Classification of Transformation groups
(変換群の分類)

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0. Introduction and Acknowledgements

Let G be a Lie group and X be a smooth manifold. A smooth map $\varphi : G \times X \to X$ is called a (smooth) G-*action on* X if it satisfies the following two properties:

(1) $\varphi(e, x) = x$ for the identity element e of G and all $x \in X$;

(2) $\varphi(g, \varphi(h, x)) = \varphi(gh, x)$ for all $g, h \in G$ and $x \in X$.

We call a triple (G, X, φ) a (smooth) *transformation group*.

A transformation group naturally appears in mathematics as the group of automorphisms of a manifold with a geometric structure such as a riemaniann metric, for instance the group of affine transformations on a Euclidean space or the group of rotations on a standard sphere. The first mathematician who recognized the importance of a transformation group from the geometrical point of view was Felix Klein. He proposed in his Erlangen program (in 1872) that geometry is the study of structures invariant under a group action. Since then, the theory of transformation groups has become one of the main research areas in mathematics.

In this thesis we consider the *classification problem* of transformation groups and its related topics. The first part of the thesis (Part 1) deals with classification of compact Lie group actions on a rational cohomology complex quadric with codimension one principal orbits. To classify those actions, we use a method developed by Wang [**Wan60**] and Uchida [**Uch78**]. This method is useful not only to construct interesting examples of compact Lie group actions with codimension one principal orbits but also to classify those actions.

The second part of this thesis (Part 2) is about *equivariant cohomology*. The equivariant cohomology $H^*_G(X)$ of a manifold X with G-action is defined to be the ordinary cohomology of $X_G := (EG \times X)/G$ where EG is a universal G-bundle and the G-action on $EG \times X$ is the diagonal one. The space X_G is called the *Borel construction* of X. Equivariant cohomology contains a lot of information about actions and is a useful invariant to distinguish transformation groups. It is not easy to compute the equivariant cohomology $H^*_G(X)$, but when G is a torus T and $H^{odd}(X) = 0$, Goresky, Kottwitz and MacPherson [**GKM98**] described the image of the restriction map $H^*_T(X) \to H^*_T(X^T)$ to the fixed poit set X^T under certain condition. Since the restriction map above is injective because $H^{odd}(X) = 0$, their result provides a method to compute $H^*_T(X)$. Motivated by this result, Guillemin and Zara [**GZ01**] introduced the notion of *GKM-graph* (Γ, α, θ) and its *equivariant graph cohomology* $H^*_T(\Gamma, \alpha)$, which is purely combinatorial, in such a way that $H^*_T(X)$ is isomorphic to $H^*_T(\Gamma, \alpha)$ where (Γ, α, θ) is the GKM graph associated with X. In Part 2, we introduce the notion of a *hypertorus graph* and its equivariant graph cohomology similarly to Guillemin-Zara's GKM graph. A hypertorus graph includes a GKM graph which is associated by the hypertoric or the cotangent bundle of the torus manifold. A hypertorus graph is not necessarily a Guillemin-Zara's GKM graph and one can expect to build a new bridge between *topology* and *combinatorics* as in [**GZ01**] and [**MMP05**].

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Part 1

Classification of compact transformation groups on complex quadrics with codimension one orbits

1. Introduction of Part 1

One of the central problems in transformation groups is to classify compact Lie group actions on a fixed smooth manifold M such as a sphere and a complex projective space. Unfortunately the problem is beyond our reach in general, but it becomes within our reach if we put some assumption on the actions. For instance, when the actions are transitive, M is a homogeneous space and the problem reduces to finding a pair of a compact Lie group G and its closed subgroup H such that $G/H = M$. As is well known, there are a rich history and an abundant work in this case (e.g. [**BH58**], [**MS43**]). In particular, the transitive actions on a sphere are completely classified. The complete list can be found in [**Aso81**] and [**HH65**].

The orbit of a transitive action is of codimension zero. So we are naturally led to study actions with codimension one principal orbits. In 1960 H. C. Wang ([**Wan60**]) initiated the work in this direction. He investigated compact Lie group actions on spheres with codimension one principal orbits. In 1977 F. Uchida ([**Uch77**]) classified compact connected Lie group actions on rational cohomology projective spaces with codimension one principal orbits. The same problem has been studied by K. Iwata on rational cohomology quaternion projective spaces ([**Iwa78**]), on rational cohomology Cayley projective planes ([**Iwa81**]) and by T. Asoh on $\mathbb{Z}_2$-cohomology spheres ([**Aso81**]).

