Hypertorus graphs and graph equivariant cohomologies

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ABSTRACT. The purpose of this paper is to establish a new graph: a *hypertorus graph* which is the analogue of the GKM-graph or the torus graph, and to study a ring structure of its *graph equivariant cohomology.* A hypertorus graph is a regular $2n$-valent graph labeled by the dual of Lie algebra of the $(n+1)$-dimensional torus $T^n \times S^1$ encoding combinatorial informations about some $4n$-dimensional manifold with $T^n \times S^1$-action. For instance, graphs induced by hypertoric varieties or cotangent bundles of toric manifolds are hypertorus graphs.

1. Introduction

Suppose a torus T acts on a space M such that the dimension of the 1-skelton is 2. Here the 1-skelton of a T-manifold M is the set of points $p \in M$, where $\dim T_p \geq \dim T - 1$. From this 1-skelton, we can construct a graph Γ: the vertex set $\mathcal{V}^\Gamma = M^T$ and two vertices are linked with an edge in Γ whenever the corresponding fixed points on M are connected by an invariant sphere. In [**GKM**] Goresky, Kottwitz and MacPherson showed that, for such manifold with some assumptions, this 1-skeleton has the structure of a "labeled" graph $(\Gamma,\ \alpha)$ labeled by the weights of the T-action on the tangent space T_pM for $p \in M^T$, and that the equivariant cohomology ring of M (over the real number $\mathbb{R}$) is isomorphic to the "cohomology ring" of this graph $(\Gamma,\ \alpha)$. For example, 1-skeltons of $2n$-dimensional non-singular toric varieties (toric manifolds) with T^n-action satisfy this property.

Motivated by the above Goresky, Kottwitz and MacPherson's theorem (we call it the *GKM theorem*), in [**GZ2**], Guillemin and Zara introduced the *GKM-graph* $(\Gamma,\ \alpha)$ and the cohomology ring $H(\Gamma, \alpha)$. If this GKM-graph (Γ, α) is associated with some torus T-action on M, then its equivariant cohomology ring is isomorphic to $H(\Gamma, \alpha)$ by the GKM theorem. So we can regard the GKM-graph as a generalized object of a space with some torus action. The above Guillemin and Zara's research has been of independent combinatorial interest since the appearance of their paper [**GZ1**] (1999), and they translated the important topological properties of Hamiltonian T-actions on M into the languages of combinatorics and applied the combinatorial theory.

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We can also define a "labeled graph" from the *torus manifold*, which is defined by Hattori and Masuda in [**HM**] as the topological generalized object of the non-singular toric variety in the algebraic geometry. The torus manifold is a $2n$-dimensional manifold with an effective T^n-action and finite fixed points, and torus manifolds do not always have an almost complex structure (this part is the most different part of the previous researches). Motivated by such torus manifold and the GKM-graph, a *torus graph* Γ and its equivariant cohomology $H_T^*(\Gamma)$ were defined by Maeda, Masuda and Panov in [**MMP**], and they showed that $H_T^*(\Gamma)$ is isomorphic to the *face ring* associated with Γ which is the ring described by the combinatorial data of Γ. According to [**MP**], if odd degree cohomologies of the torus manifold are 0 then its equivariant cohomology ring (over the integer $\mathbb{Z}$) is isomorphic to $H_T^*(\Gamma)$. Remark that GKM-graphs and torus graphs are different graphs, but the intersection of the set of GKM-graphs and the set of torus graphs include the set of graphs associated with toric manifolds.

On the other hand we can also define a labeled graph from the *hypertoric variety*. The hypertoric variety is the hyperKähler analogue of the toric variety, which is defined, in [**BD**], by the hyperKähler quotient of a torus action on a quaternion space. From this quotient, the hypertoric variety is a non-compact orbifold with the natural T^n-action. Its ordinary cohomology and equivariant cohomology were studied by Konno in [**Ko1**] and [**Ko2**]. According to Harada and Proudfoot in [**HP**], this T^n-action can extend to $T^n \times S^1$-action and they studied about its equivariant cohomology. Its equivariant cohomology ring is described by the half space arrangement induced by the hypertoric variety with $T^n \times S^1$-action. According to Harada and Holm in [**HH**], from this $T^n \times S^1$-action we can define a labeled graph, and they also define a "cohomology ring" of graph and study the correspondence between generators of equivariant cohomology of the $T^n \times S^1$-action and elements of a cohomology ring of graph.

Motivated by such hypertoric variety and the torus graph, in this paper, we define a *hypertorus graph* $\mathcal{G} = (\Gamma,\ \alpha,\ \theta)$ and its *graph equivariant cohomology* $H^*_{T^n \times S^1}(\mathcal{G})$. The goal of this paper is to show the ring structure of $H^*_{T^n \times S^1}(\mathcal{G})$ in some case (Theorem 3.1). A hypertorus graph is a generalization of graphs which appeared in [**HH**], and a remarkable difference of this graph and other graphs (GKM-graphs and torus graphs) is that some hypertorus graphs have a *leg* which is a half line from one vertex (see Figure 2). Because of the leg, we can define the neighborhood of the subgraph in $\mathcal{G}$ and apply the main theorem (Theorem 3.1) to show the Mayer-Vietoris exact sequence holds for $H^*_{T^n \times S^1}(\mathcal{G})$ in some case (Theorem 4.1).

We now give a brief outline of the contents of this paper. In Section 2, we give definitions of a *hypertorus graph* and its *graph equivariant cohomology*, and also prepare to state the main theorem (Theorem 3.1). For the main theorem, we need to prepare definitions of a *hyperfacet* which is some subgraph in a hypertorus graph, a *Thom class* of a hyperfacet, and a ring $\mathbb{Z}[\mathcal{G}]$ defined by some combinatorial data of hyperfacets. In Section 3, we prove the main theorem. In order to prove it, we divide the proof into two parts: the first part is to study an *x-forgetful graph* of a hypertorus graph, and to show its graph equivariant cohomology; in the second part, using a result of a graph equivariant cohomology of an x-forgetful graph, the main theorem is proved. In Section 4, as an application of the main theorem, we prove that there exists the Mayer-Vietoris exact sequence on the graph equivariant cohomology in some case (Theorem 4.1).

2. Definitions: hypertorus graph

The aim of this section is to define a hypertorus graph and its graph equivariant cohomology. In order to state the main theorem in Section 3, we also define a *hyperfacet* and a ring $\mathbb{Z}[\mathcal{G}]$ in the last section (Section 2.3).

2.1. Notations. First we prepare some notations. In this paper Γ is a connected graph which possibly has legs, where a *leg* means an out going half line from one vertex (see the left graph in the Figure 1). Let $\mathcal{V}^\Gamma$ be a set of vertices, E^Γ a set of edges, L^Γ a set of legs in Γ, and $\mathcal{E}^\Gamma = E^\Gamma \cup L^\Gamma$. Then Γ can be denoted by

$$\Gamma = (\mathcal{V}^\Gamma,\ \mathcal{E}^\Gamma).$$

In this paper we assume the number of $\mathcal{V}^\Gamma$ and $\mathcal{E}^\Gamma$ are finite.

Moreover we assume all edges and legs have an orientation (see Figure 1): each edge $e \in E^\Gamma$ has two possible orientations, and we denote the opposite orientation of the edge $e = pq$ by $\bar{e} = qp$; each leg $l \in L^\Gamma$ has only one orientation which is an out going direction from one vertex. We denote the initial vertex of $e = pq$ by $i(e)(= p)$ and the terminal vertex by $t(e)(= q)$, and remark that a leg l does not have a terminal vertex but an initial vertex $i(l)$. For a vertex $p \in \mathcal{V}^\Gamma$, we put the set of all out going edges and legs from $p \in \mathcal{V}^\Gamma$ by

$$\mathcal{E}_p^\Gamma = \{\epsilon \in \mathcal{E}^\Gamma \mid i(\epsilon) = p\},$$

and $|\mathcal{E}_p^\Gamma|$ denotes the number of $\mathcal{E}_p^\Gamma$. In this paper we consider only a connected graph $\Gamma = (\mathcal{V}^\Gamma,\ \mathcal{E}^\Gamma)$ which satisfies $|\mathcal{E}_p^\Gamma| = |\mathcal{E}_q^\Gamma| = m$ for all $p,\ q \in \mathcal{V}^\Gamma$. We call such graph a (*regular*) *m-valent graph*. The following two figures are examples of our graphs.

FIGURE 1. These are examples of regular graphs with legs and orientations. The left 2-valent graph has two legs, on the other hand the right 3-valent graph has no legs. Remark all edges have two orientations and all legs have only one orientation.

2.2. Hypertorus graph and Graph equivariant cohomology. We will define a *hypertorus graph* in the first section 2.2.1, and a *graph equivariant cohomology* in the next section 2.2.2.

2.2.1. *Hypertorus graph.* Let $\Gamma = (\mathcal{V}^\Gamma,\ \mathcal{E}^\Gamma)$ be an m-valent graph. In order to define a hypertorus graph, we will define a *connection* and an *axial function*.

Before we define a connection, we prepare the set $\theta = \{\theta_e \mid e \in E^\Gamma\}$ which is a collection of bijective maps

$$\theta_e : \mathcal{E}_p^\Gamma \to \mathcal{E}_q^\Gamma$$

for all edges $e = pq \in E^\Gamma$. Since Γ is an m-valent graph, we have $|\mathcal{E}_p^\Gamma| = m = |\mathcal{E}_q^\Gamma|$ for all $p,\ q \in \mathcal{V}^\Gamma$. Hence the bijective map θ_e always exists for all edges $e \in E^\Gamma$. A *connection* on Γ is the set $\theta = \{\theta_e |\ e \in E^\Gamma\}$ which satisfies the following two conditions:

- $\theta_{\bar{e}} = \theta_e^{-1}$ $(\theta_{qp} = \theta_{pq}^{-1})$;
- $\theta_e(e) = \bar{e}$ $(\theta_{pq}(pq) = qp)$.

We can easily show that an m-valent graph Γ admits different $((m-1)!)^g$ connections, where g is the number of (unoriented) edges E^Γ.

Next we define an axial function. In order to define it, we prepare some notations. Let T^n be an n-dimensional torus, that is, an n-dimensional compact commutative group. T^n is often denoted by T. In particular a 1-dimensional torus is denoted by S^1. Let $\mathfrak{t}$ be a Lie algebra of T, $\mathfrak{t}_\mathbb{Z}$ a lattice of $\mathfrak{t}$, and $\mathfrak{t}^*$ the dual algebra of $\mathfrak{t}$. $\mathrm{Hom}(T, S^1)$ means a set of all homomorphisms from the group T to S^1, and we know that it can be regarded as a lattice of the dual algebra $\mathfrak{t}_\mathbb{Z}^*$. Moreover we can identify $\mathfrak{t}_\mathbb{Z}^*$ with $H^1(T;\mathbb{Z}) = H^2(BT;\mathbb{Z})$, where BT is the base space of the universal principal T-bundle $ET \to BT$. Therefore we have the identification $\mathrm{Hom}(T, S^1) = \mathfrak{t}_\mathbb{Z}^* = H^2(BT)$. An *axial function*

$$\alpha : \mathcal{E}^\Gamma \to \mathrm{Hom}(T, S^1) = \mathfrak{t}_\mathbb{Z}^* = H^2(BT)$$

is a map which satisfies the following three conditions:

- $\alpha(\bar{e}) = \pm\alpha(e)$ for all edges $e \in E^\Gamma$;
- $\alpha(\mathcal{E}_p^\Gamma) = \{\alpha(\epsilon) \mid \epsilon \in \mathcal{E}_p^\Gamma\}$ is *pairwise linearly independent* for all $p \in \mathcal{V}^\Gamma$, that is, for two distinct elements $\epsilon_1,\ \epsilon_2 \in \mathcal{E}_p^\Gamma$, these axial functions $\alpha(\epsilon_1),\ \alpha(\epsilon_2)$ are linearly independent in $\mathfrak{t}_\mathbb{Z}^*$;
- α satisfies the *congruence relation* for all edges $e \in E^\Gamma$, that is, the next relation holds for all $\epsilon \in \mathcal{E}_{i(e)}^\Gamma$: $\alpha(\epsilon) - \alpha(\theta_e(\epsilon)) \equiv 0 \pmod{\alpha(e)}$.

Definition 2.1 (hypertorus graph). Let $\mathcal{G} = (\Gamma,\ \alpha,\ \theta)$ be a collection of a $2n$-valent graph $\Gamma = (\mathcal{V}^\Gamma,\ \mathcal{E}^\Gamma)$, a connection θ on Γ, and an axial function

$$\alpha : \mathcal{E}^\Gamma \to \mathrm{Hom}(T^n \times S^1,\ S^1) = (\mathfrak{t}^n)_\mathbb{Z}^* \oplus \mathbb{Z}x,$$

where x is a generator of $\mathbb{Z}x$ which is the dual of the Lie algebra of S^1. We call $\mathcal{G} = (\Gamma,\ \alpha,\ \theta)$ a *hypertorus graph* if it satisfies the following conditions for all $p \in \mathcal{V}^\Gamma$:

1. We can put $\mathcal{E}_p^\Gamma = \{\epsilon_1^+(p),\ \cdots,\ \epsilon_n^+(p),\ \epsilon_1^-(p),\ \cdots,\ \epsilon_n^-(p)\}$ and $(\epsilon_j^+(p),\ \epsilon_j^-(p))$ satisfies $\alpha(\epsilon_j^+(p)) + \alpha(\epsilon_j^-(p)) = x$ for all $j = 1,\ \cdots,\ n$;
2. The set $\{\alpha(\epsilon_j^+(p)),\ x \mid j = 1, \cdots, n\}$ spans $(\mathfrak{t}^n)_\mathbb{Z}^* \oplus \mathbb{Z}x$, we denote it by $\langle\alpha(\epsilon_1^+),\ \cdots,\ \alpha(\epsilon_n^+),\ x\rangle = \mathfrak{t}_\mathbb{Z}^* \oplus \mathbb{Z}x$.

We call $\{\epsilon_j^+(p),\ \epsilon_j^-(p)\}$ such that $\alpha(\epsilon_j^+(p)) + \alpha(\epsilon_j^-(p)) = x$ a *pair* in $\mathcal{E}_p^\Gamma$.

The following Figure 2 is examples of the hypertorus graph.

Because the axial function α satisfies the congruence relation, we have the following Lemma 2.2.

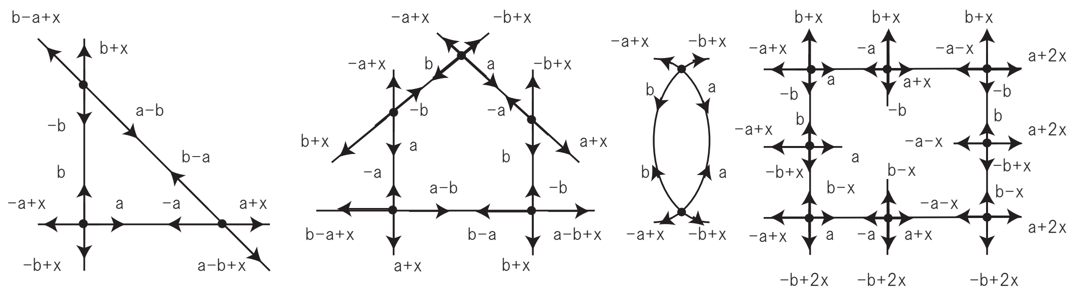

FIGURE 2. Hypertorus graphs, where $\langle \mathrm{a},\ \mathrm{b}\rangle \simeq (\mathfrak{t}^2)^*_{\mathbb{Z}}$.

LEMMA 2.2. *Let* $\{\epsilon^+,\ \epsilon^-\}$ *be a pair in* $\mathcal{E}^\Gamma_p$. *Then* $\{\theta_{pq}(\epsilon^+),\ \theta_{pq}(\epsilon^-)\}$ *is also a pair in* $\mathcal{E}^\Gamma_q$.

