quad \dim E_6/(A_2 \times A_2 \times A_2) = 54$,
$\dim E_7/(D_6 \times A_1) = 64$, $\quad \dim E_7/A_7 = 70$, $\quad \dim E_7/(A_2 \times A_5) = 90$,
$\dim E_7/(E_6 \times T^1) = 54$, $\quad \dim E_8/D_8 = 158$, $\quad \dim E_8/A_8 = 168$,
$\dim E_8/(A_4 \times A_4) = 200$, $\quad \dim E_8/(E_6 \times A_2) = 162$, $\quad \dim E_8/(E_7 \times A_1) = 112$,
$\dim F_4/(C_3 \times A_1) = 28$, $\quad \dim F_4/(A_2 \times A_2) = 36$, $\quad \dim F_4/B_4 = 16$,
$\dim G_2/A_2 = 6$, $\quad \dim G_2/(A_1 \times A_1) = 8$.

Checking whether the above dimension is $2l (= 2 \text{ rank } S)$ or not, we get the followings are equal to $2l$;

$A_l/(A_{l-1} \times T^1) \cong SU(l+1)/S(U(l) \times U(1)) \cong \mathbb{C}P(l)$
$B_l/D_l \cong SO(2l+1)/SO(2l) \cong S^{2l}$
$B_1/T^1 \cong SO(3)/SO(2) \cong S^2 \cong Sp(1)/T^1 \cong C_1/T^1$
$D_3/(A_2 \times T^1) \cong Spin(6)/U(3) \cong SU(4)/S(U(3) \times U(1)) \cong A_4/(A_3 \times T^1) \cong \mathbb{C}P^3$

If S" is not a maximal subgroup but maximal rank subgroup, then S" is a subgroup of one of the maximal subgroups S' in the above list of (S, S'). So we have $\dim S/S" > \dim S/S'$. Moreover from the list of $\dim S/S'$ we have $\dim S/S' \geq 2 \text{ rank } S$, hence such S" does not occur. Therefore we get the following lemma.

Lemma 3.2. *Let* $(G_i/H_i, T)$ *be a pair such that* $G_i \supset H_i \supset T$ *and* $\dim G_i/H_i = 2\dim T = 2l$. *Assume* G_i *is a simple, connected, compact Lie group. Then we have* $G_i/H_i \cong S^{2l}$, $\mathbb{R}P(2l)$ *or* $\mathbb{C}P(l)$.

Proof. We already have $G_i/H_i^o \cong S^{2l}$ or $\mathbb{C}P(l)$ by the above argument. It is easy to show $H_i^o \subset H_i \subset N(H_i^o; G)$, where $N(H_i^o; G_i)$ is a normalizer subgroup of H_i^o in G_i.

If $(G_i, H_i^o) \approx (A_l, A_{l-1} \times T^1)$, then $H_i^o = N(H_i^o; G_i)$ holds. Hence if $G_i/H_i^o \cong \mathbb{C}P(l)$ then we have $H_i = H_i^o$ $(l \neq 1)$. Especially we have $G_i/H_i \cong \mathbb{C}P(l)$.

If $(G_i, H_i^o) \approx (B_l, D_l)$, then $H_i^o \times \mathbb{Z}_2 \simeq N(H_i^o; G)$ holds. Hence if $G_i/H_i^o \cong S^{2l}$ then we have $H_i = H_i^o$ or $H_i = H_i^o \times \mathbb{Z}_2$. If $H_i = H_i^o$, then we have $G_i/H_i \cong S^{2l}$, and if $H_i = H_i^o \times \mathbb{Z}_2$, then we have $G_i/H_i \cong \mathbb{R}P(2l)$. □

Remark the torus manifold is an orientable manifold but $\mathbb{R}P(2l)$ is a non-orientable manifold. Hence the torus manifold (M, T) which extends to the transitive action (M, G)

is a product of S^{2l} or $\mathbb{C}P(l)$ by Lemma 3.1 and 3.2. Consequently the codimension zero case (transitive case) of our problem was solved, that is, Theorem A was proved.

4. Preparation II (from the transformation group theory)

In Section 3 we proved Theorem A (codimension zero case), so we are naturally led to study actions with codimension one orbits. We used the Lie theory for the codimension zero case, but the codimension one case belongs to the transformation group theory. In this section we recall some results from the transformation group theory which will be of use in the codimension one case.