The purpose of Part 1 is to classify compact connected Lie group actions on a rational cohomology complex quadric with codimension one principal orbits. The complex quadric Q_r of complex dimension r is a degree two hypersurface $\sum_i z_i^2 = 0$ in the complex projective space $P_{r+1}(\mathbb{C})$ of complex dimension $r+1$. The linear action of $SO(r+2)$ on $P_{r+1}(\mathbb{C})$ leaves Q_r invariant and is transitive on Q_r. When r is odd, Q_r is a rational cohomology complex projective space and this case is already treated by Uchida ([**Uch77**]) mentioned above. Therefore we assume that $r = 2n$, i.e., our rational cohomology complex quadric is of real dimension $4n$.

A pair (G, M) denotes a smooth G-action on M and we say that (G, M) is *essentially isomorphic* to (G', M') if their induced effective actions are isomorphic. Our main theorem is the following.

MAIN THEOREM 1. *Let* M *be a rational cohomology complex quadric of real dimension* $4n$ *and let* G *be a compact connected Lie group. If* (G, M) *has codimension one principal orbits, then* (G, M) *is essentially isomorphic to one of the pairs in the following list.*

n	G	M	action
$n \geq 2$	$SO(2n+1)$	Q_{2n}	$SO(2n+1) \to SO(2n+2)$
$n \geq 2$	$U(n+1)$	Q_{2n}	$U(n+1) \to SO(2n+2)$
$n \geq 2$	$SU(n+1)$	Q_{2n}	$SU(n+1) \to SO(2n+2)$
$n = 2m-1 \geq 1$	$Sp(1) \times Sp(m)$	Q_{4m-2}	$Sp(1) \times Sp(m) \to SO(4m)$
7	$Spin(9)$	Q_{14}	$Spin(9) \to SO(16)$
3	G_2	Q_6	$G_2 \to SO(7) \to SO(8)$
2	$S(U(3) \times U(1))$	Q_4	$S(U(3) \times U(1)) \to SO(6)$
2	$Sp(2)$	$S^7 \times_{Sp(1)} P_2(\mathbb{C})$	$Sp(2)$ acts transitively on S^7
3	$G_2 \times T^1$	$G_{\mathbb{R}}(2, \mathbb{O})$	G_2 acts naturally and T^1 acts by the induced action from the canonical $SO(2)$-action on $\mathbb{O}^2$

Here $S^7 \times_{Sp(1)} P_2(\mathbb{C})$ *denotes the quotient of* $S^7 \times P_2(\mathbb{C})$ *by the diagonal* $Sp(1)$*-action where* $Sp(1)$ *acts on* S^7 *canonically and on* $P_2(\mathbb{C})$ *through a double covering* $Sp(1) \to SO(3)$*. The manifold* $S^7 \times_{Sp(1)} P_2(\mathbb{C})$ *is not diffeomrophic to* Q_4 (*Proposition 6.2*)*.* $G_{\mathbb{R}}(2, \mathbb{O})$ *denotes a Grassmann manifold consisting of real 2-planes in the Cayley numbers* $\mathbb{O}$*. It is diffeomorphic to* Q_6 (*see Section 7.2*)*.*

Closed connected subgroups of $SO(r+2)$ whose restricted actions on Q_r have codimension one principal orbits are classified by Kollross [**Kol02**]. Comparing his result with our list above, we see that the action of $G_2 \times T^1$ on $G_{\mathbb{R}}(2, \mathbb{O}) \cong Q_6$ in the list does not arise through a homomorphism to $SO(8)$.

There are some works on compact connected Lie group actions with codimension two principal orbits, see [**Nak84**] and [**Uch77**], but the actions get complicated according as the codimension of principal orbit gets large. The classification of compact connected Lie group actions with codimension two principal orbits is studied by Uchida ([**?**]) on rational cohomology complex projective space. Nakanishi ([**Nak84**]) completed the classification of homology spheres with an action of $SO(n)$, $SU(n)$ or $Sp(n)$.