2.2.2. *Graph equivariant cohomology.* Let $\mathcal{G} = (\Gamma,\ \alpha,\ \theta)$ be a hypertorus graph. First we put the set of generators of $\mathfrak{t}^*_{\mathbb{Z}}$ by $\{\alpha_1,\ \cdots,\ \alpha_n\}$. Then we can identify $\mathfrak{t}^*_{\mathbb{Z}} \oplus \mathbb{Z}x$ as follows:

$$\mathfrak{t}^*_{\mathbb{Z}} \oplus \mathbb{Z}x = \langle \alpha_1,\ \cdots,\ \alpha_n,\ x\rangle = H^2(B(T^n \times S^1)),$$

and we can consider the equivariant cohomology of a point as follows:

$$H^*_{T^n\times S^1}(pt) = H^*(B(T^n \times S^1)) = \mathbb{Z}[\alpha_1,\ \cdots,\ \alpha_n,\ x].$$

Here $\mathbb{Z}[\alpha_1,\ \cdots,\ \alpha_n,\ x]$ is the polynomial ring.

DEFINITION 2.3 (graph equivariant cohomology). We define the ring $H^*_{T^n\times S^1}(\mathcal{G})$ of $\mathcal{G} = (\Gamma,\ \alpha,\ \theta)$ as follows:

$$H^*_{T^n\times S^1}(\mathcal{G}) = \{f : \mathcal{V}^\Gamma \to H^*_{T^n\times S^1}(pt) \mid f(p) - f(q) \equiv 0 \pmod{\alpha(pq)}\},$$

where $pq \in E^\Gamma$ is an edge and $\alpha(pq) \in \mathfrak{t}^*_{\mathbb{Z}} \oplus \mathbb{Z}x = H^2(B(T^n \times S^1))$. We call $H^*_{T^n\times S^1}(\mathcal{G})$ a *graph equivariant cohomology.* We also call the relation $f(p) - f(q) \equiv 0 \pmod{\alpha(pq)}$ a *congruence relation* of f.

2.3. Hyperfacet and Ring $\mathbb{Z}[\mathcal{G}]$. Before we mention the main theorem, we need to introduce a special subgraph (we call it a *hyperfacet*) of $\mathcal{G}$ and a ring $\mathbb{Z}[\mathcal{G}]$ induced by $\mathcal{G}$.

In order to define a hyperfacet, we will define a *pre-hyperfacet*, its *Thom class* and its *opposite side.*

2.3.1. *pre-hyperfacet.* First we define a *pre-hyperfacet.* Let $H = (\mathcal{V}^H,\ \mathcal{E}^H)$ be a subgraph of $\Gamma = (\mathcal{V}^\Gamma,\ \mathcal{E}^\Gamma)$, that is, H satisfies $\mathcal{V}^H \subset \mathcal{V}^\Gamma$ and $\mathcal{E}^H \subset \mathcal{E}^\Gamma$. Put $\mathcal{E}^H_p = \mathcal{E}^\Gamma_p \cap \mathcal{E}^H$ and $|\mathcal{E}^H_p|$ the number of out going edges and legs from $p \in \mathcal{V}^H$ in H. Assume $|\mathcal{E}^H_p| = 2n - 1$ or $2n$ for all $p \in \mathcal{V}^H$, where there always exists a vertex $p \in \mathcal{V}^H$ which satisfies $|\mathcal{E}^H_p| = 2n - 1$. Moreover we assume that H is *closed* by the connection θ of $\mathcal{G} = (\Gamma,\ \alpha,\ \theta)$, that is, $\theta^H_{pq} = \theta_{pq}|_{\mathcal{E}^H_p} : \mathcal{E}^H_p \to \mathcal{E}^H_q$ is bijective (if $|\mathcal{E}^H_p| = |\mathcal{E}^H_q|$) or injective (if $|\mathcal{E}^H_p| < |\mathcal{E}^H_q|$), and if $|\mathcal{E}^H_p| < |\mathcal{E}^H_q|$ then θ^H_{pq} satisfies the following congruence relation for $\{n_H(p)\} = \mathcal{E}^\Gamma_p - \mathcal{E}^H_p$ (we call $n_H(p)$ a *normal edge* or a *normal leg* of H on p):

$$\alpha(n_H(p)) - x \equiv 0 \pmod{\alpha(pq)}.$$

We call such subgraph H a *pre-hyperfacet* in $\mathcal{G} = (\Gamma,\ \alpha,\ \theta)$.

2.3.2. *Thom class.* Next we define a *Thom class* of the pre-hyperfacet. Let H be a pre-hyperfacet in $\mathcal{G}=(\Gamma,\ \alpha,\ \theta)$. We call the map $\tau_H:\mathcal{V}^\Gamma\to H^2(BT)$ such that

$$\tau_H(p)=\begin{cases}0 & \text{if } p\notin\mathcal{V}^H\\ x & \text{if } |\mathcal{E}_p^H|=2n\\ \alpha(n_H(p)) & \text{if } |\mathcal{E}_p^H|=2n-1,\end{cases}$$

the *Thom class* of a pre-hyperfacet H. For the Thom class τ_H of a pre-hyperfacet H, we have the following lemma.

LEMMA 2.4. *τ_H is an element of $H^*_{T^n\times S^1}(\mathcal{G})$.*

PROOF. Take an edge $pq\in E^\Gamma$. It suffices to prove that τ_H satisfies the congruence relation on pq, that is, $\tau_H(p)-\tau_H(q)\equiv 0\ (\mathrm{mod}\ \alpha(pq))$.

Suppose that $p\in\mathcal{V}^H$ and $q\notin\mathcal{V}^H$. Then $|\mathcal{E}_p^H|=2n-1$ and $\{pq\}=\mathcal{E}_p^\Gamma-\mathcal{E}_p^H=\{n_H(p)\}$. So we have $\alpha(pq)=\alpha(n_H(p))$. Because of the definition of the Thom class, we also have $\tau_H(p)=\alpha(n_H(p))$ and $\tau_H(q)=0$. Hence we know that

$$\tau_H(p)-\tau_H(q)=\alpha(n_H(p))-0=\alpha(pq)\equiv 0\ (\mathrm{mod}\ \alpha(pq)).$$

Suppose that $p,\ q\in\mathcal{V}^H$. If $|\mathcal{E}_p^H|=|\mathcal{E}_q^H|=2n-1$, then $\tau_H(p)=\alpha(n_H(p))$ and $\tau_H(q)=\alpha(n_H(q))$. Because $\{n_H(p)\}=\mathcal{E}_p^\Gamma-\mathcal{E}_p^H$, $\{n_H(q)\}=\mathcal{E}_q^\Gamma-\mathcal{E}_q^H$ and the map $\theta_{pq}^H=\theta_{pq}|_{\mathcal{E}_p^H}:\mathcal{E}_p^H\to\mathcal{E}_q^H$ is bijective, we have $\theta_{pq}(n_H(p))=n_H(q)$. By the congruence relation of α, we know that

$$\begin{aligned}\tau_H(p)-\tau_H(q)&=\alpha(n_H(p))-\alpha(n_H(q))\\&=\alpha(n_H(p))-\alpha(\theta_{pq}(n_H(p)))\equiv 0\ (\mathrm{mod}\ \alpha(pq)).\end{aligned}$$

If $|\mathcal{E}_p^H|=2n-1$ and $|\mathcal{E}_q^H|=2n$, then $\tau_H(p)=\alpha(n_H(p))$ and $\tau_H(q)=x$. Because of the definition of the pre-hyperfacet, we know that

$$\tau_H(p)-\tau_H(q)=\alpha(n_H(p))-x\equiv 0\ (\mathrm{mod}\ \alpha(pq)).$$

For the other cases (the cases $p,\ q\notin\mathcal{V}^H$ and $|\mathcal{E}_p^H|=|\mathcal{E}_q^H|=2n$), we can easily show that $\tau_H(p)-\tau_H(q)=0$ by the definition of the Thom class. $\square$

2.3.3. *opposite side.* Next we define the *opposite side* of the pre-hyperfacet. In order to define it, we prove the following two lemmas: Lemma 2.5 and 2.6.

LEMMA 2.5. *If $\mathcal{G}=(\Gamma,\ \alpha,\ \theta)$ is a hypertorus graph, then there is a unique $(2n-2)$-valent hypertorus subgraph of $\mathcal{G}$ containing any given $(n-1)$ pairs in $\mathcal{E}_p^\Gamma$ for all $p\in\mathcal{V}^\Gamma$.*

PROOF. Let $\mathcal{E}_p^L=\{\epsilon_1^+,\ \cdots,\ \epsilon_{n-1}^+,\ \epsilon_1^-,\ \cdots,\ \epsilon_{n-1}^-\}$ be $(n-1)$ pairs in $\mathcal{E}_p^\Gamma$. Then $\alpha(\epsilon_j^+)+\alpha(\epsilon_j^-)=x$ for $j=1,\ \cdots,\ n-1$ and $\langle\alpha(\epsilon_1^+),\ \cdots,\ \alpha(\epsilon_{n-1}^+),\ x\rangle\simeq(\mathfrak{t}^{n-1})^*_{\mathbb{Z}}\oplus\mathbb{Z}x$ by the definition of the hypertorus graph. We put that

$$\langle\alpha(\epsilon_1^+),\ \cdots,\ \alpha(\epsilon_{n-1}^+),\ x\rangle=\mathfrak{k}^*_{\mathbb{Z}}\oplus\mathbb{Z}x.$$

Take any edge e from $E_p^L=\mathcal{E}_p^L\cap E_p^\Gamma$. By Lemma 2.2, through the connection θ_e, $\mathcal{E}_p^L$ maps to some $(n-1)$ pairs $\theta_e(\mathcal{E}_p^L)$ in $\mathcal{E}_{t(e)}^\Gamma$. The α-images of these $(n-1)$ pairs in $\mathcal{E}_{t(e)}^\Gamma$ and x span the same subspace $\mathfrak{k}^*_{\mathbb{Z}}\oplus\mathbb{Z}x$, because α satisfies the congruence relation on e. We can translate the given $(n-1)$ pairs in $\mathcal{E}_p^\Gamma$ along all edges in E_p^L in this way. Then we have the following $(2n-2)$-valent graph

$$L_1 = \bigcup_{e \in E_p^L} \theta_e(\mathcal{E}_p^L) \cup \mathcal{E}_p^L.$$

We continue this operation along all edges $E^{L_1} - E_p^L$, where E^{L_1} is the set of edges in L_1, then we get a graph L_2. This graph L_2 is also a $(2n-2)$-valent graph, because the α-images of all $(n-1)$ pairs in $\mathcal{E}_p^{L_2}$, for all $p \in \mathcal{V}^{L_2}$, and x also span $\mathfrak{t}_\mathbb{Z}^* \oplus \mathbb{Z}x$. Continuing this process, finally we can get a $(2n-2)$-valent hypertorus subgraph in $\mathcal{G}$.

Suppose that L and L' are $(2n-2)$-valent hypertorus subgraphs and $\mathcal{E}_p^L = \mathcal{E}_p^{L'}$. Similarly we translate $\mathcal{E}_p^{L'}$ along each edge in E_p^L and translate $\mathcal{E}_p^L$ along each edge in $E_p^{L'}$ by the connection θ of $\mathcal{G}$. Then we have two same graphs

$$L_1 = \bigcup_{e \in E_p^L} \theta_e(\mathcal{E}_p^{L'}) \cup \mathcal{E}_p^{L'} \text{and } L_1' = \bigcup_{e \in E_p^{L'}} \theta_e(\mathcal{E}_p^L) \cup \mathcal{E}_p^L,$$

because $E_p^L = E_p^{L'}$. We continue this operation along edges of L_1 and L_1'. Then we also have two same graphs L_2 and L_2'. Moreover we continue this operation, finally we know that $L = L'$. □

Next we will prove Lemma 2.6. In order to prove it, we define a *boundary* of a pre-hyperfacet. Let H be a pre-hyperfacet. By the definition of a pre-hyperfacet, there is a vertex $p \in \mathcal{V}^H$ such that $|\mathcal{E}_p^H| = 2n-1$. Then $\mathcal{E}_p^H$ has $(n-1)$ pairs $\{\epsilon_1^+, \cdots, \epsilon_{n-1}^+, \epsilon_1^-, \cdots, \epsilon_{n-1}^-\}$. Because of Lemma 2.5, we can get the unique $(2n-2)$-valent hypertorus subgraph L in $\mathcal{G}$ such that $p \in \mathcal{V}^L$ and $\mathcal{E}_p^L = \{\epsilon_1^+, \cdots, \epsilon_{n-1}^+, \epsilon_1^-, \cdots, \epsilon_{n-1}^-\}$. Then we call the union of all such L a *boundary* of H, and we denote it by ∂H. Note that a boundary of H need not be connected.

LEMMA 2.6. *Let H be a pre-hyperfacet in the hypertorus graph $\mathcal{G} = (\Gamma, \alpha, \theta)$ and x be a generator of $\mathbb{Z}x \subset \mathfrak{t}_\mathbb{Z}^* \oplus \mathbb{Z}x$. Then there is a unique pre-hyperfacet I which satisfies the following conditions:*

- $H \cup I = \Gamma$;
- $\tau_H + \tau_I = \chi \in H^*_{T^n \times S^1}(\mathcal{G})$,

where χ is a map $\chi(\mathcal{V}^\Gamma) = \{x\}$.

PROOF. Set $H = (\mathcal{V}^H, \mathcal{E}^H)$ and $\hat{I} = (\mathcal{V}^\Gamma - \mathcal{V}^H, \mathcal{E}^\Gamma - \mathcal{E}^H)$. Define

$$I = \hat{I} \cup \partial H.$$

Then we can easily see that $H \cup I = (\mathcal{V}^\Gamma, \mathcal{E}^\Gamma) = \Gamma$.

Next we prove I is a pre-hyperfacet. Take $p \in \mathcal{V}^I$. If $p \in \mathcal{V}^{\hat{I}} = \mathcal{V}^\Gamma - \mathcal{V}^H$, then $\mathcal{E}_p^I = \mathcal{E}_p^\Gamma$, that is, $|\mathcal{E}_p^I| = 2n$. If $p \in \mathcal{V}^{\partial H}$, then $\mathcal{E}_p^I = \mathcal{E}_p^{\partial H} \cup \{n_H(p)\}$, that is, $|\mathcal{E}_p^I| = 2n-1$. Here $n_H(p)$ is a normal edge (leg) of H on p. So $\theta_{pq}^I : \mathcal{E}_p^I \to \mathcal{E}_q^I$ is bijective (if $|\mathcal{E}_p^I| = |\mathcal{E}_q^I|$) or injective (if $|\mathcal{E}_p^I| < |\mathcal{E}_q^I|$). Put $n_H(p) = \epsilon^+$, then the normal edge (leg) of I on p can be denoted by $\epsilon^- = n_I(p)$ where $\{\epsilon^+, \epsilon^-\} = \mathcal{E}_p^\Gamma - \mathcal{E}_p^{\partial H}$ is one of the pair in $\mathcal{E}_p^\Gamma$. So we have the following equation:

$$\alpha(n_I(p)) = \alpha(\epsilon^-) = x - \alpha(\epsilon^+) = x - \alpha(n_H(p)).$$

If $n_H(p) = pq \in E^I$ such that $|\mathcal{E}_q^\Gamma| = |\mathcal{E}_q^I| = 2n$, then we see that $\alpha(n_I(p)) - x = -\alpha(n_H(p)) \equiv 0 \pmod{\alpha(n_H(p)) = \alpha(pq)}$ by the above equation. Hence I is a pre-hyperfacet. Moreover we have that $\tau_H + \tau_I = \chi$, because of the above equation and the definition of the Thom class of the pre-hyperfacet.