Let (M, G) be a smooth transformation group which has codimension one orbits, and M be a compact manifold whose first $\mathbb{Z}_2$ cohomology is 0, i.e., $H^1(M; \mathbb{Z}_2) = 0$. Then the following structure theorem holds.

Theorem 4.1 (Uchida[Uch77] Lemma 1.2.1)**.** *Let G be a compact connected Lie group and M a compact connected manifold without boundary. Assume* $H^1(M; \mathbb{Z}_2) = 0$*, and G acts smoothly on M with codimension one orbits* $G(x)$*. Then* $G(x) \cong G/K$ *is a principal orbit and* (G, M) *has just two singular orbits* $G(x_1) \cong G/K_1$ *and* $G(x_2) \cong G/K_2$*. Moreover there exists a closed invariant tubular neighborhood* X_s *of* $G(x_s)$ *such that*

$$M = X_1 \cup X_2 \quad \text{and} \quad X_1 \cap X_2 = \partial X_1 = \partial X_2.$$

In our case $H^1(M; \mathbb{Z}_2) = 0$, hence this structure theorem holds for (M, G). The tubular neighborhoods of two singular orbits are understood from the following theorem (for example it is in [Bre72]).

Theorem 4.2 (differentiable slice theorem)**.** *Let* G *be a compact Lie group and* M *be a smooth* G*-manifold. Then for all* $x_i \in M$ *there is a closed invariant tubular neighborhood* X_i *of the orbit* $G(x_i) \cong G/K_i$ *and* $X_i \cong G \times_{K_i} D_{x_i}$ *as a* G*-manifold, where* D_{x_i} *is a closed disk which has an orthogonal* K_i*-action via the representation* $\sigma_i : K_i \to O(D_{x_i})$*.*

We call the representation σ_i in this theorem a *slice representation* of K_i at $x_i \in M$. Put the codimension of two singular orbits by m_i $(i = 1, 2)$. Then two tubular neighborhoods of G/K_1 and G/K_2 are denoted by $X_1 \cong G \times_{K_1} D^{m_1}$ and $X_2 \cong G \times_{K_2} D^{m_2}$ such that K_i acts on D^{m_i} through the slice representation $\sigma_i : K_i \to O(m_i)$ for $i = 1, 2$.

Moreover we have $K_i/K \cong S^{m_i-1} \subset D^{m_i}$ for $i = 1, 2$, that is, K_i acts on $S^{m_i-1} \subset D^{m_i}$ transitively because (M, G) has a codimension one principal orbit G/K. The transitive actions on sphere have been already studied by Montgomery and Samelson, Borel and Poncet. The following results were proved in a series of papers [MS43], [Bor50] and [Pon59].

Theorem 4.3. *Let* G *be a compact connected Lie group acting effectively and transitively on a homotopy sphere* Σ^m *and* H *be the isotropy subgroup, namely,* $G/H \cong \Sigma^m$*. Then there always exists a simple normal subgroup* G_1 *of* G *that is already transitive on* Σ^m*, i.e.,* $G_1/(G_1 \cap H) = \Sigma^m$*.*

Theorem 4.4. *Let* G_1 *be a one of the simple groups in* G *such that* $G_1/H_1 \cong \Sigma^m$*. Then, as in* [Bor50], [MS43], *we have that*

(1) *If* m *is even,* $G_1 = SO(m+1)$ *or the exceptional Lie group* G_2 *in case* $m = 6$.
(2) *If* $m = 2l-1$ *and* l *odd,* $G_1 = SO(m+1)$ *or* $SU(l)$.
(3) *If* $m = 2l-1$ *and* l *even,* $G_1 = SO(m+1)$, $SU(l)$, $Sp(l/2)$, $Spin(9)$ *(if* $m = 15$, $l = 8$*), or* $Spin(7)$ *(if* $m = 7$, $l = 4$*).*

As we may check easily in each of the above cases, we have a unique embedding of G_1 in $SO(m+1)$ such that $H_1 = G_1 \cap SO(m)$. Hence Σ^m is diffeomorphic to the standerd sphere S^m.