The organization of Part 1 is as follows. In Section 2 we review a key theorem by F. Uchida on compact connected Lie group actions on M with codimension one principal orbits. It says that if $H^1(M; \mathbb{Z}_2) = 0$, then there are exactly two singular orbits and M decomposes into a union of closed invariant tubular neighborhoods of the singular orbits. In Section 3 we compute the Poincaré polynomials of the singular orbits. To do this, we distinguish three cases according to orientability of singular orbits. In Section 4 we determine the possible transformation groups G from the Poincaré polynomials using a well known fact on Lie theory([**TM**]). We also recall some facts used in later sections and state an outline of our steps to the classification. Section 5 through 10 are devoted to classifying the pairs (G, M). By looking at the slice representations of the singular orbits, we completely determine the transformation groups G and the tubular neighborhood of singular orbits. Then we check whether the G-manifold obtained by gluing those two tubular neighborhoods along their boundary is a rational cohomology complex quadric. Finally we give all actions in Section 11.

2. Preliminary

In this section, we present some basic facts on a complex quadric and the key theorem to solve the classification problem on a rational cohomology complex quadric. Let us recall the definition of complex quadric.

Definition(complex quadric Q_r).

$$\begin{aligned} Q_r &= \{\mathbf{z} \in P_{r+1}(\mathbb{C}) \mid z_0^2 + z_1^2 + \cdots + z_{r+1}^2 = 0\} \\ &\cong SO(r+2)/SO(r) \times SO(2), \end{aligned}$$

where $\mathbf{z} = [z_0 : z_1 : \ldots : z_{r+1}] \in P_{r+1}(\mathbb{C})$.

A simply connected closed manifold of dimension $2r$ is called a *rational cohomology complex quadric* if it has the same cohomology ring as Q_r with $\mathbb{Q}$ coefficient. It is well known that the rational cohomology ring of Q_{2n} is given by

$$H^*(Q_{2n};\mathbb{Q}) = \mathbb{Q}[c,x]/(c^{n+1} - cx,\ x^2,\ c^{2n+1}),$$

where $\deg(x) = 2n,\ \deg(c) = 2$.

Let us recall the key theorem about the structure of (G, M).

THEOREM 2.1 (Uchida[**Uch77**] Lemma 1.2.1). *Let G be a compact connected Lie group and M a compact connected manifold without boundary. Assume*

$$H^1(M;\mathbb{Z}_2) = 0,$$

and G acts smoothly on M with codimension one orbits $G(x)$. *Then* $G(x) \cong G/K$ *is a principal orbit and* (G, M) *has just two singular orbits* $G(x_1) \cong G/K_1$ *and* $G(x_2) \cong G/K_2$. *Moreover there exists a closed invariant tubular neighborhood* X_s *of* $G(x_s)$ *such that*

$$M = X_1 \cup X_2 \quad \text{and} \quad X_1 \cap X_2 = \partial X_1 = \partial X_2.$$

Note that X_s is a k_s-dimensional disk bundle over G/K_s ($k_s \geq 2$). The following Figure 2.1 is an image of our manifold.

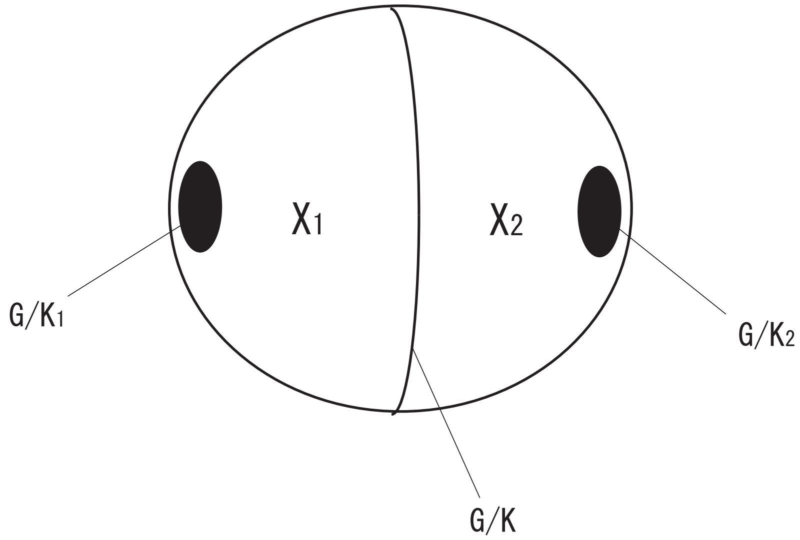

FIGURE 2.1. The image of Theorem 2.1

3. Poincaré polynomial

Let M be a rational cohomology complex quadric and G a compact connected Lie group which acts smoothly on M with codimension one principal orbits. Then the pair (G, M) satisfies the assumptions of Theorem 2.1. Therefore M is devided into X_1 and X_2 where X_i is the tubular neighborhood of singular orbit G/K_i $(i = 1,\ 2)$. Let us calculate the Poincaré polynomial of the singular orbits G/K_1 and G/K_2.