Finally we prove the uniqueness of I. By two conditions $H \cup I = \Gamma$, $\tau_H + \tau_I = \chi$ and the definition of the Thom class, we see $\partial I = \partial H$ and $I = (\mathcal{V}^\Gamma - \mathcal{V}^H,\ \mathcal{E}^\Gamma - \mathcal{E}^H) \cup \partial I$. From Lemma 2.5, the boundary $\partial H = \partial I$ is unique. So we know the uniqueness of I. $\Box$

We call I in Lemma 2.6 an *opposite side* of H and denote it by $\overline{H}$. Note that

$$H \cap \overline{H} = \partial H$$

by the proof of Lemma 2.6.

2.3.4. *Hyperfacet and Ring* $\mathbb{Z}[\mathcal{G}]$. Under the above preparations, we can define the hyperfacet.

Definition 2.7 (hyperfacet). We call a connected pre-hyperfacet a *hyperfacet*, if its opposite side is also connected.

Remark 2.8. For the hyperfacet H, its opposite side $\overline{H}$ is also a hyperfacet.

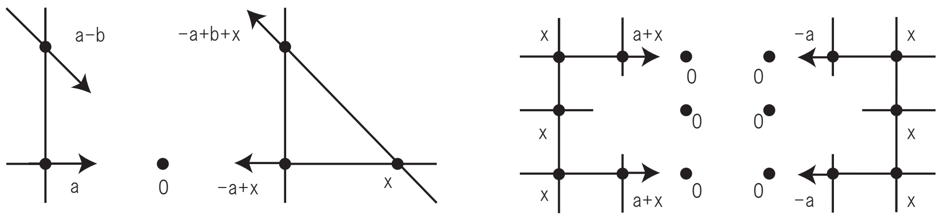

Figure 3. The above figures are hyperfacets and their opposite side of the left and right examples in Figure 2. Labels on vertices mean values of their Thom classes on vertices. Note that the boundary $\partial H = H \cap \overline{H}$ of the left example is connected, but the right one is not connected.

Let $\mathcal{H}$ be the set of all hyperfacets in $\mathcal{G}$, then we can put

$$\mathcal{H} = \{H_1,\ \cdots,\ H_m,\ \overline{H_1},\ \cdots,\ \overline{H_m}\}$$

by the finiteness of $\mathcal{V}^\Gamma$, Lemma 2.6, and Remark 2.8. Put

$$\mathbb{Z}[X,\ \mathcal{H}] = \mathbb{Z}[X,\ H_1,\ \cdots,\ H_m,\ \overline{H_1},\ \cdots,\ \overline{H_m}]$$

where $\mathbb{Z}[X,\ H_1,\ \cdots,\ H_m,\ \overline{H_1},\ \cdots,\ \overline{H_m}]$ is a polynomial ring which is generated by X and all elements in $\mathcal{H}$, and put

$$\mathcal{I} = \left\langle H_i + \overline{H_i} - X,\ \prod_{H \in \mathcal{H}'} H \ \middle|\ i = 1,\ \cdots,\ m,\ \mathcal{H}' \in \mathfrak{I}(\mathcal{H}) \right\rangle$$

which is the ideal in $\mathbb{Z}[X,\ \mathcal{H}]$ generated by $H_i + \overline{H_i} - X$ $(i = 1,\ \cdots,\ m)$ and $\prod_{H \in \mathcal{H}' \in \mathfrak{I}(\mathcal{H})} H$, where $\mathfrak{I}(\mathcal{H}) = \{\mathcal{H}' \subset \mathcal{H} \mid \cap \mathcal{H}' = \emptyset\}$. We define a ring $\mathbb{Z}[\mathcal{G}]$ as follows:

$$\mathbb{Z}[\mathcal{G}] = \mathbb{Z}[X,\ \mathcal{H}] \big/ \mathcal{I}.$$

3. Main theorem: ring structures of graph equivariant cohomologies

The aim of this section is to prove the following main theorem.

THEOREM 3.1. *Let $\mathcal{G}$ be a $2n$-valent hypertorus graph and $\mathcal{L} = \{L_1, \cdots, L_m\}$ a set of all connected $(2n-2)$-valent hypertorus subgraphs in $\mathcal{G}$. If $\mathcal{G}$ satisfies the following two assumptions:*

(1) *For each $L \in \mathcal{L}$, there are a hyperfacet H and its opposite side $\overline{H}$ such that $H \cap \overline{H} = L$, and such H, $\overline{H}$ are unique;*
(2) *For all subsets $\mathcal{L}' \subset \mathcal{L}$, its intersection $\cap\mathcal{L}'$ is empty or connected.*

Then $H^*_{T^n \times S^1}(\mathcal{G}) \simeq \mathbb{Z}[\mathcal{G}]$.

Henceforth in this section the hypertorus graph $\mathcal{G} = (\Gamma, \alpha, \theta)$ satisfies assumptions (1), (2) of Theorem 3.1. For example two left hypertorus graphs in Figure 2 satisfy these assumptions. However the third example does not satisfy the assumption (2) and the right (fourth) example does not satisfy the assumption (1).

In order to prove Theorem 3.1, we will prove that the following map is an isomorphism:

$$\Psi : \mathbb{Z}[\mathcal{G}] \to H^*_{T^n \times S^1}(\mathcal{G})$$

such that $\Psi(X) = \chi$ and $\Psi(H) = \tau_H$, where $\chi(p) = x$ for all $p \in \mathcal{V}^\Gamma$ and H is a hyperfacet. Because $\tau_H + \tau_{\overline{H}} = \chi$ and $\prod_{H \in \mathcal{H}'} \tau_H = 0$ ($\mathcal{H}'$ is a set of hyperfacets whose intersection is empty), the map Ψ is well-defined. From the next section we start to prove the bijectivity of Ψ. The proof will be divided into two steps:

(I) To study an equivariant graph cohomology of an *x-forgetful graph* $\widetilde{\mathcal{G}}$ and to prove $H^*_{T^n}(\widetilde{\mathcal{G}}) \simeq \mathbb{Z}[\widetilde{\mathcal{G}}]$;
(II) To prove Ψ is surjective and injective.

In the first step, we will use the method of [**MMP**] (or [**MP**]) which was used to show ring structures of graph equivariant cohomologies of torus graphs. In the second step, we will use the method of [**HP**] which was applied to show ring structures of equivariant cohomologies of hypertoric varieties.

3.1. x-forgetful graph $\widetilde{\mathcal{G}}$.

In order to prove Theorem 3.1, as the first step, we introduce an *x-forgetful graph* $\widetilde{\mathcal{G}} = (\Gamma, \widetilde{\alpha}, \theta)$ of the hypertorus graph $\mathcal{G} = (\Gamma, \alpha, \theta)$. Here an *$x$-forgetful axial function*

$$\widetilde{\alpha} = F \circ \alpha : \mathcal{E}^\Gamma \to (\mathfrak{t}^n)^*_{\mathbb{Z}}$$

is defined by the x-forgetful map $F : (\mathfrak{t}^n)^*_{\mathbb{Z}} \oplus \mathbb{Z}x \to (\mathfrak{t}^n)^*_{\mathbb{Z}}$.

Moreover we define a graph equivariant cohomology of $\widetilde{\mathcal{G}}$ as follows:

$$H^*_{T^n}(\widetilde{\mathcal{G}}) = \{f : \mathcal{V}^\Gamma \to H^*_{T^n}(pt) \mid f(p) - f(q) \equiv 0 \ (\mathrm{mod}\ \widetilde{\alpha}(pq))\}.$$

Fix $\{H_1, \cdots, H_m\}$ in the set of all hyperfacets $\mathcal{H} = \{H_1, \cdots, H_m, \overline{H}_1, \cdots, \overline{H}_m\}$, and define the *Thom class* of L by

$$\tau_L = F \circ \tau_H$$

for the $(2n-2)$-valent hypertorus subgraph $L = H \cap \overline{H}$. From the assumption (1) of Theorem 3.1 and Lemma 2.6, there is a one to one corresponding between H and $L = H \cap \overline{H}$. Therefore we can put a set of all connected $(2n-2)$-valent hypertorus

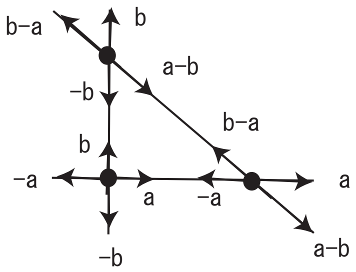

FIGURE 4. An example of the x-forgetful graph for the left hypertorus graph in Figure 2.

subgraphs by $\mathcal{L} = \{L_1, \cdots, L_m\}$ where $L_i = H_i \cap \overline{H}_i$ for all $i = 1, \cdots, m$. Moreover we have

$$\tau_L(p) = \begin{cases} 0 & p \notin \mathcal{V}^L \\ \widetilde{\alpha}(n_H(p)) & p \in \mathcal{V}^L \end{cases}$$

by the definitions of τ_H and the x-forgetful map F, where $n_H(p)$ is a normal edge (leg) of H on p. We also have $F \circ \tau_{\overline{H}} = -\tau_L$ by the definition of the opposite side of H. Since $\tau_H \in H^*_{T^n \times S^1}(\mathcal{G})$ (Lemma 2.4), we have $\tau_L \in H^*_{T^n}(\widetilde{\mathcal{G}})$.

Next we define the following ring:

$$\mathbb{Z}[\widetilde{\mathcal{G}}] = \mathbb{Z}[L_1, \cdots, L_m] \Big/ \Big\langle \prod_{L \in \mathcal{L}'} L \ \Big| \ \mathcal{L}' \in \mathfrak{I}(\mathcal{L}) \Big\rangle,$$

where $\mathfrak{I}(\mathcal{L}) = \{\mathcal{L}' \subset \mathcal{L} \mid \cap \mathcal{L}' = \emptyset\}$ and $\langle \prod_{L \in \mathcal{L}'} L \mid \mathcal{L}' \in \mathfrak{I}(\mathcal{L})\rangle$ is an ideal which is generated by the product $\prod_{L \in \mathcal{L}'} L$ for all $\mathcal{L}' \in \mathfrak{I}(\mathcal{L})$.

The goal of this section (the first step (I) of the proof of Theorem 3.1) is to prove $H^*_{T^n}(\widetilde{\mathcal{G}}) \simeq \mathbb{Z}[\widetilde{\mathcal{G}}]$. Define the map

$$\Psi' : \mathbb{Z}[\widetilde{\mathcal{G}}] \to H^*_{T^n}(\widetilde{\mathcal{G}})$$

by $\Psi'(L) = \tau_L$. Obviously Ψ' is a well-defined homomorphism. We will prove this homomorphism is bijective.

First we prove the injectivity of Ψ'. In order to prove the injectivity of Ψ', we study the following ring:

$$\mathbb{Z}[\widetilde{\mathcal{G}}]_p = \mathbb{Z}[L_1, \cdots, L_m]/\langle L \mid p \notin \mathcal{V}^L\rangle,$$

where $\langle L \mid p \notin \mathcal{V}^L\rangle$ is an ideal which is generated by L such that $p \notin \mathcal{V}^L$. As a beginning, we prove the following lemma.

LEMMA 3.2. *For the x-forgetful graph $\widetilde{\mathcal{G}} = (\Gamma, \widetilde{\alpha}, \theta)$, we have*

$$I_p : \mathbb{Z}[\widetilde{\mathcal{G}}]_p \simeq \mathbb{Z}[L \mid p \in \mathcal{V}^L] = \mathbb{Z}[L_1, \cdots, L_n] \overset{\iota_p}{\simeq} H^*_{T^n}(pt),$$

where the last isomorphism ι_p is defined by $\iota_p : L \mapsto \tau_L(p)$.

PROOF. The first equivalence $\mathbb{Z}[\widetilde{\mathcal{G}}]_p \simeq \mathbb{Z}[L \mid p \in \mathcal{V}^L]$ is trivial by the definition of $\mathbb{Z}[\widetilde{\mathcal{G}}]_p$. We prove $\mathbb{Z}[L \mid p \in \mathcal{V}^L] = \mathbb{Z}[L_1, \cdots, L_n] \overset{\iota_p}{\simeq} H_{T^n}(pt)$.

Because of the definition of the hypertorus graph, we can put

$$\mathcal{E}^\Gamma_p = \{e^+_1(p), \cdots, e^+_n(p), e^-_1(p), \cdots, e^-_n(p)\}$$

for all $p \in \mathcal{V}^{\Gamma}$. There is a unique L_i such that $\tau_{L_i}(p) = \widetilde{\alpha}(e_i^+(p)) = -\widetilde{\alpha}(e_i^-(p))$ for all $i = 1, \cdots, n$ by Lemma 2.5. Hence we have $\mathbb{Z}[L \mid p \in \mathcal{V}^L] = \mathbb{Z}[L_1, \cdots, L_n]$. By the definition of the axial function of the hypertorus graph, we see

$$\langle \alpha(e_1^+(p)), \cdots, \alpha(e_n^+(p)), x \rangle \simeq \mathfrak{t}_{\mathbb{Z}}^* \oplus \mathbb{Z}x = H^2(B(T^n \times S^1)).$$

Hence, by the definition of $\widetilde{\alpha}$, we have that

$$\mathbb{Z}[\widetilde{\alpha}(e_1^+(p)), \cdots, \widetilde{\alpha}(e_n^+(p))] \simeq H^*_{T^n}(pt).$$

This means that ι_p is an isomorphism. □

Next we will define ρ and prove Lemma 3.3.

Since there is a $L \in \mathcal{L}'$ such that $p \notin \mathcal{V}^L$ for the set $\mathcal{L}'$ such that $\cap_{L \in \mathcal{L}'} L = \emptyset$, we have

$$\langle L \mid p \notin \mathcal{V}^L \rangle \supset \langle \prod_{L \in \mathcal{L}'} L \mid \mathcal{L}' \in \mathfrak{I}(\mathcal{L}) \rangle.$$

Therefore there exists the natural projection

$$\rho_p : \mathbb{Z}[\widetilde{\mathcal{G}}] \to \mathbb{Z}[\widetilde{\mathcal{G}}]_p,$$

and for this ρ_p we can easily see that

$$\text{Ker } \rho_p = \langle L \mid p \notin \mathcal{V}^L \rangle / \langle \prod_{L \in \mathcal{L}'} L \mid \mathcal{L}' \in \mathfrak{I}(\mathcal{L}) \rangle.$$

Define the homomorphism

$$\rho = \bigoplus_{p \in \mathcal{V}^{\Gamma}} \rho_p : \mathbb{Z}[\widetilde{\mathcal{G}}] \to \bigoplus_{p \in \mathcal{V}^{\Gamma}} \mathbb{Z}[\widetilde{\mathcal{G}}]_p$$

by $\rho(Y) = \bigoplus_{p \in \mathcal{V}^{\Gamma}} \rho_p(Y)$ for $Y \in \mathbb{Z}[\widetilde{\mathcal{G}}]$. Then we have the following lemma.

