Equivariant cohomology distinguishes the geometric structures of toric hyperKähler manifolds

Shintarô Kuroki

Abstract. Toric hyperKähler manifolds are the hyperKähler analogue of symplectic toric manifolds. The theory of Bielawski and Dancer tells us that, while a symplectic toric manifold is determined by a Delzant polytope, a toric hyperKähler manifold is determined by a smooth hyperplane arrangement. The purpose of this paper is to show that a toric hyperKähler manifold up to weak hyperhamiltonian T-isometry is determined by not only a smooth hyperplane arrangement up to weak linear equivalence but also by its equivariant cohomology $H^*_T(M;\mathbb{Z})$ with a point $\widehat{a}$ in $H^2(M;\mathbb{R}) \setminus \{0\}$ up to weak $H^*(BT;\mathbb{Z})$-algebra isomorphism preserving $\widehat{a}$.

1. Introduction

A *toric hyperKähler manifold* is defined by the hyperKähler quotient of the torus action on the quaternionic space $\mathbb{H}^m$. This space was introduced as the hyperKähler analogue of symplectic toric manifolds by Goto and Bielawski-Dancer in [**BiDa, Go**]. The resulting manifolds are $4n$-dimensional manifolds with quarter-dimensional torus actions, i.e., with T^n-actions (see Section 2), while symplectic toric manifolds are $2n$-dimensional manifolds with half-dimensional torus actions. Toric hyperKähler manifolds and symplectic toric manifolds are rather different-looking spaces. However, in [**BiDa**], Bielawski-Dancer showed a correspondence between toric hyperKähler manifolds (geometry) and smooth hyperplane arrangements (combinatorics) (see Section 3). A correspondence was also shown between symplectic toric manifolds and Delzant polytopes by Delzant in [**De**]. In particular, using this correspondence between the geometry and combinatorics, we find that the equivariant cohomology of toric manifolds and toric hyperKähler manifolds is isomorphic to the Stanley-Reisner face ring of the corresponding combinatorial objects (see [**Fu, Od**] for toric manifolds and [**Ko1, Ko2**] for toric hyperKähler manifolds). Here, the *equivariant cohomology* of T-manifold M, denoted by $H^*_T(M)$, is defined by the ordinary cohomology of the *Borel construction* $ET \times_T M$, i.e.,

$$H^*_T(M) = H^*(ET \times_T M),$$

where ET is the total space of the universal T-bundle and $ET \times_T M$ is the orbit space of $ET \times M$ by the diagonal T-action (see [**Br, Hs, Ka**] for details, and also see Section 5). In this paper, if we omit the coefficient ring of the cohomology, it means that we take the integer $\mathbb{Z}$ as the coefficient ring.

Equivariant cohomology $H^*_T(M)$ is not only a ring but also an algebra over $H^*(BT)$ through the projection map from $ET \times_T M$ onto $ET/T = BT$. The $H^*(BT)$-algebra structure on $H^*_T(M)$ contains more geometric information about M. For example, as is well known, the ring structure of $H^*_T(M)$ is determined by their orbit structure M/T only; on the other hand, for the $H^*(BT)$-algebra structure, Masuda proves the following important theorem in [**Ma2**]:

2000 *Mathematics Subject Classification.* Principal: 55N91; 57S25, Secondly: 14M25; 32S22; 37J05.

Key words and phrases. Cohomological rigidity, Equivariant cohomology, Hyperhamiltonian isometry, Hyperplane arrangement, Toric hyperKähler manifold (hypertoric manifold).

The author was supported in part by Basic Science Research Program through the NRF of Korea funded by the Ministry of Education, Science and Technology (2010-0001651) and the Fujukai Foundation.

Theorem 1 (Masuda). *Two toric manifolds are isomorphic as varieties if and only if their equivariant cohomology algebras are weakly isomorphic.*

Roughly speaking, Masuda's theorem together with the fundamental correspondence between toric manifolds and combinatorial objects (i.e., fans or polytopes) means that the geometric, combinatorial and algebraic pieces of data in toric manifolds are identical. Motivated by Masuda's theorem, the purpose of this paper is to show that the geometric, combinatorial and algebraic pieces of data are also the same in the case of toric hyperKähler manifolds.

In order to state the main theorem, we quickly introduce some notations. For two G-spaces X and Y, a map $f : X \to Y$ is termed a *weak G-equivariant map* if there is a group isomorphism $\varphi : G \to G$ such that $f(x{\cdot}t) = f(x){\cdot}\varphi(t)$ for all $t \in G$ and $x \in X$; if this isomorphism φ is an identity map, then f is called a *G-equivariant map.* Let $(M_\alpha, T^n, \mu_{\widehat{\alpha}})$ and $(M_\beta, T, \mu_{\widehat{\beta}})$ be toric hyperKähler manifolds M_α, M_β with hyperKähler moment maps $\mu_{\widehat{\alpha}}$, $\mu_{\widehat{\beta}}$, where $\alpha \in (\mathfrak{k}_1)^*$, $\widehat{\alpha} \in (\mathfrak{t}^{m_1})^*$, $\beta \in (\mathfrak{k}_2)^*$ and $\widehat{\beta} \in (\mathfrak{t}^{m_2})^*$ (see Section 2 for details). Here, M_α, M_β are defined by the hyperKähler quotient on $\mathbb{H}^{m_1}$, $\mathbb{H}^{m_2}$, respectively. We refer to two such pairs $(M_\alpha, T^n, \mu_{\widehat{\alpha}})$ and $(M_\beta, T, \mu_{\widehat{\beta}})$ as (*weakly*) *hyperhamiltonian T^n-isometric* and denote them as $(M_\alpha, T, \mu_{\widehat{\alpha}}) \equiv_w (M_\beta, T, \mu_{\widehat{\beta}})$, if there exists a weak T^n-equivariant diffeomorphism which preserves hyperKähler structures and hyperKähler moment maps (see Definition 4.1 for details). Two hyperplane arrangements $\mathcal{H}_{\widehat{\alpha}}$ and $\mathcal{H}_{\widehat{\beta}}$ are said to be *weakly (linear) equivalent* if there exists a linear isomorphism $\varphi^* : (\mathfrak{t}^n)^* \to (\mathfrak{t}^n)^*$, induced from an isomorphism $\varphi : T^n \to T^n$, such that φ^* preserves $\mathcal{H}_{\widehat{\alpha}}$ to $\mathcal{H}_{\widehat{\beta}}$. We denote such hyperplane arrangements as $\mathcal{H}_{\widehat{\alpha}} \equiv_w \mathcal{H}_{\widehat{\beta}}$ (see Section 4.2 for details). Two equivariant cohomology $H^*_T(M_\alpha)$ and $H^*_T(M_\beta)$ are said to be *weakly $H^*(BT)$-algebra isomorphic* if there exists an isomorphism map f^*_T between $H^*_T(M_\alpha)$ and $H^*_T(M_\beta)$ which preserves the $H^*(BT)$-algebra structures of $H^*_T(M_\alpha)$ and $H^*_T(M_\beta)$ up to an automorphism on $H^*(BT)$ (see Section 5.1 for details). We denote them as $H^*_T(M_\alpha) \simeq_w H^*_T(M_\beta)$. Because M_α, M_β are defined by the hyperKähler quotient on $\mathbb{H}^{m_1}$, $\mathbb{H}^{m_2}$ (respectively), two isomorphisms $J^*_{m_1} : (\mathfrak{t}^{m_1})^* \to H^2_T(M_\alpha; \mathbb{R})$ and $J^*_{m_2} : (\mathfrak{t}^{m_2})^* \to H^2_T(M_\beta; \mathbb{R})$ are defined by the correspondences between the standard basis of $\mathfrak{t}^{m_1}$ and $\mathfrak{t}^{m_2}$ to the canonical basis of $H^2_T(M_\alpha; \mathbb{R})$ and $H^2_T(M_\beta; \mathbb{R})$, respectively (see Section 7.1.2 for details). Let $\widehat{a} = J^*_{m_1}(\widehat{\alpha}) \in H^2_T(M_\alpha; \mathbb{R})$, $\widehat{b} = J^*_{m_2}(\widehat{\beta}) \in H^2_T(M_\beta; \mathbb{R})$, and $(f^*_T)_{\mathbb{R}} : H^*_T(M_\alpha; \mathbb{R}) \to H^*_T(M_\beta; \mathbb{R})$ be the natural isomorphism induced from $f^*_T : H^*_T(M_\alpha) \simeq_w H^*_T(M_\beta)$. At this point, we may state our main theorem, as follows (see Theorems 4.3 and 8.1 for details):

Theorem 1.1. *Let $(M_\alpha, T, \mu_{\widehat{\alpha}})$ and $(M_\beta, T, \mu_{\widehat{\beta}})$ be toric hyperKähler manifolds M_α, M_β with hyperKähler moment maps $\mu_{\widehat{\alpha}}$, $\mu_{\widehat{\beta}}$, and let $\mathcal{H}_{\widehat{\alpha}}$, $\mathcal{H}_{\widehat{\beta}}$ be their respective hyperplane arrangements. Then, the following three statements are equivalent:*

(1) $(M_\alpha, T, \mu_{\widehat{\alpha}}) \equiv_w (M_\beta, T, \mu_{\widehat{\beta}})$;
(2) $\mathcal{H}_{\widehat{\alpha}} \equiv_w \mathcal{H}_{\widehat{\beta}}$;
(3) *there exists a weak $H^*(BT)$-algebra isomorphism $f^*_T : H^*_T(M_\alpha; \mathbb{Z}) \to H^*_T(M_\beta; \mathbb{Z})$ such that $(f^*_T)_{\mathbb{R}}(\widehat{a}) = \widehat{b}$.*

Theorem 1.1 tells us that, as far as toric hyperKähler manifolds are concerned, the following three objects belong in different areas: a toric hyperKähler manifold with a hyperKähler moment map $(M_\alpha, T, \mu_{\widehat{\alpha}})$ (geometry); a smooth hyperplane arrangement $\mathcal{H}_{\widehat{\alpha}} \subset (\mathfrak{t}^n)^*$ (combinatorics); and an equivariant cohomology with a point in ordinary cohomology $(H^*_T(M_\alpha), \pi^*, \widehat{a})$ (algebra) are identical (see the following triangle diagram).

$$\begin{tikzcd} & \boxed{\{(M_\alpha, T, \mu_{\widehat{\alpha}})\}/_{\equiv_w}} \arrow[dl, Leftrightarrow] \arrow[dr, Leftrightarrow] & \\ \boxed{\{\mathcal{H}_{\widehat{\alpha}} \subset (\mathfrak{t}^n)^*\}/_{\equiv_w}} \arrow[rr, Leftrightarrow] & & \boxed{\{(H^*_T(M_\alpha), \pi^*, \widehat{a})\}/_{\simeq_w}} \end{tikzcd}$$

The organization of this paper and the idea of the proof of Theorem 1.1 are as follows. We first recall toric hyperKähler manifolds and their basic facts in Section 2, and we then introduce hyperplane arrangements defined by toric hyperKähler manifolds in Section 3. In Section 4, we introduce the equivalence relation of toric hyperKähler manifolds and that of hyperplane arrangements, and prove these two equivalence relations are compatible (in Theorem 4.3). In Section 5, we recall the basic facts about the equivariant cohomology of toric hyperKähler manifolds. The idea of the proof of Theorem 1.1 is to translate the original hyperplane arrangements defined in the dual of Lie algebra $(\mathfrak{t}^n)^*$ into the hyperplane arrangements defined in the equivariant cohomology $H^2(BT^n;\mathbb{R})$. Thus, in Section 6, we define the hyperplane arrangement in $H^2(BT^n;\mathbb{R})$, while in Section 7, we prove that the two hyperplane arrangements in $(\mathfrak{t}^n)^*$ and $H^2(BT^n;\mathbb{R})$ are equivalent up to the level of weak equivalence defined in Section 4 (in Proposition 7.11). Finally, in Section 8, we prove that the weak $H^*(BT)$-algebra isomorphism between the two equivariant cohomology $H_T^*(M_\alpha)$ and $H_T^*(M_\beta)$ induces a weak equivalence between two hyperplane arrangements in $\pi_1^* : H^*(BT;\mathbb{R}) \to H_T^*(M_\alpha;\mathbb{R})$ and $\pi_2^* : H^*(BT;\mathbb{R}) \to H_T^*(M_\beta;\mathbb{R})$.

2. Toric hyperKähler varieties

In this section, we recall the basic facts of the *toric hyperKähler variety* needed later (see [**BiDa, HaSt, Ko3, Pr**], for detail). We assume throughout this paper that $\mathbb{Z}$ is the integer, $\mathbb{R}$ is the real, $\mathbb{C}$ is the complex and $\mathbb{H}$ is the quaternionic numbers, i.e., $\mathbb{H} \simeq \mathbb{R}^4$ as the $\mathbb{R}$-vector space whose basis are $1,\ \mathbf{i},\ \mathbf{j},\ \mathbf{k}$ and they satisfy the following quaternionic relations:

$$\mathbf{ijk} = \mathbf{i}^2 = \mathbf{j}^2 = \mathbf{k}^2 = -1.$$

2.1. Definition of toric hyperKähler manifolds and their geometric structures. In this subsection, we define a *toric hyperKähler manifold* and introduce its geometric structure.

2.1.1. *HyperKähler structure on* $\mathbb{H}^m$. In the beginning, let us recall a geometric structure on the quaternionic vector space. Assume $\mathbb{H}^m$ is the m-dimensional quaternionic vector space with the left $\mathbb{H}$-scalar product. Then the complex structure $I_1 : \mathbb{H}^m \to \mathbb{H}^m$ with $I_1^2 = -\mathrm{Id}_{\mathbb{H}^m}$ (resp. I_2 and I_3) on $\mathbb{H}^m$ is defined by the left multiplication of $\mathbf{i}$ (resp. $\mathbf{j}$ and $\mathbf{k}$). We now put the flat Riemannian metric g on $\mathbb{H}^m$ arising from the standard Euclidean scalar product on $\mathbb{H}^m \simeq \mathbb{R}^{4m} = \mathbb{R}^m \oplus \mathbf{i}\mathbb{R}^m \oplus \mathbf{j}\mathbb{R}^m \oplus \mathbf{k}\mathbb{R}^m$. Using these structures, we can define three Kähler forms on $\mathbb{H}^m$ as follows:

$$\omega_i(X,Y) = g(I_iX,Y) \quad i = 1,\ 2,\ 3 \tag{2.1}$$

where $X,\ Y$ are tangent vectors on a point in $\mathbb{H}^m$. The metric g is called a *hyperKähler metric*, i.e., it is a Kähler metric with respect to three complex structures I_1, I_2 and I_3 which satisfy the quaternionic relations. The automorphism group which preserves the hyperKähler structure (i.e., g, I_1, I_2 and I_3) is called *the symplectic group* and denoted by $Sp(m)$, i.e., the subgroup of special orthogonal group $SO(4m)$ which commutes with I_1, I_2 and I_3, or equivalently preserves the Kähler forms ω_1, ω_2 and ω_3. Note that $Sp(m)$ acts on $\mathbb{H}^m$ from the right. In this paper, if the group G acts on a hyperKähler manifold M with preserving its hyperKähler structure, we call G acts on (M, g_M, I_M, J_M, K_M), where g_M is a Riemannian structure and I_M, J_M and K_M are three complex structures which define the hyperKähler structure on M.

2.1.2. *HyperKähler moment map of* $(\mathbb{H}^m, T^m)$. We next recall properties of the torus action on $(\mathbb{H}^m, g, I_1, I_2, I_3)$. Because $Sp(m)$ acts on $(\mathbb{H}^m, g, I_1, I_2, I_3)$, a maximal torus in $Sp(m)$ also preserves the hyperKähler structure on $\mathbb{H}^m$. Let T^m be the diagonal abelian subgroup in $Sp(m)$, i.e., m-dimensional torus. This torus T^m is a maximal torus in $Sp(m)$, and the T^m-action on $\mathbb{H}^m$ is defined by right multiplication:

$$\begin{array}{ccc} \mathbb{H}^m & \longrightarrow & \mathbb{H}^m \\ \cup & & \cup \\ z + w\mathbf{k} & \stackrel{\cdot t}{\longmapsto} & zt + wt^{-1}\mathbf{k} \end{array} \tag{2.2}$$

for $z,\ w \in \mathbb{C}^m$ and $t \in T^m$. By using this action, we may regard $\mathbb{H}^m$ as $T^*\mathbb{C}^m$, i.e., the cotangent bundle of $\mathbb{C}^m$; or equivalently $\mathbb{C}^m \oplus \overline{\mathbb{C}^m}$, where $\overline{\mathbb{C}^m}$ is isomorphic to $\mathbb{C}^m$ with reversed orientation.

Regard a symplectic form on $\mathbb{H}^m$ as $\omega_{\mathbb{R}} = \omega_1$ and a holomorphic symplectic form on $\mathbb{H}^m$ as $\omega_{\mathbb{C}} = \omega_2 + \sqrt{-1}\omega_3$. Then the T^m-action defined in (2.2) preserves $\omega_{\mathbb{R}}$ and $\omega_{\mathbb{C}}$, and induces the *hyperKähler moment map*

$$\mu_{\mathbb{R}} \oplus \mu_{\mathbb{C}} : \mathbb{H}^m \longrightarrow (\mathfrak{t}^m)^* \oplus (\mathfrak{t}^m_{\mathbb{C}})^*$$

such that

$$\mu_{\mathbb{R}}(z, w) = \frac{1}{2} \sum_{i=1}^{m} (|z_i|^2 - |w_i|^2) \partial_i$$

and

$$\mu_{\mathbb{C}}(z, w) = 2\sqrt{-1} \sum_{i=1}^{m} z_i w_i \partial_i,$$

where $z = (z_1, \ldots, z_m) \in \mathbb{C}^m$ and $w = (w_1, \ldots, w_m) \in \overline{\mathbb{C}^m}$ for $\mathbb{H}^m = \mathbb{C}^m \oplus \overline{\mathbb{C}^m}$, and ∂_i ($i = 1, \ldots, m$) is the basis in $(\mathfrak{t}^m)^*$ and $(\mathfrak{t}^m_{\mathbb{C}})^*$.

2.1.3. *Definition of a toric hyperKähler variety.* In order to define a toric hyperKähler variety, we explain the *hyperKähler quotient.*

Let K be a connected subgroup T^m. There is the following sequence:

$$K \xrightarrow{\iota} T^m \xrightarrow{\rho} T^m/K \simeq T^n, \tag{2.3}$$

where ι is the inclusion, ρ is the projection to the cokernel of ι, and put $n = m - \dim K$. This sequence induces the following exact sequence of Lie algebras:

$$\{0\} \longrightarrow \mathfrak{k} \xrightarrow{\iota_*} \mathfrak{t}^m \xrightarrow{\rho_*} \mathfrak{t}^n \longrightarrow \{0\}. \tag{2.4}$$

Taking its dual, we have the following exact sequence:

$$\{0\} \longrightarrow (\mathfrak{t}^n)^* \xrightarrow{\rho^*} (\mathfrak{t}^m)^* \xrightarrow{\iota^*} \mathfrak{k}^* \longrightarrow \{0\}. \tag{2.5}$$

By using ι^* and its complexification $\iota^*_{\mathbb{C}}$, we can define *the hyperKähler moment map* of K-action on $\mathbb{H}^m$ as follows:

$$\mu_{HK} : \mathbb{H}^m \xrightarrow{\mu_{\mathbb{R}} \oplus \mu_{\mathbb{C}}} (\mathfrak{t}^m)^* \oplus (\mathfrak{t}^m_{\mathbb{C}})^* \xrightarrow{\iota^* \oplus \iota^*_{\mathbb{C}}} \mathfrak{k}^* \oplus \mathfrak{k}^*_{\mathbb{C}}. \tag{2.6}$$

By the definition of μ_{HK}, an element $(\alpha, 0) \in \mathfrak{k}^* \oplus \mathfrak{k}^*_{\mathbb{C}}$ for each non-zero $\alpha \in \mathfrak{k}^*$ is a regular value of μ_{HK}. Hence, its inverse image $\mu^{-1}_{HK}(\alpha, 0)$ has the almost free K-action. Therefore, its quotient space $\mu^{-1}_{HK}(\alpha, 0)/K$ is a $4n$-dimensional orbifold with the induced T^m/K ($\simeq T^n$) action from T^m-action on $\mu^{-1}_{HK}(\alpha, 0)$. This quotient is called the *hyperKähler quotient.*

Put

$$M_\alpha = \mu^{-1}_{HK}(\alpha, 0)/K.$$

We call M_α a *toric hyperKähler variety.* If M_α is non-singular, then we call it a *toric hyperKähler manifold.* The following proposition gives the necessary and sufficient condition for the smoothness of a toric hyperKähler variety (see [**Ko1**, Proposition 2.2]).