First we prepare some notations. Let $f_s^* : H^*(M;\mathbb{Q}) \to H^*(X_s;\mathbb{Q})$ be the homomorphism induced by the inclusion $f_s : X_s \to M$ and n_s a non-negative integer such that $f_s^*(c^{n_s}) \neq 0$ and $f_s^*(c^{n_s+1}) = 0$ where $c \in H^2(M;\mathbb{Q})$ is a generator. The following theorem is the goal of this section. The result in the case where the two singular orbits are orientable is due to an unpublished note by S. Kikuchi.

THEOREM 3.1. *If the two singular orbits are both orientable, then these singular orbits satisfy one of the following.*

(1) $G/K_s \sim P_n(\mathbb{C})$, $k_1 = 2n = k_2$, $n_1 = n = n_2$.
(2) $G/K_1 \sim P_{2n-1}(\mathbb{C})$, $G/K_2 \sim S^{2n}$, $k_1 = 2$, $k_2 = 2n$, $n_1 = 2n-1$, $n_2 = 0$.
(3) $P(G/K_s;\ t) = (1+t^{k_r-1})(1+t^2+\cdots+t^{2n})$, $k_1+k_2 = 2n+1$, $n_1 = n = n_2$, $s+r = 3$.

If G/K_1 *is orientable and* G/K_2 *is non-orientable, then*

- $G/K_1 \sim P_{2n-1}(\mathbb{C})$,
- $P(G/K_2;\ t) = (1+t^{2n})$, $P(G/K_2^o;\ t) = (1+t^n)(1+t^{2n})$,
- $G/K^o \sim S^{4n-1}$,

for $n_1 = 2n-1$, $n_2 = 0$, $k_1 = 2$, $k_2 = n$.

If the two singular orbits are both non-orientable, then

- $P(G/K_s;\ t) = 1 + t^2 + t^4$,
- $P(G/K_s^o;\ t) = (1 + t^2)(1 + t^2 + t^4)$,
- $P(G/K;\ t) = P(G/K^o;\ t) = (1 + t^3)(1 + t^2 + t^4)$,

for $n = k_1 = k_2 = 2$ *and* $n_1, n_2 = 1,\ 2$.

Here $M \sim N$ *means* $P(M;\ t) = P(N;\ t)$, $P(X;\ t)$ *is the Poincaré polynomial of* X, *and* K^o *is the identity component of* K.

To prove Theorem 3.1, we will consider three cases according to orientability of two singular orbits. Before we consider three cases, we shall show the following general proposition.

PROPOSITION 3.1.

(1) $n_1 + n_2 + \epsilon_1 + \epsilon_2 = 2n$.
(2) *If* $\epsilon_1 = \epsilon_2 = 0$ *then* $n_1 = n_2 = n$.
(3) *If* $\epsilon_1 = \epsilon_2 = 1$ *then* $n_1 = n_2 = n - 1$.

First we show the following three lemmas to prove Proposition 3.1.

LEMMA 3.1. *If we put* $P(\text{Ker } f_s^*; t) = \sum t^q \dim(\text{Ker } f_s^q)$ *and* $P(\text{Im } f_s^*; t) = \sum t^q \dim(\text{Im } f_s^q)$ *where* $\text{Ker } f_s^q = \text{Ker}(f_s^*) \cap H^q(M; \mathbb{Q})$ *and* $\text{Im}(f_s^q) = \text{Im}(f_s^*) \cap H^q(X_s; \mathbb{Q})$, *then the equation* $P(X_{3-s}, \partial X_{3-s}; t) - tP(X_s; t) = P(\text{Ker } f_s^*; t) - tP(\text{Im } f_s^*; t)$ *holds.*

PROOF. We get $\dim(H^q(X_{3-s}, \partial X_{3-s})) = \dim(H^q(M, X_s))$ by the excision isomorphism. From this equality and the cohomology exact sequence of (M, X_s)

$$\cdots \longrightarrow H^{q-1}(X_s; \mathbb{Q}) \xrightarrow{\delta^{q-1}} H^q(M, X_s; \mathbb{Q}) \xrightarrow{j^q} H^q(M; \mathbb{Q}) \xrightarrow{f_s^*} H^q(X_s; \mathbb{Q}) \longrightarrow \cdots,$$

we get

$$\begin{aligned} \dim(H^q(X_{3-s}, \partial X_{3-s})) &= \dim(\text{Im } \delta^{q-1}) + \dim(\text{Ker } f_s^q) \\ &= \dim(H^{q-1}(X_s)) - \dim(\text{Im } f_s^{q-1}) + \dim(\text{Ker } f_s^q). \end{aligned}$$

Hence we have this lemma. □

From Lemma 3.1, we can show the following lemma.