Lemma 3.3. *ρ is injective.*

Proof. Obviously we have

$$\text{Ker } \rho = \left(\bigcap_{p \in \mathcal{V}^{\Gamma}} \langle L \mid p \notin \mathcal{V}^L \rangle \right) \Big/ \left\langle \prod_{L \in \mathcal{L}'} L \;\middle|\; \mathcal{L}' \in \mathfrak{I}(\mathcal{L}) \right\rangle.$$

We may only show that

$$\bigcap_{p \in \mathcal{V}^{\Gamma}} \langle L \mid p \notin \mathcal{V}^L \rangle \subset \langle \prod_{L \in \mathcal{L}'} L \mid \mathcal{L}' \in \mathfrak{I}(\mathcal{L}) \rangle (\subset \mathbb{Z}[L_1, \cdots, L_m]).$$

Take a non-zero element

$$\begin{aligned} A &= \sum_{a_1, \cdots, a_m \in \mathbb{N} \cup \{0\}} k(a_1, \cdots, a_m) L_1^{a_1} \cdots L_m^{a_m} \\ &\in \bigcap_{p \in \mathcal{V}^{\Gamma}} \langle L \mid p \notin \mathcal{V}^L \rangle \subset \mathbb{Z}[L_1, \cdots, L_m], \end{aligned}$$

where $k(a_1, \cdots, a_m)(= k) \in \mathbb{Z} - \{0\}$. Because $\sum k(a_1, \cdots, a_m) L_1^{a_1} \cdots L_m^{a_m} \in \langle L \mid p \notin \mathcal{V}^L \rangle$, we have that each term $k L_1^{a_1} \cdots L_m^{a_m} \in \langle L \mid p \notin \mathcal{V}^L \rangle$ for all $p \in \mathcal{V}^{\Gamma}$. Hence we have each term

$$k L_1^{a_1} \cdots L_m^{a_m} \in \bigcap_{p \in \mathcal{V}^{\Gamma}} \langle L \mid p \notin \mathcal{V}^L \rangle.$$

This means, for each term $kL_1^{a_1}\cdots L_m^{a_m}$ and all vertices $p \in \mathcal{V}^\Gamma$, there is $r \in \{1, \cdots, m\}$ such that $p \notin \mathcal{V}^{L_r}$ and $a_r \neq 0$. We put the set of such L_r by $\mathcal{L}''$. Because for all $p \in \mathcal{V}$ there is $L_r \in \mathcal{L}''$ such that $p \notin \mathcal{V}^{L_r}$, we have $\cap\mathcal{L}'' = \emptyset$. This means $\mathcal{L}'' \in \mathfrak{I}(\mathcal{L})$. Therefore each term $kL_1^{a_1}\cdots L_m^{a_m}$ is in $\langle \prod_{L\in\mathcal{L}'} L \mid \mathcal{L}' \in \mathfrak{I}(\mathcal{L})\rangle$. We conclude $A \in \langle \prod_{L\in\mathcal{L}'} L \mid \mathcal{L}' \in \mathfrak{I}(\mathcal{L})\rangle$. $\square$

From the above two lemmas: Lemma 3.2 and 3.3, we can prove the injectivity of Ψ'.

Proposition 3.4. Ψ' is injective.

Proof. Define $\rho' : H^*_{T^n}(\widetilde{\mathcal{G}}) \to \bigoplus_{p\in\mathcal{V}^\Gamma} H^*_{T^n}(pt)$ by $\rho'(f) = \bigoplus_{p\in\mathcal{V}^\Gamma} f(p)$. Then we can show the following diagram is commutative:

$$\begin{array}{ccc} \mathbb{Z}[\widetilde{\mathcal{G}}] & \xrightarrow{\rho} & \bigoplus_{p\in\mathcal{V}^\Gamma} \mathbb{Z}[\widetilde{\mathcal{G}}]_p \\ \Psi' \downarrow & & \downarrow \oplus_p I_p \\ H^*_{T^n}(\widetilde{\mathcal{G}}) & \xrightarrow{\rho'} & \bigoplus_{p\in\mathcal{V}^\Gamma} H^*_{T^n}(pt) \end{array}$$

by chasing generators $L_1, \cdots, L_m$. Now ρ is injective by Lemma 3.3, and the right map $\oplus_p I_p$ is bijective by Lemma 3.2. Consequently Ψ' is injective. $\square$

Next we prove the surjectivity of Ψ'. In order to prove the surjectivity, we will define an ideal $I(K)$ of $H^*_{T^n}(pt)$.

Let K be an intersection $L_1\cap\cdots\cap L_b (\neq \emptyset)$. Note that the graph K is connected because of the assumption (2) of Theorem 3.1. K is also a $(2n-2b)$-valent subgraph of $\widetilde{\mathcal{G}} = (\Gamma, \widetilde{\alpha}, \theta)$, that is, the restricted bijection $\theta^K_{pq} = \theta_{pq}|_{\mathcal{E}^K_p} : \mathcal{E}^K_p \to \mathcal{E}^K_q$ is well-defined for $pq \in E^K_p$. We define an ideal $I(K)$ (in $H^*_{T^n}(pt)$) on K as follows:

$$I(K) = \langle \widetilde{\alpha}(\epsilon) \mid \epsilon \in \mathcal{E}^K \rangle,$$

that is, this ideal is generated by all x-forgetful axial functions of edges and legs in K. The following lemma holds for this $I(K)$.

Lemma 3.5. *Let f be an element in $H^*_{T^n}(\widetilde{\mathcal{G}})$. If $f(p) \notin I(K)$ for some $p \in \mathcal{V}^K$, then $f(q) \notin I(K)$ for all $q \in \mathcal{V}^K$.*

Proof. Assume $f(p) \notin I(K)$ and $f(q) \in I(K)$. Since K is connected, there is a path in K from q to p, which is constructed by edges

$$qr_1,\ r_1r_2,\ \cdots,\ r_{s-1}r_s,\ r_sp \in E^K \subset \mathcal{E}^K.$$

Because of the congruence relation, we have

$$\begin{aligned} & f(q) - f(p) \\ = \ & (f(q) - f(r_1)) + (f(r_1) - f(r_2)) + \cdots + (f(r_{s-1}) - f(r_s)) + (f(r_s) - f(p)) \\ = \ & A_1\widetilde{\alpha}(qr_1) + A_2\widetilde{\alpha}(r_1r_2)\cdots + A_s\widetilde{\alpha}(r_{s-1}r_s) + A'\widetilde{\alpha}(r_sp) \\ \in \ & I(K), \end{aligned}$$

where $A_1, \cdots, A_s, A' \in H^*_{T^n}(pt)$ and the last relation is known by the definition of $I(K)$. Since $f(q),\ A_1\widetilde{\alpha}(qr_1), \cdots, A_s\widetilde{\alpha}(r_{s-1}r_s),\ A'\widetilde{\alpha}(r_sp) \in I(K)$, we have $f(p) \in I(K)$. This gives a contradiction. $\square$

From the above lemma, we can prove the surjectivity of Ψ'.

Proposition 3.6. Ψ' is surjective.

PROOF. Assume that $f(p) \in H^*_{T^n}(pt)$ has a non-zero constant term $k \in \mathbb{Z} - \{0\}$ for some $p \in \mathcal{V}^\Gamma$, that is, $f(p) = k + g(p)$ where $g(p) \in H^{>0}_{T^n}(pt) \cup \{0\}$. Because $f \in H^*_{T^n}(\widetilde{\mathcal{G}})$ satisfies the congruence relation, we see that $f(q) = k + g(q)$ for all $q \in \mathcal{V}^\Gamma$ and $g \in H^{>0}_{T^n}(\widetilde{\mathcal{G}}) \cup \{0\}$, where $H^{>0}_{T^n}(\widetilde{\mathcal{G}}) \cup \{0\}$ is the set of $g \in H^*_{T^n}(\widetilde{\mathcal{G}})$ whose constant term is 0. So we can take $k \in \mathbb{Z} \subset \mathbb{Z}[\widetilde{\mathcal{G}}]$ such that $f = \Psi'(k) + g$. Hence it suffices to prove the case that $g \in H^*_{T^n}(\widetilde{\mathcal{G}}) - \{0\}$ satisfies $g(p) = 0$ or $g(p) \in H^{>0}_{T^n}(pt)$ for all $p \in \mathcal{V}^\Gamma$, that is, $g \in H^{>0}_{T^n}(\widetilde{\mathcal{G}})(= H^{>0}_{T^n}(\widetilde{\mathcal{G}}) \cup \{0\} - \{0\})$.

Assume that $g \in H^{>0}_{T^n}(\widetilde{\mathcal{G}})$. Put $Z(g) = \{p \in \mathcal{V}^\Gamma \mid g(p) = 0\}$. If $Z(g) = \emptyset$, then $g(p) \in H^*_{T^n}(pt) = \mathbb{Z}[\tau_{L_1}(p), \cdots, \tau_{L_n}(p)]$ because of Lemma 3.2, where L_i is the $(2n-2)$-valent hypertorus subgraph such that $p \in \mathcal{V}^{L_i}$. So there is an element $A \in \mathbb{Z}[\widetilde{\mathcal{G}}]$ such that $\Psi'(A)(p) = g(p)$. Hence $p \in Z(g - \Psi'(A))$.

So we can assume $Z(g) \neq \emptyset$. Take $p \in \mathcal{V}^\Gamma \backslash Z(g)$, that is, $g(p) \neq 0$. Let $k\tau^{a_1}_{L_1} \cdots \tau^{a_n}_{L_n}(p)$ be a monomial appearing in $g(p)$, where k is a non-zero integer, $p \in \mathcal{V}^{L_i}$ and $a_i \geq 0$ $(i = 1, \cdots, n)$. Since $g(p) \in H^{>0}_{T^n}(pt)$, we can assume $a_1, \cdots, a_b \neq 0$ and $a_{b+1} = \cdots = a_n = 0$. Put $K = \cap^b_{i=1} L_i$. Then $g(p) \notin I(K)$ because $g(p)$ contains the monomial $k\tau^{a_1}_{L_1} \cdots \tau^{a_b}_{L_b}(p)$ and $\tau_{L_i}(p)$ $(i = 1, \cdots, b)$ is defined by the axial function of the normal edge or leg of K on p. Hence $g(q) \notin I(K)$ for all $q \in \mathcal{V}^K$, because of Lemma 3.5. In particular $g(q) \neq 0$ for all $q \in \mathcal{V}^K$. Let $r \notin \mathcal{V}^K$. Then we see $k\tau^{a_1}_{L_1} \cdots \tau^{a_b}_{L_b}(r) = 0$. Put $g' = g - k\tau^{a_1}_{L_1} \cdots \tau^{a_b}_{L_b} = g - \Psi'(kL^{a_1}_1 \cdots L^{a_b}_b)$, then $g'(r) = g(r)$ for all $r \notin \mathcal{V}^K$.

Therefore we see that $g(q) \neq 0$ for all $q \in \mathcal{V}^K$ and $g'(r) = g(r)$ for all $r \notin \mathcal{V}^K$. Hence $Z(g') \supset Z(g)$ holds. Note that the number of monomials in $g'(p)$ is strictly smaller than that in $g(p)$. Again we apply the same argument for $g' = g - \Psi'(kL^{a_1}_1 \cdots L^{a_b}_b)$. Then we get $g" = g' - \Psi'(k'L^{a'_1}_{i_1} \cdots L^{a'_c}_{i_c})$ such that $Z(g") \supset Z(g')$ and the number of monomials in $g"(p)$ is strictly smaller than that in $g'(p)$, where $\{L_{i_1}, \cdots, L_{i_c}\} \subset \{L_1, \cdots, L_n\}$ and k' is non-zero integer. Repeating this argument, we obtain $B \in \mathbb{Z}[\widetilde{\mathcal{G}}]$ such that $Z(g - \Psi'(B))$ contains $Z(g)$ as a proper subset. Moreover repeating this procedure, we can take $C \in \mathbb{Z}[\widetilde{\mathcal{G}}]$ such that $Z(g - \Psi'(C)) = \mathcal{V}^\Gamma$. This gives $g - \Psi'(C) = 0$. Therefore we conclude that Ψ' is surjective. $\square$

Consequently Ψ' is an isomorphic map by Proposition 3.4 and 3.6, and we have

$$H^*_{T^n}(\widetilde{\mathcal{G}}) \simeq \mathbb{Z}[\widetilde{\mathcal{G}}].$$

REMARK 3.7. From the above argument, we know that the assumption (2) of Theorem 3.1 does not need to prove the "injectivity" of Ψ', but it needs to prove the "surjectivity" of Ψ'. Hence the assumption (2) of Theorem 3.1 means that $H^*_{T^n}(\widetilde{\mathcal{G}})$ (resp. $H^*_{T^n \times S^1}(\mathcal{G})$) is generated by elements of $H^2_{T^n}(\widetilde{\mathcal{G}})$ (resp. $H^2_{T^n \times S^1}(\mathcal{G})$), that is, $\tau_L \in H^2_{T^n}(\widetilde{\mathcal{G}})$ (resp. $\tau_H, \chi \in H^2_{T^n \times S^1}(\mathcal{G})$). In fact, there exists a generator which is not in $H^2_{T^n \times S^1}(\mathcal{G})$ for the graph equivariant cohomology of the third example in Figure 2.1, note that this example does not satisfy the assumption (2) of Theorem 3.1 because the intersection of two lines (2-valent hypertorus subgraphs) are disconnected two vertices. In the future paper, we hope to find the generalized result of Theorem 3.1 without the assumption (2).

3.2. Proof of Main theorem. Let $\mathcal{G}$ be a hypertorus graph which satisfies the assumptions (1), (2) of Theorem 3.1, that is, $\mathcal{G}$ satisfies the following two assumptions:

(1) For each $L \in \mathcal{L}$, there are a hyperfacet H and its opposite side $\overline{H}$ such that $H \cap \overline{H} = L$;
(2) For all subsets $\mathcal{L}' \subset \mathcal{L}$, its intersection $\cap \mathcal{L}'$ is empty or connected.

Let us recall the homomorphism $\Psi : \mathbb{Z}[\mathcal{G}] \to H^*_{T^n \times S^1}(\mathcal{G})$. This Ψ is defined as follows:

$$\Psi(X) = \chi, \quad \Psi(H) = \tau_H.$$

The goal of this section is to prove Ψ is the isomorphism.

By the assumption (1), we can put the set of all hyperfacets by

$$\mathcal{H} = \{H_1, \cdots, H_m, \overline{H_1}, \cdots, \overline{H_m}\},$$

and the set of all connected $(2n-2)$-valent hypertorus subgraphs by

$$\mathcal{L} = \{L_1, \cdots, L_m\}$$

such that $L_i = H_i \cap \overline{H_i}$.

First we consider the following diagram:

$$\begin{array}{ccc} \mathbb{Z}[X, H_1, \cdots, \overline{H_m}] & \xrightarrow{\hat{\pi}} & \mathbb{Z}[\mathcal{G}] \\ \phi' \downarrow & & \downarrow \Psi \\ \mathbb{Z}[X, H_1, \cdots, H_m] & \xrightarrow{\pi} & H^*_{T^n \times S^1}(\mathcal{G}) \\ \phi \downarrow & & \downarrow \mathcal{F} \\ \mathbb{Z}[L_1, \cdots, L_m] & \xrightarrow{\pi'} & H^*_{T^n}(\widetilde{\mathcal{G}}). \end{array}$$

We explain about the above diagram. $\hat{\pi}$ is the natural projection, ϕ' is defined by $\phi'(X) = X$, $\phi'(H_i) = H_i$, $\phi'(\overline{H_i}) = X - H_i$ and $\pi(X) = \chi$, $\pi(H_i) = \tau_{H_i}$. So the top diagram is commutative. On the other hand π' is defined by $\pi'(L_i) = \tau_{L_i}$ (π' is surjective from Proposition 3.6), ϕ is defined by $\phi(X) = 0$, $\phi(H_i) = L_i$ ($i = 1, \cdots, m$) and $\mathcal{F}(f) = F \circ f$ where $f \in H^*_{T^n \times S^1}(\mathcal{G})$ and $F : H^*_{T^n \times S^1}(pt) \to H^*_{T^n}(pt)$ is the x-forgetful map (x is the generator of $\mathbb{Z}x \subset (\mathfrak{t}^n)^*_{\mathbb{Z}} \oplus \mathbb{Z}x = H^2_{T^n \times S^1}(pt)$). So the bottom diagram is also commutative.

We will use the above diagram to prove Ψ is the isomorphism.

3.2.1. *Surjectivity of* Ψ. Let us start to prove the surjectivity of Ψ. In order to prove the surjectivity, we will prove Proposition 3.11 which states the surjectivity of π (in the diagram). Moreover, in order to prove the surjectivity of π (Proposition 3.11), we need to prove the following three lemmas Lemma 3.8, 3.9 and 3.10.