Proposition 2.1. *The following two statements are equivalent.*

1. *The action of K on $\mu^{-1}_{HK}(\alpha, 0)$ is free, i.e., $M_\alpha = \mu^{-1}_{HK}(\alpha, 0)/K$ is a manifold.*
2. *For any $J \subset \{1, \ldots, m\}$ such that $\{\iota^* u_j \mid j \in J\}$ forms a basis of $\mathfrak{k}^*$,*

$$\mathfrak{t}^m_{\mathbb{Z}} = \mathfrak{k}_{\mathbb{Z}} \oplus \sum_{j \in J^c} \mathbb{Z}\partial_j \quad \text{as a } \mathbb{Z}\text{-module,}$$

where we regard $\mathfrak{k}$ as the subset of $\mathfrak{t}^m$ via ι_*, u_j is an element of $(\mathfrak{t}^m)^*$, and $\mathfrak{k}_{\mathbb{Z}}$ and $\mathfrak{t}^m_{\mathbb{Z}}$ are lattice subgroups of $\mathfrak{k}$ and $\mathfrak{t}^m$ respectively.

One can easily show the following proposition (also see [**Ko3**, Remark 4.2]):

Proposition 2.2. *Let $u_m \in (\mathfrak{t}^m)^*$ be a basis. If $\iota^*(u_m) = 0$, then there exists the $(4n-4)$-dimensional toric hyperKähler manifold M such that*

$$\mu^{-1}_{HK}(\alpha, 0)/K = M \times \mathbb{H},$$

where $M \times \mathbb{H}$ has the diagonal $T^{n-1} \times S^1$-action and the S^1-action on $\mathbb{H}$ is the standard action.

2.1.4. *HyperKähler moment map of* (M_α, T^n). By the definition of the toric hyperKähler variety M_α, the T^n-action on the smooth part of M_α preserves three Kähler forms $\widetilde{\omega}_i$ induced from ω_i, $i = 1,\ 2,\ 3$ (see (2.1)), i.e., this action preserves the hyperKähler structure on the smooth part of M_α. Hence, this T^n-action also preserves the real symplectic form $\widetilde{\omega}_\mathbb{R} = \widetilde{\omega}_1$ and the holomorphic symplectic form $\widetilde{\omega}_\mathbb{C} = \widetilde{\omega}_2 + \sqrt{-1}\widetilde{\omega}_3$ on the smooth part of M_α. Define a hyperKähler moment map $\mu_{\widehat{\alpha}} = \widetilde{\mu}_\mathbb{R} \oplus \widetilde{\mu}_\mathbb{C}$ as follows:

$$\begin{aligned} \widetilde{\mu}_\mathbb{R}[z,w] \oplus \widetilde{\mu}_\mathbb{C}[z,w] &= \left(\frac{1}{2}\sum_{i=1}^{m}(|z_i|^2 - |w_i|^2) - \alpha_i\right)\partial_i \oplus 2\sqrt{-1}\sum_{i=1}^{m} z_i w_i \partial_i \\ &\in \ \ker \iota^* \oplus \ker \iota^*_\mathbb{C} \simeq (\mathfrak{t}^n)^* \oplus (\mathfrak{t}^n_\mathbb{C})^*, \end{aligned} \tag{2.7}$$

where $[z,w] \in M_\alpha$ and α_i is the i^{th} element of $\widehat{\alpha} = (\alpha_1, \dots, \alpha_m) \in (\iota^*)^{-1}(\alpha) \subset (\mathfrak{t}^m)^*$; we call $\widehat{\alpha} \in (\mathfrak{t}^m)^*$ a *lift* of $\alpha \in (\mathfrak{t}^n)^*$.

We note the following remark (assumption) needed in Section 8.

Remark 2.3. Suppose $\iota^*(u_m) = 0$. Then, by Proposition 2.2, the toric hyperKähler manifold is $\mu_{HK}(\alpha, 0)/K = M \times \mathbb{H}$. Note that S^1 acts on $\mathbb{H}$ standardly. In this paper, we assume that the hyperKähler moment map of the standard S^1-action on $\mathbb{H}$ is always defined by the map $\mu = \mu_\mathbb{R} \oplus \mu_\mathbb{C} : \mathbb{H} \to \mathbb{R} \oplus \mathbb{C}$ in Section 2.1.2. Therefore, by the definition of the hyperKähler moment map $\mu_{\widehat{\alpha}}$, it is easy to check that there exists the following decomposition:

$$\mu_{\widehat{\alpha}} = \mu_{\widehat{\alpha}'} \oplus \mu,$$

for some lift $\widehat{\alpha}' \in (\mathfrak{t}^{m-1})^*$ of $\alpha \in \mathfrak{k}^*$ (note that $K \subset T^{m-1}$ in this case). Namely, if $\iota^*(u_m) = 0$ then we always assume a lift $\widehat{\alpha} \in (\mathfrak{t}^m)^*$ of α as follows:

$$(\widehat{\alpha}', 0) \in (\mathfrak{t}^{m-1})^* \oplus (\mathfrak{t}^1)^*.$$

2.2. Example and Remark. Here, we give the standard example of toric hyperKähler varieties.

Example 2.4. Let Δ be the diagonal subgroup in T^{n+1}. Then, we get the following exact sequence by using the inclusion $\Delta \subset T^{n+1}$:

$$(\mathfrak{t}^n)^* \xrightarrow{\rho^*} (\mathfrak{t}^{n+1})^* \xrightarrow{\iota^*} \mathbb{R} \tag{2.8}$$

such that

$$\iota^*(\alpha_1, \dots, \alpha_{n+1}) = \alpha_1 + \cdots + \alpha_{n+1} \in \mathbb{R},$$

where $(\alpha_1, \dots, \alpha_{n+1}) \in (\mathfrak{t}^{n+1})^* \simeq \mathbb{R}^{n+1}$ and $\mathbb{R}$ is the dual of Lie algebra of Δ.

Put $\alpha = n+1 \in \mathbb{R}$. Then the toric hyperKähler manifold $\mu_{HK}^{-1}(\alpha, 0)/\Delta$ is T^n-equivariantly diffeomorphic to $T^*\mathbb{C}P^n$, where the T^n-action on $T^*\mathbb{C}P^n$ is induced from the standard T^n-action on $\mathbb{C}P^n$.

We finish this section by the following remark (assumption).

Remark 2.5. Let $\{\mathbf{e}_1, \dots, \mathbf{e}_m\}$ be the standard basis of $\mathfrak{t}^m$ whose dual basis are $\{\partial_1, \dots, \partial_m\}$. If $\rho_*(\mathbf{e}_i) = 0$, then we can easily show that K and T^m can be decomposed into $K = K' \times S_i$ and $T^m = T^{m-1} \times S_i$ by using (2.3) and (2.4), where $K' \subset T^{m-1}$ and S_i is the i^{th} coordinate circle of T^m. Because the hyperKähler variety constructed by the S_i-action on $\mathbb{H}$ is the 0-dimensional manifold, we may regard that the toric hyperKähler variety in this case (i.e., $\rho_*(\mathbf{e}_i) = 0$) is constructed by the hyperKähler quotient of K'-action on $\mathbb{H}^{m-1}$. Hence, throughout this paper, we assume $\rho_*(\mathbf{e}_i) \neq 0$ for all $i = 1,\ \dots,\ m$.

3. Hyperplane arrangements

One of the most important properties of toric hyperKähler manifolds is the correspondence between toric hyperKähler manifolds and hyperplane arrangements, established by Bielawski-Dancer in [**BiDa**] (also see [**HaPr, Ko3, Pr**]). In this section, we recall some facts of the hyperplane arrangement induced from the toric hyperKähler variety.

3.1. Hyperplane arrangements induced from toric hyperKähler varieties. In order to define toric hyperKähler varieties, it is enough to use the exact sequence (2.5) and the non-zero element $\alpha \in \mathfrak{k}^*$ (see Section 2). By the exactness of (2.5), there is a non-zero lift $\widehat{\alpha}$ of α, i.e., $\iota^*(\widehat{\alpha}) = \alpha$. This $\widehat{\alpha}$ determines m affine hyperplanes in $(\mathfrak{t}^n)^*$ as follows:

$$H_i = \{x \in (\mathfrak{t}^n)^* \mid \langle \rho^*(x) + \widehat{\alpha}, \mathbf{e}_i \rangle = 0\}$$

where $\{\mathbf{e}_1, \ldots, \mathbf{e}_m\}$ is the standard basis of $\mathfrak{t}^m \simeq \mathbb{R}^m$ such that $\rho_*(\mathbf{e}_i) \neq 0$ for all $i = 1, \ldots, m$ (see Remark 2.5). Because of the assumption $\rho_*(\mathbf{e}_i) \neq 0$, the hyperplane H_i defined as above is always codimension-one affine subspace in $(\mathfrak{t}^n)^*$, i.e., $\dim H_i = n - 1$.

REMARK 3.1. We may regard H_i $(i = 1, \ldots, m)$ as a *weighted, cooriented, affine hyperplane* by regarding $\rho_*(\mathbf{e}_i) \in \mathfrak{t}^n_{\mathbb{Z}}$ as a nonzero integer, cooriented, normal vector of H_i. Here, "weighted" means that $\rho_*(\mathbf{e}_i)$ is not required to be primitive. We call $\rho_*(\mathbf{e}_i)$ the *normal vector of H_i in $\mathfrak{t}^n$*.

We call the set of hyperplanes

$$\mathcal{H}_{\widehat{\alpha}} = \{H_1, \ldots, H_m\}$$

the *hyperplane arrangement induced from* (M_α, T^n) or *hyperplane arrangement of* (M_α, T^n).

REMARK 3.2. Let $\widehat{\alpha}_1$ and $\widehat{\alpha}_2$ be elements in $(\mathfrak{t}^n)^*$ which satisfy that $\iota^*(\widehat{\alpha}_1) = \alpha = \iota^*(\widehat{\alpha}_2)$. It is easy to check that $\mathcal{H}_{\widehat{\alpha}_1}$ and $\mathcal{H}_{\widehat{\alpha}_2}$ are the same up to parallel translation; moreover, their intersection posets are also the same. Namely, the choice of a lift $\widehat{\alpha}$ does not change the combinatorial structure (intersection posets) of $\mathcal{H}_{\widehat{\alpha}}$. However, geometrically, $\widehat{\alpha}$ determines the hyperKähler moment map because of the definition in (2.7). This implies that the date $(M_\alpha, T^n, \mu_{\widehat{\alpha}})$ gives more precise structure of hyperplane arrangements than the combinatorial structure (see Section 4.2, 4.3).

We also note that all hyperplane arrangements do not appear as hyperplane arrangements of (M_α, T^n). We shall explain it in the following Lemma 3.3. Before that, we prepare two notations. A hyperplane arrangement $\mathcal{H} = \{H_1, \ldots, H_m\}$ is called *simple*, if every subset of $k(\leq n)$ hyperplanes in $\mathcal{H}$ intersect in codimension k whenever their intersection is non-empty and there exists n hyperplanes whose intersection is non-empty set (i.e., one point). Furthermore, a simple hyperplane arrangement is called *smooth*, if some subset of n hyperplanes H_i has non-empty intersection, the corresponding normal vectors $\rho_*(\mathbf{e}_i)$ form a $\mathbb{Z}$-basis for $\mathfrak{t}^n_{\mathbb{Z}} \subset \mathfrak{t}^n$ (See Figure 1 and 2).

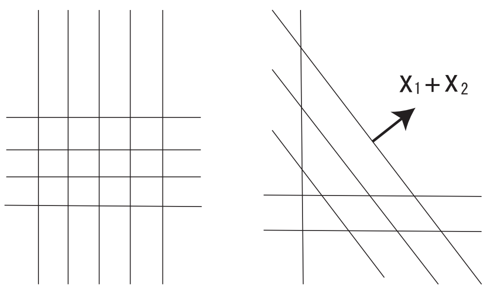

FIGURE 1. Smooth hyperplane arrangements in $(\mathfrak{t}^2)^*$, where horizontal lines represent the hyperplanes whose normal vectors in $\mathfrak{t}^2$ are $\mathbf{x}_2$ and vertical lines represent those whose normal vectors in $\mathfrak{t}^2$ are $\mathbf{x}_1$. Here, $\mathbf{x}_1$ and $\mathbf{x}_2$ are the standard basis of $\mathfrak{t}^2$.

The following lemma is fundamental (see [**BiDa, HaPr**] for detail).

LEMMA 3.3. *The toric hyperKähler variety M is an orbifold if and only if a hyperplane arrangement of (M, T) is simple. Furthermore, M is smooth (i.e., non-singular) if and only if the hyperplane arrangement of (M, T) is smooth.*

The following example is one of the standard examples.

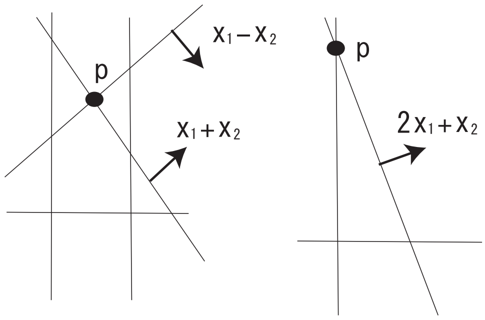

FIGURE 2. Non-smooth hyperplane arrangements. Both of the hyperplanes do not satisfy the condition of smooth hyperplane arrangements around the vertex p; therefore, both of them are not smooth (but simple).

EXAMPLE 3.4. As seen in Example 2.4, the toric hyperKähler manifold which is defined by the diagonal subgroup Δ in T^{n+1} is T^n-equivariantly diffeomorphic to $T^*\mathbb{C}P^n$.

By using the exactness of (2.8), we may define a representation ρ^* as follows:

$$\rho^*(x_1, \dots, x_n) = (x_1, \dots, x_n, -(x_1 + \cdots + x_n)) \in (\mathfrak{t}^{n+1})^*$$

Because we took a lift of α as $(1, \dots, 1) \in (\mathfrak{t}^{n+1})^*$ (see Example 2.4), by definition, we get the following hyperplanes:

$$\begin{aligned} H_1 &= \{(x_1, \dots, x_n) \in (\mathfrak{t}^n)^* \mid x_1 = -1\}; \\ &\vdots \\ H_n &= \{(x_1, \dots, x_n) \in (\mathfrak{t}^n)^* \mid x_n = -1\}; \\ H_{n+1} &= \{(x_1, \dots, x_n) \in (\mathfrak{t}^n)^* \mid x_1 + \cdots + x_n = 1\}. \end{aligned}$$

Figure 3 illustrates the case when $n = 2$.

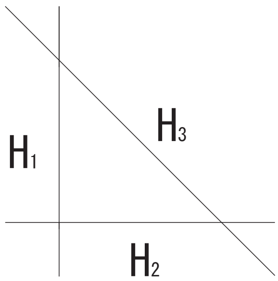

FIGURE 3. The hyperplane arrangement $\mathcal{H}_{(1,1,1)}$ of $T^*\mathbb{C}P(2)$

Henceforth, we assume all toric hyperKähler varieties are non-singular, i.e., all toric hyperKähler varieties are smooth manifolds (see Proposition 2.1), and all hyperplane arrangements are smooth hyperplane arrangements (see Lemma 3.3). Moreover, we note the following remark (assumption).

REMARK 3.5. If $m = n$, i.e., $\dim \mathbb{H}^m = 4 \dim T^n$, then the toric hyperKähler manifold is $\mathbb{H}^n$ itself and the hyperKähler moment map is given by $\mu_{\mathbb{R}} \oplus \mu_{\mathbb{C}}$ (see Section 2.1). Therefore, this case is uniquely characterized as the quaternionic space $(\mathbb{H}^n, T^n, \mu_{\mathbb{R}} \oplus \mu_{\mathbb{C}})$. If $n = 0$, then the toric hyperKähler manifold is the one point and the hyperKähler moment map is given by the zero map to $\{0\} \oplus \{0\}$. Therefore, this case is also uniquely characterized as the one point with the constant hyperKähler moment map. Hence, in this paper, we assume $m \geq n + 1$ and $n \geq 1$.

3.2. Toric hyperKähler manifolds induced from smooth hyperplane arrangements. We next demonstrate the construction of the toric hyperKähler manifold with the hyperKähler moment map $(M^{4n}_\alpha, T^n, \mu_{\widehat{\alpha}})$ from the smooth hyperplane arrangement in $(\mathfrak{t}^n)^*$.

Let $\mathcal{H}$ be the set of weighted, cooriented, affine hyperplanes $\{H_1, \dots, H_m\}$ such that

$$H_i = \{x \in (\mathfrak{t}^n)^* \mid \langle x, v_i \rangle + \alpha_i = 0\}$$

and

$$\dim H_i = n - 1,$$

where $v_i \in \mathfrak{t}^n_{\mathbb{Z}}$ $(i = 1, \dots, m)$ regarded as the weighted coorientation (normal vector) of H_i and $\alpha_i \in \mathbb{R}$ determines the position in $(\mathfrak{t}^n)^*$. Because $\mathcal{H}$ is smooth and $m \geq n+1$ (see Remark 3.5), the surjective homomorphism $\rho_* : \mathfrak{t}^m \to \mathfrak{t}^n$ can be defined by

$$\rho_*(\mathbf{e}_i) = v_i \in \mathfrak{t}^n_{\mathbb{Z}}, \tag{3.1}$$

where $\mathbf{e}_i$ is the i^{th} standard basis of $\mathfrak{t}^m \simeq \mathbb{R}^m$. Because $\dim H_i = n-1$, we have that $v_i \neq 0$ (also see Remark 2.5). Put $\mathfrak{k} = \ker \rho_*$. Then, there is the following exact sequence:

$$\{0\} \longrightarrow \mathfrak{k} = \ker \rho_* \xrightarrow{\iota_*} \mathfrak{t}^m \xrightarrow{\rho_*} \mathfrak{t}^n \longrightarrow \{0\}.$$

Taking its dual, we can define the following sequence such as the sequence (2.5):

$$\{0\} \longrightarrow (\mathfrak{t}^n)^* \xrightarrow{\rho^*} (\mathfrak{t}^m)^* \xrightarrow{\iota^*} \mathfrak{k}^* \longrightarrow \{0\}.$$

Now we may regard $\alpha_i = \langle \widehat{\alpha}, \mathbf{e}_i \rangle$ by taking $\widehat{\alpha} = (\alpha_1, \dots, \alpha_m) \in (\mathfrak{t}^m)^*$ with respect to the dual basis ∂_i of $\mathbf{e}_i$. Using two assumptions that $\mathcal{H}$ is smooth and $m \geq n+1$, it is easy to check that $\iota^*(\widehat{\alpha}) = \alpha \neq 0$ in $\mathfrak{k}^*$. Therefore, with the method similar to that demonstrated in Section 2.1, we can construct the toric hyperKähler variety $M_\alpha = \mu^{-1}_{HK}(\alpha, 0)/K$ from the above exact sequence and the non-zero element $\alpha \in \mathfrak{k}^*$, where K is the connected torus whose Lie algebra is $\mathfrak{k}$. Then, M_α is non-singular by using the smoothness of $\mathcal{H}$ and Proposition 2.1. We also define the hyperKähler moment map $\mu_{\widehat{\alpha}}$ by the equation (2.7). Consequently, from the smooth, weighted, cooriented, affine hyperplane arrangement, the toric hyperKähler manifold with the hyperKähler moment map $(M_\alpha, T^n, \mu_{\widehat{\alpha}})$ can be constructed.

3.3. Geometric meaning of weighted, cooriented, affine hyperplanes. In this subsection, we quickly review the correspondence between hyperplanes and submanifolds in M_α.