Theorem 4.5. *Let* $G_1 (\subset G)$ *be the simple subgroup in Theorem 4.4. Then* $G_1 \subset G \subset N(G_1)^o \subset SO(m+1)$*, where* $N(G_1)^o$ *is the identity componenet of the normalizer of* G_1 *in* $SO(m+1)$*, and the following holds;*

(1) *In case* $G_1 = SO(m+1)$, G_2 $(m = 6)$, $Spin(7)$ $(m = 7,\ l = 4)$*, or* $Spin(9)$ $(m = 15,\ l = 8)$*, we have that* $N(G_1)^o = G_1$*, hence* $G_1 = G$.
(2) *In case* $G_1 = SU(l)$*, we see that* $N(G_1)^o = U(l)$*; hence we may have either* $G = G_1$ *or* $U(l)$.
(3) *In case* $G_1 = Sp(\frac{l}{2})$, $N(G_1)^o$ *is the subgroup of* $SO(m+1)$ *generated by* $Sp(\frac{l}{2})$*, and the* S^3*-subgroup realized as right multiplications of unit quaternions. As a group,* $N(Sp(\frac{l}{2}))^o$ *is isomorphic to* $Sp(\frac{l}{2}) \times_{\mathbb{Z}_2} S^3$*, where* $\mathbb{Z}_2$ *is the subgroup generated by* $(-\mathrm{Id},\ -1)$*. Hence* G *is either* $Sp(\frac{l}{2})$ *or* $Sp(\frac{l}{2}) \times_{\mathbb{Z}_2} S^1$ *or* $Sp(\frac{l}{2}) \times_{\mathbb{Z}_2} S^3$.

The above reuslts are also refered in a paper [HH65].
In particular the following lemma comes from Theorem 4.4 and 4.5 directly.

Lemma 4.6. *Assume the connected subgroup* H *in* $O(2l)$ *acts on* S^{2l-1} *transitively and its rank is* l*, i.e.,* rank $H = l$*. Then* $H \simeq U(l)$ *or* $SO(2l)$ *in* $O(2l)$.

The above results (from Theorem 4.2 to Lemma 4.6) were about the local structure of singular orbits G/K_1 and G/K_2. On the other hand, Theorem 4.1 and the following lemma are about the global structure of $(M,\ G)$.

Lemma 4.7 ([Uch77] Lemma 5.3.1)**.** *Let* $f, f' : \partial X_1 \to \partial X_2$ *be G-equivariant diffeomorphisms, where* ∂X_i *means a boundary of* X_i*. Then* $M(f)$ *is equivariantly diffeomorphic to* $M(f')$ *as G-manifolds, if one of the following conditions are satisfied (where* $M(f) = X_1 \cup_f X_2$*);*

(1) f *is G-diffeotopic to* f'.
(2) $f^{-1}f'$ *is extendable to a G-equivariant diffeomorphism on* X_1.
(3) $f'f^{-1}$ *is extendable to a G-equivariant diffeomorphism on* X_2.

The story of the classification is first we pick up all tubular neighborhoods X_i which have possible to occur. Then we consider the attaching map f from ∂X_1 to ∂X_2. Because the boundary of tubular neighborhoods are a principal orbit G/K, we can take an attaching map f from $N(K;\ G)/K$. Using this attaching map f, we construct a manifold $M(f) = X_1 \cup_f X_2$ with G-action. Finally Lemma 4.7 is a criterion whether $M(f)$ and $M(f')$ are same or not. From the next section we will start to classify the action which has codimension one orbits along the above story.

5. Prelude to classify the codimension one case

Let $(M,\ G)$ be an extension of the torus manifold $(M,\ T)$ such that rank $G = \dim T$. Assume $H^1(M,\ \mathbb{Z}_2) = 0$ and M has a codimension one orbits. Then $(M,\ G)$ has just two singular orbits by Theorem 4.1. Put these two orbits G/K_1 and G/K_2. Then they have the following structure by Theorem 4.1;

$$M \cong X_1 \cup X_2, \quad X_1 \cap X_2 = G/K$$

where X_i is a tubular neighborhood of G/K_i for $i = 1,\ 2$ and G/K is a principal orbit. In this section we show some fundamental results for such $(M,\ G)$. In particular we will know the structure of the singular orbit G/K_1 in Lemma 5.4 and 5.5.

5.1. Singular orbit.

First we study about the singular orbits. The following lemma holds.

Lemma 5.1. *The orbit of the fixed point (in M^T) is the singular orbits, that is,* $G(p) = G/K_1$ *or* G/K_2 *for all* $p \in M^T$.