LEMMA 3.8. *Let χ be in $H^*_{T^n \times S^1}(\mathcal{G})$ such that $\chi(p) = x$ for all $p \in \mathcal{V}^\Gamma$. Then we have* Ker $\mathcal{F} = H^*_{T^n \times S^1}(\mathcal{G})\chi$,

PROOF. Let $f \in$ Ker $\mathcal{F}$. By the definition of $\mathcal{F}$, we have $\mathcal{F}(f)(p) = F \circ f(p) = 0$ for all $p \in \mathcal{V}^\Gamma$. Since $F : H^*_{T^n \times S^1}(pt) = \mathbb{Z}[x, \alpha_1, \cdots, \alpha_n] \to \mathbb{Z}[\beta_1, \cdots, \beta_n] = H^*_{T^n}(pt)$ is defined by $F(x) = 0$ and $F(\alpha_i) = \beta_i$ for all $i = 1, \cdots, n$, we have Ker $F = \langle x \rangle \subset H^*_{T^n \times S^1}(pt)$. So $f(p) = g(p)x$ for all $p \in \mathcal{V}^\Gamma$, where $g(p)$ is a polynomial in $H^*_{T^n \times S^1}(pt)$. Because $f \in H^*_{T^n \times S^1}(\mathcal{G})$, it satisfies the congruence relation

$$f(p) - f(q) = g(p)x - g(q)x = (g(p) - g(q))x \equiv 0 \pmod{\alpha(pq)}$$

for all edges pq. By the definition of the hypertorus graph, we see $\alpha(pq) \neq x$. Hence $g \in H_{T^n \times S^1}(\mathcal{G})$. Therefore for all $f \in$ Ker $\mathcal{F}$, there exists some $g \in H^*_{T^n \times S^1}(\mathcal{G})$ such that $f = g\chi$. Hence Ker $\mathcal{F} \subset H^*_{T^n \times S^1}(\mathcal{G})\chi$. On the other hand, we can easily show Ker $\mathcal{F} \supset H^*_{T^n \times S^1}(\mathcal{G})\chi$. So we have that Ker $\mathcal{F} = H^*_{T^n \times S^1}(\mathcal{G})\chi$. $\square$

LEMMA 3.9. *Let $f \in H^*_{T^n \times S^1}(\mathcal{G})$. Then there exists $f_{2i} \in H^*_{T^n \times S^1}(\mathcal{G})$ which satisfies $f_{2i}(p) = 0$ or a $2i$ degree homogeneous polynomial in $H^*_{T^n \times S^1}(pt)$ such that $f = f_0 + f_2 + \cdots + f_{2l}$.*

PROOF. Since $f(p) \in H^*_{T^n \times S^1}(pt)$, we have that $f(p) = f_0(p) + \cdots + f_{2l}(p)$ where $f_{2i}(p) \in H^{2i}_{T^n \times S^1}(pt)$ for all $p \in \mathcal{V}^\Gamma$. Because f satisfies the congruence relation for all edges qr, we see that

$$f(q) - f(r) = (f_0(q) - f_0(r)) + \cdots + (f_{2l}(q) - f_{2l}(r)) = A\alpha(qr)$$

for some $A \in H^*_{T^n \times S^1}(pt)$. Moreover A can be divided into $A = A_0 + \cdots + A_{2l-2}$, where $A_{2i} \in H^{2i}_{T^n \times S^1}(pt)$. Hence $f_{2i}(q) - f_{2i}(r) = A_{2i-2}\alpha(qr)$. This concludes $f_{2i} \in H^*_{T^n \times S^1}(\mathcal{G})$ for all $i = 0, \cdots, l$. □

We call each f_{2i} in Lemma 3.9 a $2i$ *degree homogeneous class* ($i = 0, \cdots, l$). We denote deg $f_{2i} = 2i$ and $f_{2i} \in H^{2i}_{T^n \times S^1}(\mathcal{G})$.

LEMMA 3.10. *If $f \notin \mathrm{Im}\ \pi$, then there are $A \in \mathbb{Z}[X, H_1, \cdots, H_m]$ and $g \notin \mathrm{Im}\ \pi$ such that*

$$\pi(A) - f = g\chi,$$

where g can be denoted by $g = \sum_k g_{2j_k}$ whose all homogeneous classes $g_{2j_k} \notin \mathrm{Im}\ \pi$.

PROOF. Assume $f \notin \mathrm{Im}\ \pi$. Since ϕ is surjective by the assumption (1) of Theorem 3.1 and π' is surjective by Proposition 3.6, there is a non-zero element $B \in \mathbb{Z}[X, H_1, \cdots, H_m]$ such that $\mathcal{F}(f) = \pi' \circ \phi(B)$. Because $\pi' \circ \phi = \mathcal{F} \circ \pi$, we have

$$\pi' \circ \phi(B) = \mathcal{F} \circ \pi(B) = \mathcal{F}(f).$$

Hence $\pi(B) - f \in \mathrm{Ker}\ \mathcal{F}$. Because of Lemma 3.8, there is a $g' \in H^*_{T^n \times S^1}(\mathcal{G})$ such that

$$\pi(B) - f = g'\chi.$$

Since $f \notin \mathrm{Im}\ \pi$ and $\pi(X) = \chi$, we have

$$g' \notin \mathrm{Im}\ \pi.$$

Form Lemma 3.9, we can divide g' into $g' = g_0 + \cdots + g_{2l}$ where g_{2i} is a $2i$ degree homogeneous class. If $g_{2i} \in \mathrm{Im}\ \pi$, then $g' - g_{2i} \notin \mathrm{Im}\ \pi$. Therefore g' can be divided into two terms $(0 \neq) g = \sum_k g_{2j_k} \notin \mathrm{Im}\ \pi$ and $h = \sum_{k'} g_{2i_{k'}} \in \mathrm{Im}\ \pi$ such that

$$g' = g + h,$$

where homogeneous classes $g_{2j_k} \notin \mathrm{Im}\ \pi$ and $g_{2i_{k'}} \in \mathrm{Im}\ \pi$. Since

$$g'\chi = g\chi + h\chi = g\chi + \pi(CX)$$

for some $C \in \mathbb{Z}[X, H_1, \cdots, H_m]$, we see that there is an element $A = B - CX \in \mathbb{Z}[X, H_1, \cdots, H_m]$ such that $\pi(A) - f = g\chi$. □

Next we prove Proposition 3.11.

PROPOSITION 3.11. The homomorphism $\pi : \mathbb{Z}[X, H_1, \cdots, H_m] \to H^*_{T^n \times S^1}(\mathcal{G})$ is surjective.

PROOF. If we show all homogeneous classes $f \in H^*_{T^n\times S^1}(\mathcal{G})$ are in Im π, then we have that the graph equivariant cohomology $H^*_{T^n\times S^1}(\mathcal{G})$ is included in Im π by Lemma 3.9, that is, $\pi : \mathbb{Z}[X, H_1, \cdots, H_m] \to H^*_{T^n\times S^1}(\mathcal{G})$ is surjective. We prove all homogeneous classes are in Im π.

Let f be a minimal degree homogeneous class in $H^*_{T^n\times S^1}(\mathcal{G})\backslash\text{Im } \pi$. Then $f(p)$ is a non-zero homogeneous polynomial or $f(p) = 0$ for all $p \in \mathcal{V}^\Gamma$ by the definition of the homogeneous class. Because of Lemma 3.10, we have the following two cases for all $p \in \mathcal{V}^\Gamma$ and $f \notin$ Im π:

$$\pi(A)(p) - f(p) = g(p)x \quad (\text{if } f(p) \neq 0); \tag{3.1}$$

$$\pi(A)(p) = g(p)x \quad (\text{if } f(p) = 0), \tag{3.2}$$

for some $A \in \mathbb{Z}[X, H_1, \cdots, H_m]$ and $g \notin$ Im π whose all homogeneous classes are not in Im π.

Because $f \notin$ Im π, in particular $f \neq 0$, we can take a vertex $p \in \mathcal{V}^\Gamma$ such that $f(p) \neq 0$. Then we have the above case (3.1):

$$\pi(A)(p) - f(p) = g(p)x.$$

Here, put the minimal degree homogeneous class of $\pi(A)$ by $\pi(A') \in H_{T^n\times S^1}(\mathcal{G})$. If $\pi(A')(p) \neq f(p)$, then there is a term in $g(p)x$ whose degree is less than or equal to deg f. So there is a non-zero homogeneous terms $g_{2j_k}(p)$ of $g(p)$ such that

$$\text{deg } g_{2j_k}(p) = 2j_k = \text{deg } g_{2j_k} = \text{deg } g_{2j_k}x - 1 < \text{deg } f.$$

Moreover this homogeneous class $g_{2j_k} \notin$ Im π from Lemma 3.10. This gives a contradiction, since f is a minimal degree homogeneous class in $H^*_{T^n\times S^1}(\mathcal{G})\backslash\text{Im } \pi$. Hence the minimal degree homogeneous class $\pi(A')$ satisfies $\pi(A')(p) = f(p)$, when $f(p) \neq 0$. We also have deg $\pi(A') =$ deg f.

If $f(p) \neq 0$ for all $p \in \mathcal{V}^\Gamma$, then $f = \pi(A')$ by the above argument. This gives a contradiction, by the assumption $f \notin$ Im π. Therefore we can take a vertex $q \in \mathcal{V}^\Gamma$ such that $f(q) = 0$. Then we have the case (3.2): $\pi(A)(q) = g(q)x$. Because $\pi(A')$ is the minimal degree homogeneous class of $\pi(A)$, we also have

$$\pi(A')(q) = g'(q)x$$

holds for some homogeneous class $g' \notin$ Im π of g (by Lemma 3.10). If $g'(q)x = 0$ for all such q, then we have $f = \pi(A')$. This gives a contradiction, by the assumption $f \notin$ Im π. Hence there is some $q \in \mathcal{V}^\Gamma$ such that $f(q) = 0$ and $g'(q)x \neq 0$. However deg $g'(q) =$ deg $g' =$ deg $g'\chi - 1 <$ deg $\pi(A') =$ deg f. This also gives a contradiction, because g' is a homogeneous class in $H^*_{T^n\times S^1}(\mathcal{G})\backslash\text{Im } \pi$ and f is a minimal degree homogeneous class in $H^*_{T^n\times S^1}(\mathcal{G})\backslash\text{Im } \pi$. Consequently all homogeneous classes of $H^*_{T^n\times S^1}(\mathcal{G})$ are in Im π. □

Now $\hat{\pi}$ is surjective by the definition of $\mathbb{Z}[\mathcal{G}]$, and ϕ' is also surjective by its definition. Moreover π is surjective by Proposition 3.11. Since $\Psi \circ \hat{\pi} = \pi \circ \phi'$, we have the following corollary.

COROLLARY 3.12. Ψ is surjective.

3.2.2. *Injectivity of* Ψ. Finally we will prove the injectivity of Ψ. In this section we put

$$I_j = \{1, \cdots, l\} - \{j\}$$

for $j = 1, \cdots, l$.

First we need to prove Proposition 3.14. In order to prove Proposition 3.14, we prepare the following lemma.

LEMMA 3.13. *If* $\cap_{k=1}^{l} L_k = \emptyset$ *and* $L_k = H_k \cap \overline{H_k}$ $(k = 1, \cdots, l)$*, then we can take a hyperfacet* H_j *as* $H_j \cap (\cap_{k \in I_j} L_k) = \emptyset$ *for all* $j = 1, \cdots, l$*.*

PROOF. Assume $\cap_{k=1}^{l} L_k = \emptyset$. If $L_j \cap (\cap_{k \in I_j} L_k) = \cap_{k \in I_j} L_k = \emptyset$, then $H_j \cap (\cap_{k \in I_j} L_k) = \overline{H_j} \cap (\cap_{k \in I_j} L_k) = \emptyset$ for each H_j and $\overline{H_j}$ such that $H_j \cap \overline{H_j} = L_j$. So we may assume, for all $j = 1, \cdots, l$,

$$\bigcap_{k \in I_j} L_k \neq \emptyset.$$

If $p \in \mathcal{V}^{\cap_{k \in I_j} L_k}$ satisfies $\tau_{H_j}(p) = \alpha(n_{H_j}(p))$, then $p \in \mathcal{V}^{L_j}$ by the definition of the Thom class τ_{H_j}. However this gives a contradiction, since $L_j \cap (\cap_{k \in I_j} L_k) = \cap_{k=1}^{l} L_k = \emptyset$. Hence all vertices $p \in \mathcal{V}^{\cap_{k \in I_j} L_k}$ satisfy

$$\tau_{H_j}(p) = \begin{cases} 0 & (\text{if } p \notin \mathcal{V}^{H_j}) \\ x & (\text{if } p \in \mathcal{V}^{H_j}) \end{cases}$$

by the definition of the Thom class τ_{H_j}.

Assume $\tau_{H_1}(p) = x$ for all $p \in \mathcal{V}^{\cap_{k=2}^{l} L_k}$. Retaking $\overline{H_1}$ as H_1, we can put $\tau_{H_1}(p) = 0$ for all $p \in \mathcal{V}^{\cap_{k=2}^{l} L_k}$ by the equation $\tau_{H_1}(p) + \tau_{\overline{H_1}}(p) = x$. This implies that H_1 satisfies $H_1 \cap (\cap_{k=2}^{l} L_k) = \emptyset$. So we may assume that there is a vertex $p \in \mathcal{V}^{\cap_{k=2}^{l} L_k}$ such that $\tau_{H_1}(p) = 0$, that is, $p \notin \mathcal{V}^{H_1}$. Because we assume the assumption (2) of Theorem 3.1, we can take edges from p to the other vertex $q \in \mathcal{V}^{\cap_{k=2}^{l} L_k}$ as follows:

$$pr_1,\ r_1 r_2,\ \cdots,\ r_{u-1} r_u,\ r_u q \in E^{\cap_{k=2}^{l} L_k}.$$

Because $r_t \in \mathcal{V}^{\cap_{k=2}^{l} L_k}$ for $t = 0, \cdots, u$, we see $\tau_{H_1}(r_1) = 0$ or x by the above argument. Hence we have $\tau_{H_1}(p) - \tau_{H_1}(r_1) = 0$ or $-x$. Moreover we have $\tau_{H_1}(p) - \tau_{H_1}(r_1) \equiv 0 \pmod{\alpha(pr_1)}$ by the congruence relation. Because of the definition of the hypertorus graph, we also have $\alpha(e) \neq x$ for all edges $e \in E^{\Gamma}$. So we have $\tau_{H_1}(r_1) = 0$. Inductively we have $\tau_{H_1}(q) = 0$ for all $q \in \mathcal{V}^{\cap_{k=2}^{l} L_k}$. This implies that we can take H_1 as $H_1 \cap (\cap_{k=2}^{l} L_k) = \emptyset$. Since we can apply the same argument for the other H_j $(j = 2, \cdots, l)$, we can take H_j as

$$H_j \cap (\cap_{k \in I_j} L_k) = \emptyset$$

for all $j = 1, \cdots, l$. □

From Lemma 3.13, we have the following key fact.

PROPOSITION 3.14. Assume the hypertorus graph $\mathcal{G}$ satisfies two assumptions (1), (2) of Theorem 3.1. If $\cap_{k=1}^{l} L_k = \emptyset$ and $L_k = H_k \cap \overline{H_k}$ $(k = 1, \cdots, l)$, then we can take a hyperfacet H_k such that $\cap_{k=1}^{l} H_k = \emptyset$.

PROOF. If $\cap_{k=1}^{l} L_k = \emptyset$ and $L_k = H_k \cap \overline{H_k}$ $(k = 1, \cdots, l)$, we can take H_j as $H_j \cap (\cap_{k \in I_j} L_k) = \emptyset$ for all $j = 1, \cdots, l$ from Lemma 3.13. Set

$$\mathcal{H}' = \left\{ H_1, \cdots, H_l \mid H_j \cap (\cap_{k \in I_j} L_k) = \emptyset,\ j = 1, \cdots, l \right\}.$$

Let us show $\cap \mathcal{H}' = \emptyset$. In order to show it, we use the inductive argument. The first step of the induction has already shown in Lemma 3.13. Set $R = \{1, \cdots, l\}$, $S \subset R$

and $T = R - S$. Assume that the hyperfacet H_j satisfies $(\cap_{j\in S} H_j) \cap (\cap_{k\in T} L_k) = \emptyset$ for all $S \subset R$ whose number $|S| \leq s-1$ ($s \geq 2$).