According to [**BiDa**, Theorem 3.1 (1)], the hyperKähler moment map $\mu_{\widehat{\alpha}}$ in (2.7) is surjective. Therefore, we have the following surjective map onto $(\mathfrak{t}^n)^*$:

$$\Psi_{\widehat{\alpha}} : M_\alpha \xrightarrow{\mu_{\widehat{\alpha}}} (\mathfrak{t}^n)^* \oplus (\mathfrak{t}^n_{\mathbb{C}})^* \longrightarrow (\mathfrak{t}^n)^*,$$

where the second map is the projection onto $(\mathfrak{t}^n)^*$.

Let M_i be a *characteristic submanifold* of (M_α, T^n), i.e., M_i is an invariant connected submanifold in M_α which is fixed by some circle subgroup in T^n. By [**HaHo**, Section 3], we have the following proposition.

Proposition 3.6. *Let N_i $(i = 1, \dots, m)$ be the subset in $\mu^{-1}_{HK}(\alpha, 0) \subset \mathbb{C}^m \oplus \overline{\mathbb{C}^m}$ such that $z_i = 0 = w_i$, where z_i and w_i are the i^{th} coordinate in $\mathbb{C}^m$ and $\overline{\mathbb{C}^m}$, respectively. For the given characteristic submanifold M_i, there exists the subset N_i such that*

$$M_i = N_i/K.$$

Equivalently, M_i is the hyperKähler quotient of the restricted K-action on the $(m-1)$-dimensional subspace $\mathbb{H}^{m-1}_i$, where $\mathbb{H}^{m-1}_i$ is the subspace of $\mathbb{H}^m$ whose i^{th} coordinate is 0.

Due to Proposition 3.6, we have $\dim M_i = 4n - 4$ for all $i = 1, \dots, m$. By the definition of $\mu_{\widehat{\alpha}}$ and hyperplanes H_i (for $i = 1, \dots, m$), we have the following relation:

$$\Psi_{\widehat{\alpha}}(M_i) = H_i \subset (\mathfrak{t}^n)^*. \tag{3.2}$$

Consequently, there exists the one-to-one correspondence between hyperplanes in $\mathcal{H}_{\widehat{\alpha}}$ and characteristic submanifolds in $(M_\alpha, T^n, \mu_{\widehat{\alpha}})$.

Note that the normal vector $\rho_*(\mathbf{e}_i)(\neq 0)$ of H_i is a primitive vector in $\mathfrak{t}^n$, because a hyperplane arrangement is smooth. Hence, by using Proposition 3.6, the circle subgroup T_i which fixes M_i is induced from the i^{th} coordinate circle subgroup S_i in T^m, i.e., $T_i = S_i/K \subset T^m/K$, and can be obtained by the exponent of $\rho_*(\mathbf{e}_i) \in \mathfrak{t}^n$. In other words, T_i can be determined by the vector $\rho_*(\mathbf{e}_i) \in \mathfrak{t}^n$ up to sign. Moreover, it follows from Proposition 3.6 that the circle subgroup T_i acts on the normal bundle ν_i of M_i and this action is induced by the right scalar multiplications on fibres, where fibers are isomorphic to $\mathbb{H} = \mathbb{C} \oplus \overline{\mathbb{C}}$. Namely, two choices of signs of $\rho_*(\mathbf{e}_i) \in \mathfrak{t}^n$ correspond to two orientations of ν_i (we often call a collection of orientations of $\nu_1, \ldots, \nu_m, M_\alpha$ an *omni-orientation* of M_α, also see [**HaMa**]). Summing up, we have the following corollary as a geometric meaning of the normal vector $\rho_*(\mathbf{e}_i)$ of H_i.

Corollary 3.7. *Let $\rho_*(\boldsymbol{e}_i)$ be the weighted, cooriented, normal vector of H_i. Then, the circle subgroup which fixes the characteristic submanifold M_i such that $\Psi_{\widehat{\alpha}}(M_i) = H_i$ is given the following subgroup:*

$$T_i = \{\exp r\rho_*(\boldsymbol{e}_i) \mid r \in \mathbb{R}\}$$

for all $i = 1, \ \ldots, \ m$.

Furthermore, two signs of $\rho_*(\boldsymbol{e}_i)$ correspond to two orientations of the normal bundle ν_i of M_i.

4. Equivalence relations on toric hyperKähler manifolds

In this section, we define the equivalence relations on toric hyperKähler manifolds and the hyperplane arrangements precisely, and prove that these equivalence relations are compatible (see Theorem 4.3).

4.1. Equivalence relations of toric hyperKähler manifolds. Let $(M_\alpha, T^n, \mu_{\widehat{\alpha}})$ and $(M_\beta, T^n, \mu_{\widehat{\beta}})$ be two toric hyperKähler manifolds with hyperKähler moment maps, where we put $\alpha \in \mathfrak{k}_1^*$, $\beta \in \mathfrak{k}_2^*$ and their lifts $\widehat{\alpha} \in \mathfrak{t}_1^*$, $\widehat{\beta} \in \mathfrak{t}_2^*$, respectively. Here, $\dim T_1 - \dim K_1 = n = \dim T_2 - \dim K_2$ ($\dim T_1 = m_1$ and $\dim T_2 = m_2$).

Definition 4.1. We say a weakly T^n-equivariantly isometric map $f : M_\alpha \to M_\beta$ a *weak hyperhamiltonian T^n-isometry* or *weak T^n-isomorphism* simply, if a weak T^n-equivariant diffeomorphism f preserves the hyperKähler structures and satisfies that

$$\mu_{\widehat{\alpha}} = \varphi^*_{\mathbb{R}\oplus\mathbb{C}} \circ \mu_{\widehat{\beta}} \circ f,$$

that is, the following diagram is commute:

$$\begin{tikzcd} M_\alpha \arrow[r, "\mu_{\widehat{\alpha}}"] \arrow[d, "f"] & (\mathfrak{t}^n)^* \oplus (\mathfrak{t}^n)^*_{\mathbb{C}} \\ M_\beta \arrow[r, "\mu_{\widehat{\beta}}"] & (\mathfrak{t}^n)^* \oplus (\mathfrak{t}^n)^*_{\mathbb{C}} \arrow[u, "\varphi^*_{\mathbb{R}\oplus\mathbb{C}}"'] \end{tikzcd} \tag{4.1}$$

where $\varphi^*_{\mathbb{R}\oplus\mathbb{C}} : (\mathfrak{t}^n)^* \oplus (\mathfrak{t}^n_{\mathbb{C}})^* \to (\mathfrak{t}^n)^* \oplus (\mathfrak{t}^n_{\mathbb{C}})^*$ is the induced isomorphism from $\varphi : T^n \to T^n$ such that $f(x \cdot t) = f(x) \cdot \varphi(t)$. If φ is the identity map, then f is called a *hyperhamiltonian T^n-isometry* or *T^n-isomorphism* simply.

If there is a (weak) T^n-isomorphism between two triples $(M_\alpha, T^n, \mu_{\widehat{\alpha}})$ and $(M_\beta, T^n, \mu_{\widehat{\beta}})$, then we say that such triples are (*weakly*) *hyperhamiltonian T^n-isometric* or (*weakly*) *T^n-isomorphic*.

Remark 4.2. In the paper [**Bi**], if the above φ is identity then M_α and M_β are called *isomorphic as tri-Hamiltonian hyperKähler T-manifolds*.

In this paper, the symbol $(M_\alpha, T^n, \mu_{\widehat{\alpha}}) \equiv_w (M_\beta, T^n, \mu_{\widehat{\beta}})$ (resp. $(M_\alpha, T^n, \mu_{\widehat{\alpha}}) \equiv (M_\beta, T^n, \mu_{\widehat{\beta}})$) represents that $(M_\alpha, T^n, \mu_{\widehat{\alpha}})$ and $(M_\beta, T^n, \mu_{\widehat{\beta}})$ are weakly T^n-isomorphic (resp. T^n-isomorphic).

4.2. Equivalence relations of hyperplane arrangements. In this subsection, we introduce the equivalence relations of weighted, cooriented, smooth hyperplane arrangements in $(\mathfrak{t}^n)^*$. Let $\mathcal{H}_{\widehat{\alpha}} = \{H_1^{(1)}, \ldots, H_{m_1}^{(1)}\}$ and $\mathcal{H}_{\widehat{\beta}} = \{H_1^{(2)}, \ldots, H_{m_2}^{(2)}\}$ be such hyperplane arrangements consist of

$$H_i^{(1)} = \{x \in (\mathfrak{t}^n)^* \mid \langle x, v_i^{(1)} \rangle + \widehat{\alpha}_i = 0\}, \tag{4.2}$$

$$H_j^{(2)} = \{x \in (\mathfrak{t}^n)^* \mid \langle x, v_j^{(2)} \rangle + \widehat{\beta}_j = 0\}, \tag{4.3}$$

where $v_i^{(1)} \in \mathfrak{t}^n$ $(i = 1, \ \ldots, \ m_1)$ and $v_j^{(2)} \in \mathfrak{t}^n$ $(j = 1, \ \ldots, \ m_2)$ are weighted, cooriented vectors, $\widehat{\alpha}_i, \ \widehat{\beta}_j \in \mathbb{R}$ represent positions of hyperplanes, respectively

Now we may introduce the equivalence relation on the weighted, cooriented, smooth hyperplane arrangements. Two hyperplane arrangements $\mathcal{H}_{\widehat{\alpha}}$ and $\mathcal{H}_{\widehat{\beta}}$ are called *weakly (linear) equivalent* if there exists a linear isomorphism $\varphi^* : (\mathfrak{t}^n)^* \to (\mathfrak{t}^n)^*$, induced from an isomorphism $\varphi : T^n \to T^n$ (or equivalently φ^* preserves $(\mathfrak{t}^n_{\mathbb{Z}})^*$), such that φ^* sends $\mathcal{H}_{\widehat{\alpha}}$ to $\mathcal{H}_{\widehat{\beta}}$, i.e., $m_1 = m_2 = m$ and there is a permutation $\sigma : [m] \to [m]$ such that

$$\varphi^*(H_i^{(1)}) = H_{\sigma(i)}^{(2)}$$

for all $i \in [m]$; in this paper, the symbol $[m]$ for some $m \in \mathbb{N}$ represents the finite set $\{1, \ldots, m\}$, and we denote such hyperplane arrangements by $\mathcal{H}_{\widehat{\alpha}} \equiv_w \mathcal{H}_{\widehat{\beta}}$. Moreover, if we can take such φ^* as the identity map, then $\mathcal{H}_{\widehat{\alpha}}$ and $\mathcal{H}_{\widehat{\beta}}$ are said to be *equivalent*, and we denote them by $\mathcal{H}_{\widehat{\alpha}} \equiv \mathcal{H}_{\widehat{\beta}}$.

4.3. Relations between equivalent toric hyperKähler manifolds and their hyperplane arrangements. The goal of this subsection is to prove the following theorem:

THEOREM 4.3. *Let $(M_\alpha, T^n, \mu_{\widehat{\alpha}})$, $(M_\beta, T^n, \mu_{\widehat{\beta}})$ be two toric hyperKähler manifolds, and $\mathcal{H}_{\widehat{\alpha}}$, $\mathcal{H}_{\widehat{\beta}}$ be their hyperplane arrangements, respectively. Then, the following two statements are equivalent:*

(1) *$(M_\alpha, T^n, \mu_{\widehat{\alpha}}) \equiv_w (M_\beta, T^n, \mu_{\widehat{\beta}})$;*
(2) *$\mathcal{H}_{\widehat{\alpha}} \equiv_w \mathcal{H}_{\widehat{\beta}}$.*

We first show the direction (1) $\Rightarrow$ (2) in Theorem 4.3.

PROPOSITION 4.4. *Under the hypothesis of Theorem 4.3, if $(M_\alpha, T^n, \mu_{\widehat{\alpha}}) \equiv_w (M_\beta, T^n, \mu_{\widehat{\beta}})$, then $\mathcal{H}_{\widehat{\alpha}} \equiv_w \mathcal{H}_{\widehat{\beta}}$.*

PROOF. Assume $(M_\alpha, T^n, \mu_{\widehat{\alpha}}) \equiv_w (M_\beta, T^n, \mu_{\widehat{\beta}})$. By definition, there exists a weak T^n-isomorphism $f : M_\alpha \to M_\beta$ such that the following diagram commutes (see Section 3.3 and 4.1):

$$\begin{tikzcd} M_\alpha \arrow[r, "f"] \arrow[d, "\Psi_{\widehat{\alpha}}"] & M_\beta \arrow[d, "\Psi_{\widehat{\beta}}"] \\ (\mathfrak{t}^n)^* & (\mathfrak{t}^n)^* \arrow[l, "\varphi^*"] \end{tikzcd}$$

where φ^* is induced from the isomorphism $\varphi : T^n \to T^n$ such that $f(x \cdot t) = f(x) \cdot \varphi(t)$. Because f preserves the characteristic submanifolds, the cardinality of characteristic submanifolds of M_α and that of M_β are identical, say m. Define the permutation $\sigma : [m] \to [m]$ by $f(M_i^{(1)}) = M_{\sigma(i)}^{(2)}$. Then, we have that

$$\begin{aligned} H_i^{(1)} &= \Psi_{\widehat{\alpha}}(M_i^{(1)}) && \text{(by the relation (3.2))} \\ &= \varphi^* \circ \Psi_{\widehat{\beta}} \circ f(M_i^{(1)}) && \text{(by the commutativity of the above diagram)} \\ &= \varphi^* \circ \Psi_{\widehat{\beta}}(M_{\sigma(i)}^{(2)}) && \text{(by the definition of } \sigma) \\ &= \varphi^*(H_{\sigma(i)}^{(2)}) && \text{(by the relation (3.2))}, \end{aligned}$$

for all $i \in [m]$. This implies that $\varphi^* : (\mathfrak{t}^n)^* \to (\mathfrak{t}^n)^*$ is a linear map which gives $\mathcal{H}_{\widehat{\beta}} \equiv_w \mathcal{H}_{\widehat{\alpha}}$. $\square$

The following lemma is a key lemma to prove Theorem 4.3:

LEMMA 4.5. *Let $\mathcal{H}_{\widehat{\alpha}}$ and $\mathcal{H}_{\widehat{\beta}}$ be the smooth hyperplane arrangements induced from toric hyperKähler manifolds $(M_\alpha, T^n, \mu_{\widehat{\alpha}})$ and $(M_\beta, T^n, \mu_{\widehat{\beta}})$, respectively. Assume that $\mathcal{H}_{\widehat{\beta}} \equiv_w \mathcal{H}_{\widehat{\alpha}}$ given by a linear isomorphism $\varphi^* : (\mathfrak{t}^n)^* \to (\mathfrak{t}^n)^*$ which is induced from an isomorphism $\varphi : T^n \to T^n$. Then, there exists the lift $\widetilde{\varphi}^*$ such that the following diagram commutes:*

$$\begin{tikzcd}
(\mathfrak{t}^n)^* \arrow[r, "\rho_2^*"] \arrow[d, "\varphi^*"] & (\mathfrak{t}^m)^* \arrow[d, "\widetilde{\varphi}^*"] \\
(\mathfrak{t}^n)^* \arrow[r, "\rho_1^*"] & (\mathfrak{t}^m)^*
\end{tikzcd}$$

and

$$\widetilde{\varphi}^*(\widehat{\beta}) = \widehat{\alpha},$$

where $\rho_1^*, \rho_2^* : (\mathfrak{t}^n)^* \to (\mathfrak{t}^m)^*$ are the injective representations defined in (2.5) for M_α, M_β, respectively.

Furthermore, the above $\widetilde{\varphi}^*$ can be represented as a following matrix:

$$\begin{pmatrix} \epsilon_1 & 0 & \cdots & 0 \\ 0 & \epsilon_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \epsilon_m \end{pmatrix} \cdot \Sigma$$

with respect to the basis ∂_i $(i \in [m])$ of $(\mathfrak{t}^m)^*$, where $\epsilon_i = \pm 1$ for $i \in [m]$ and Σ is the $m \times m$ matrix induced from the permutation $\sigma : [m] \to [m]$.

PROOF. Assume $\mathcal{H}_{\widehat{\beta}} \equiv_w \mathcal{H}_{\widehat{\alpha}}$. Let m be their cardinality. Then, by definition, there exists a linear isomorphism $\varphi^* : (\mathfrak{t}^n)^* \to (\mathfrak{t}^n)^*$ such that

$$\varphi^*(H_j^{(2)}) = H_{\sigma(j)}^{(1)},$$

for all $j \in [m]$ and some permutation $\sigma : [m] \to [m]$, where $H_{\sigma(j)}^{(1)}$ and $H_j^{(2)}$ are hyperplanes in $\mathcal{H}_{\widehat{\alpha}}$ and $\mathcal{H}_{\widehat{\beta}}$, respectively. By definition of hyperplanes (see Section 3.1), we also have the following relations:

$$\begin{aligned} \varphi^*(H_j^{(2)}) &= \varphi^*(\{x \in (\mathfrak{t}^n)^* \mid \langle \rho_2^*(x) + \widehat{\beta}, \mathbf{e}_j \rangle = 0\}) \\ = \quad H_{\sigma(j)}^{(1)} &= \{y \in (\mathfrak{t}^n)^* \mid \langle \rho_1^*(y) + \widehat{\alpha}, \mathbf{e}_{\sigma(j)} \rangle = 0\}. \end{aligned}$$

Hence, for $x \in H_j^{(2)}$, we have the following equations:

$$\begin{aligned} &\langle \rho_2^*(x), \mathbf{e}_j \rangle = -\langle \widehat{\beta}, \mathbf{e}_j \rangle; \\ &\langle \rho_1^* \circ \varphi^*(x), \mathbf{e}_{\sigma(j)} \rangle = -\langle \widehat{\alpha}, \mathbf{e}_{\sigma(j)} \rangle. \end{aligned} \tag{4.4}$$

One can easily show that the following two statements are equivalent:

- $\langle \widehat{\beta}, \mathbf{e}_j \rangle = 0$;
- $H_j^{(2)}$ is a linear subspace in $(\mathfrak{t}^n)^*$.