Proof. Let p be a fixed point ($p \in M^T$). Consider the G-isotropy group on p and denote it G_p. Then we have $T \subset G_p$ because p is fixed by $T \subset G$. Hence rank G_p coincides with rank G. So G/G_p has an even dimension. Therefore we have the G-orbit on p is a singular orbit. □

From this lemma we expect G/K_1 (or G/K_2) is a torus manifold. We shall show it is true. Before to show it, the following proposition holds in general.

Proposition 5.2. *Let M' be a T-invariant submanifold in the torus manifold $(M,\ T)$. If M' has a fixed point $p \in M^T$, then $(M',\ T/T")$ is a torus manifold for some subtorus $T" \subset T$.*

Proof. Because M' is a T-invariant submanifold, the tangent space $T_p(M)$ on $p \in M' \cap M^T$ divides into;

$$T_p(M) \simeq T_p(M') \oplus N_p(M'),$$

where $N_p(M')$ is a normal space of M' on p. Since $T_p(M')$ has a T-action through the tangential representation, we have the irreducible decomposition

$$T_p(M') \simeq V(\alpha_1) \oplus \cdots \oplus V(\alpha_k)$$

where $V(\alpha_i) \simeq \mathbb{C}$, $\alpha_i : T \to S^1$ $(i = 1, \cdots, k)$ and $2k = \dim M'$. Put $T" = \cap_{i=1}^k \mathrm{Ker}\ \alpha_i \subset T$. Then $T"$ acts on $T_p(M')$ trivially. Hence $T"$ acts $N_p(M')$ by the restricted T-action on $T_p(M)$ and we also have the irreducible decomposition;

$$N_p(M') \simeq V(\alpha_{k+1}) \oplus \cdots \oplus V(\alpha_n),$$

where $\dim T = n$. So $\dim T" = n - k$. Therefore k-dimensional torus $T/T"$ acts on $2k$-dimensional manifold M'. This is a torus manifold, because M is a torus manifold. □

Consequently, form Lemma 5.1 and Proposition 5.2, the following corollary holds.

Corollary 5.3. *At least one of the singular orbits of $(M,\ G)$ is a torus manifold.*

We can assume $(G/K_1,\ T/T")$ is a torus manifold by some $T" \subset T$.

5.2. **Structure of G and K_1.** Next we study the structure of G and K_1 by making use of Theorem A, because G/K_1 is a torus manifold and it extends to a transitive action.

From Lemma 2.2, we see the decomposition $G = G_1' \times G"_1$ and $K_1 = K_1' \times G"_1$ for G and K_1, where $G"_1$ is the normal subgroup of G and G_1' acts on G/K_1 almost effectively and transitively. Moreover the following decomposition holds for the maximal tori;

$$G_1' \times G"_1 \supset K_1' \times G"_1 \supset T' \times T" \simeq T,$$

where T' is a maximal tori of G_1'.

Put $k_1 = \text{rank } G"_1 = \dim T"$ and $p \in M^T$. Then we have $\dim N_p(G/K_1) = 2k_1$ from the proof of Proposition 5.2. So the equations

$$\dim G/K_1 = 2n - 2k_1 = 2n - 2\text{rank } G"_1$$

holds.

Since the torus manifold $(G_1'/K_1',\ T') = (G/K_1,\ T)$ extends to the transitive action $(G_1'/K_1',\ G_1')$, we have from Theorem A;

$$G_1'/K_1' \cong \prod_{i=1}^{a} \mathbb{C}P(l_i) \times \prod_{j=1}^{b} S^{2m_j},$$

$$(G_1',\ K_1') \approx \left(\prod_{i=1}^{a} SU(l_i+1) \times \prod_{j=1}^{b} SO(2m_j+1),\ \prod_{i=1}^{a} S(U(1) \times U(l_i)) \times \prod_{j=1}^{b} SO(2m_j) \right),$$

where $\sum_{i=1}^{a} l_i + \sum_{j=1}^{b} m_j = n - k_1 = n - \text{rank } G"_1$.

5.3. **Slice representation of K_1.** Denote the slice representation of $G_p = K_1$ on $p \in M^T$ by

$$\sigma_1 : K_1 = K_1' \times G"_1 \to O(2k_1).$$

Remark the image of σ_1 is in $SO(2k_1)$ because of the connectedness of K_1 (see Section 5.2).