Assume $(\cap_{j=1}^{s} H_j) \cap (\cap_{k=s+1}^{l} L_k) \neq \emptyset$ and take $p \in \mathcal{V}^{(\cap_{j=1}^{s} H_j)\cap(\cap_{k=s+1}^{l} L_k)}$. Then we have $\prod_{j=1}^{s} \tau_{H_j}(p) \neq 0$. Because $\cap_{k=1}^{l} L_k = (\cap_{j\in S} H_j) \cap (\cap_{k\in T} L_k) = \emptyset$ for all $S \subset R$ such that $|S| \leq s-1$, we see that $p \notin \mathcal{V}^{L_j}$ for all $j = 1, \cdots, s$. So $\tau_{H_j}(p) \neq \alpha(n_{H_j}(p))$ for all $j = 1, \cdots, s$. Hence we have, for all $p \in \mathcal{V}^{(\cap_{j=1}^{s} H_j)\cap(\cap_{k=s+1}^{l} L_k)}$,

$$\prod_{j=1}^{s} \tau_{H_j}(p) = x^s.$$

Therefore we have, for $q \in \mathcal{V}^{\cap_{k=s+1}^{l} L_k}$,

$$\prod_{j=1}^{s} \tau_{H_j}(q) = \begin{cases} 0 & (\text{if } q \notin \mathcal{V}^{(\cap_{j=1}^{s} H_j)\cap(\cap_{k=s+1}^{l} L_k)}) \\ x^s & (\text{if } q \in \mathcal{V}^{(\cap_{j=1}^{s} H_j)\cap(\cap_{k=s+1}^{l} L_k)}). \end{cases}$$

Because of the assumption (2) of Theorem 3.1, for all $q \in \mathcal{V}^{\cap_{k=s+1}^{l} L_k}$, we can take edges from q to $p \in \mathcal{V}^{(\cap_{j=1}^{s} H_j)\cap(\cap_{k=s+1}^{l} L_k)}$. Applying the similar argument in the proof of Lemma 3.13, we have $\prod_{j=1}^{s} \tau_{H_j}(q) = x^s$ for all $q \in \mathcal{V}^{\cap_{k=s+1}^{l} L_k}$. This implies $\cap_{k=s+1}^{l} L_k \subset \cap_{j=1}^{s} H_j$. So we have

$$\bigcap_{k=s}^{l} L_k \subset \bigcap_{k=s+1}^{l} L_k \subset \bigcap_{j=1}^{s} H_j \subset \bigcap_{j=1}^{s-1} H_j.$$

This gives a contradiction, since we assume $(\cap_{j=1}^{s-1} H_j) \cap (\cap_{k=s}^{l} L_k) = \emptyset$ as an assumption of the induction. Therefore we have $(\cap_{j=1}^{s} H_j) \cap (\cap_{k=s+1}^{l} L_k) = \emptyset$. From the above argument, we also have $(\cap_{j\in S'} H_j) \cap (\cap_{k\in T'} L_k) = \emptyset$ for all $S' \subset R$ such that $|S'| = s$ and $T' = R - S'$.

Inductively, we see that $(\cap_{j\in S} H_j) \cap (\cap_{k\in T} L_k) = \emptyset$ for all $S \subset R$ such that $|S| \leq l-1$ and $T = R - S$. So we have

$$\prod_{j=1}^{l} \tau_{H_j}(p) = \begin{cases} 0 & (\text{if } p \notin \mathcal{V}^{\cap_{j=1}^{l} H_j}) \\ x^l & (\text{if } p \in \mathcal{V}^{\cap_{j=1}^{l} H_j}), \end{cases}$$

because $\cap_{k=1}^{l} L_k = \emptyset$. If $\cap_{j=1}^{l} H_j \neq \emptyset$, then there is a vertex $p \in \mathcal{V}^{\Gamma}$ such that $\prod_{j=1}^{l} \tau_{H_j}(p) = x^l$. By the definition of the hyperfacet, we see $\cap_{j=1}^{l} H_j \neq \Gamma$. So we can take a vertex $q \in \mathcal{V}^{\Gamma}$ such that $\prod_{j=1}^{l} \tau_{H_j}(q) = 0$. Since Γ is connected, we can take edges from p to q. Similarly this gives a contradiction by the congruence relation. Consequently we conclude that if $\cap_{j=1}^{l} L_j = \emptyset$ then we can take H_j such that $\cap_{j=1}^{l} H_j = \emptyset$. □

Next we will prove Proposition 3.17. In order to prove it, we prepare some notations and lemmas: Lemma 3.15 and 3.16.

Let $\widetilde{\pi} : \mathbb{Z}[X, H_1, \cdots, H_m] \to \mathbb{Z}[\mathcal{G}]$ be the natural homomorphism such that $\widetilde{\pi}(X) = X$, $\widetilde{\pi}(H_i) = H_i$ for $i = 1, \cdots, m$. Because $\overline{H_i} = X - H_i$ in $\mathbb{Z}[\mathcal{G}]$, we have

$$\widetilde{\pi} \circ \phi' = \hat{\pi} : \mathbb{Z}[X, H_1, \cdots, \overline{H_m}] \to \mathbb{Z}[\mathcal{G}].$$

Since $\hat{\pi}$ is surjective, $\widetilde{\pi}$ is also surjective. Moreover we have

$$\Psi \circ \widetilde{\pi} = \pi : \mathbb{Z}[X, H_1, \cdots, H_m] \to H^*_{T^n \times S^1}(\mathcal{G})$$

by definitions of Ψ and π. Hence we have the following commutative diagram:

$$\begin{array}{ccc} \mathbb{Z}[X,H_1,\cdots,\overline{H_m}] & \xrightarrow{\hat{\pi}} & \mathbb{Z}[\mathcal{G}] \\ \phi'\downarrow & \overset{\widetilde{\pi}}{\nearrow} & \downarrow\Psi \\ \mathbb{Z}[X,H_1,\cdots,H_m] & \xrightarrow{\pi} & H^*_{T^n\times S^1}(\mathcal{G}) \\ \phi\downarrow & & \downarrow\mathcal{F} \\ \mathbb{Z}[L_1,\cdots,L_m] & \xrightarrow{\pi'} & H^*_{T^n}(\widetilde{\mathcal{G}}). \end{array}$$

Then we have the following lemma.

LEMMA 3.15. *For the ideal (in $\mathbb{Z}[X,H_1,\cdots,\overline{H_m}]$)*

$$\mathcal{I}=\Big\langle H_i+\overline{H_i}-X,\ \prod_{H\in\mathcal{H}'}H\ \Big|\ i=1,\ \cdots,\ m,\ \mathcal{H}'\in\mathfrak{I}(\mathcal{H})\Big\rangle,$$

where $\mathfrak{I}(\mathcal{H})=\{\mathcal{H}'\subset\mathcal{H}\mid\cap\mathcal{H}'=\emptyset\}$, it satisfies the following two properties:

(i) Ker $\widetilde{\pi}=\phi'(\mathcal{I})$;
(ii) Ker $\pi'=\phi\circ\phi'(\mathcal{I})$.

PROOF. Since $\hat{\pi}$ is the natural projection, we know $\mathcal{I}=$ Ker $\hat{\pi}$. So we see $\widetilde{\pi}(\phi'(\mathcal{I}))=\hat{\pi}(\mathcal{I})=\hat{\pi}(\text{Ker }\hat{\pi})=\{0\}$. Hence $\phi'(\mathcal{I})\subset$ Ker $\widetilde{\pi}$. Let A be an element in Ker $\widetilde{\pi}$. Because ϕ' is surjective, there is an element $B\in\mathbb{Z}[X,\ H_1,\ \cdots,\ \overline{H_m}]$ such that $\phi'(B)=A$. Now $\hat{\pi}(B)=\widetilde{\pi}\circ\phi'(B)=\widetilde{\pi}(A)=0$. So $B\in$ Ker $\hat{\pi}=\mathcal{I}$. Hence $A=\phi'(B)\in\phi'(\mathcal{I})$, that is, Ker $\widetilde{\pi}\subset\phi'(\mathcal{I})$. Therefore we conclude the first property: Ker $\widetilde{\pi}=\phi'(\mathcal{I})$.

From Proposition 3.4 and 3.6, we know Ker $\pi'=\langle\prod_{L\in\mathcal{L}'}L\mid\mathcal{L}'\in\mathfrak{I}(\mathcal{L})\rangle$ where $\mathfrak{I}(\mathcal{L})=\{\mathcal{L}'\subset\mathcal{L}\mid\cap\mathcal{L}'=\emptyset\}$. Take a generator $\prod_{L\in\mathcal{L}'}L\in$ Ker π'. From Proposition 3.14, for $\mathcal{L}'=\{L_1,\ \cdots,\ L_l\}\in\mathfrak{I}(\mathcal{L})$, there is a set of hyperfacets $\mathcal{H}'=\{H_1,\ \cdots,\ H_l\}\in\mathfrak{I}(\mathcal{H})$ such that $H_k\cap\overline{H_k}=L_k$. By the definition of the ideal $\mathcal{I}$, a product $\prod_{k=1}^{l}H_k$ is one of the generators of $\mathcal{I}$. Moreover we see $\phi\circ\phi'(\mathcal{I})\ni\phi\circ\phi'(\prod_{k=1}^{l}H_k)=\pm\prod_{k=1}^{l}L_k$ by definitions of ϕ' and ϕ. Hence we have Ker $\pi'\subset\phi\circ\phi'(\mathcal{I})$. Moreover we have $\pi'\circ\phi\circ\phi'(A)=\{0\}$ for all $A\in\mathcal{I}$, because $\phi'(H+\overline{H}-X)=0$ and $\phi\circ\phi'(\prod_{H\in\mathcal{H}'}H)=\pm\prod_{L\in\mathcal{L}'}L\in$ Ker π'. So we have Ker $\pi'\supset\phi\circ\phi'(\mathcal{I})$. Therefore we conclude the second property: Ker $\pi'=\phi\circ\phi'(\mathcal{I})$. □

In order to prove Proposition 3.17, we also prepare the following lemma.

LEMMA 3.16. *Let $\mathcal{I}\subset\mathbb{Z}[x_1,\ \cdots,\ x_l]$ be an ideal generated by homogeneous polynomials, that is, $\mathcal{I}=\langle p_1,\ \cdots,\ p_m\rangle$ where p_i is a homogeneous polynomial of $\mathbb{Z}[x_1,\ \cdots,\ x_l]$ such that* deg $p_i\leq$ deg p_j *for $i<j$. For $A\in\mathcal{I}$, we denote $A=A_1+\cdots+A_n$ where A_i is a homogeneous term ($i=1,\ \cdots,\ n$) and* deg $A_i<$ deg A_j *for $i<j$. Then $A_i\in\mathcal{I}$ for all $i=1,\ \cdots,\ n$.*

PROOF. Because $A\in\mathcal{I}$, we can denote $A=X_1p_1+\cdots+X_mp_m$ where $X_i\in\mathbb{Z}[x_1,\ \cdots,\ x_l]$. Then we can put $X_k=X_{k1}+X_{k2}+\cdots+X_{ks_k}$ where X_{ki} is a homogeneous term ($i=1,\ \cdots,\ s_k$) and deg $X_{ki}<$ deg X_{kj} for $i<j$. Hence we see that

$$\begin{aligned} A &= (X_{11}+\cdots+X_{1s_1})p_1+\cdots+(X_{m1}+\cdots+X_{ms_m})p_m \\ &= A_1+\cdots+A_n. \end{aligned}$$

Because A_i is a homogeneous term, we have $A_i = \sum_{j\in\mathcal{D}_i} X_{jh_j}p_j$ where $\mathcal{D}_i = \{j \mid \deg X_{jh_j} + \deg p_j = \deg A_i\}$. Therefore $A_i \in \mathcal{I}$ for all $i = 1, \cdots, n$. $\square$

Using the above two lemmas, we have the following lemma.

PROPOSITION 3.17. $\mathrm{Ker}\ \widetilde{\pi} = \mathrm{Ker}\ \pi$.

PROOF. By Lemma 3.15 (i), the commutativity of the previous diagram and $\mathrm{Ker}\ \hat{\pi} = \mathcal{I}$, we see $\pi(\mathrm{Ker}\ \widetilde{\pi}) = \pi\circ\phi'(\mathcal{I}) = \Psi\circ\hat{\pi}(\mathcal{I}) = 0$. Hence $\mathrm{Ker}\ \widetilde{\pi} = \phi'(\mathcal{I}) \subset \mathrm{Ker}\ \pi$.

Let $A(\in \mathbb{Z}[X, H_1, \cdots, H_m])$ be a minimal degree homogeneous polynomial in $\mathrm{Ker}\ \pi\backslash\phi'(\mathcal{I})$. By the previous diagram, $\pi'\circ\phi(A) = \mathcal{F}\circ\pi(A) = 0$. So $\phi(A) \in \mathrm{Ker}\ \pi' = \phi\circ\phi'(\mathcal{I})$ by Lemma 3.15 (ii). Therefore we can take $B \in \phi'(\mathcal{I})(\subset \mathrm{Ker}\ \pi)$ such that $\phi(A) = \phi(B)$. By the definition of ϕ, we have

$$A - B \in \mathrm{Ker}\ \phi = \mathbb{Z}[X, H_1, \cdots, H_m]X.$$

So there is a polynomial $C \in \mathbb{Z}[X, H_1, \cdots, H_m]$ such that

$$A - B = CX.$$

Since $A, B \in \mathrm{Ker}\ \pi$, we have $\pi(A-B) = 0 = \pi(CX) = \pi(C)\chi$. Because $0 = \pi(C)\chi \in H^*_{T^n\times S^1}(\mathcal{G})$, we have that $0 = \pi(C)(p)x \in H^*_{T^n\times S^1}(pt)$ for all $p \in \mathcal{V}^\Gamma$. Since $H^*_{T^n\times S^1}(pt)$ is a free $H^*_{S^1}(pt)(=\mathbb{Z}[x])$-module, we see that $\pi(C)(p) = 0$ for all $p \in \mathcal{V}^\Gamma$. Hence $\pi(C) = 0$, that is, $C \in \mathrm{Ker}\ \pi$. If $C \in \phi'(\mathcal{I})$, then $CX \in \phi'(\mathcal{I})$ because $\mathcal{I}$ is the ideal and $\phi'(X) = X$. So we have $A = B + CX \in \phi'(\mathcal{I})$. This gives a contradiction, since $A \in \mathrm{Ker}\ \pi\backslash\phi'(\mathcal{I})$. Hence we have

$$C \in \mathrm{Ker}\ \pi\backslash\phi'(\mathcal{I}).$$

Now we can denote $C = \sum_{i=1}^{l} C_i \in \mathrm{Ker}\ \pi\backslash\phi'(\mathcal{I})$ where C_i is a homogeneous polynomial and $\deg C_i < \deg C_j$ for $i < j$. Then we see that there is a term $C_k \in \mathrm{Ker}\ \pi\backslash\phi'(\mathcal{I})$ of $C = \sum_{i=1}^{l} C_i$. If there is a term $C_i \in \phi'(\mathcal{I})$, then $B' = B + C_iX \in \phi'(\mathcal{I})$ because $B \in \phi'(\mathcal{I})$ and $\phi'(X) = X$. Moreover we have $\phi(B') = \phi(A)$ because $\phi(X) = 0$ and $\phi(A) = \phi(B)$. Hence we can put

$$A = B + CX \in \mathrm{Ker}\ \pi\backslash\phi'(\mathcal{I})$$

where $C = \sum_{k=1}^{l'} C_{j_k}$ such that $C_{j_k} \in \mathrm{Ker}\ \pi\backslash\phi'(\mathcal{I})$ for all $k = 1, \cdots, l'$ and $B \in \phi'(\mathcal{I})$ such that $\phi(A) = \phi(B)$. From the definitions of $\mathcal{I}$ and ϕ', we have that the ideal $\phi'(\mathcal{I})$ is generated by homogeneous polynomials of the polynomial ring $\mathbb{Z}[X, H_1, \cdots, H_m]$. Hence we have that all homogeneous terms B_h of $B \in \phi'(\mathcal{I})$ are elements in $\phi'(\mathcal{I})$ from Lemma 3.16. Since $B_h \in \phi'(\mathcal{I})$ and $C_{j_k} \in \mathrm{Ker}\ \pi\backslash\phi'(\mathcal{I})$, we have $C_{j_k} \neq -B_h$. Therefore, since A is a homogeneous polynomial, we have that B and C are also homogeneous polynomials. In particular $C \in \mathrm{Ker}\ \pi\backslash\phi'(\mathcal{I})$ and $\deg A = \deg CX > \deg C$. This gives a contradiction, since A is a minimal degree homogeneous polynomial in $\mathrm{Ker}\ \pi\backslash\phi'(\mathcal{I})$. Hence we have $\mathrm{Ker}\ \pi\backslash\phi'(\mathcal{I}) = \emptyset$, that is, $\mathrm{Ker}\ \pi = \phi'(\mathcal{I}) = \mathrm{Ker}\ \widetilde{\pi}$ by Lemma 3.15 (i). $\square$

So we can prove the injectivity of Ψ.