This implies that $\langle \widehat{\beta}, \mathbf{e}_j \rangle = 0$ if and only if $\langle \widehat{\alpha}, \mathbf{e}_{\sigma(j)} \rangle = 0$, because φ^* is a linear isomorphism. Therefore, because $\langle \widehat{\alpha}, \mathbf{e}_{\sigma(j)} \rangle$ and $\langle \widehat{\beta}, \mathbf{e}_j \rangle$ are real numbers, there exists $\epsilon_j' \in \mathbb{R} \setminus \{0\}$ such that

$$\langle \widehat{\alpha}, \mathbf{e}_{\sigma(j)} \rangle = \epsilon_j' \langle \widehat{\beta}, \mathbf{e}_j \rangle. \tag{4.5}$$

Using the equations (4.4) and (4.5) as above, it is easy to check that, for all $x \in H_j^{(2)}$,

$$\langle x, (\rho_2)_*(\epsilon_j' \mathbf{e}_j) \rangle = \langle x, \varphi_* \circ (\rho_1)_*(\mathbf{e}_{\sigma(j)}) \rangle,$$

where $\varphi_* : \mathfrak{t}^n \to \mathfrak{t}^n$ is the dual linear isomorphism of φ^*. Therefore, by using the fact that $H_j^{(2)}$ is a codimension one hyperplane, we can easily show that two non-zero vectors $(\rho_2)_*(\epsilon_j' \mathbf{e}_j)$ and

$\varphi_* \circ (\rho_1)_*(\mathbf{e}_{\sigma(j)})$ lie in the same 1-dimensional linear subspace in $\mathfrak{t}^n$. Hence, there exists $\epsilon_j \in \mathbb{R}\backslash\{0\}$ such that

$$\epsilon_j(\rho_2)_*(\mathbf{e}_j) = \varphi_* \circ (\rho_1)_*(\mathbf{e}_{\sigma(j)}). \tag{4.6}$$

We also have $\epsilon_j = \epsilon'_j$ by using (4.4), (4.5) and (4.6). Define the linear map $\widetilde{\varphi}_* : \mathfrak{t}^m \to \mathfrak{t}^m$ as the following $(m \times m)$-square matrix with respect to the standard basis $\{\mathbf{e}_1, \ldots, \mathbf{e}_m\}$:

$$\begin{pmatrix} \epsilon_1 & 0 & \cdots & 0 \\ 0 & \epsilon_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \epsilon_m \end{pmatrix} \cdot \Sigma^{-1},$$

where the $(m \times m)$-matrix Σ^{-1} is the inverse of the matrix Σ induced from the permutation $\sigma : [m] \to [m]$. Using (4.6), we have that the linear map $\widetilde{\varphi}_*$ satisfies that

$$\varphi_* \circ (\rho_1)_* = (\rho_2)_* \circ \widetilde{\varphi}_*,$$

i.e., the following diagram is commute:

$$\begin{tikzcd} \mathfrak{t}^n & \mathfrak{t}^m \arrow[l, "(\rho_2)_*"'] \\ \mathfrak{t}^n \arrow[u, "\varphi_*"] & \mathfrak{t}^m \arrow[l, "(\rho_1)_*"'] \arrow[u, "\widetilde{\varphi}_*"'] \end{tikzcd}$$

We claim $\epsilon_j = \pm 1$ for all $j = 1, \ldots, m$. Because $\mathcal{H}_{\widehat{\beta}}$ is a smooth hyperplane arrangements, for all $j \in [m]$, there exists $I_j \subset [m]$ such that $|I_j| = n$ (i.e., the cardinality of I_j is n), $j \in I_j$, $\cap_{i \in I_j} H_i^{(2)} \neq \emptyset$ and

$$\{(\rho_2)_*(\mathbf{e}_i) \mid i \in I_j\}$$

spans $\mathfrak{t}^n_{\mathbb{Z}}$. Therefore, because φ^* gives weak equivalence between two smooth hyperplane arrangements $\mathcal{H}_{\widehat{\beta}}$ and $\mathcal{H}_{\widehat{\alpha}}$, the vectors

$$\{(\rho_1)_*(\mathbf{e}_{\sigma(i)}) \mid i \in I_j\}$$

also spans $\mathfrak{t}^n_{\mathbb{Z}}$. By the definition of $\widetilde{\varphi}_*$ and the commutativity of the diagram above, we have that

$$\begin{aligned} \varphi_* \circ (\rho_1)_*(\mathbf{e}_{\sigma(i)}) &= (\rho_2)_* \circ \widetilde{\varphi}_*(\mathbf{e}_{\sigma(i)}) \\ &= (\rho_2)_*(\epsilon_i \mathbf{e}_i). \end{aligned}$$

Because the isomorphism $\varphi_* : \mathfrak{t}^n \to \mathfrak{t}^n$ is induced from an isomorphism $\varphi : T^n \to T^n$, the restriction of φ_* to $\mathfrak{t}^n_{\mathbb{Z}}$ induces the isomorphism between $\mathfrak{t}^n_{\mathbb{Z}}$ and $\mathfrak{t}^n_{\mathbb{Z}}$. This implies that $\epsilon_i = \pm 1$ for all $i \in I_j$. Because this satisfies for all I_j ($j \in [m]$), we have $\epsilon_j = \pm 1$ for all $j = 1, \ldots, m$. By taking the dual of the above homomorphisms, it is easy to check the statement of Lemma 4.5. □

Now we may prove the direction (2) ⇒ (1) in Theorem 4.3.

PROPOSITION 4.6. *Under the hypothesis of Theorem 4.3, if $\mathcal{H}_{\widehat{\alpha}} \equiv_w \mathcal{H}_{\widehat{\beta}}$, then $(M_\alpha, T^n, \mu_{\widehat{\alpha}}) \equiv_w (M_\beta, T^n, \mu_{\widehat{\beta}})$.*

PROOF. Assume $\mathcal{H}_{\widehat{\alpha}} \equiv_w \mathcal{H}_{\widehat{\beta}}$ and this is given by $\varphi^* : (\mathfrak{t}^n)^* \to (\mathfrak{t}^n)^*$. In this case, by using the arguments in Section 3.3, the cardinality of characteristic submanifolds of M_α and that of M_β are identical, say m. Therefore, it also follows from Proposition 3.6 that M_α and M_β are defined by the hyperKähler quotient of torus actions on $\mathbb{H}^m$, i.e., there exist the same dimensional subtori $K_1, K_2 \subset T^m$ such that

$$\begin{aligned} M_\alpha &= \mu_{HK}^{-1}(\alpha, 0)/K_1, \\ M_\beta &= \mu_{HK}^{-1}(\beta, 0)/K_2, \end{aligned}$$

where K_s, $s = 1, 2$, can be defined by the exponent of Lie algebra $\mathfrak{k}_s$ whose dual is $\mathfrak{k}_s^* = (\mathfrak{t}^m)^*/\mathrm{Im}\,\rho_s^*$. By using Lemma 4.5, there exists the lift $\widetilde{\varphi}^* : (\mathfrak{t}^m)^* \to (\mathfrak{t}^m)^*$ of φ^* such that

$$\widetilde{\varphi}^*(\widehat{\beta}) = \widehat{\alpha},$$

and we also have its matrix representation with respect to the basis ∂_i ($i \in [m]$) of $(\mathfrak{t}^m)^*$ such as the matrix in Lemma 4.5. We denote this matrix representation by $X \in O(m)$.

Let $\mu = \mu_{\mathbb{R}} \oplus \mu_{\mathbb{C}}$ be the hyperKähler moment map of the standard T^m-action on $\mathbb{H}^m$ (see Section 2.1). Then, by the definition of μ, it is easy to check that the following diagram is commutative:

$$\begin{tikzcd} \mathbb{H}^m \arrow[r, "\mu"] & (\mathfrak{t}^m)^* \oplus (\mathfrak{t}^m)^*_{\mathbb{C}} \arrow[d, "\widetilde{\varphi}^*_{\mathbb{R}\oplus\mathbb{C}}"'] \\ \mathbb{H}^m \arrow[u, "\Phi_X"] \arrow[r, "\mu"] & (\mathfrak{t}^m)^* \oplus (\mathfrak{t}^m)^*_{\mathbb{C}} \end{tikzcd}$$

where $\widetilde{\varphi}^*_{\mathbb{R}\oplus\mathbb{C}} = \widetilde{\varphi}^* \oplus \widetilde{\varphi}^*_{\mathbb{C}} (= X \oplus X)$ and $\Phi_X : \mathbb{H}^m \to \mathbb{H}^m$ is defined by the following matrix $X_{\mathbb{H}}$ in $Sp(m)$:

$$\begin{pmatrix} h_1 & 0 & \cdots & 0 \\ 0 & h_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & h_m \end{pmatrix} \cdot \Sigma_{\mathbb{H}}^{-1}$$

such that

$$\begin{cases} h_i = 1 & \text{if } \epsilon_i = 1 \\ h_i = \mathbf{k} & \text{if } \epsilon_i = -1, \end{cases}$$

and $\Sigma_{\mathbb{H}}^{-1}$ is the $(m \times m)$-matrix induced by the permutation $\sigma^{-1} : [m] \to [m]$ acting on $\mathbb{H}^m$. Here, $\mathbf{k}$ is the basis of $\mathbb{H}$, i.e., $\{1, \mathbf{i}, \mathbf{j}, \mathbf{k}\}$, and $X_{\mathbb{H}}$ acts on $\mathbb{H}^m$ from the right hand side (note that X acts on $(\mathfrak{t}^m)^* \oplus (\mathfrak{t}^m)^*_{\mathbb{C}}$ from the left hand side). Because $\Phi_X : \mathbb{H}^m \to \mathbb{H}^m$ is represented by $X_{\mathbb{H}} \in Sp(m)$ as above and $r\mathbf{k} = \mathbf{k}r^{-1}$ for all $r \in S^1$, we see that Φ_X gives the weak T^m-isomorphism on $(\mathbb{H}^m, T^m, \mu)$ such that $\Phi_X((z, w)t) = \Phi_X(z, w)\widetilde{\varphi}(t)$, where $\widetilde{\varphi} : T^m \to T^m$ is the induced isomorphism from $\widetilde{\varphi}^* : (\mathfrak{t}^m)^* \to (\mathfrak{t}^m)^*$ (also see $\widetilde{\varphi}_*$ in the proof of Lemma 4.5).

Because $\widetilde{\varphi}^*(\widehat{\beta}) = \widehat{\alpha}$, one can easily show that

$$\Phi_X(\mu_{HK}^{-1}(\alpha, 0)) = \mu_{HK}^{-1}(\beta, 0).$$

Recall that, in the first paragraph of this proof, we define K_s, $s = 1, 2$, as the induced torus from $\mathfrak{k}_s^* = (\mathfrak{t}^m)^*/\mathrm{Im}\rho_s^*$. Because $\widetilde{\varphi}^*$ is a lift of φ^*, i.e., $\widetilde{\varphi}^* \circ \rho_2^* = \rho_1^* \circ \varphi^*$, this lift $\widetilde{\varphi}^*$ induces the natural isomorphism between $\mathfrak{k}_2^*$ and $\mathfrak{k}_1^*$. This implies that

$$\widetilde{\varphi}(K_1) = K_2.$$

Therefore, Φ_X induces the weak T^n-equivariant diffeomorphism between $M_\alpha = \mu_{HK}^{-1}(\alpha, 0)/K_1$ and $M_\beta = \mu_{HK}^{-1}(\beta, 0)/K_2$; moreover, Φ_X preserves their hyperKähler structures because $X_{\mathbb{H}} \in Sp(m)$. Note that $\widetilde{\varphi} : T^m \to T^m$ induces $\widetilde{\varphi}/K : T^m/K_1 \to T^m/K_2$ and $\widetilde{\varphi}/K$ coincides with $\varphi : T^n \to T^n$, where φ is the isomorphism which induces $\varphi^* : (\mathfrak{t}^n)^* \to (\mathfrak{t}^n)^*$ in the first paragraph (also see the definition of $\mathcal{H}_\beta \equiv_w \mathcal{H}_\alpha$ in Section 4.2). Moreover, by using $\widetilde{\varphi}^*(\widehat{\beta}) = \widehat{\alpha}$ and the definitions of the hyperKähler moment maps $\mu_{\widehat{\alpha}}$ and $\mu_{\widehat{\beta}}$ (see Section 2.1.4), we have that the induced diffeomorphism from Φ_X also preserves hyperKähler moment maps $\mu_{\widehat{\alpha}}$ and $\mu_{\widehat{\beta}}$. This establishes that $(M_\alpha, T^n, \mu_{\widehat{\alpha}}) \equiv_w (M_\beta, T^n, \mu_{\widehat{\beta}})$. $\square$

By Propositions 4.4 and 4.6, we have Theorem 4.3.

5. Equivariant cohomology of toric hyperKähler manifolds

Using the combinatorial data of the smooth hyperplane arrangement induced from the toric hyperKähler manifold (M, T), we can describe the ring structure of the equivariant cohomology $H_T^*(M)$ of (M, T). In this section, we recall this important fact.

5.1. Notations and $H_T^*(M)$ as the $H^*(BT)$-algebra. We first recall the $H^*(BT)$-algebra structure of $H_T^*(M)$. In order to define $H_T^*(M)$, we need to take a space $ET \times_T M$ called the *Borel construction* (or *homotopy quotient*). Because T acts freely on the ET-factor in $ET \times M$, the Borel construction may be regarded as the fibre bundle over the classifying space $BT(= ET/T)$ with fibre M, i.e., there is the following fibration:

$$M \xrightarrow{j} ET \times_T M \xrightarrow{\pi} BT, \tag{5.1}$$

where π is the projection and j is the injection to the fixed fibre. We call the ordinary cohomology $H^*(ET \times_T M)$ the *equivariant cohomology of* (M, T), and denote it by $H_T^*(M)$. By using the fibration (5.1), we have the following induced homomorphism:

$$\pi^* : H^*(BT) \longrightarrow H_T^*(M).$$

Hence, the equivariant cohomology $H_T^*(M)$ can be regarded as the $H^*(BT)$-algebra via π^*.

Remark 5.1. As we mentioned in Section 1, if we do not mention the coefficient of the cohomology, it means that we take the integer $\mathbb{Z}$ as the coefficient ring.

The following proposition is well known (see e.g. [**MiTo**, Chapter 3]).

Proposition 5.2. *If* $\dim T = n$, *the cohomology ring* $H^*(BT; \mathcal{R})$ *is isomorphic to the polynomial ring, i.e.,*

$$H^*(BT; \mathcal{R}) \simeq \mathcal{R}[x_1, \ldots, x_n]$$

for $\mathcal{R} = \mathbb{Z}$ *or* $\mathbb{R}$, *where* $\deg x_i = 2$ $(i = 1, \ \ldots, \ n)$.

By using the Serre spectral sequence and Proposition 5.2, we also have the following well-known proposition (see e.g. [**MiTo**, Chapter 3]).

Proposition 5.3. *If* $H^{odd}(M; \mathcal{R}) = 0$ *and* M *is simply connected, then the induced homomorphism* π^* *is injective and the induced homomorphism* $j^* : H_T^*(M; \mathcal{R}) \to H^*(M; \mathcal{R})$ *is surjective for* $\mathcal{R} = \mathbb{Z}$ *or* $\mathbb{R}$.

In closing this subsection, we recall the equivalence relations on equivariant cohomology. Let (M, T) and (N, T) be two manifolds with T-actions. If there exists ring isomorphisms $f_T^* : H_T^*(M) \to H_T^*(N)$ and $\varphi^* : H^*(BT) \to H^*(BT)$ such that the following diagram commutes:

$$\begin{array}{ccc} H^*(BT) & \xrightarrow{\pi_1^*} & H_T^*(M) \\ \downarrow{\scriptstyle \varphi^*} & & \downarrow{\scriptstyle f_T^*} \\ H^*(BT) & \xrightarrow{\pi_2^*} & H_T^*(N) \end{array}$$

then we call $H_T^*(M)$ and $H_T^*(N)$ are *weakly* $H^*(BT)$*-algebra isomorphic*, and denote them by $H_T^*(M) \simeq_w H_T^*(N)$. If the above φ^* is the identity, then we call $H_T^*(M)$ and $H_T^*(N)$ are $H^*(BT)$*-algebra isomorphic*, and denote them by $H_T^*(M) \simeq H_T^*(N)$.

5.2. Equivariant cohomology of toric hyperKähler manifolds. In this subsection, we review the ring structure of $H_T^*(M_\alpha)$ of toric hyperKähler manifold (M_α, T) (see e.g. [**Ko3, HaHo, HaPr, Pr**] for details).

First, we introduce the ring generators of $H_T^*(M_\alpha)$ which are defined by the 1st Chern classes of line bundles along the characteristic submanifolds. Let M_i, $i = 1, \ \ldots, \ m$, be a characteristic submanifold of (M_α, T) (see Section 3.3). The symbol ν_i represents its normal bundle in M_α. Then, we may regard the total space $E(\nu_i)$ of ν_i as follows:

$$E(\nu_i) = N_i \times_K \mathbb{H}_i, \tag{5.2}$$

by using Proposition 3.6. Here, the 1-dimensional $\mathbb{H}$-vector space $\mathbb{H}_i (\simeq \mathbb{H} \simeq \mathbb{R}^4)$ is the representation space of K by the following representation:

$$\iota_i : K \xrightarrow{\iota} T^m \xrightarrow{p_i} S_i, \tag{5.3}$$

where p_i is the projection to the i^{th} coordinate subgroup $S_i \simeq S^1$ of T^m. Thus, we may regard the normal bundle ν_i as the pull-back bundle induced from the following diagram:

$$\begin{array}{ccc} E(\nu_i) & \longrightarrow & \mu_{HK}^{-1}(\alpha,0) \times_K \mathbb{H}_i \\ \downarrow & & \downarrow \\ M_i & \longrightarrow & M_\alpha = \mu_{HK}^{-1}(\alpha,0)/K \end{array} \tag{5.4}$$

where the bottom $M_i \to M_\alpha$ is the inclusion.

Now we may regard the product manifold $\mu_{HK}^{-1}(\alpha,0) \times \mathbb{H}_i$ as the T^m-manifold, i.e., T^m acts on the $\mu_{HK}^{-1}(\alpha,0)$-factor naturally and on the $\mathbb{H}_i$-factor by the representation p_i (see (5.3)). Then, the subgroup $K(\subset T^m)$ acts on $\mu_{HK}^{-1}(\alpha,0) \times \mathbb{H}_i$ freely, because K acts on the $\mu_{HK}^{-1}(\alpha,0)$-factor freely. Therefore, the induced $T^n = T^m/K$-action on $\mu_{HK}^{-1}(\alpha,0) \times_K \mathbb{H}_i$ is well-defined. Similarly, we can define T^n-action on $E(\nu_i)$. Thus, by taking the Borel construction of each factor in (5.4), we have the following commutative diagram:

$$\begin{array}{ccc} ET^n \times_{T^n} E(\nu_i) & \longrightarrow & ET^n \times_{T^n} (\mu_{HK}^{-1}(\alpha,0) \times_K \mathbb{H}_i) \\ \downarrow & & \downarrow \\ ET^n \times_{T^n} M_i & \longrightarrow & ET^n \times_{T^n} M_\alpha \end{array} \tag{5.5}$$

Because $\mathbb{H} = \mathbb{C} \oplus \overline{\mathbb{C}}$ (see Section 2.1), the bundle in (5.5) splits into the following line bundles:

$$ET^n \times_{T^n} (\mu_{HK}^{-1}(\alpha,0) \times_K \mathbb{H}_i) \equiv ET^n \times_{T^n} (\mu_{HK}^{-1}(\alpha,0) \times_K (\mathbb{C}_i \oplus \overline{\mathbb{C}_i})),$$

where $\mathbb{C}_i$ is the complex 1-dimensional representation space with K-representation via ι_i defined in (5.3), and $\overline{\mathbb{C}}_i$ is isomorphic to $\mathbb{C}_i$ with reversed orientation. Let $\mathbb{L}_i$ be the following line bundle over $ET^n \times_{T^n} M_\alpha$:

$$E(\mathbb{L}_i) = ET^n \times_{T^n} (\mu_{HK}^{-1}(\alpha,0) \times_K \mathbb{C}_i), \tag{5.6}$$

and let τ_i be the 1^{st} Chern class of $\mathbb{L}_i$, i.e.,

$$\tau_i = c_1(\mathbb{L}_i) \in H_T^2(M_\alpha). \tag{5.7}$$

The following Konno's theorem says that the set of such 1^{st} Chern classes $\{\tau_1, \ldots, \tau_m\}$ gives the canonical generator of $H_T^*(M_\alpha)$:

Theorem 5.4 (Konno). *Let (M,T) be a toric hyperKähler manifold and $\mathcal{H} = \{H_1, \ldots, H_m\}$ its hyperplane arrangement. Then, the equivariant cohomology $H_T^*(M)$ satisfies the following isomorphism:*

$$H_T^*(M;\mathbb{Z}) \simeq \mathbb{Z}[\tau_1, \ldots, \tau_m]/\mathcal{I}$$

where $\mathcal{I}$ is the ideal in the polynomial ring $\mathbb{Z}[\tau_1, \ldots, \tau_m]$ generated by

$$\prod_{i\in I} \tau_i$$

for all $I \subset [m]$ such that $\cap_{i\in I} H_i = \emptyset$.

Due to the correspondence between characteristic submanifolds and hyperplanes mentioned in Section 3.3 and the definition of τ_i in (5.7), we have the following correspondence among the characteristic submanifolds $M_i \subset M_\alpha$, the hyperplanes $H_i \in \mathcal{H}_{\widehat{\alpha}}$ and the canonical generators $\tau_i \in H_T^2(M_\alpha)$:

$$\begin{array}{ccccc} & & \boxed{M_i \subset M_\alpha} & & \\ & \swarrow\!\!\nearrow & & \searrow\!\!\nwarrow & \\ \boxed{H_i \in \mathcal{H}_{\widehat{\alpha}}} & & \Longleftrightarrow & & \boxed{\tau_i \in H_T^2(M_\alpha)} \end{array}$$

We finish this section by the following example.