Since $T" \subset G"_1$ acts effectively on the normal space of $p \in G/K_1$ in M (because M is a torus manifold), we have $\sigma_1(T") = T^{k_1} \subset SO(2k_1)$ where $\dim T" = k_1$. In particular the restricted representation $\sigma_1|_{G"_1}$ to $G"_1$ is non-trivial. Moreover we see $\sigma_1(K_1') \subset \sigma_1(G"_1)$ because $\sigma_1(K_1')$ is in the centralizer of $\sigma_1(T") = T^{k_1} \subset SO(2k_1)$ (the centralizer is isomorphic to T^{k_1}). Therefore $G"_1$ acts transitively on S^{2k_1-1} through the slice representation $\sigma_1 : K_1' \times G"_1 \to SO(2k_1)$. Because rank $\sigma_1(G"_1) = k_1$, we have from Lemma 4.6;

$$\sigma_1(G"_1) \simeq G"_1 \simeq U(k_1),\ \text{or } SO(2k_1)$$

Because $\sigma_1(K_1')$ is also in the centralizer of $\sigma_1(G"_1)$, we see

$$\sigma_1(K_1') \simeq S^1 \ (\text{if } \sigma_1(G"_1) \simeq U(k_1)) \text{ or}$$
$$\sigma_1(K_1') \simeq \{e\} \ (\text{if } \sigma_1(G"_1) \simeq SO(2k_1)).$$

Let us gather the above arguments, then we have the following two lemmas.

Lemma 5.4. *Let* G/K_1 *be a singular orbit in* $(M,\ G)$ *which is an extension of a torus manifold* $(M,\ T)$. *Then we can put* $(G/K_1,\ T/T")$ *is a torus manifold and the followings are satisfied;*

$$(G,\ K_1) = (G_1' \times G"_1,\ K_1' \times G"_1),$$
$$(G_1',\ K_1') \approx \left(\prod_{i=1}^{a} SU(l_i+1) \times \prod_{j=1}^{b} SO(2m_j+1),\ \prod_{i=1}^{a} S(U(1)\times U(l_i)) \times \prod_{j=1}^{b} SO(2m_j)\right),$$
$$G"_1 \approx SU(k_1) \times T^1 \text{ or } SO(2k_1),$$

where $T"$ *is a maximal tori* $(\dim T" = k_1)$ *in* $G"_1$, *and* $T/T" = T'$ *is a maximal tori* $(\dim T' = \sum_{i=1}^{a} l_i + \sum_{j=1}^{b} m_j)$ *in* G_1'

Lemma 5.5. *Under the hypothesis in Lemma 5.4, we have the followings for the slice representation* $\sigma_1 : K_1 \simeq K_1' \times G"_1 \to SO(2k_1)$ *of* K_1*;*

1. *if* $G"_1 \approx SO(2k_1)$, *then* $\sigma_1(G"_1) = SO(2k_1)$ *and* $\sigma_1(K_1') = \{e\}$,
2. *if* $G"_1 \approx SU(k_1) \times T^1$, *then* $\sigma_1(G"_1) = U(k_1) \subset SO(2k_1)$ *and* $\sigma_1(K_1') = S^1$ *which is the diagonal subgroup in* $U(k_1)$.

Above these lemmas is corresponding to the structue of G/K_1 and of its closed invariant tubular neighborhood X_1.

In the next section we will begin to classify the codimension one case.

6. Codimension one case (for quasi-toric manifold)

In this paper we only classify the (quasi-)toric manifold[2]. First of all we define the (quasi-)toric manifold. We call M is a *(quasi-)toric manifold* if M is a toric manifold or a quasi-toric manifold as follows;

Definition[toric manifold] A *toric manifold* is a smooth algebraic variety M containing the algebraic torus $(\mathbb{C}^*)^n$ as a Zariski open dence subset such that the natural $(\mathbb{C}^*)^n$-action on itself extends to an action on M

Definition[quasi-toric manifold] Given a combinatorial simple polytope P^n, a T^n-manifold M^{2n} is called a *quasi-toric manifold over* P^n if the following two conditions are satisfied;

1. the T^n-action is *locally standard* (equivariant to the standerd $(\mathbb{C}^n,\ T^n)$),
2. there is a projective map $\pi : M^{2n} \to P^n$ constant on T^n-orbits which maps every k-dimensional orbit to a point in the interior of a codimension-k face of P^n, $k = 1, \cdots, n$.

Remark a (quasi-)toric manifold is a torus manifold, especially $\mathbb{C}P(m)$ is a (quasi-)toric, but S^{2m} $(m \geq 2)$ is not a (quasi-)toric.