COROLLARY 3.18. Ψ is injective.

PROOF. Let A be in $\mathrm{Ker}\ \Psi$. Then there is an element $B \in \mathbb{Z}[X, H_1, \cdots, H_m]$ such that $\widetilde{\pi}(B) = A$, because $\widetilde{\pi}$ is surjective. So we have $\pi(B) = \Psi\circ\widetilde{\pi}(B) =$

$\Psi(A) = 0$. Hence $B \in \mathrm{Ker}\ \pi = \mathrm{Ker}\ \widetilde{\pi}$ by Proposition 3.17. Therefore we have $\widetilde{\pi}(B) = 0 = A$. Hence we conclude that Ψ is injective. □

Because of Corollary 3.12 and 3.18, we have that Ψ is the isomorphism. Consequently the proof of Theorem 3.1 is complete, that is, we get $H^*_{T^n\times S^1}(\mathcal{G}) \simeq \mathbb{Z}[\mathcal{G}]$.

4. Application: Mayer-Vietoris exact sequence of graph

Let $\mathcal{G} = (\Gamma,\ \alpha,\ \theta)$ be a hypertorus graph which satisfies the assumptions (1) and (2) of Theorem 3.1. In this final section, we give one application of Theorem 3.1. The purpose of this section is to prove the following theorem.

THEOREM 4.1. *If $\mathcal{G}$ is not a minimal hypertorus graph, then there is the following exact sequence:*

$$\{0\} \longrightarrow H^*_{T^n\times S^1}(\mathcal{G}) \xrightarrow{\rho_1} H^*_{T^n\times S^1}(\mathcal{G}_1) \oplus H^*_{T^n\times S^1}(\mathcal{G}_2) \xrightarrow{\rho_2} H^*_{T^n\times S^1}(\mathcal{G}_3) \longrightarrow \{0\}.$$

We start to prepare notations which appear in Theorem 4.1.

4.1. Neighborhood and Minimal hypertorus graph. In order to define a *minimal hypertorus graph*, we define a *neighborhood* $N(H) = (\mathcal{V}^{N(H)},\ \mathcal{E}^{N(H)})$ of the subgraph $H = (\mathcal{V}^H,\ \mathcal{E}^H)$ in $\Gamma = (\mathcal{V}^\Gamma,\ \mathcal{E}^\Gamma)$.

DEFINITION 4.2 (neighborhood of subgraph). Let H be a subgraph of Γ. Let $N(H)$ be a $2n$-valent graph which satisfies the following properties:

$$\mathcal{V}^{N(H)} = \mathcal{V}^H;$$
$$\mathcal{E}^{N(H)}_p = \mathcal{E}^H_p \text{ if } |\mathcal{E}^H_p| = 2n;$$
$$\mathcal{E}^{N(H)}_q = \mathcal{E}^H_q \cup \{l(n(q)_1), \cdots, l(n(q)_k)\} \text{ if } |\mathcal{E}^H_q| = 2n-k,$$

where $\{n(q)_1, \cdots, n(q)_k\} = \mathcal{E}^\Gamma_q - \mathcal{E}^H_q$. Here if $n(q)$ is a leg (in Γ) then $l(n(q)) = n(q)$, if not so then we regard the edge $n(q)$ (in Γ) as a leg (in $N(H)$) whose initial vertex is q. We call $N(H)$ a *neighborhood* of the subgraph H in Γ.

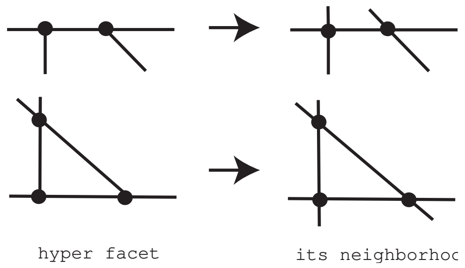

FIGURE 5. Two neighborhoods in the left graph in Figure 2.

REMARK 4.3. We *do not* call a neighborhood $N(H)$ a subgraph of Γ, if $N(H)$ has a leg $l(n(q))$ such that $n(q)$ is an edge in Γ. Of course the neighborhoods $N(H)$ is a $2n$-valent hypertorus graph for every hypertorus subgraph H.

Let us define a minimal hypertorus graph.

Definition 4.4 (minimal hypertorus graph). We call $\mathcal{G}$ is a *minimal hypertorus graph*, if we can put a set of all hyperfacets in $\mathcal{G}$ as follows:

$$\mathcal{H}=\{H_1,\ \cdots,\ H_m,\ \overline{H}_1,\ \cdots,\ \overline{H}_m\}$$

such that $N(\overline{H}_i)=\Gamma$ for all $i=1,\cdots,m$.

For example left two hypertorus graphs in Figure 2 are minimal hypertorus graphs. However the following graph in Figure 6 is not a minimal hypertorus graph, because we can divide Γ into the upper graph H and the lower graph $\overline{H}$ for the middle line (2-valent hypertorus subgraph) in Figure 6.

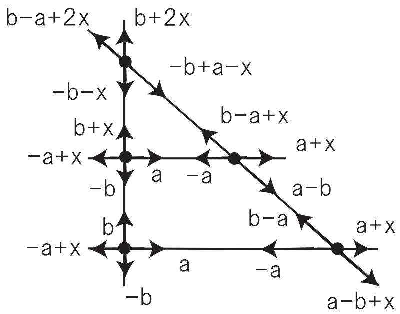

Figure 6. An example which is not a minimal hypertorus graph.

Remark 4.5. For an n-valent torus graph in [**MMP**], if we put n legs on all vertices, then we can construct a $2n$-valent hypertorus graph. We can regard this hypertorus graph as a *tangent bundle over a torus graph*. We can easily check this hypertorus graph is a minimal hypertorus graph.

Next we define $\mathcal{G}_1$, $\mathcal{G}_2$ and $\mathcal{G}_3$ in Theorem 4.1. If a hypertorus graph $\mathcal{G}=(\Gamma,\ \alpha,\ \theta)$ is not minimal hypertorus graph, then there are a hyperfacet H and its opposite side $\overline{H}$ such that

$$N(H),\ N(\overline{H})\neq\Gamma.$$

Put $N(H)=\Gamma_1$, $N(\overline{H})=\Gamma_2$ and $N(H)\cap N(\overline{H})=N(L)=\Gamma_3$, where $L=H\cap\overline{H}$. Then we can easily check that all Γ_i are $2n$-valent graph, all restricted connections $\theta_i=\{\theta_e\mid e\in E^{\Gamma_i}\subset E^{\Gamma}\}$ are well-defined connections on Γ_i and all restricted axial functions $\alpha_i=\alpha|_{\mathcal{E}^{\Gamma_i}}$ are also well-defined axial functions on Γ_i $(i=1,\ 2,\ 3)$. Therefore $\mathcal{G}_i=(\Gamma_i,\ \alpha_i,\ \theta_i)$ is a hypertorus graph for all $i=1,\ 2,\ 3$, and we can define each graph equivariant cohomologies $H^*_{T^n\times S^1}(\mathcal{G}_i)$.

ρ_1, ρ_2 and ρ_3 will define in the next section.

4.2. Proof of Theorem 4.1. Henceforth in this section a non-minimal hypertorus graph $\mathcal{G}=(\Gamma,\ \alpha,\ \theta)$ satisfies assumptions (1), (2) of Theorem 3.1. Take some hyperfacet H in $\mathcal{G}$ such that $N(H)$, $N(\overline{H})\neq\Gamma$. Put $N(H)=\Gamma_1$, $N(\overline{H})=\Gamma_2$ and $N(H)\cap N(\overline{H})=N(L)=\Gamma_3$, where $L=H\cap\overline{H}$.

4.2.1. *Preliminary.* In order to prove Theorem 4.1, first we study some properties for $\mathcal{G}_i = (\Gamma_i, \alpha_i, \theta_i)$ $(i = 1, 2, 3)$.

The following lemma shows that all $(2n-2)$-valent hypertorus subgraphs in $\mathcal{G}_i$ come from $(2n-2)$-valent hypertorus subgraphs in $\mathcal{G}$.

LEMMA 4.6. *Put* $\Gamma_1 = N(H)$, $\Gamma_2 = N(\overline{H})$, $\Gamma_3 = N(L)$ *and* $L = H \cap \overline{H}$, *where* H *is a hyperfacet in* $\mathcal{G}$. *The following statements hold for* $(2n-2)$*-valent hypertorus subgraphs in* $\mathcal{G}_i$.

(1) *If* L' *is a* $(2n-2)$*-valent hypertorus subgraph in* $\mathcal{G}_i$, *then there is unique* $(2n-2)$*-valent hypertorus subgraph* $\hat{L}$ *in* $\mathcal{G}$ *such that* $L' = \Gamma_i \cap \hat{L}$.
(2) *If a* $(2n-2)$*-valent hypertorus graph* L' *in* $\mathcal{G}_i$ *satisfies* $L' \cap L \neq \emptyset$, *then* $L' \cap L = \hat{L} \cap L$. *In particular* $L' \cap L$ *is connected in* $\mathcal{G}_i$.
(3) *If* $L'_a = \Gamma_i \cap \hat{L}_a$ *and* $L'_b = \Gamma_i \cap \hat{L}_b$ *are two distinct* $(2n-2)$*-valent hypertorus subgraphs in* $\mathcal{G}_i$, *then* $\hat{L}_a \neq \hat{L}_b$, *where* $\hat{L}_a$, $\hat{L}_b$ *are* $(2n-2)$*-valent hypertorus subgraphs in* $\mathcal{G}$.

PROOF. First we prove the statement (1). Let L' be a $(2n-2)$-valent hypertorus subgraph in $\mathcal{G}_i$. Then we can take $(n-1)$-pairs $\mathcal{E}_p^{L'} (\subset \mathcal{E}_p^{\Gamma_i})$ for $p \in \mathcal{V}^{L'}$. Conversely, for such $(n-1)$-pairs, a $(2n-2)$-valent hypertorus subgraph which is through this $(n-1)$-pairs is L', by Lemma 2.5. Now we see $\mathcal{E}_p^{\Gamma_i} = \mathcal{E}_p^{\Gamma}$, and we can also take a unique $(2n-2)$-valent hypertorus subgraph $\hat{L}$ in $\mathcal{G}$, by Lemma 2.5. Moreover $L' = \Gamma_i \cap \hat{L}$ by their uniqueness.

Next we prove the statement (2). Assume a $(2n-2)$-valent hypertorus subgraph L' in $\mathcal{G}_i$ satisfies $L' \cap L \neq \emptyset$. By the statement (1), we can take $(2n-2)$-valent hypertorus subgraph $\hat{L}$ in $\mathcal{G}$ such that $L' = \Gamma_i \cap \hat{L}$. Because $L \subset \Gamma_i$, we have $L' \cap L = \hat{L} \cap L$. By the assumption (2) of Theorem 3.1, we also have $L' \cap L = \hat{L} \cap L$ is connected.

Finally we prove the statement (3). Assume two $(2n-2)$-valent hypertorus subgraphs L'_a and L'_b satisfy that $L'_a \neq L'_b$ in $\mathcal{G}_i$. By the statement (1), we have $L'_a = \Gamma_i \cap \hat{L}_a$ and $L'_b = \Gamma_i \cap \hat{L}_b$ for $(2n-2)$-valent hypertorus subgraphs $\hat{L}_a$, $\hat{L}_b$ in $\mathcal{G}$. If $L'_a = \hat{L}_a$ or $L'_b = \hat{L}_b$, that is, $\hat{L}_a$ or $\hat{L}_b$ is a hyperfacet in $\mathcal{G}_i$, then we can easily prove $\hat{L}_a \neq \hat{L}_b$ because $L'_a \neq L'_b$. Hence we can assume $L'_a \neq \hat{L}_a$ and $L'_b \neq \hat{L}_b$. In this case, we also see $L'_a \cap L \neq \emptyset$ and $L'_b \cap L \neq \emptyset$. If $L'_a \cap L'_b \neq \emptyset$ in Γ_i, then $\mathcal{E}_p^{\hat{L}_a} = \mathcal{E}_p^{L'_a} \neq \mathcal{E}_p^{L'_b} = \mathcal{E}_p^{\hat{L}_b}$ for $p \in \mathcal{V}^{L'_a \cap L'_b}$, by Lemma 2.5. Hence $\hat{L}_a \neq \hat{L}_b$ because of Lemma 2.5. If $L'_a \cap L'_b = \emptyset$ in Γ_i, then $L'_a \cap L$ separates from $L'_b \cap L$ in Γ_i and Γ. By the statement (2), we have that $L'_a \cap L = \hat{L}_a \cap L$ and $L'_b \cap L = \hat{L}_b \cap L$. Hence, if $L_a = L_b$, there are two connected components $L'_a \cap L$ and $L'_b \cap L$ in $L_a \cap L = L_b \cap L$. This gives a contradiction, since the assumption (2) of Theorem 3.1 holds for $\mathcal{G}$. Therefore we have $\hat{L}_a \neq \hat{L}_b$. □

The following lemma is the key lemma to prove Theorem 4.1.

LEMMA 4.7. *Assume* $\mathcal{G}$ *satisfies the assumptions (1) and (2) of Theorem 3.1. Then the following statements hold:*

(1) *All hyperfacets in* $\mathcal{G}_i$ *can be denoted by* $\Gamma_i \cap F$ *for unique hyperfacet* F *of* $\mathcal{G}$ *and its opposite side is* $\Gamma_i \cap \overline{F}$, *for* $i = 1, 2, 3$;
(2) $\mathcal{G}_i$ *also satisfies the assumptions (1) and (2) of Theorem 3.1.*

PROOF. First we prove the statement (1). Take hyperfacet F' and its opposite side $\overline{F'}$ in $\mathcal{G}_i$. For its boundary $L' = F' \cap \overline{F'}$, there is a unique $(2n-2)$-valent hypertorus subgraph $\hat{L}$ in $\mathcal{G}$ such that $L' = \Gamma_i \cap \hat{L}$ by Lemma 4.6 (1). Because $\mathcal{G}$ satisfies the assumption (1) of Theorem 3.1, we can take unique hyperfacet F and its opposite side $\overline{F}$ such that $F \cap \overline{F} = \hat{L}$. Since $F \cup \overline{F} = \Gamma$, we can assume $F \cap F' - L' \neq \emptyset$. Moreover we have $\Gamma_i \cap (F \cup \overline{F}) = (\Gamma_i \cap F) \cup (\Gamma_i \cap \overline{F}) = \Gamma_i = F' \cup \overline{F'}$. We can easily show $\Gamma_i \cap F$ is a hyperfacet in Γ_i and $\Gamma_i \cap \overline{F}$ is its opposite side. Now L' divides Γ_i into two hyperfacets F', $\overline{F'}$, and $\Gamma_i \cap \hat{L}$ also divides Γ_i into two hyperfacets $\Gamma_i \cap F$, $\Gamma_i \cap \overline{F}$. Therefore we have $\Gamma_i \cap F = F'$ and $\Gamma_i \cap \overline{F} = \overline{F'}$, because $L' = \Gamma_i \cap \hat{L}$ and $F \cap F' - L' \neq \emptyset$.