LEMMA 3.2. $P(\text{Ker } f_1^*; t) - tP(\text{Im } f_1^*; t) = t^{4n} P(\text{Im } f_2^*; t^{-1}) - t^{4n+1} P(\text{Ker } f_2^*; t^{-1})$.

PROOF. By the Poincaré-Lefschetz duality and the universal coefficient theorem we get $H^q(X_s) \simeq H^{4n-q}(X_s, \partial X_s)$. Hence $P(X_s; t) = t^{4n} P(X_s, \partial X_s; t^{-1})$. From Lemma 3.1 we get

$$\begin{aligned} P(\text{Ker } f_1^*; t) - tP(\text{Im } f_1^*; t) &= P(X_2, \partial X_2; t) - tP(X_1; t) \\ &= t^{4n} P(X_2; t^{-1}) - t^{4n+1} P(X_1, \partial X_1; t^{-1}) \\ &= -t^{4n+1}\{P(X_1, \partial X_1; t^{-1}) - t^{-1} P(X_2; t^{-1})\} \\ &= -t^{4n+1}\{P(\text{Ker } f_2^*; t^{-1}) - t^{-1} P(\text{Im } f_2^*; t^{-1})\}. \end{aligned}$$

□

Because $H^*(M;\mathbb{Q}) \simeq H^*(Q_{2n};\mathbb{Q})$, we get the following equations.

LEMMA 3.3. *Put* $\epsilon_s = 1$ *if* $f_s^*(x) \neq \lambda f_s^*(c^n)$ *for all* $\lambda \in \mathbb{Q}$, $\epsilon_s = 0$ *otherwise. Then we have*

$$\begin{aligned} P(\mathrm{Im}\ f_s^*, t) &= 1 + t^2 + \cdots + t^{2n_s} + \epsilon_s t^{2n} \text{ and} \\ P(\mathrm{Ker}\ f_s^*, t) &= t^{2n_s+2} + \cdots + t^{4n} + (1-\epsilon_s)t^{2n}. \end{aligned}$$

So we can prove Proposition 3.1.

Proof of Proposition 3.1. From Lemma 3.2 and 3.3, we get the following equation

$$\begin{aligned} &t^{2n_1+2}(1 + t^2 + \cdots + t^{4n-2n_1-2}) + (1-\epsilon_1)t^{2n} - t(1 + t^2 + \cdots + t^{2n_1}) - \epsilon_1 t^{2n+1} \\ &= t^{4n}(1 + t^{-2} + \cdots + t^{-2n_2}) + \epsilon_2 t^{2n} - t(t^{4n-2n_2-2} + \cdots + t^2 + 1) - (1-\epsilon_2)t^{2n+1}. \end{aligned}$$

Put $t = 1$ then we get the first statement in Proposition 3.1. Moreover put $\epsilon_1 = \epsilon_2 = 0$ and compare the degree of this obtained equation by using the first statement then we get the second statement. The third statement can be proved similarly.

□

Let us consider three cases according to orientability of two singular orbits.

3.1. Both singular orbits are orientable.

Suppose the two singular orbits G/K_1 and G/K_2 are orientable. First we prove the following equality.

LEMMA 3.4. *Assume* k_s *is the dimension of the normal bundle of* G/K_s *and* $s + r = 3$, *then the following equation holds.*

$$\begin{aligned} &(1 - t^{k_1+k_2-2})P(G/K_s;t) \\ = \ &(1 + t^{-1})\{P(\mathrm{Im}\ f_s^*;t) + t^{k_r-1}P(\mathrm{Im}\ f_r^*;t)\} - t^{-1}(1 + t^{k_r-1})P(M;t). \end{aligned}$$