Example 5.5. By Example 2.4, the cotangent bundle $T^*\mathbb{C}P^n$ over $\mathbb{C}P^n$ is a toric hyperKähler manifold. Using Example 3.4, we may put the hyperplane arrangement of $T^*\mathbb{C}P^n$ as $\mathcal{H} = \{H_1, \ldots, H_{n+1}\}$. Again by Example 3.4, if $I \subset [n+1]$ satisfies that $\cap_{i\in I} H_i = \emptyset$ then $I = [n+1]$ only. Therefore, by using Theorem 5.4, the equivariant cohomology ring of $T^*\mathbb{C}P^n$ is given by

$$H_T^*(T^*\mathbb{C}P^n) \simeq \mathbb{Z}[\tau_1,\ \ldots,\ \tau_{n+1}]/\langle \tau_1 \cdots \tau_{n+1}\rangle,$$

where $\deg \tau_i = 2$.

In this case, characteristic submanifolds are given by $T^*\mathbb{C}P_i$, where $\mathbb{C}P_i \cong \mathbb{C}P^{n-1}$ is the complex projective space in $\mathbb{C}P^n$ whose i^{th} projective coordinate is 0.

6. Hyperplane arrangements induced from the equivariant cohomology

One of the key points to prove Theorem 1.1 is to translate the hyperplane arrangements defined in $(\mathfrak{t}^n)^*$ into the equivariant cohomology $H^2(BT;\mathbb{R}) \simeq \mathbb{R}^n \simeq (\mathfrak{t}^n)^*$. In this section, we define the hyperplane arrangement induced from the equivariant cohomology (with fixed generators).

Let (M_α, T) be a toric hyperKähler manifold. Then, by using the homotopy exact sequence for the fibration (5.1), i.e.,

$$M_\alpha \xrightarrow{j} ET \times_T M_\alpha \xrightarrow{\pi} BT,$$

we have that the fundamental group $\pi_1(ET \times_T M_\alpha)$ is trivial; therefore, $H_1(ET \times_T M_\alpha) = \{0\}$. Hence, by using the universal coefficient theorem for cohomology, we have

$$H_T^2(M_\alpha;\mathbb{Z}) \simeq \mathrm{Hom}(H_2(ET \times_T M_\alpha), \mathbb{Z}). \tag{6.1}$$

Here, the homology $H_*(ET \times_T M_\alpha)$ is called the *equivariant homology*, and often denoted by $H_*^T(M_\alpha)$.

Because of Proposition 5.3 and Theorem 5.4, we have that the following induced exact sequence from the fibration (5.1):

$$\{0\} \longrightarrow H^2(BT;\mathbb{Z}) \xrightarrow{\pi^*} H_T^2(M_\alpha;\mathbb{Z}) \xrightarrow{j^*} H^2(M_\alpha;\mathbb{Z}) \longrightarrow \{0\}. \tag{6.2}$$

Take the canonical generator in $H_T^2(M_\alpha;\mathbb{Z})$ as $\{\tau_1, \ldots, \tau_m\}$ by virtue of Theorem 5.4. With the method similar to that demonstrated by Masuda in the proof of [**Ma2**, Proposition 2.2], it is easy to show that the homomorphism π^* in (6.2) can be expressed as that in the following proposition:

Proposition 6.1. *To each $i \in [m]$, there exists a unique element $v_i \in H_2(BT;\mathbb{Z})$ such that*

$$\pi^*(x) = \sum_{i=1}^m \langle x, v_i\rangle_H \tau_i$$

for any $x \in H^2(BT;\mathbb{Z})$.

Here, in Proposition 6.1, the symbol $\langle , \rangle_H$ represents the pairing of the cohomology and homology defined by $H^2(BT) \simeq \mathrm{Hom}(H_2(BT);\mathbb{Z})$ (see (6.1) with $M_\alpha = \{*\}$). We also note that Theorem 5.4 and Proposition 6.1 give the $H^*(BT)$-algebra structure on $H_T^*(M_\alpha)$.

Taking the tensor product $H_T^*(M_\alpha) \otimes_\mathbb{Z} \mathbb{R}$, the sequence (6.2) induces the following exact sequence:

$$\{0\} \longrightarrow H^2(BT;\mathbb{R}) \xrightarrow{\pi_\mathbb{R}^*} H_T^2(M_\alpha;\mathbb{R}) \xrightarrow{j_\mathbb{R}^*} H^2(M_\alpha;\mathbb{R}) \longrightarrow \{0\}, \tag{6.3}$$

where $\pi_\mathbb{R}^*$ in (6.3) is induced by $\pi^* \otimes id : H^*(BT) \otimes_\mathbb{Z} \mathbb{R} \to H_T^*(M_\alpha) \otimes_\mathbb{Z} \mathbb{R}$. Using Proposition 6.1, the injective homomorphism $\pi_\mathbb{R}^*$ can be expressed by the following formula:

$$\pi_\mathbb{R}^*(x) = \sum_{i=1}^m \langle x, v_i\rangle_H \tau_i, \tag{6.4}$$

where $x \in H^2(BT;\mathbb{R})$, $v_i \in H_2(BT;\mathbb{Z}) \subset H_2(BT;\mathbb{R})$ and $\tau_i \in H_T^2(M_\alpha;\mathbb{Z}) \subset H_T^2(M_\alpha;\mathbb{R})$.

Take an element $\widehat{a} \in H_T^2(M_\alpha;\mathbb{R})$, and define $\mathcal{H}_{\widehat{a}}^{eq}$ as follows:

$$\mathcal{H}_{\widehat{a}}^{eq} = \{H_1^{eq}, \ldots, H_m^{eq}\}$$

such that each hyperplane $H_i^{eq} \subset H^2(BT;\mathbb{R})$, $i=1,\ \ldots,\ m$, is defined by

$$H_i^{eq} = \{x \in H^2(BT;\mathbb{R}) \mid \langle \pi_{\mathbb{R}}^*(x) + \widehat{a}, \mathbf{u}_i \rangle_H = 0\}.$$

Here, $\mathbf{u}_i$, $i=1,\ \ldots,\ m$, is the linear basis in $H_2^T(M;\mathbb{Z})(\subset H_2^T(M;\mathbb{R}))$ such that

$$\langle \tau_j, \mathbf{u}_i \rangle_H = \begin{cases} 1 & \text{if } i=j \\ 0 & \text{if } i \neq j \end{cases} \tag{6.5}$$

where the paring $\langle , \rangle_H$ is defined by (6.1). We call $\mathcal{H}_{\widehat{a}}^{eq}$ a *hyperplane arrangement of* $H_T^*(M_\alpha)$.

We finish this section by the following two remarks.

Remark 6.2. The hyperplane arrangement of equivariant cohomology $\mathcal{H}_{\widehat{a}}^{eq}$ is determined by the triple $(H_T^2(M_\alpha), \pi^*, \widehat{a})$ for $\widehat{a} \in H_T^2(M;\mathbb{R})$, while the hyperplane arrangement of toric hyperKähler manifold $\mathcal{H}_{\widehat{\alpha}}$ is determined by the triple $(M_\alpha, T^n, \mu_{\widehat{\alpha}})$ (see Section 2.2 and 3.3). Therefore, we may think of the inclusion $\pi^* : H^2(BT) \to H_T^2(M)$ as the algebraic counterpart of the T-action on M_α and the fixed element $\widehat{a} \in H_T^2(M_\alpha;\mathbb{R})$ as that of the hyperKähler moment map $\mu_{\widehat{\alpha}} : M_\alpha \to (\mathfrak{t}^n)^* \oplus (\mathfrak{t}^n)_{\mathbb{C}}^*$ (also see [**AtBo**] and the *equivariant symplectic form* in [**GuSt**, Chapter 9]).

Remark 6.3. Using Proposition 5.3, if $H^{odd}(M)=0$ then a simply connected T-manifold (M,T) (not only toric hyperKähler manifolds) satisfies all conditions mentioned as above. Therefore, for more general class of T-manifolds, we can define a hyperplane arrangement of $H_T^*(M)$ as above.

7. Equivalence between two hyperplane arrangements in $(\mathfrak{t}^n)^*$ and $H^2(BT^n;\mathbb{R})$

Henceforth, we assume $(M_\alpha^{4n}, T^n, \mu_{\widehat{\alpha}})$ is a triple of $4n$-dimensional toric hyperKähler manifold M_α, its T^n-action and its hyperKähler moment map $\mu_{\widehat{\alpha}}$, where $\alpha \in \mathfrak{k}^*$ is a non-zero element and $\widehat{\alpha} \in (\mathfrak{t}^m)^*$ is its lift (see Section 2). In this section, we prove that two hyperplane arrangements induced from M_α and the equivariant cohomology $H_T^*(M_\alpha)$ are weak equivalent.

7.1. Equivalence of two exact sequences. We first recall the following two exact sequences defined in (2.5) and (6.3):

$$\{0\} \longrightarrow (\mathfrak{t}^n)^* \xrightarrow{\rho^*} (\mathfrak{t}^m)^* \xrightarrow{\iota^*} \mathfrak{k}^* \longrightarrow \{0\}; \tag{7.1}$$

and

$$\{0\} \longrightarrow H^2(BT;\mathbb{R}) \xrightarrow{\pi_{\mathbb{R}}^*} H_T^2(M_\alpha;\mathbb{R}) \xrightarrow{j_{\mathbb{R}}^*} H^2(M_\alpha;\mathbb{R}) \longrightarrow \{0\}. \tag{7.2}$$

In this subsection, we will define the following three natural isomorphisms:

$$\begin{array}{ll} J_n^* : (\mathfrak{t}^n)^* \longrightarrow H^2(BT^n;\mathbb{R}) & \text{(see Section 7.1.1);} \\ J_m^* : (\mathfrak{t}^m)^* \longrightarrow H_T^2(M_\alpha;\mathbb{R}) & \text{(see Section 7.1.2);} \\ J_K^* : \mathfrak{k}^* \longrightarrow H^2(M_\alpha;\mathbb{R}) & \text{(see Section 7.1.4),} \end{array}$$

and prove the following proposition:

Proposition 7.1. *The following diagram* (7.3) *is a commutative diagram.*

$$\begin{tikzcd} 0 \arrow[r] & (\mathfrak{t}^n)^* \arrow[r, "\rho^*"] \arrow[d, "J_n^*"] & (\mathfrak{t}^m)^* \arrow[r, "\iota^*"] \arrow[d, "J_m^*"] & \mathfrak{k}^* \arrow[r] \arrow[d, "J_K^*"] & 0 \\ 0 \arrow[r] & H^2(BT^n;\mathbb{R}) \arrow[r, "\pi_{\mathbb{R}}^*"] & H_T^2(M_\alpha;\mathbb{R}) \arrow[r, "j_{\mathbb{R}}^*"] & H^2(M_\alpha;\mathbb{R}) \arrow[r] & 0 \end{tikzcd} \tag{7.3}$$

7.1.1. *Definition of the* 1st *isomorphism.* We first define the 1st isomorphism

$$J_n^* : (\mathfrak{t}^n)^* \longrightarrow H^2(BT^n;\mathbb{R}). \tag{7.4}$$

Let $f \in \mathrm{Hom}(S^1, T^n)$. The homomorphism f can be denoted by

$$\begin{array}{ccc} S^1 & \xrightarrow{f} & T^n \\ \cup & & \cup \\ t & \longmapsto & (t^{c_1}, \ldots, t^{c_n}), \end{array} \tag{7.5}$$

for some integers $c_1, \ldots, c_n$. Using this (7.5), we have the isomorphism $\mathrm{Hom}(S^1, T^n) \simeq \mathbb{Z}^n$. Moreover, f induces the continuous map $Bf : BS^1 \to BT^n$, and this map induces the homomorphism $Bf_* : H_2(BS^1;\mathbb{Z}) \to H_2(BT^n;\mathbb{Z})$. We fix a generator $\kappa \in H_2(BS^1;\mathbb{Z}) \simeq \mathbb{Z}$. Because $H_2(BT^n;\mathbb{Z}) \simeq \mathbb{Z}^n \simeq \mathrm{Hom}(S^1, T^n)$, the homomorphism defined as follow is the isomorphism:

$$\begin{array}{ccc} \mathrm{Hom}(S^1, T^n) & \xrightarrow{\simeq} & H_2(BT^n;\mathbb{Z}) \\ \cup & & \cup \\ f & \longmapsto & Bf_*(\kappa). \end{array} \tag{7.6}$$

By using (7.5), we have that the Lie algebra homomorphism $\widehat{f} : \mathbb{R} \to \mathfrak{t}^n$ defined by $\widehat{f}(r) = (c_1 r, \ldots, c_n r)$ for $r \in \mathbb{R}$ is the lift of f, i.e., we have the following commutative diagram

$$\begin{tikzcd} \mathbb{R} \arrow[r, "\widehat{f}"] \arrow[d] & \mathfrak{t}^n \arrow[d] \\ S^1 \arrow[r, "f"] & T^n \end{tikzcd}$$

where two vertical maps are the exponential maps from Lie algebras to Lie groups. Because $(c_1, \ldots, c_n) \in \mathbb{Z}^n$, the lift $\widehat{f}$ preserve lattices, i.e., $\widehat{f}(\mathbb{Z}) \subset \mathfrak{t}^n_{\mathbb{Z}}$, where $\mathfrak{t}^n_{\mathbb{Z}} \simeq \mathbb{Z}^n$ is the lattice in $\mathfrak{t}^n$. Hence, there is the following isomorphism:

$$\begin{array}{ccc} \mathrm{Hom}(S^1, T^n) & \xrightarrow{\simeq} & \mathfrak{t}^n_{\mathbb{Z}} \\ \cup & & \cup \\ f & \longmapsto & \widehat{f}(1) = (c_1, \ldots, c_n). \end{array} \tag{7.7}$$

Using the composition of two isomorphisms (7.6) and (7.7) as above together with taking their tensor products with $\mathbb{R}$, we have the following isomorphism:

$$(J_n)_* : H_2(BT^n;\mathbb{R}) \xrightarrow{\simeq} \mathfrak{t}^n. \tag{7.8}$$

The isomorphism (7.4) is define by taking the dual of $(J_n)_*$.

7.1.2. *Definition of the* 2nd *isomorphisms.* We next define the 2nd isomorphism

$$J_m^* : (\mathfrak{t}^m)^* \longrightarrow H^2_T(M_\alpha;\mathbb{R}) \tag{7.9}$$

as follows:

$$\begin{array}{ccc} (\mathfrak{t}^m)^* & \xrightarrow{\simeq} & H^2_T(M_\alpha;\mathbb{R}) \\ \cup & & \cup \\ \partial_i & \overset{J_m^*}{\longmapsto} & \tau_i, \end{array}$$

where ∂_i is the dual basis in $(\mathfrak{t}^m)^*$ ($i = 1, \ldots, m$) of the basis $\mathbf{e}_i$ in $\mathfrak{t}^m \simeq \mathbb{R}^m$ (see Remark 2.5), and τ_i is the canonical generator of the equivariant cohomology $H^*_T(M_\alpha;\mathbb{Z})$ (see (5.7) and Theorem 5.4); here, we may regard it as the basis of $H^2_T(M_\alpha;\mathbb{R})$.

Note that the isomorphism J_m^* in (7.9) is also defined by the dual of the following isomorphism:

$$\begin{array}{ccc} H_2^T(M_\alpha;\mathbb{R}) & \underset{\simeq}{\xrightarrow{(J_m)_*}} & \mathfrak{t}^m \\ \cup & & \cup \\ \mathbf{u}_i & \longmapsto & \mathbf{e}_i, \end{array}$$

where $\mathbf{u}_i$ is the basis which satisfies (6.5).

7.1.3. *Preparation to define the* $3^{\rm rd}$ *isomorphisms: the geometric interpretation of* v_i. In order to define the $3^{\rm rd}$ isomorphisms, we recall the geometric meaning of the unique element $v_i \in H_2(BT;\mathbb{Z}) \subset H_2(BT;\mathbb{R})$ in the following formulation (6.4) (also see Proposition 6.1);

$$\pi_{\mathbb{R}}^*(u) = \sum_{i=1}^{m} \langle u, v_i\rangle_H \tau_i \in H_T^2(M_\alpha;\mathbb{R}),$$

for $u \in H^2(BT;\mathbb{R})$. The purpose of Section 7.1.3 is to prove the following proposition.

PROPOSITION 7.2. *Let T_i be the circle subgroup in T^n appeared in Corollary 3.7, i.e., the circle subgroup which fixes the characteristic submanifold M_i. Let $\lambda_{v_i} : S^1 \to T^n$ be the homomorphism which corresponds to the unique element $v_i \in H_2(BT;\mathbb{Z})$ in Proposition 6.1 via the inverse of isomorphism* (7.6)*. Then, the following relation holds:*

$$\lambda_{v_i}(S^1) = T_i.$$

We note that the arguments we will use in this subsection (Section 7.1.3) is almost similar to the proof of [**Ma1**, Lemma 1.10].

First, we recall the tangent spaces of fixed points of toric hyperKähler manifolds (see [**HaHo**, Section 3] for details). Let p be a fixed point, i.e., $p \in M_\alpha^T$, and $M_i \subset M_\alpha$ be the characteristic submanifold (see Section 3.2) ($i = 1,\dots,m$). Put $I(p) = \{i \mid p \in M_i^T\}$. By the definitions of toric hyperKähler manifolds and their characteristic submanifolds (also see Proposition 3.6), we have that the cardinality of $I(p)$ is just n for all $p \in M_\alpha^T$, i.e., $|I(p)| = n$, and $\{p\} = \cap_{i\in I(p)} M_i$. Using (5.2), there exists the following decomposition:

$$T_pM_\alpha = \bigoplus_{i\in I(p)} \nu_i|_p = \bigoplus_{i\in I(p)} V(\chi_i) \oplus V(-\chi_i), \tag{7.10}$$

where $V(\chi_i) \oplus V(-\chi_i) \simeq \mathbb{C} \oplus \overline{\mathbb{C}} \simeq \mathbb{H}$, the representation $\chi_i : T^n \to S^1$ is induced from the normal representation of M_i on p and $-\chi_i$ is its orientation-reversing representation, i.e., $-\chi_i(t) = \chi_i(t)^{-1}$.

Let us interpret the above $\chi_i \in \mathrm{Hom}(T^n, S^1)$ as the element in the (equivariant) cohomology $H^2(BT^n;\mathbb{Z})$ (see Proposition 7.4). Taking the dual of (7.6), we first define the following isomorphism:

$$\begin{array}{ccc} H^2(BT^n;\mathbb{Z}) & \xrightarrow{\simeq} & \mathrm{Hom}(T^n, S^1) \\ \Psi & & \Psi \\ u & \longmapsto & \chi^u. \end{array} \tag{7.11}$$

Then, we have the following lemma:

LEMMA 7.3. *Let $(B\chi^u)^* : H^*(BS^1;\mathbb{Z}) \to H^*(BT^n;\mathbb{Z})$ be the induced representation. Then, the following equation holds:*

$$(B\chi^u)^*(\kappa^*) = u$$

for the dual basis $\kappa^* \in H^2(BS^1;\mathbb{Z})$ of a basis $\kappa \in H_2(BS^1;\mathbb{Z})$.