Let $(M,\ T)$ be a (quasi-)toric manifold, and G be a compact, connected, Lie group. Assume $(M,\ T)$ extends to $(M,\ G)$ which has codimension one orbits. The aim of this section is to prove the following theorem (Corollary B).

[2]For the general torus manifold case, I will write soon as a generalized version of this paper.

Theorem B′ . $(M,\ G)$ *is an extension of a (quasi-)toric manifold* $(M,\ T)$ *where* $\operatorname{rank} G = \dim T$. *Then* $(M,\ G)$ *is as follows;*

$$M \cong G \times_H P\left(\mathbb{C}^{k_1} \oplus \mathbb{C}^{k_2}\right),\ G \simeq \prod_{i=1}^{h} SU(l_i+1) \times (SU(k_1) \times SU(k_2)) \times T^1,$$

where the subgroup $H \simeq (\prod_{i=1}^{h} S(U(1)\times U(l_i)))\times(SU(k_1)\times SU(k_2))\times T^1$ *acts on* $P(\mathbb{C}^{k_1}\oplus\mathbb{C}^{k_2})$ *by the following representation;*

$$H \ni \left(\begin{pmatrix} t_1 & 0 \\ 0 & A_1 \end{pmatrix}, \cdots, \begin{pmatrix} t_h & 0 \\ 0 & A_h \end{pmatrix}, B_1,\ B_2,\ t\right) \mapsto (t_1^{\alpha_1}\cdots t_h^{\alpha_h} t^{\alpha} B_1,\ B_2) \in U(k_1) \times SU(k_2).$$

Here G *acts on* M *canonically and* $G/H \cong \prod_{i=1}^{h} \mathbb{C}P(l_i)$, $\sum_{i=1}^{h} l_i + k_1 + k_2 - 1 = n = \dim T$, $(\alpha_1,\ \cdots,\ \alpha_h,\ \alpha) \in \mathbb{Z}^h \oplus (\mathbb{Z} - 0)$.

First the following lemma is proved.

Lemma 6.1. *Two singular orbits* G/K_1 *and* G/K_2 *in* $(M,\ G)$ *are quasi-toric manifolds.*

Proof. Since M is a (quasi-)toric manifold, $M_1 = G/K_1$ is also a (quasi-)toric manifold by Corollary 5.3. So we see M/T is a polytope. Then M_1/T is a face in the polytope M/T. Therefore there exists $p \in M^T$ which is not in M_1. Hence its orbit $G(p)$ is the other singular orbit G/K_2 by Lemma 5.1. Because M is a (quasi-)toric manifold, G/K_2 is also a (quasi-)toric manifold by Proposition 5.2. □

Therefore for the other singular orbit G/K_2 in $(M,\ G)$ we can apply the same argument in Lemma 5.4 and 5.5. Hence we can put

$$(G,\ K_1) =$$

$$\left(\prod_{i=1}^{h} SU(l_i+1) \times Spin(2k_1),\ \prod_{i=1}^{h} S(U(1)\times U(l_i)) \times Spin(2k_1)\right)\ (k_1 \neq 2),\ \text{or} \tag{6.1}$$

$$\left(\prod_{i=1}^{h} SU(l_i+1) \times SU(k_1) \times T^1,\ \prod_{i=1}^{h} S(U(1)\times U(l_i)) \times SU(k_1) \times T^1\right) \tag{6.2}$$

and

$$(G,\ K_2) =$$

$$\left(\prod_{j=1}^{k} SU(m_j+1) \times Spin(2k_2),\ \prod_{j=1}^{k} S(U(1)\times U(m_j)) \times Spin(2k_2)\right)\ (k_2 \neq 2),\ \text{or} \tag{6.3}$$

$$\left(\prod_{j=1}^{k} SU(m_j+1) \times SU(k_2) \times T^1,\ \prod_{j=1}^{k} S(U(1)\times U(m_j)) \times SU(k_2) \times T^1\right) \tag{6.4}$$

because S^{2m} $(m \geq 4)$ is not a (quasi-)toric.

Therefore we only consider the following three cases.