Next we prove $\mathcal{G}_i$ satisfies the assumption (1) of Theorem 3.1. Put a $(2n-2)$-valent hypertorus subgraph $L' = \Gamma_i \cap \hat{L}$ in Γ_i, by Lemma 4.6 (1). Since $\mathcal{G}$ satisfies the assumption (1) of Theorem 3.1, for a $(2n-2)$-valent hypertorus subgraph $\hat{L}$ there is a unique hyperfacet F and its opposite side $\overline{F}$ such that $\hat{L} = F \cap \overline{F}$. Then we can easily show that $\Gamma_i \cap F$ is a hyperfacet, $\Gamma_i \cap \overline{F}$ is its opposite side and $(\Gamma_i \cap F) \cap (\Gamma_i \cap \overline{F}) = L'$. Conversely, take hyperfacet F' in Γ_i such that $F' \cap \overline{F'} = L'$. Then we see $F' = \Gamma_i \cap F$ and $\overline{F'} = \Gamma_i \cap \overline{F}$ by the statement (1). Hence $\mathcal{G}_i$ satisfies the assumption (1) of Theorem 3.1.

Finally we prove $\mathcal{G}_i$ satisfies the assumption (2) of Theorem 3.1. Take some distinct $(2n-2)$-valent hypertorus subgraphs $L'_1, \cdots, L'_k$ in $\mathcal{G}_i$, and put $\mathcal{L}' = \{L'_1, \cdots, L'_k\}$. By Lemma 4.6 (1), (3), we can take distinct $(2n-2)$-valent hypertorus subgraphs $\hat{L}_1, \cdots, \hat{L}_k$ in $\mathcal{G}$ such that $\Gamma_i \cap \hat{L}_j = L'_j$, and put $\mathcal{L} = \{\hat{L}_1, \cdots, \hat{L}_k\}$. Assume $\cap\mathcal{L}' = L'_1 \cap \cdots \cap L'_k = \Gamma_i \cap (\cap\mathcal{L}) \neq \emptyset$. If $\cap\mathcal{L} \subset \Gamma_i$, we can easily show $\cap\mathcal{L} = \cap\mathcal{L}'$. Hence $\cap\mathcal{L}'$ is connected by the assumption (2) of Theorem 3.1. If $\cap\mathcal{L} \not\subset \Gamma_i$, that is, $L \cap (\cap\mathcal{L}) \neq \emptyset$ and $\cap\mathcal{L} \not\subset L$. By the assumption (2) of Theorem 3.1, $L \cap (\cap\mathcal{L})$ is connected. For $\Gamma_3 = N(L)$, because Γ_3 is a hypertorus graph which attaches two legs on each vertex $\mathcal{V}^L$ for L, the intersection $\cap\mathcal{L}' = \Gamma_3 \cap (\cap\mathcal{L})$ is the graph which attaches two legs on each vertex of the connected subgraph $L \cap (\cap\mathcal{L})$. Hence $\cap\mathcal{L}'$ is connected in Γ_3. Next, for $\Gamma_1 = N(H)$ and $\Gamma_2 = N(\overline{H})$, we have that two subgraphs $\Gamma_1 \cap (\cap\mathcal{L})$ and $\Gamma_2 \cap (\cap\mathcal{L})$ are connected, because $H \cup \overline{H} = \Gamma$, $H \cap \overline{H} = L$ and $L \cap (\cap\mathcal{L})$ is connected by the assumption (2) of Theorem 3.1. Therefore $\cap\mathcal{L}' = \Gamma_i \cap (\cap\mathcal{L})$ is connected in Γ_i $(i = 1, 2)$. Consequently $\mathcal{G}_i$ satisfies the assumption (2) of Theorem 3.1. □

Because of Theorem 3.1 and Lemma 4.7, we have the following isomorphism:

$$\Psi_i : \mathbb{Z}[\mathcal{G}_i] \to H^*_{T^n \times S^1}(\mathcal{G}_i),$$

for all $\mathcal{G}_i = (\Gamma_i,\ \alpha_i,\ \theta_i)$, such that $\Psi_i(X) = \chi|_{\mathcal{V}^{\Gamma_i}}$ and $\Psi_i(\Gamma_i \cap F) = \tau_F|_{\mathcal{V}^{\Gamma_i}}$, where $\chi|_{\mathcal{V}^{\Gamma_i}}$ and $\tau_F|_{\mathcal{V}^{\Gamma_i}}$ are the restricted to $\mathcal{V}^{\Gamma_i}$ of χ, $\tau_F \in H^*_{T^n \times S^1}(\mathcal{G})$ $(i = 1,\ 2,\ 3)$, and F is a hyperfacet of $\mathcal{G}$.

4.2.2. *Exact sequence on $H^*_{T^n \times S^1}(\mathcal{G})$.* Next we consider the graph equivariant cohomologies $H^*_{T^n \times S^1}(\mathcal{G})$ and $H^*_{T^n \times S^1}(\mathcal{G}_i)$ $(i = 1,\ 2,\ 3)$.

Define the homomorphisms

$$\rho_1 : H^*_{T^n \times S^1}(\mathcal{G}) \to H^*_{T^n \times S^1}(\mathcal{G}_1) \oplus H^*_{T^n \times S^1}(\mathcal{G}_2)$$

by $\rho_1(f) = f|_{\mathcal{V}^{\Gamma_1}} \oplus f|_{\mathcal{V}^{\Gamma_2}}$, and

$$\rho_2 : H^*_{T^n \times S^1}(\mathcal{G}_1) \oplus H^*_{T^n \times S^1}(\mathcal{G}_2) \to H^*_{T^n \times S^1}(\mathcal{G}_3)$$

by $\rho_2(g \oplus h) = g|_{\mathcal{V}^{\Gamma_3}} - h|_{\mathcal{V}^{\Gamma_3}}$.

We have the following lemma.

LEMMA 4.8. *The following sequence is exact:*

$$\{0\} \longrightarrow H^*_{T^n\times S^1}(\mathcal{G}) \xrightarrow{\rho_1} H^*_{T^n\times S^1}(\mathcal{G}_1)\oplus H^*_{T^n\times S^1}(\mathcal{G}_2) \xrightarrow{\rho_2} H^*_{T^n\times S^1}(\mathcal{G}_3).$$

PROOF. First we prove ρ_1 is injective. If $\rho_1(f) = f|_{\mathcal{V}^{\Gamma_1}} \oplus f|_{\mathcal{V}^{\Gamma_2}} = 0$, then $f(p) = 0$ for all $p \in \mathcal{V}^{\Gamma_1}\cup\mathcal{V}^{\Gamma_2} = \mathcal{V}^{\Gamma}$, that is, $f = 0 \in H^*_{T^n\times S^1}(\mathcal{G})$. This means $\mathrm{Ker}\ \rho_1 = \{0\}$, hence ρ_1 is injective.

Next we have $\rho_2\circ\rho_1(f) = f|_{\mathcal{V}^{\Gamma_3}} - f|_{\mathcal{V}^{\Gamma_3}} = 0$. Hence $\mathrm{Im}(\rho_1)\subset\mathrm{Ker}(\rho_2)$.

Finally we take $g\oplus h\in\mathrm{Ker}(\rho_2)$, then $g|_{\mathcal{V}^{\Gamma_3}} = h|_{\mathcal{V}^{\Gamma_3}}$. Hence the following map $f:\mathcal{V}^{\Gamma}\to H^*_{T^n\times S^1}(pt)$ is well-defined and in $H^*_{T^n\times S^1}(\mathcal{G})$:

$$f(p) = \begin{cases} g(p) & \text{if } p\in\mathcal{V}^{\Gamma_1} \\ h(p) & \text{if } p\in\mathcal{V}^{\Gamma_2}. \end{cases}$$

So we have $\mathrm{Im}(\rho_1)\supset\mathrm{Ker}(\rho_2)$. □

4.2.3. *Proof of Theorem 4.1.* To prove Theorem 4.1, we prepare some facts for $\mathbb{Z}[\mathcal{G}]$.

Let $A\in\mathbb{Z}[X,\ H_1,\ \cdots,\ H_{2m}]$. Then we can denote it by

$$A = \sum_{(a_1,\cdots,a_{2m},a)} k_{(a_1,\cdots,a_{2m},a)} H_1^{a_1}\cdots H_{2m}^{a_{2m}} X^a$$

for some $k_{(a_1,\cdots,a_{2m},a)}\in\mathbb{Z}$ and $(a_1,\cdots,a_{2m},a)\in(\mathbb{N}\cup\{0\})^{2m+1}$, where $H_1,\cdots,H_{2m}$ are all hyperfacets in $\mathcal{G}$. Define $\Gamma_i\cap A$ as follows:

$$\Gamma_i\cap A = \sum_{(a_1,\cdots,a_{2m},a)} k_{(a_1,\cdots,a_{2m},a)}(\Gamma_i\cap H_1)^{a_1}\cdots(\Gamma_i\cap H_{2m})^{a_{2m}}X^a,$$

where if $\Gamma_i\cap H_j = \emptyset$, $\Gamma_i\cap\overline{H}_j = \Gamma_i$ (resp. $\Gamma_i\cap H_j = \Gamma_i$, $\Gamma_i\cap\overline{H}_j = \emptyset$) as a subgraph in Γ then $\Gamma_i\cap H_j = 0$, $\Gamma_i\cap\overline{H}_j = X$ (resp. $\Gamma_i\cap H_j = X$, $\Gamma_i\cap\overline{H}_j = 0$) for $j = 1,\cdots,2m$. We define the homomorphism $\hat{\rho}_1$ as follows:

$$\hat{\rho}_1 : \mathbb{Z}[\mathcal{G}]\to\mathbb{Z}[\mathcal{G}_1]\oplus\mathbb{Z}[\mathcal{G}_2]$$

such that $\hat{\rho}_1([A]) = [\Gamma_1\cap A]_1\oplus[\Gamma_2\cap A]_2$, where $[A]\in\mathbb{Z}[\mathcal{G}]$ and $[B]_i\in\mathbb{Z}[\mathcal{G}_i]$ for $i = 1,\ 2,\ 3$. Because of the definition of $\Gamma_i\cap A$ and Lemma 4.7 (1), we can easily see that $\hat{\rho}_1([H_i+\overline{H_i}-X]) = [0]_1\oplus[0]_2$ and $\hat{\rho}_1([\prod_{F\in\mathcal{H}'}F]) = [0]_1\oplus[0]_2$, where a subset $\mathcal{H}'\subset\mathcal{H}$ satisfies $\cap\mathcal{H}' = \emptyset$. Therefore $\hat{\rho}_1$ is well-defined homomorphism. Because $\Gamma_3 = \Gamma_1\cap\Gamma_2$, we can similarly show that the following homomorphism is well-defined:

$$\hat{\rho}_2 : \mathbb{Z}[\mathcal{G}_1]\oplus\mathbb{Z}[\mathcal{G}_2]\to\mathbb{Z}[\mathcal{G}_3]$$

such that $\hat{\rho}_2([B]_1\oplus[C]_2) = [\Gamma_3\cap B]_3 - [\Gamma_3\cap C]_3\in\mathbb{Z}[\mathcal{G}_3]$.

By the definitions of $\hat{\rho}_j$ and ρ_j $(j = 1, 2)$, we can easily show the following lemma.

LEMMA 4.9. *The following diagram is commutative:*

$$\begin{array}{ccccccccc} & & \mathbb{Z}[\mathcal{G}] & \xrightarrow{\hat{\rho}_1} & \mathbb{Z}[\mathcal{G}_1]\oplus\mathbb{Z}[\mathcal{G}_2] & \xrightarrow{\hat{\rho}_2} & \mathbb{Z}[\mathcal{G}_3] & \longrightarrow & \{0\} \\ & & \Psi\downarrow & & \Psi_1\oplus\Psi_2\downarrow & & \Psi_3\downarrow & & \\ \{0\} & \longrightarrow & H^*_T(\mathcal{G}) & \xrightarrow{\rho_1} & H^*_T(\mathcal{G}_1)\oplus H^*_T(\mathcal{G}_2) & \xrightarrow{\rho_2} & H^*_T(\mathcal{G}_3) & & \end{array}$$

From Theorem 3.1 and Lemma 4.7, we have that Ψ, $\Psi_1\oplus\Psi_2$ and Ψ_3 are isomorphic maps.

Moreover we have the following lemma.

LEMMA 4.10. *The following sequence is exact:*

$$\mathbb{Z}[\mathcal{G}] \xrightarrow{\hat{\rho}_1} \mathbb{Z}[\mathcal{G}_1] \oplus \mathbb{Z}[\mathcal{G}_2] \xrightarrow{\hat{\rho}_2} \mathbb{Z}[\mathcal{G}_3] \longrightarrow \{0\}.$$

PROOF. Because all hyperfacets of $\mathcal{G}_3$ are induced from $\mathcal{G}_1$ (by Lemma 4.7 (1)), we can get all generators (hyperfacets) $[F']_3$ and $[X]_3$ of $\mathbb{Z}[\mathcal{G}_3]$ by $\hat{\rho}_2([F]_1 \oplus [0]_2) = [\Gamma_3 \cap F]_3 = [F']_3$ for some generator $[F]_1 \in \mathbb{Z}[\mathcal{G}_1]$ such that $\Gamma_3 \cap F = F'$ in Γ_3, and $\hat{\rho}_2([X]_1 \oplus [0]_2) = [\Gamma_3 \cap X]_3 = [X]_3$. Hence we see $\hat{\rho}_2$ is surjective.

Finally $\mathrm{Im}(\hat{\rho}_1) = \mathrm{Ker}(\hat{\rho}_2)$ comes from the commutative diagram (Lemma 4.9)

$$\begin{array}{ccccccccc}
 & & \mathbb{Z}[\mathcal{G}] & \xrightarrow{\hat{\rho}_1} & \mathbb{Z}[\mathcal{G}_1] \oplus \mathbb{Z}[\mathcal{G}_2] & \xrightarrow{\hat{\rho}_2} & \mathbb{Z}[\mathcal{G}_3] & \longrightarrow & \{0\} \\
 & & \Psi \downarrow & & \Psi_1 \oplus \Psi_2 \downarrow & & \Psi_3 \downarrow & & \\
\{0\} & \longrightarrow & H_T^*(\mathcal{G}) & \xrightarrow{\rho_1} & H_T^*(\mathcal{G}_1) \oplus H_T^*(\mathcal{G}_2) & \xrightarrow{\rho_2} & H_T^*(\mathcal{G}_3) & &
\end{array}$$

where its bottom sequence is exact by Lemma 4.8 and Ψ, $\Psi_1 \oplus \Psi_2$, Ψ_3 are isomorphic maps. □

From Lemma 4.8 and 4.10, we have that the top and bottom sequences in Lemma 4.9 are exact. Hence we have that $\hat{\rho}_1$ is injective and ρ_2 is surjective by the snake lemma. Therefore we have the following two sequences are exact:

$$\begin{array}{ccccccccc}
\{0\} & \longrightarrow & \mathbb{Z}[\mathcal{G}] & \xrightarrow{\hat{\rho}_1} & \mathbb{Z}[\mathcal{G}_1] \oplus \mathbb{Z}[\mathcal{G}_2] & \xrightarrow{\hat{\rho}_2} & \mathbb{Z}[\mathcal{G}_3] & \longrightarrow & \{0\}; \\
\{0\} & \longrightarrow & H_T^*(\mathcal{G}) & \xrightarrow{\rho_1} & H_T^*(\mathcal{G}_1) \oplus H_T^*(\mathcal{G}_2) & \xrightarrow{\rho_2} & H_T^*(\mathcal{G}_3) & \longrightarrow & \{0\}.
\end{array}$$

In particular, we conclude Theorem 4.1.

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