PROOF. Let $f \in \mathrm{Hom}(S^1, T^n)$. By definition, $\chi^u \circ f : S^1 \to S^1$ can be defined by the equation $\chi^u \circ f(r) = r^{\langle u, Bf_*(\kappa)\rangle_H}$, where $r \in S^1$. By using the arguments demonstrated in Section 7.1.1 (in this case, $n = 1$), this equation implies that

$$\langle u, Bf_*(\kappa)\rangle_H = \langle \kappa^*, (B\chi^u \circ f)_*(\kappa)\rangle_H.$$

Because this equation holds for all $f \in \mathrm{Hom}(S^1, T^n)$, we have the statement of this lemma. $\square$

Let ι_p be the inclusion $\iota_p : \{p\} \to M_\alpha$ and ι_p^T be its induced inclusion $\iota_p^T : ET \times_T \{p\} \to ET \times_T M_\alpha$. The inclusion ι_p^T induces the following representation:

$$\iota_p^* : H_T^*(M_\alpha) \to H_T^*(\{p\}) = H^*(BT).$$

We have the following proposition:

PROPOSITION 7.4. *Let $\chi_i : T^n \to S^1$ be the representation in* (7.10) *and $\chi^{\iota_p^*(\tau_i)}$ be the representation corresponding to $\iota_p^*(\tau_i)$ by* (7.11)*. Then, for all $i \in I(p)$, $\chi_i = \chi^{\iota_p^*(\tau_i)}$.*

PROOF. The representation $\chi_i : T^n \to S^1$ in (7.10) induces the continuous map $B\chi_i : BT \to BS^1$. Now we may regard (by changing the sign of κ if we need) the basis $\kappa^* \in H^*(BS^1;\mathbb{Z})$ as the 1st Chern class of the canonical line bundle η:

$$E(\eta) = ES^1 \times_{S^1} \mathbb{C} \longrightarrow BS^1,$$

where S^1 acts on $\mathbb{C}$ by the scalar multiplication (rotated by one time only), i.e.,

$$\kappa^* = c_1(\eta).$$

Let $ET \times_T V(\chi_i)$ be the Borel construction of the representation space $V(\chi_i)$. Then we may regard $ET \times_T V(\chi_i) \to BT$ as the line bundle over BT and denote this line bundle as γ_i. It is easy to see that γ_i is the pull-back of η along $B\chi_i : BT \to BS^1$, i.e.,

$$\begin{tikzcd} ET \times_T V(\chi_i) \arrow[r] \arrow[d] & ES^1 \times_{S^1} \mathbb{C} \arrow[d] \\ BT \arrow[r, "B\chi_i"] & BS^1. \end{tikzcd}$$

Therefore, $B\chi_i^*(\kappa^*) = B\chi_i^* c_1(\eta) = c_1(\gamma_i)$. Hence, it follows from Lemma 7.3 that $B\chi_i^* = (B\chi^{c_1(\gamma_i)})^*$. Thus, together with the definition of J_n^* in Section 7.1.1, we have

$$\chi_i = \chi^{c_1(\gamma_i)}.$$

Hence, in order to complete the proof, i.e., to prove $\chi_i = \chi^{\iota_p^*(\tau_i)}$, it is sufficient to show that $c_1(\gamma_i) = \iota_p^*(\tau_i)$.

Using (5.5) and (7.10), one can easily show that γ_i is the pull-back of $\mathbb{L}_i$ (see (5.6)) along the following inclusion:

$$\iota_p^T : ET \times_T \{p\} \hookrightarrow ET \times_T M_i \hookrightarrow ET \times_T M_\alpha.$$

Because $c_1(\mathbb{L}_i) = \tau_i$, we have $c_1(\gamma_i) = \iota_p^*(\tau_i)$. This establishes Proposition 7.4. $\square$

By the definition of the toric hyperKähler manifold M_α, we see that T^n acts on M_α effectively. Hence, using the decomposition (7.10) and Proposition 7.4 we can easily show that the following corollary:

COROLLARY 7.5. *The set $\{\iota_p^*(\tau_i) \mid i \in I(p)\}$ is a basis of $H^2(BT;\mathbb{Z})$ for all $p \in M_\alpha^T$.*

Using the definition of $\mathbb{L}_i$ (see (5.6)), we have that the restricted bundle $\mathbb{L}_i|_p$ to $p \in M^T \backslash M_i^T$ is the trivial line bundle over BT. Therefore, it follows from Corollary 7.5 that we have the following relations:

$$\begin{cases} \iota_p^*(\tau_i) = 0 & \text{if } p \in M^T \backslash M_i^T \\ \iota_p^*(\tau_i) \neq 0 & \text{if } p \in M_i^T. \end{cases} \tag{7.12}$$

The following lemma tells us the relation between $\iota_p^*(\tau_i) \in H^2(BT)$ and $v_i \in H_2(BT)$ in Proposition 6.1.

LEMMA 7.6. *Let v_i be the element appearing in Proposition 6.1. Then, the set $\{\iota_p^*(\tau_i) \mid i \in I(p)\} \subset H^2(BT;\mathbb{Z})$ is the dual basis of $\{v_i \mid i \in I(p)\} \subset H_2(BT;\mathbb{Z})$.*

PROOF. By using Proposition 6.1 and (7.12), we have the following equation:

$$\iota_p^* \circ \pi^*(\iota_p^*(\tau_i)) = \sum_{j \in I(p)} \langle \iota_p^*(\tau_i), v_j \rangle_H \iota_p^*(\tau_j).$$

Because $ET \times_T \{p\} \cong BT \xrightarrow{\iota_p} ET \times_T M \xrightarrow{\pi} BT$ can be regarded as the identity map, the induced map $\iota_p^* \circ \pi^*$ is the identity map. Therefore, we have

$$\iota_p^*(\tau_i) = \sum_{j \in I(p)} \langle \iota_p^*(\tau_i), v_j \rangle_H \iota_p^*(\tau_j).$$

By Corollary 7.5, there are no linear relations among $\iota_p^*(\tau_j)$'s. Therefore we have the following equation:

$$\begin{cases} \langle \iota_p^*(\tau_i), v_j\rangle_H = 0 & \text{if } i \neq j \\ \langle \iota_p^*(\tau_i), v_j\rangle_H = 1 & \text{if } i = j. \end{cases}$$

This establishes that the element $\iota_p^*(\tau_i)$ is the dual basis of v_i for all $i \in I(p)$. $\square$

In order to prove Proposition 7.2, we prepare one more lemma:

LEMMA 7.7. *Let $\chi^u \in \mathrm{Hom}(T, S^1)$ (resp. $\lambda_v \in \mathrm{Hom}(S^1, T)$) be the corresponding homomorphism to $u \in H^2(BT;\mathbb{Z})$ (resp. $v \in H_2(BT;\mathbb{Z})$) via (7.11) (resp. (7.6)). Then*

$$\chi^u \circ \lambda_v(r) = r^{\langle u,v\rangle_H}$$

for all $r \in S^1$.

PROOF. Let $\kappa^* \in H^2(BS^1;\mathbb{Z}) \subset H^2(BS^1;\mathbb{R})$ be the dual basis of $\kappa \in H_2(BS^1;\mathbb{Z}) \subset H_2(BS^1;\mathbb{R})$. Put $(B\chi^u \circ B\lambda_v)_* : H_*(BS^1;\mathbb{R}) \to H_*(BS^1;\mathbb{R})$ the induced homomorphism by $\chi^u \circ \lambda_v : S^1 \to S^1$. Because $\kappa \in H_2(BS^1;\mathbb{Z})$ is a generator, we can put $(B\chi^u \circ B\lambda_v)_*(\kappa) = a(u,v)\kappa$ for some $a(u,v) \in \mathbb{Z}$. Using (7.6) and Lemma 7.3, we have

$$\begin{aligned} \langle u, v\rangle_H &= \langle (B\chi^u)^*(\kappa^*), (B\lambda_v)_*(\kappa)\rangle_H = \langle \kappa^*, B\chi^u_* \circ (B\lambda_v)_*(\kappa)\rangle_H \\ &= \langle \kappa^*, (B\chi^u \circ B\lambda_v)_*(\kappa)\rangle_H = \langle \kappa^*, a(u,v)\kappa\rangle_H \\ &= a(u,v). \end{aligned}$$

It follows that

$$(B\chi^u \circ B\lambda_v)_*(\kappa) = \langle u, v\rangle_H \kappa.$$

Therefore, by using the isomorphism (7.6), (7.7) for $n = 1$, we have that

$$\chi^u \circ \lambda_v(r) = r^{\langle u,v\rangle_H}$$

for $r \in S^1$. $\square$

At this point, we can prove Proposition 7.2.

PROOF OF PROPOSITION 7.2. Using Lemma 7.6 and 7.7, we have $\chi^{\iota_p^*(\tau_i)} \circ \lambda_{v_i}(r) = r$ for all $p \in M_i^T$ and $r \in S^1$. By Proposition 7.4, we also have $\chi_i \circ \lambda_{v_i}(r) = r$ for all $r \in S^1$. It follows from the decomposition (7.10) that $\lambda_{v_i}(S^1)$ acts on the normal space of M_i on each $p \in M_i^T$ effectively. Therefore, one can easily show that the circle subgroup $\lambda_{v_i}(S^1) \subset T^n$ acts trivially on M_i. Thus, we have that $\lambda_{v_i}(S^1) = T_i$. This establishes Proposition 7.2. $\square$

7.1.4. *Definition of the 3rd isomorphisms.* We finally define the 3rd isomorphism

$$J_K^* : \mathfrak{k}^* \longrightarrow H^2(M_\alpha;\mathbb{R}), \tag{7.13}$$

and prove Proposition 7.1.

In order to do that, we first prove the following proposition:

PROPOSITION 7.8. *The following diagram is a commutative diagram:*

$$\begin{tikzcd} (\mathfrak{t}^n)^* \arrow[r, "\rho^*"] \arrow[d, "J_n^*"] & (\mathfrak{t}^m)^* \arrow[d, "J_m^*"] \\ H^2(BT^n;\mathbb{R}) \arrow[r, "\pi_{\mathbb{R}}^*"] & H_T^2(M_\alpha;\mathbb{R}) \end{tikzcd} \tag{7.14}$$

where ρ^*, $\pi_{\mathbb{R}}^*$, J_n^* and J_m^* are defined by (2.5) (or see (7.1)), (6.4), (7.4) and (7.9), respectively.

In order to prove Proposition 7.8, we prepare the following lemma:

LEMMA 7.9. *The following equation holds for all $i = 1, \ \ldots, \ m$,*

$$(J_n)_*(v_i) = \rho_*(\boldsymbol{e}_i),$$

where v_i is the unique element appearing in Proposition 6.1.

Proof. Let $\lambda_{v_i} \in \mathrm{Hom}(S^1, T)$ be the element which corresponds to $v_i \in H_2(BT;\mathbb{Z})$ via the isomorphism (7.6). By the definition of $(J_n)_*$ in Section 7.1.1, we have that

$$(J_n)_*(v_i) = \widehat{\lambda}_{v_i}(1),$$

where $\widehat{\lambda}_{v_i} : \mathbb{R} \to \mathfrak{t}^n$ is the lift of λ_{v_i}. Therefore, it is enough to show that $\widehat{\lambda}_{v_i}(1) = \rho_*(\mathbf{e}_i)$.

Because $\mathbf{e}_i \in \mathfrak{t}^m_{\mathbb{Z}}$, we have that $\rho_*(\mathbf{e}_i) \in \mathfrak{t}^n_{\mathbb{Z}}$. Let $\lambda_{\rho_*(\mathbf{e}_i)} \in \mathrm{Hom}(S^1, T)$ be the element which corresponds to $\rho_*(\mathbf{e}_i) \in \mathfrak{t}^n_{\mathbb{Z}}$ via the isomorphism (7.7), i.e., $\widehat{\lambda}_{\rho_*(\mathbf{e}_i)}(1) = \rho_*(\mathbf{e}_i)$ for the lift $\widehat{\lambda}_{\rho_*(\mathbf{e}_i)} : \mathbb{R} \to \mathfrak{t}^n$. Hence, using Corollary 3.7 and Proposition 7.2, we have that

$$\lambda_{\rho_*(\mathbf{e}_i)}(S^1) = \{\exp r\rho_*(\mathbf{e}_i) \mid r \in \mathbb{R}\} = T_i = \lambda_{v_i}(S^1).$$

It follows that $\widehat{\lambda}_{\rho_*(\mathbf{e}_i)}(1) = \rho_*(\mathbf{e}_i) = \widehat{\lambda}_{v_i}(1)$. This establishes the statement of Lemma 7.9. $\square$

Now we may prove prove Proposition 7.8.

Proof of Proposition 7.8. Let $x \in (\mathfrak{t}^n)^*$. Using the formula (6.4), we have the equation

$$\pi_{\mathbb{R}}^* \circ J_n^*(x) = \sum_{i=1}^m \langle J_n^*(x), v_i\rangle_H \tau_i,$$

where $v_i \in H_2(BT;\mathbb{Z}) \subset H_2(BT;\mathbb{R})$ and $\tau_i \in H^2_T(M;\mathbb{Z}) \subset H^2_T(M;\mathbb{R})$ $(i = 1, \ldots, m)$. By the definition of J_n^* in Section 7.1, we also have the equation

$$\langle J_n^*(x), v_i\rangle_H = \langle x, (J_n)_*(v_i)\rangle,$$

where $(J_n)_* : H_2(BT;\mathbb{R}) \to \mathfrak{t}^n$ is the isomorphism defined in (7.8). Note that, in the above equation, the left $\langle,\rangle_H$ is the pairing of the cohomology and homology and the right $\langle,\rangle$ is the pairing of the dual Lie algebra and Lie algebra. Using the above two equations, we have

$$\pi_{\mathbb{R}}^* \circ J_n^*(x) = \sum_{i=1}^m \langle x, (J_n)_*(v_i)\rangle_H \tau_i. \tag{7.15}$$

On the other hand, the homomorphism $\rho^* : (\mathfrak{t}^n)^* \to (\mathfrak{t}^m)^*$ can be denoted as follows:

$$\begin{aligned} \rho^*(x) &= \sum_{i=1}^m \langle \rho^*(x), \mathbf{e}_i\rangle \partial_i \\ &= \sum_{i=1}^m \langle x, \rho_*(\mathbf{e}_i)\rangle \partial_i \in (\mathfrak{t}^m)^*, \end{aligned} \tag{7.16}$$

where $\mathbf{e}_i$ is the natural basis in $\mathfrak{t}^m$ and ∂_i is its dual basis in $(\mathfrak{t}^m)^*$. Therefore, by the definition of J_m^* in Section 7.1.2 and (7.16), we have that

$$J_m^* \circ \rho^*(x) = \sum_{i=1}^m \langle x, \rho_*(\mathbf{e}_i)\rangle_H \tau_i \tag{7.17}$$

Using (7.15), (7.17) and Lemma 7.9, we have Proposition 7.8. $\square$

Let us define the 3rd isomorphism J_K^* in (7.13). First, we regard the two homomorphisms in (7.1) and (7.2) as the quotient homomorphism:

$$\iota^* : (\mathfrak{t}^m)^* \to \mathfrak{k}^* \simeq (\mathfrak{t}^m)^*/\mathrm{Im}(\rho^*)$$

and

$$j_{\mathbb{R}}^* : H^2_T(M;\mathbb{R}) \to H^2(M;\mathbb{R}) \simeq H^2_T(M;\mathbb{R})/\mathrm{Im}(\pi_{\mathbb{R}}^*).$$

Then, using Proposition 7.8, we get the well-defined isomorphism $J_K^* : \mathfrak{k}^* \to H^2(M;\mathbb{R})$ as the quotient homomorphism of the isomorphism $J_m^* : (\mathfrak{t}^m)^* \to H^2_T(M;\mathbb{R})$ defined by (7.9). This gives the definition of the 3rd isomorphism (7.13). Using Proposition 7.8 and the definition of J_K^*, it is easy to check Proposition 7.1.

7.2. Equivalence of two hyperplane arrangements. In this subsection, we prove that the hyperplane arrangement $\mathcal{H}_{\widehat{\alpha}} \subset (\mathfrak{t}^n)^*$ defined by $(M_\alpha, T^n, \mu_{\widehat{\alpha}})$ is weak equivalent to the hyperplane arrangement $\mathcal{H}^{eq}_{\widehat{a}} \subset H^2(BT;\mathbb{R})$ defined by the triple $(H^2_T(M_\alpha), \pi^*, \widehat{a})$, where $\widehat{a}$ is the element in $H^2_T(M_\alpha)$ such that $\widehat{a} = J^*_m(\widehat{\alpha})$. Namely, we prove that the isomorphism $J^*_n : (\mathfrak{t}^n)^* \to H^2(BT;\mathbb{R})$ defined in Section 7.1.1 preserves these two hyperplane arrangements.

We first recall two hyperplane arrangements (see Section 3 and Section 6). Using the top exact sequence in (7.3), the hyperplane arrangement $\mathcal{H}_{\widehat{\alpha}} = \{H_1, \dots, H_m\}$ of $(M_\alpha, T^n, \mu_{\widehat{\alpha}})$ satisfies

$$H_i = \{x \in (\mathfrak{t}^n)^* \mid \langle \rho^*(x) + \widehat{\alpha}, \mathbf{e}_i\rangle = 0\},$$

where $\widehat{\alpha} \in (\mathfrak{t}^m)^*$ is a lift of the non-zero element $\alpha \in \mathfrak{k}^*$ and $\mathbf{e}_i \in \mathfrak{t}^m$ is the basis whose dual is $\partial_i \in (\mathfrak{t}^m)^*$. On the other hand, using the bottom exact sequence in (7.3), we have the hyperplane arrangement $\mathcal{H}^{eq}_{\widehat{a}} = \{H^{eq}_1, \dots, H^{eq}_m\}$ of $(H^*_T(M_\alpha), \pi^*, \widehat{a})$ for $\widehat{a}(= J^*_m(\widehat{\alpha})) \in H^2_T(M_\alpha;\mathbb{R})$ as follows:

$$H^{eq}_i = \{y \in H^2(BT;\mathbb{R}) \mid \langle \pi^*_{\mathbb{R}}(y) + \widehat{a}, \mathbf{u}_i\rangle = 0\},$$

where $\mathbf{u}_i \in H^T_2(M_\alpha;\mathbb{R})$ is the basis whose dual is tha canonical generator $\tau_i \in H^2_T(M)$.

REMARK 7.10. Using the following three facts: $(J_m)_*(\mathbf{u}_i) = \mathbf{e}_i$ (see Section 7.1.2); $\rho_*(\mathbf{e}_i) \neq 0$ (by the assumption of hyperplanes in Section 3.1); and the commutativity of the dual of the diagram (7.3), we have

$$(\pi_{\mathbb{R}})_*(\mathbf{u}_i) \neq 0$$

for all $i = 1, \dots, m$. This gives that $\dim H^{eq}_i = n - 1$.

Let us prove the equivalence of $\mathcal{H}_{\widehat{\alpha}}$ and $\mathcal{H}^{eq}_{\widehat{a}}$.

PROPOSITION 7.11. *Let $\mathcal{H}_{\widehat{\alpha}} = \{H_1, \dots, H_m\}$ be the hyperplane arrangement of $(M_\alpha, T^n, \mu_{\widehat{\alpha}})$ and $\mathcal{H}^{eq}_{\widehat{a}} = \{H^{eq}_1, \dots, H^{eq}_m\}$ that of $(H^*_T(M_\alpha), \pi^*, \widehat{a})$, where $\widehat{a} = J^*_m(\widehat{\alpha})$. Then, the isomorphism $J^*_n : (\mathfrak{t}^n)^* \to H^2(BT;\mathbb{R})$ satisfies that*

$$J^*_n(H_i) = H^{eq}_i$$

for all $i = 1, \dots, m$. In particular, $\mathcal{H}_{\widehat{\alpha}} \equiv_w \mathcal{H}^{eq}_{\widehat{a}}$.

PROOF. Let $x \in H_i \in \mathcal{H}_{\widehat{\alpha}}$. Then, $J^*_n(x) \in H^2(BT;\mathbb{R})$ satisfies the following equations:

$$\begin{array}{lll} & \langle \pi^*_{\mathbb{R}} \circ J^*_n(x) + \widehat{a}, \mathbf{u}_i\rangle_H & \\ = & \langle J^*_m \circ \rho^*(x) + J^*_m(\widehat{\alpha}), \mathbf{u}_i\rangle_H & \text{(by the commutativity of (7.3))} \\ = & \langle \rho^*(x) + \widehat{\alpha}, (J_m)_*(\mathbf{u}_i)\rangle & \text{(by the definition of } J^*_m) \\ = & \langle \rho^*(x) + \widehat{\alpha}, \mathbf{e}_i\rangle & \text{(by } (J_m)_*(\mathbf{u}_i) = \mathbf{e}_i) \\ = & 0. & \text{(by definition of } H_i) \end{array}$$

Therefore, $J^*_n(H_i) \subset H^{eq}_i$. Because J^*_n is the isomorphism and $\dim H_i = \dim H^{eq}_i = n - 1$ (by the assumption of hyperplanes in Section 3.1 and Remark 7.10), we have that $J^*_n(H_i) = H^{eq}_i$ for all $i = 1, \dots, m$.