6.1. **When (6.1) and (6.3) occur.** First we assume $k_1 \geq 4$. Then we easily see $k_1 = k_2$, $h = k$ and $l_i = m_j$ for $(i = j)$, by comparing each simple group factor in (G, K_1) and (G, K_2). Moreover the slice representations σ_1 and σ_2 are unique from Lemma 5.5. Hence we have unique tubular neighborhoods $X_i \cong G \times_{K_i} D^{2k_i}$ of G/K_i for $i = 1, 2$. Let us consider the attaching map $f : \partial X_1 \to \partial X_2$. Since f can be considered as an equivariantly map from $G/K \to G/K$, so the map f is in $N(K; G)/K$. Now $K = \sigma_1^{-1}(SO(2k_1 - 1)) = \sigma_2^{-1}(SO(2k_2 - 1))$. So we have $K = \prod_{i=1}^{h} SU(l_i + 1) \times Spin(2k_1 - 1)$. From Lemma 4.7 (1), if two attaching maps f and f' are in $N(K; G)^o$ then the manifolds $M(f) = X_1 \cup_f X_2$ and $M(f') = X_1 \cup_{f'} X_2$ are equivariantly diffeomorphic. So we may take $f \in N(K; G)/N(K; G)^o$. As is easily seen that $N(K; G)/N(K; G)^o \simeq S(O(1) \times O(2k_1 - 1))/SO(2k_1 - 1) \simeq \mathbb{Z}_2$. Put $f \in (S(O(1) \times O(2k_1 - 1))/SO(2k_1 - 1)) - \{e\}$, then we can put

$$f = \begin{pmatrix} -1 & 0 \\ 0 & -I_{2k_1 - 1} \end{pmatrix}.$$

This f is in the center of K in K_1. Hence the following diagram is well-defined and commute;

$$\begin{array}{ccccc} \partial X_1 & = & G \times_{K_1} K_1/K & \xrightarrow{\pi} & G/K \\ & & 1 \times F \downarrow & & \downarrow R(f) \\ \partial X_1 & = & G \times_{K_1} K_1/K & \xrightarrow{\pi} & G/K \end{array}$$

where $\pi([g, k_1K]) = gk_1K$, $R(f)(kK) = kfK$ and $(1 \times F)([g, k_1K]) = [g, k_1fK]$. Here $F : K_1/K \cong S^{2k_1-1} \to S^{2k_1-1} \cong K_1/K$ can take from the special orthogonal group $SO(2k_1)$. So F is extendable to the diffeomorphism from D^{2k_1} to itself. Hence from Lemma 4.7 (2), we see $M(f) \cong M(e)$ for $\{e, f\} = N(K; G)/N(K; G)^o$. Therefore this case has unique (M, G). We can put such (M, G) as

$$M \cong \prod_{i=1}^{h} \mathbb{C}P(l_i) \times S^{2k_1}, \quad G \simeq \prod_{i=1}^{h} SU(l_i + 1) \times SO(2k_1)$$

up to essential isomorphism, because the above (M, G) has same orbits in this case (where $k_1 = k_2$ and $\sum_{i=1}^{h} l_i + k_1 = n$). But this is not a (quasi-)toric manifold, because $k_1 \geq 4$ and S^{2k_1} is not a (quasi-)toric manifold.

Next $k_1 = 3$. Then there are two cases $Spin(2k_1) = Spin(2k_2)$ and $Spin(2k_1) = SU(m_k + 1)$ because of $Spin(6) \simeq SU(4)$. The first case is similar to the $k_1 \geq 4$ case, so it is not a (quasi-)toric manifold. Hence $Spin(2k_1) = SU(4) = SU(m_k + 1)$ holds, easily we have $Spin(2k_2) = SU(4) = SU(l_h + 1)$. However we can easily show this is a contradiction from $\sigma_1^{-1}(SO(5)) \simeq K \simeq \sigma_2^{-1}(SO(5))$ (as a conjugation).

Finally $k_1 = 1$. Then easily we have

$$G = \prod_{i=1}^{h} SU(l_i + 1) \times T^1, \quad K_1 = \prod_{i=1}^{h} S(U(l_i) \times U(1)) \times T^1 = K_2,$$

and $k_1 = k_2 = 1$. The slice representations are as follows;

$$\sigma_1 : K_1 = \prod_{i=1}^{h} S(U(l_i) \times U(1)) \times T^1 \longrightarrow T^1, \quad \sigma_1(t_1, \cdots, t_h, t) = t_1^{\alpha_1} \cdots t_h^{\alpha_h} t^{\alpha},$$
$$\sigma_2 : K_2 = \prod_{i=1}^{h} S(U(l_i) \times U(1)) \times T^1 \longrightarrow T^1, \quad \sigma_2(t_1, \cdots, t_h, t) = t_1^{\beta_1} \cdots t_h^{\beta_h} t^{\beta}.$$