Because of the definition of J^*_n (in Section 7.1.1), we may regarded J^*_n as the linear isomorphism induced from an isomorphism $T^n \to T^n$. By the definition of the weak equivalence of hyperplane arrangements (in Section 4.2) and using the arguments as above, it is easy to check that $\mathcal{H}_{\widehat{\alpha}} \equiv_w \mathcal{H}^{eq}_{\widehat{a}}$. □

8. T^n-equivariant cohomological rigidity

In this final section, we prove the following main theorem of this paper:

THEOREM 8.1. *Let $(M_\alpha, T, \mu_{\widehat{\alpha}})$ and $(M_\beta, T, \mu_{\widehat{\beta}})$ be toric hyperKähler manifolds with hyperKähler moment maps. The following two statements are equivalent:*

1. *$(M_\alpha, T, \mu_{\widehat{\alpha}}) \equiv_w (M_\beta, T, \mu_{\widehat{\beta}})$;*
2. *there exists a weak $H^*(BT)$-algebra isomorphism $f^*_T : H^*_T(M_\alpha;\mathbb{Z}) \to H^*_T(M_\beta;\mathbb{Z})$ such that $f^*_T(\widehat{a}) = \widehat{b}$,*

where $\widehat{a} = J^*_{m_1}(\widehat{\alpha})$ and $\widehat{b} = J^*_{m_2}(\widehat{\beta})$ for the isomorphisms $J^*_{m_1} : (\mathfrak{t}^{m_1})^* \to H^2_T(M_\alpha;\mathbb{R})$ and $J^*_{m_2} : (\mathfrak{t}^{m_2})^* \to H^2_T(M_\beta;\mathbb{R})$ defined in Section 7.1.2.

Let $(M_\alpha, T^n, \mu_{\widehat{\alpha}})$, $(M_\beta, T^n, \mu_{\widehat{\beta}})$ be two toric hyperKähler manifolds, and let $\mathcal{H}^{eq}_{\widehat{a}}$, $\mathcal{H}^{eq}_{\widehat{b}}$ be their respective hyperplane arrangements induced from the equivariant cohomology, where $\widehat{a} = J^*_{m_1}(\widehat{\alpha})$ and $\widehat{b} = J^*_{m_2}(\widehat{\beta})$. Here, $M_\alpha = \mu^{-1}_{HK}(\alpha, 0)/K_1$ such that $\mu_{HK} : \mathbb{H}^{m_1} \to (\mathfrak{k}_1)^* \oplus (\mathfrak{k}_1)^*_{\mathbb{C}}$ and $M_\beta = \mu^{-1}_{HK}(\beta, 0)/K_2$ such that $\mu_{HK} : \mathbb{H}^{m_2} \to (\mathfrak{k}_2)^* \oplus (\mathfrak{k}_2)^*_{\mathbb{C}}$. By using Theorem 4.3 and Proposition 7.11, in order to prove Theorem 8.1, it is enough to show the following two statements are equivalent:

(1) $\mathcal{H}^{eq}_{\widehat{a}} \equiv_w \mathcal{H}^{eq}_{\widehat{b}}$;
(2) there exists a weak $H^*(BT)$-algebra isomorphism $f^*_T : H^*_T(M_\alpha; \mathbb{Z}) \to H^*_T(M_\beta; \mathbb{Z})$ such that $f^*_T(\widehat{a}) = \widehat{b}$.

8.1. Proof of (1) $\Rightarrow$ (2)**.** We first prove the following proposition:

Proposition 8.2. *If $\mathcal{H}^{eq}_{\widehat{a}} \equiv_w \mathcal{H}^{eq}_{\widehat{b}}$, then there exists a weak $H^*(BT)$-algebra isomorphism $f^*_T : H^*_T(M_\alpha; \mathbb{Z}) \to H^*_T(M_\beta; \mathbb{Z})$ such that $f^*_T(\widehat{a}) = \widehat{b}$.*

Proof. Let $\varphi^* : H^2(BT) \to H^2(BT)$ be an isomorphism such that $\varphi^*_{\mathbb{R}}$ gives $\mathcal{H}^{eq}_{\widehat{a}} \equiv_w \mathcal{H}^{eq}_{\widehat{b}}$. Then, we may put m as the cardinalities of $\mathcal{H}^{eq}_{\widehat{a}}$ and $\mathcal{H}^{eq}_{\widehat{b}}$. Due to Proposition 7.1, two exact sequences appearing in Proposition 7.1 are identical. Therefore, with the method similar to that demonstrated in the proof of Lemma 4.5, there exists the lift f^*_T such that the following diagram commutes:

$$
\begin{array}{ccc}
H^2(BT;\mathbb{R}) & \xrightarrow{(\pi_1^*)_{\mathbb{R}}} & H^2_T(M_\alpha;\mathbb{R}) \\
\Big\downarrow{\scriptstyle \varphi^*_{\mathbb{R}}} & & \Big\downarrow{\scriptstyle f^*_T} \\
H^2(BT;\mathbb{R}) & \xrightarrow{(\pi_2^*)_{\mathbb{R}}} & H^2_T(M_\beta;\mathbb{R})
\end{array}
\tag{8.1}
$$

and

$$f^*_T(\widehat{a}) = \widehat{b},$$

where $\pi_1 : ET \times_T M_\alpha \to BT$ and $\pi_2 : ET \times_T M_\beta \to BT$ are the projections of the Borel constructions. Furthermore, the above f^*_T can be represented as the following matrix:

$$
\begin{pmatrix}
\epsilon_1 & 0 & \cdots & 0 \\
0 & \epsilon_2 & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & \epsilon_m
\end{pmatrix} \cdot \Sigma
$$

with respect to the canonical basis $\tau^{(1)}_i$ of $H^2_T(M_\alpha)$ and $\tau^{(2)}_i$ of $H^2_T(M_\beta)$ ($i \in [m]$), where $\epsilon_i = \pm 1$ and Σ is the $(m \times m)$-matrix induced from the permutation $\sigma : [m] \to [m]$. Namely, for the canonical basis of $H^*_T(M_\alpha)$ and $H^*_T(M_\beta)$, we have

$$f^*_T(\tau^{(1)}_i) = \epsilon_i \tau^{(2)}_{\sigma(i)}.$$

We claim that this f^*_T extends to the weak $H^*(BT)$-algebra isomorphism. Because $\varphi^*_{\mathbb{R}}(H^{(1)}_i) = H^{(2)}_{\sigma(i)}$ for $H^{(1)}_i \in \mathcal{H}^{eq}_{\widehat{a}}$ and $H^{(2)}_{\sigma(i)} \in \mathcal{H}^{eq}_{\widehat{b}}$, we have that the following two statements are equivalent for $I \subset [m]$:

- $\cap_{i \in I} H^{(1)}_i = \emptyset$;
- $\cap_{\sigma(i) \in \sigma(I)} H^{(2)}_{\sigma(i)} = \emptyset$.

Therefore, due to Theorem 5.4, the linear isomorphism $f^*_T : H^2_T(M_\alpha) \to H^2_T(M_\beta)$ naturally extends to the ring isomorphism

$$f^*_T : H^*_T(M_\alpha) \longrightarrow H^*_T(M_\beta).$$

Note that $H^*(BT)$ is the polynomial ring (see Proposition 5.2), i.e., there is no relations among generators $\{x_1, \ldots, x_n\}$ in $H^*(BT)$. Therefore, the linear isomorphism $\varphi^* : H^2(BT) \to H^2(BT)$

also naturally extends to the ring isomorphism

$$\varphi^* : H^*(BT) \longrightarrow H^*(BT).$$

By using Proposition 6.1 and the commutativity of the diagram (8.1), it is easy to check that, for every degree, the following diagram commutes:

$$\begin{tikzcd}
H^*(BT) \arrow[r, "\pi_1^*"] \arrow[d, "\varphi^*"] & H_T^*(M_\alpha) \arrow[d, "f_T^*"] \\
H^*(BT) \arrow[r, "\pi_2^*"] & H_T^*(M_\beta)
\end{tikzcd}$$

i.e., f_T^* is a weak $H^*(BT)$-algebra isomorphism. $\square$

8.2. Proof of (2) $\Rightarrow$ (1)**.** We next prove the converse of Proposition 8.2. Assume that $f_T^* : H_T^*(M_\alpha) \to H_T^*(M_\beta)$ is a weak $H^*(BT)$-algebra isomorphism such that $f_T^*(\widehat{a}) = \widehat{b}$. Let $\mathcal{T}^{(1)} = \{\tau_1^{(1)}, \ldots, \tau_{m_1}^{(1)}\}$ and $\mathcal{T}^{(2)} = \{\tau_1^{(2)}, \ldots, \tau_{m_2}^{(2)}\}$ be the canonical generators of $H_T^*(M_\alpha)$ and $H_T^*(M_\beta)$, respectively.

Because $H_T^*(M_\alpha) \simeq H_T^*(M_\beta)$, we have that $m_1 = m_2 = m$. The goal of this final subsection is to prove the following proposition:

Proposition 8.3. *If there exists a weak $H^*(BT)$-algebra isomorphism $f_T^* : H_T^*(M_\alpha;\mathbb{Z}) \to H_T^*(M_\beta;\mathbb{Z})$ such that $f_T^*(\widehat{a}) = \widehat{b}$, then $\mathcal{H}_{\widehat{a}}^{eq} \equiv_w \mathcal{H}_{\widehat{b}}^{eq}$.*

In order to prove Proposition 8.3, the most part of this subsection is devoted to the proof of the following key lemma:

Lemma 8.4. *For any weak $H^*(BT)$-algebra isomorphism $f_T^* : H_T^*(M_\alpha) \to H_T^*(M_\beta)$ such that $f_T^*(\widehat{a}) = \widehat{b}$, there exists a weak $H^*(BT)$-algebra isomorphism $g_T^* : H_T^*(M_\alpha) \to H_T^*(M_\beta)$ such that $g_T^*(\widehat{a}) = \widehat{b}$ and g_T^* preserves the canonical generators up to sign, i.e., there exists a permutation $\sigma : [m] \to [m]$ and $\epsilon_i = \pm 1$ such that*

$$g_T^*(\tau_i^{(1)}) = \epsilon_i \tau_{\sigma(i)}^{(2)},$$

for all $i \in [m]$.

In order to prove this lemma, we prepare some notations and facts (also see [**Ma2**]).

Let M_α^T be the set of T-fixed points in M_α. Because $p \in M_\alpha^T$ can be represented by

$$\{p\} = \cap_{j=1}^n M_{i_j}$$

for some characteristic submanifolds M_{i_j} ($j = 1, \ldots, n$, where $4n = \dim M_\alpha$), the fixed point set M_α^T consists of finitely many points.

For $\xi \in H_T^2(M_\alpha)$, we denote its restriction to $p \in M_\alpha^T$ by $\xi|_p$, i,e,

$$\xi|_p = \iota_p^*(\xi) \in H_T^*(\{p\}) = H^*(BT),$$

where ι_p^* is the induced homomorphism from the natural projection $\iota_p : \{p\} \to M_\alpha$. Set

$$Z(\xi) := \{p \in M_\alpha^T \mid \xi|_p = 0\}.$$

We call the cardinality of $Z(\xi)$ the *zero-length of* ξ, denote it by $|Z(\xi)|$. The following proposition tells us that the zero-length $|Z(\xi)|$ is invariant under an algebra isomorphism.

Proposition 8.5. *Let $f_T^* : H_T^*(M_\alpha) \to H_T^*(M_\beta)$ be a weak $H^*(BT)$-algebra isomorphism. Then, the following equation holds for all $\xi \in H_T^*(M_\alpha)$:*

$$|Z(\xi)| = |Z(f_T^*(\xi))|.$$

Proof. Let $S = H^*(BT) \setminus \{0\}$ and let $S^{-1}H_T^*(M)$ denote the localized ring of $H_T^*(M)$ by S, i.e.,

$$S^{-1}H_T^*(M) = \{\frac{r}{s} \mid r \in H_T^*(M),\ s \in S\}/\sim$$

where

$$\frac{r_1}{s_1} \sim \frac{r_2}{s_2} \iff (r_1 s_2 - r_2 s_1)t = 0 \ \text{ for some } t \in S.$$

Due to Theorem 5.4 and Proposition 6.1, $H_T^*(M_\alpha; \mathbb{Z})$ is free as a module over $H^*(BT; \mathbb{Z})$. Hence, because of the localization theorem in equivariant cohomology (see [**Hs**, p.40]), the natural map

$$H_T^*(M_\alpha) \longrightarrow S^{-1}H_T^*(M_\alpha) \simeq S^{-1}H_T^*(M_\alpha^T) = \bigoplus_{p \in M_\alpha^T} S^{-1}H_T^*(\{p\})$$

is injective (also see [**HaPr**, Theorem 2.5]), where the isomorphism appearing in the above maps is induced from the inclusion map from M_α^T to M_α. Therefore, we may regard an element $\xi \in H_T^*(M_\alpha)$ as an element in $S^{-1}H_T^*(M_\alpha)$. Then, the annihilator

$$\mathrm{Ann}(\xi) := \{\eta \in S^{-1}H_T^*(M_\alpha) \mid \eta\xi = 0\} \subset \bigoplus_{p \in M_\alpha^T} S^{-1}H_T^*(\{p\})$$

of ξ is nothing but the sum of $S^{-1}H_T^*(\{p\})$ over p with $\xi|_p = 0$, because $\eta|_p = 0$ if $\xi|_p \neq 0$. Therefore, $\mathrm{Ann}(\xi)$ is a free $S^{-1}H^*(BT)$-module of rank $|Z(\xi)|$. Since f_T^* is a weak $H^*(BT)$-algebra isomorphism, we have

$$\mathrm{Ann}(\xi) \simeq \mathrm{Ann}(f_T^*(\xi))$$

as a free $S^{-1}H^*(BT)$-module. This implies that $|Z(\xi)| = |Z(f_T^*(\xi))|$. □

Let $\mathcal{T}_0^{(s)} \subset \mathcal{T}^{(s)}$, for $s = 1, 2$, be the canonical generators whose zero-length are zero. Let $\mathcal{T}_1^{(s)}$ be the set in $\mathcal{T}^{(s)} \setminus \mathcal{T}_0^{(s)}$ with largest zero-length, and let $\mathcal{T}_2^{(s)}$ be the set in $\mathcal{T}^{(s)} \setminus \mathcal{T}_0^{(s)}$ second largest zero-length, and so on.

Remark 8.6. In toric manifolds, all of canonical generators satisfy $|Z(\tau_i)| \neq 0$, i.e., $\mathcal{T}_0^{(s)} = \emptyset$. However, in toric hyperKähler manifolds, there exists a canonical generator τ_i such that $|Z(\tau_i)| = 0$, i.e., $\mathcal{T}_0^{(s)} \neq \emptyset$. For example, let $M = M' \times \mathbb{H}$. Then, the characteristic submanifold M' contains all fixed points of M. This implies that the canonical generator τ which corresponds to M' satisfies $|Z(\tau)| = 0$. This gives one of the difference between the proof of the Masuda's theorem (see Theorem 1 in Section 1) for toric manifolds proved in [**Ma2**] and that of Theorem 1.1 in Section 1 for toric hyperKähler manifolds.

We first show the following property for $\mathcal{T}_0^{(s)}$ $(s = 1, 2)$:

Proposition 8.7. *If there exists $\tau_i \in \mathcal{T}_0^{(s)}$ $(s = 1, 2)$, then there exists the element $x_i \in H^2(BT)$ such that*

$$\pi_s^*(x_i) = \tau_i.$$

Proof. Assume $\tau_i \in \mathcal{T}_0^{(1)}$. By definition, we have that $\tau_i|_p \neq 0$ for all $p \in M^T$. Recall that the canonical generator τ_i corresponds to the characteristic submanifold M_i, and M_i corresponds to the hyperplane $H_i \subset H^2(BT; \mathbb{R})$ (see Section 5.2). Using $\tau_i|_p \neq 0$ for all $p \in M^T$, it is easy to check that $(M_i)^T = M_\alpha^T$. Therefore, because fixed points correspond to the intersection of just n hyperplanes, we have that $H_i \cap H_j \neq \emptyset$ for all $j \in [m] \setminus \{i\}$, where $m \geq n + 1$ (see Remark 3.5). Moreover, we have that there are no intersection points of just n hyperplanes in the complement of $H_i(\subset H^2(BT; \mathbb{R}))$. Using the smoothness of hyperplane arrangements and $m \geq n+1$, it is easy to show that there exists a primitive $x_i \in H^2(BT)$ such that all of the hyperplanes H_j (where $j \neq i$) satisfy

$$H_j = L_j \times \mathbb{R}x_i,$$

where $L_j = H_i \cap H_j$ is the $(n-2)$-dimensional hyperplane (see Figure 4). This means that, for

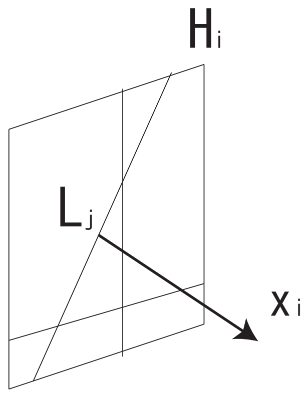

FIGURE 4. The hyperplane $H_j = L_j \times \mathbb{R}x_i$. Three lines in the hyperplane H_i represent the intersections with the other hyperplanes and $H_j \cap H_i = L_i$.

all $x \in H_j$ $(j \neq i)$, there exists $x' \in L_j = H_i \cap H_j$ and $r \in \mathbb{R}$ such that

$$x = x' + rx_i.$$

Hence, we have the following relations:

$$\begin{aligned} 0 &= \langle \pi_1^*(x) + \widehat{a}, \mathbf{u}_j \rangle_H && \text{(by } x \in H_j) \\ &= \langle \pi_1^*(x' + rx_i) + \widehat{a}, \mathbf{u}_j \rangle_H && \text{(by } x = x' + rx_i) \\ &= \langle \pi_1^*(rx_i), \mathbf{u}_j \rangle_H && \text{(by } x' \in H_j). \end{aligned}$$

Therefore, we have that

$$\langle \pi_1^*(x_i), \mathbf{u}_j \rangle_H = 0 \quad \text{for all } j \neq i.$$

Because π_1^* is injective, this also implies that $\langle \pi_1^*(x_i), \mathbf{u}_i \rangle_H \neq 0$. Because τ_i is the dual basis of $\mathbf{u}_i$ (see Section 7.2), we have that

$$\pi_1^*(x_i) = r'\tau_i \in H_T^2(M_\alpha)$$

for $r' = \langle \pi_1^*(x_i), \mathbf{u}_i \rangle \in \mathbb{Z} \setminus \{0\}$. Therefore, one can easily show that

$$r'\tau_i|_p = x_i,$$

for all $p \in M_\alpha^T$ (e.g. by using the localization theorem, also see [**HaHo, HaPr**]). Because x_i is a primitive vector in $H^2(BT)$, we have that $r' = \pm 1$. Therefore, by changing the sign if we need, we have that there exists $x_i \in H^2(BT)$ such that $\pi_1^*(x_i) = \tau_i$. This establishes the statement for $s = 1$.

Similarly, we have the statement for $s = 2$. □

Next, we prove that f_T^* preserves $\mathcal{T}_k^{(1)}$ to $\mathcal{T}_k^{(2)}$ for $k \neq 0$ (see Lemma 8.9). In order to do that, we need the following lemma:

LEMMA 8.8. *Let $\xi \in H_T^2(M_\alpha)$ be an element with $|Z(\xi)| \neq 0$. Express $\xi = \sum_{i=1}^m a_i\tau_i$ for some integers a_i. If $a_i \neq 0$ for some i, then $Z(\xi) \subset Z(\tau_i)$ for $\tau_i \in \mathcal{T}^{(1)} \setminus \mathcal{T}_0^{(1)}$.*

Furthermore, if $a_i \neq 0$ and $a_j \neq 0$ for some different i and j, then $Z(\xi) \subsetneq Z(\tau_i)$ for $\tau_i \in \mathcal{T}^{(1)} \setminus \mathcal{T}_0^{(1)}$.

PROOF. Let $p \in M_\alpha^T$ and $p \in Z(\xi)$. Then $0 = \xi|_p = \sum_{i=1}^m a_i\tau_i|_p$. Using Corollary 7.5, we have that if $a_i \neq 0$ then $\tau_i|_p = 0$. This establishes that if $a_i \neq 0$ then $Z(\xi) \subset Z(\tau_i)$; moreover, if both a_i and a_j are non-zero, then $Z(\xi) \subset Z(\tau_i) \cap Z(\tau_j)$. Therefore, it suffices to prove that $Z(\tau_i) \cap Z(\tau_j)$ is properly contained in $Z(\tau_i)$.

Suppose that $Z(\tau_i) \cap Z(\tau_j) = Z(\tau_i)$. Then $Z(\tau_j) \supset Z(\tau_i)$. By (7.12), we have that

$$\tau_i|_q = 0 \quad \text{if and only if} \quad q \notin M_i^T. \tag{8.2}$$

Therefore, $M_j^T \subset M_i^T$. Recall that fixed points of (M_α, T) correspond to the intersections of just n hyperplanes, and denote the corresponding hyperplane with M_k by H_k for $k \in [m]$. Hence, the condition $M_j^T \subset M_i^T$ gives that H_j satisfies that

$$H_j = L_j \times \mathbb{R}x_i,$$

for the $(n-2)$-dimensional hyperplane $L_j = H_j \cap H_i$ and a primitive vector x_i which is not parallel to H_i (also see the proof of Proposition 8.7). Because we assume $\tau_i|_p = 0$ for some $p \in M_\alpha^T$, there exists $x \notin H_i \cap H_j$ in $H^2(BT; \mathbb{R})$ such that $\{x\} = \cap_{k \in I(p)} H_k$ by using (8.2) (also see Figure 5), where $I(p) \subset [m] \setminus \{i,j\}$ is the set satisfying that $\{p\} = \cap_{k \in I(p)} M_k$, i.e., $x \in H^2(BT;\mathbb{R})$ is the corresponding intersection point to $p \in M_\alpha^T$. Because the hyperplane arrangement is smooth, it is easy to check that there exists $l \in I(p)$ such that, for $I'(p) = I(p) \setminus \{l\}$, the intersection $\cap_{k \in I'(p)} H_k$ is the 1-dimensional affine subspace which goes through both of x and H_j. It also follows from $I(p) \subset [m] \setminus \{i,j\}$ that the 1-dimensional affine subspace $\cap_{k \in I'(p)} H_k$ does not intersect with $L_j = H_i \cap H_j$. Therefore, there is the point $\{y\} = (\cap_{k \in I'(p)} H_k) \cap H_j$ such that $y \notin H_i$ (see Figure 5). This gives a contradiction to the assumption that $M_j^T \subset M_i^T$. Therefore, $Z(\tau_i) \cap Z(\tau_j) \neq Z(\tau_i)$.

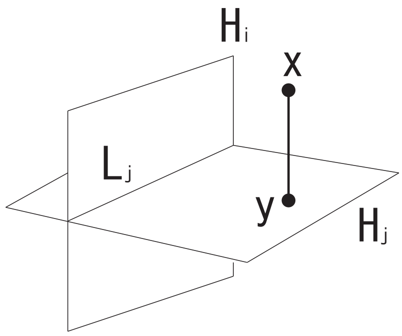

FIGURE 5

This establishes that $Z(\xi) \subsetneq Z(\tau_i)$. $\square$

Now we may prove Lemma 8.9.

LEMMA 8.9. *Every weak $H^*(BT)$-algebra isomorphism f_T^* preserves $\mathcal{T}_k^{(1)}$ to $\mathcal{T}_k^{(2)}$ up to sign for $k \neq 0$.*

PROOF. Let $m_k^{(1)}$ (resp. $m_k^{(2)}$) be the zero-length of elements in $\mathcal{T}_k^{(1)}$ (resp. $\mathcal{T}_k^{(2)}$).

By using Lemma 8.8, if $\xi = \sum_{i=1}^m a_i \tau_i^{(1)}$ satisfies $a_i, a_j \neq 0$ then $|Z(\xi)| < |Z(\tau_i^{(1)})|$. Therefore, we have that $m_1^{(1)}$ is the highest zero-length in $H_T^2(M_\alpha)$, and if $|Z(\xi)| = m_1^{(1)}$ then $\xi = a_i \tau_i^{(1)}$ for some non-zero integer a_i and $\tau_i^{(1)} \in \mathcal{T}_1^{(1)}$. Similarly, we have that $m_1^{(2)}$ is the highest zero-length in $H_T^2(M_\beta)$, and if $|Z(\xi)| = m_1^{(2)}$ then $\xi = b_i \tau_i^{(2)}$ for some non-zero integer b_i and $\tau_i^{(2)} \in \mathcal{T}_1^{(2)}$. Take an element $\tau_i^{(1)} \in \mathcal{T}_1^{(1)}$. Due to Proposition 8.5, $f_T^*(\tau_i^{(1)})$ has the zero-length $m_1^{(1)}$. Moreover, $f_T^*(\tau_i^{(1)})$ has the highest zero-length in $f_T^*(H_T^2(M_\alpha)) = H_T^2(M_\beta)$. Therefore, we have $m_1^{(1)} = m_1^{(2)}$. By using the arguments above, we have that $f_T^*(\tau_i^{(1)}) = b\tau_j^{(2)}$ for some $\tau_j^{(2)} \in \mathcal{T}_1^{(2)}$ and non-zero integer b. Because f_T^* is isomorphism, we also have that $b = \pm 1$, i.e., f_T^* maps $\mathcal{T}_1^{(1)}$ to $\mathcal{T}_1^{(2)}$ bijectively up to sign.

Take an element $\tau_i^{(1)} \in \mathcal{T}_2^{(1)}$. Because $\mathcal{T}_1^{(1)}$ and $\mathcal{T}_1^{(2)}$ are preserved under f_T^* and $(f_T^*)^{-1}$, $f_T^*(\tau_i^{(1)})$ does not have a term described by a linear combination of elements in $\mathcal{T}_1^{(2)}$. Therefore, by using Proposition 8.5 and Lemma 8.8, we have that $f_T^*(\tau_i^{(1)})$ has the second highest zero-length $m_2^{(1)}$ in $f_T^*(H_T^2(M_\alpha)) = H_T^2(M_\beta)$. With the method similar to that demonstrated as above, we also have that $m_2^{(1)} = m_2^{(2)}$ and f_T^* maps $\mathcal{T}_2^{(1)}$ to $\mathcal{T}_2^{(2)}$ bijectively up to sign. By repeating this argument, we have that f_T^* preserves $\mathcal{T}_k^{(1)}$ to $\mathcal{T}_k^{(2)}$ up to sign for $k \neq 0$. $\square$

Let us prove Lemma 8.4.

Proof of Lemma 8.4. Because of Lemma 8.9, f_T^* preserves $\mathcal{T}^{(1)} \setminus \mathcal{T}_0^{(1)}$ to $\mathcal{T}^{(2)} \setminus \mathcal{T}_0^{(2)}$. Together with $|\mathcal{T}^{(1)}| = |\mathcal{T}^{(2)}| = m$, we may put

$$|\mathcal{T}_0^{(1)}| = |\mathcal{T}_0^{(2)}| = m_0.$$

Let $\mathcal{T}_0^{(s)} = \{\tau_i^{(s)} \mid i = 1, \ldots, m_0\}$ for $s = 1, 2$. By Proposition 8.7, there exist elements $x_i, y_i \in H^2(BT)$ such that $\pi_1^*(x_i) = \tau_i^{(1)}$ and $\pi_2^*(y_i) = \tau_i^{(2)}$ for $i = 1, \ldots, m_0$. Therefore, by the exactness of the sequence (6.2), we have that $j^*(\tau_i^{(s)}) = 0$. Hence, it follows from the assumption mentioned in Remark 2.3 that $a_i = b_i = 0$, where a_i, b_i are the i^{th} coordinate of $\widehat{a}$ and $\widehat{b}$, respectively. Moreover, using Proposition 2.2, we have that $M_\alpha = M'_\alpha \times \mathbb{H}^{m_0}$ and $M_\beta = M'_\beta \times \mathbb{H}^{m_0}$ for some $4(n - m_0)$-dimensional toric hyperKähler manifolds M'_α and M'_β; therefore, we have

$$H_T^*(M_\alpha) \simeq H^*_{T^{n-m_0}}(M'_\alpha) \otimes H^*_{T^{m_0}}(\mathbb{H}^{m_0}),$$
$$H_T^*(M_\beta) \simeq H^*_{T^{n-m_0}}(M'_\beta) \otimes H^*_{T^{m_0}}(\mathbb{H}^{m_0})$$

such that $\mathcal{T}_0^{(s)}$ becomes the canonical generators in $H^*_{T^{m_0}}(\mathbb{H}^{m_0})$ and $\mathcal{T}^{(1)} \setminus \mathcal{T}_0^{(1)}$ (resp. $\mathcal{T}^{(2)} \setminus \mathcal{T}_0^{(2)}$) becomes those in $H^*_{T^{n-m_0}}(M'_\alpha)$ (resp. $H^*_{T^{n-m_0}}(M'_\beta)$). Because $\pi_1^*(x_i) = \tau_i^{(1)}$ and $\pi_2^*(y_i) = \tau_i^{(2)}$, we may identify $H^*(BT^{m_0})$ and $H^*_{T^{m_0}}(\mathbb{H}^{m_0})$ by π_s^*. By using the assumption that f_T^* is a weak $H^*(BT)$-algebra isomorphism, f_T^* decomposes into

$$f_T^* = f^*_{n-m_0} \otimes \varphi^*_{m_0}.$$

where

$$\varphi^*_{m_0} : H^*_{T^{m_0}}(\mathbb{H}^{m_0}) \to H^*_{T^{m_0}}(\mathbb{H}^{m_0})$$

is an isomorphism and

$$f^*_{n-m_0} : H^*_{T^{n-m_0}}(M'_\alpha) \to H^*_{T^{n-m_0}}(M'_\beta)$$

is a weak $H^*(BT^{n-m_0})$-algebra isomorphism which preserves $\mathcal{T}^{(1)} \setminus \mathcal{T}_0^{(1)}$ to $\mathcal{T}^{(2)} \setminus \mathcal{T}_0^{(2)}$. Moreover, φ^* decomposes into

$$\varphi^* = \varphi^*_{n-m_0} \otimes \varphi^*_{m_0}$$

for some isomorphism $\varphi^*_{n-m_0} : H^*(BT^{n-m_0}) \to H^*(BT^{n-m_0})$ such that that the following diagram is commutative for each factor:

$$\begin{array}{ccc}
H^*(BT^{n-m_0}) \otimes H^*(BT^{m_0}) & \xrightarrow{\pi_1^*} & H^*_{T^{n-m_0}}(M'_\alpha) \otimes H^*_{T^{m_0}}(\mathbb{H}^{m_0}) \\
\downarrow {\scriptstyle \varphi^* = \varphi^*_{n-m_0} \otimes \varphi^*_{m_0}} & & \downarrow {\scriptstyle f_T^* = f^*_{n-m_0} \otimes \varphi^*_{m_0}} \\
H^*(BT^{n-m_0}) \otimes H^*(BT^{m_0}) & \xrightarrow{\pi_2^*} & H^*_{T^{n-m_0}}(M'_\beta) \otimes H^*_{T^{m_0}}(\mathbb{H}^{m_0})
\end{array}$$

i.e., $\varphi^*_{m_0} \circ \pi_1^* = \pi_2^* \circ \varphi^*_{m_0}$ and $f^*_{n-m_0} \circ \pi_1^* = \pi_2^* \circ \varphi^*_{n-m_0}$.

In order to prove Lemma 8.4, it is enough to change $\varphi^*_{m_0}$ into some isomorphism which preserves $\mathcal{T}_0^{(1)}$ to $\mathcal{T}_0^{(2)}$. Because $\varphi^*_{m_0}$ is an isomorphism and $\{x_i\}$ and $\{y_i\}$ are generators of $H^*(BT^{m_0})$ (see the proof of Proposition 8.7), it is easy to check that there exists an isomorphism $X : H^*(BT^{m_0}) \to H^*(BT^{m_0})$ such that $X \circ \varphi^*_{m_0}(x_i) = y_i$. Then, $g_T^* : H_T^*(M_\alpha) \to H_T^*(M_\beta)$ defined by

$$g_T^* = f^*_{n-m_0} \otimes X \circ \varphi^*_{m_0}$$

is a weak $H^*(BT)$-algebra isomorphism which preserves $\mathcal{T}_k^{(1)}$ to $\mathcal{T}_k^{(2)}$ for $k \geq 0$. Moreover, $g_T^*(\widehat{a}) = \widehat{b}$ because $a_i = b_i = 0$ for the coordinates in $H^2_{T^{m_0}}(\mathbb{H}^{m_0})$. This establishes Lemma 8.4. $\square$

At this point, we can prove Proposition 8.3:

PROOF OF PROPOSITION 8.3. Let $H_i^{(1)}$ and $H_j^{(2)}$ be hyperplanes in $\mathcal{H}_{\widehat{a}}$ and $\mathcal{H}_{\widehat{b}}$, respectively. Due to Lemma 8.4, there exists a weak $H^*(BT)$-algebra isomorphism g_T^* such as Lemma 8.4. Because g_T^* is a weak algebraic isomorphism, there exists an isomorphism $\varphi^* : H^*(BT) \to H^*(BT)$ such that $g_T^* \circ \pi_1^* = \pi_2^* \circ \varphi^*$. We claim $\varphi^*(H_i^{(1)}) = H_{\sigma(i)}^{(2)}$ for some permutation $\sigma : [m] \to [m]$. Because $g_T^*(\tau_i^{(1)}) = \epsilon_i \tau_{\sigma(i)}^{(2)}$, we have

$$(g_T)_*(\mathbf{u}_{\sigma(i)}^{(2)}) = \epsilon_i \mathbf{u}_i^{(1)}, \tag{8.3}$$

for the dual basis $\mathbf{u}_i^{(1)} \in H_2^T(M_\alpha)$ and $\mathbf{u}_{\sigma(i)}^{(2)} \in H_2^T(M_\beta)$ of $\tau_i^{(1)}$ and $\tau_{\sigma(i)}^{(2)}$, respectively. Therefore, we have the following relations for $x \in H_i^{(1)}$:

$$\begin{array}{lll} & \langle \pi_2^* \circ \varphi^*(x) + \widehat{b}, \mathbf{u}_{\sigma(i)}^{(2)} \rangle_H & \\ = & \langle g_T^* \circ \pi_1^*(x) + g_T^*(\widehat{a}), \mathbf{u}_{\sigma(i)}^{(2)} \rangle_H & \text{(by assumptions)} \\ = & \langle \pi_1^*(x) + \widehat{a}, (g_T)_*(\mathbf{u}_{\sigma(i)}^{(2)}) \rangle_H & \text{(by dual)} \\ = & \langle \pi_1^*(x) + \widehat{a}, \epsilon_i \mathbf{u}_i^{(1)} \rangle_H & \text{(by (8.3))} \\ = & 0. & \text{(by } x \in H_i^{(1)}\text{)} \end{array}$$

It follows from the relations above that we can easily prove $\varphi^*(H_i^{(1)}) = H_{\sigma(i)}^{(2)}$ for all $i \in [m]$. This establishes Proposition 8.3. □

By using Propositions 8.2 and 8.3, we have Theorem 8.1. Because of Theorems 4.3 and 8.1, we get Theorem 1.1.

Acknowledgments

Finally the author would like to thank organizers of "GEOMETRY, TOPOLOGY, ALGEBRA, and APPLICATIONS dedicated to the 120th anniversary of Boris Delone" held at Steklov Mathematical Institute, Moscow, in August 2010, for giving him the opportunity to give a talk. He would like to express his gratitude to Yasuzo Nishimura for invaluable advises and comments. He also would like to thank Professors DongYoup Suh and Mikiya Masuda for providing him excellent circumstances to do research.

References

[AtBo] M. Atiyah, R. Bott, *The moment map and equivariant cohomology*, Topology, **23** (1984), 1–28.

[Bi] R. Bielawski, *Complete hyperkähler $4n$-manifolds with a local tri-Hamiltonian $\mathbb{R}^n$-action*, Math. Ann., **314** (1999), 505–528.

[BiDa] R. Bielawski, A. Dancer, *The geometry and topology of toric hyperKäler*, Comm. Anal. Geom., **8** (2000), 727–759.

[Br] G.E. Bredon, Introduction to compact transformation groups, Academic Press, 1972.

[BuPa] V.M. Buchstaber, T.E. Panov, Torus Actions and Their Applications in Topology and Combinatorics, University Lecture, 24, Amer. Math. Soc., Providence, R.I., 2002.

[De] T. Delzant, *Hamiltoniens périodiques et images convexes de l'application moment*, Bull. Soc. Math. France, **116** (1988), 315–339.

[Fu] W. Fulton, An Introduction to Toric Varieties, Ann. of Math. Studies, 113, Princeton Univ. Press, Princeton, N. J., 1993.

[Go] R. Goto, *On toric hyper-Kähler manifolds given by the hyper-Kähler quotient method*, Infinite analysis, Part A, B (Kyoto, 1991), 317–338, Adv. Ser. Math. Phys., 16, World Sci. Publ., River Edge, NJ, 1992.

[GuSt] V. Guillemin, S. Sternberg, Supersymmetry and equivariant de Rham theory, Springer Berlin, 1999.

[HaHo] M. Harada, T. Holm, *The equivariant cohomology of hypertoric varieties and their real loci*, Comm. Anal. Geom., **13** (2005), 645–677.

[HaMa] A. Hattori, M. Masuda, *Theory of Multi-fans*, Osaka. J. Math., **40** (2003), 1–68.

[HaPr] M. Harada, N. Proudfoot, *Properties of the residual circle action on a hypertoric variety*, Pacific J. of Math., **214** (2004), 263–284.

[HaSt] T. Hausel, B. Sturmfels, *Toric hyperkähler varieties*, Doc. Math., **7** (2002), 495–534.

[Hi] F. Hirzebruch, *Über eine Klasse von einfachzusammenhängenden komplexen Mannigfaltigkeiten*, Math. Ann., **124** (1951) 77–86.

[Hs] W.Y. Hsiang, Cohomology Theory of Topological Transformation Groups, Ergeb. Math., 85, Springer-Verlag, Berlin, 1975.

[Ka] K. Kawakubo, The theory of transformation groups, Oxford Univ. Press, London, 1991.

[Ko1] H. Konno, *Equivariant cohomology rings of toric hyperkahler manifolds*, Quaternionic structures in mathematics and physics (Rome, 1999), 231–240 (electronic), Univ. Studi Roma “La Sapienza”, Rome, 1999.

[Ko2] H. Konno, *Cohomology rings of toric hyperKähler manifolds*, Int. J. of Math., **11** (2000), 1001–1026.

[Ko3] H. Konno, *The geometry of toric hyperkahler varieties. Toric topology*, Contemp. Math., **460** (2008), 241–260.

[Ku] S. Kuroki, *A topological definition of hypertoric manifolds and its equivariant cohomology*, Trends in Mathematics - New Series Vol 12 No 1, 135–138 (2010).

[Ma1] M. Masuda, *Unitary toric manifolds, multi-fans and equivariant index*, Tohoku Math. J., **51** (1999), 237–265.

[Ma2] M. Masuda, *Equivariant cohomology distinguishes toric manifolds*, Adv. Math., **218** (2008), 2005–2012.

[MaSu] M. Masuda, D. Y. Suh, *Classification problems of toric manifolds via topology*, Proc. of Toric Topology, Contemp. Math., **460** (2008), 273-286.

[MiTo] M. Mimura, H. Toda, Topology of Lie Groups, I and II, Amer. Math. Soc., 1991.

[Od] T. Oda, Convex Bodies and Algebraic Geometry. An Introduction to the Theory of Toric Varieties, Ergeb. Math. Grenzgeb. (3), **15**, Springer-Verlag, Berlin, 1988.

[Pr] N. Proudfoot, *A survey of hypertoric geometry and topology*, Proc. of Toric topology, Contemp. Math., **460** (2008), 323–338.

Department of Mathematical Sciences, KAIST, Daejeon 305-701, R. Korea
E-mail address: `kuroki@kaist.ac.kr`