Beause of Ker $\sigma_1 \simeq K$, we see $K \cap (\{e\} \times T^1) \simeq \mathbb{Z}_{\alpha}$. On the other hand Ker $\sigma_2 \simeq K$, so we see $K \cap (\{e\} \times T^1) \simeq \mathbb{Z}_{\beta}$. Hence $|\alpha| = |\beta|$. Because T^1 acts on K_i/K transitively by Lemma 5.5, we also have $|\alpha| = |\beta| \neq 0$. Similarly for each factor, we have $|\alpha_i| = |\beta_i|$. Denote such representations σ_1, σ_2 by $(\alpha_1, \cdots, \alpha_h, \alpha)$ and $(\alpha_1', \cdots, \alpha_h', \alpha')$, where $|\alpha_i| = |\alpha_i'|$ and $|\alpha| = |\alpha'|$. Moreover comparing $K \cap (SU(l_i + 1) \times SU(l_j + 1))$, we have $(\alpha_1', \cdots, \alpha_h', \alpha') = \pm(\alpha_1, \cdots, \alpha_h, \alpha)$. Since the slice representation is identified up to the equivalence of representation in $\sigma_i : K_i \to O(2k_i)$, we can put

$$(\alpha_1', \cdots, \alpha_h', \alpha') = -(\alpha_1, \cdots, \alpha_h, \alpha).$$

Hence two tubular neighborhoods are

$$X_1(\alpha_1, \cdots, \alpha_h, \alpha) \cong G \times_{K_1} D^2, \quad X_2(-\alpha_1, \cdots, -\alpha_h, -\alpha) \cong G \times_{K_2} D^2$$

such that K_1 acts on $D^2 \subset \mathbb{C}$ by the representation $\sigma_1 = (\alpha_1, \cdots, \alpha_h, \alpha)$ and K_2 acts on $D^2 \subset \mathbb{C}$ by the representation $\sigma_2 = (-\alpha_1, \cdots, -\alpha_h, -\alpha)$.

Now $N(K; G) = N(K; G)^o$, so we have unique $M(f)$ for each tubular neighborhood (X_1, X_2) from Lemma 4.7 (1).

We can construct such (M, G) as follows;

$$M \cong G \times_{K_1} P(\mathbb{C}_1 \oplus \mathbb{C}_2),$$
$$G = \prod_{i=1}^{h} SU(l_i + 1) \times T^1,$$
$$K_1 = \prod_{i=1}^{h} S(U(l_i) \times U(1)) \times T^1,$$

where K_1 acts on $\mathbb{C}_1$ by σ_1. This is the case $k_1 = k_2 = 1$ in Theorem B′.

6.2. **When (6.2) and (6.4) occur.** If $k_1 = 1$, then we have already studied in Section 6.1 $k_1 = 1$ case. So we put $k_1 \geq 2$. Comparing (G, K_1) and (G, K_2) we have two cases in this case; one of them is $SU(k_1) = SU(k_2)$ and the other is $SU(k_1) = SU(m_k + 1)$, $SU(k_2) = SU(l_h + 1)$. We can similarly show the former case is not a (quasi-)toric manifold.

Hence we have

$$G = \prod_{i=1}^{h-1} SU(m_i+1) \times SU(k_1) \times SU(k_2) \times T^1,$$
$$K_1 = \prod_{i=1}^{h-1} S(U(m_i) \times U(1)) \times SU(k_1) \times S(U(k_2) \times U(1)) \times T^1,$$
$$K_2 = \prod_{i=1}^{h-1} S(U(m_i) \times U(1)) \times S(U(k_1) \times U(1)) \times SU(k_2) \times T^1,$$

and the slice representation

$$\sigma_i K_i \to U(k_i) \subset SO(2k_i) \subset O(2k_i)$$

is $\sigma_i(SU(k_i)) = SU(k_i) \subset U(k_i)$ and the other factor $N = K_i/SU(k_i)$ goes to the diagonal subgroup T^1 in $U(k_i)$. From similar argument with the Section 6.1 $k_1 = 1$ case, we see this case satisfies Theorem B′.

6.3. **When (6.1) and (6.4) (or (6.2) and (6.3)) occur.** We can show this case does not become a (quasi-)toric manifold, similary as the Section 6.1.

Consequently we get Theorem B′.

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