PROOF. By the Thom isomorphism, we get $t^{k_s}P(G/K_s;t) = P(X_s, \partial X_s;t)$. Since X_s is a deformation retract to G/K_s, $P(X_s;t) = P(G/K_s;t)$. Hence by Lemma 3.1, $t^{k_r}P(G/K_r;t) - tP(G/K_s;t) = P(\mathrm{Ker}\ f_s^*;t) - tP(\mathrm{Im}\ f_s^*;t)$. Moreover we get $P(G/K_r;t) = t^{k_s-1}P(G/K_s;t) - t^{-1}P(\mathrm{Ker}\ f_r^*;t) + P(\mathrm{Im}\ f_r^*;t)$. Using these equations and $P(\mathrm{Ker}\ f_s^*;t) = P(M;t) - P(\mathrm{Im}\ f_s^*;t)$, we can easily check the above equation. □

Putting $t = -1$ in Lemma 3.4, we get $(1 - (-1)^{k_1+k_2})\chi(G/K_s) = (1 - (-1)^{k_r})\chi(M)$ where $\chi(X)$ is the Euler characteristic of X. From this equation, we see

LEMMA 3.5. *If* $k_1 - k_2$ *is even, then* k_1 *and* k_2 *are even. Hence the case* $k_1 \equiv k_2 \equiv 1 (\mathrm{mod}\ 2)$ *does not occur.*

Next we consider two cases.

3.1.1. *The case* $\epsilon_1 = \epsilon_2 = 0$ *and* $\epsilon_1 = \epsilon_2 = 1$.

If $\epsilon_1 = \epsilon_2 = 0$ then $n_1 = n_2 = n$ and if $\epsilon_1 = \epsilon_2 = 1$ then $n_1 = n_2 = n-1$ by Proposition 3.1. By Lemma 3.3 and 3.4 we have the following equation

$$f_s(t) = (1 + t^{k_r-1})(1 - t^{2n-1})a(n) \tag{3.1}$$

where $a(n) = 1 + t^2 + \cdots + t^{2n}$ and $f_s(t) = (1 - t^{k_1+k_2-2})P(G/K_s; t)$.

Suppose $k_1 \equiv k_2 \equiv 0 (\text{mod } 2)$. Dividing both sides of the equation (3.1) by $1 + t$ and putting $t = -1$, we get $\chi(G/K_s) \neq 0$ for $s = 1, 2$. Now we have the following lemma.

LEMMA 3.6. *If* $\chi(G/K_s) \neq 0$, *then the Poincaré polynomials* $P(G/K_s; t)$ *are even functions for* $s = 1, 2$, *that is,* $P(G/K_s; t) = P(G/K_s; -t)$.

PROOF. If $\chi(G/K_s) \neq 0$, then we have rank K_s^o = rank G (see **[TM]** Chapter III). Hence $H^{odd}(G/K_s^o; \mathbb{Q}) = 0$ from **[TM]** Theorem 3.21 in Chapter VII. Sine the induced map

$$H^*(G/K_s; \mathbb{Q}) \to H^*(G/K_s^o; \mathbb{Q})$$

is injective, the Poincaré polynomials $P(G/K_1; t)$ and $P(G/K_2; t)$ are even functions. □

From this lemma, we see $(1 + t^{k_r-1})(1 - t^{2n-1}) = (1 - t^{k_r-1})(1 + t^{2n-1})$ by the equation (3.1). Consequently we have $k_1 = k_2 = 2n$ and $G/K_s \sim P_n(\mathbb{C})$ because $P(P_n(\mathbb{C}); t) = 1 + t^2 + \cdots + t^{2n}$. This means Theorem 3.1 (1).

Suppose k_1 is even and k_2 is odd. Then both sides of the equation (3.1) are divisible by $1 - t$. Hence we have $\chi(G/K_1) \neq 0$. So $P(G/K_1; t)$ is an even function. Compare even degree terms and odd degree terms of the equation (1) then we have $k_1 + k_2 = 2n + 1$, $P(G/K_1; t) = (1 + t^{k_2-1})a(n)$ and $P(G/K_2; t) = (1 + t^{k_1-1})a(n)$. This means Theorem 3.1 (3). If k_1 is odd and k_2 is even, then we get a similar result.

By Lemma 3.5, there does not exist the case that k_1 and k_2 are odd.

3.1.2. *The case* $\epsilon_1 = 0$ *and* $\epsilon_2 = 1$.

By Proposition 3.1, Lemma 3.3, Lemma 3.4 and $n_1 + n_2 = 2n - 1$, we easily get

$$f_1(t) = (1 - t^{2n_2+k_2})a(n_1) + (t^{k_2-1} - t^{2n_1+1})a(n_2) - t^{2n-1}(1 - t^{k_2}), \tag{3.2}$$

$$f_2(t) = (1 - t^{2n_1+k_1})a(n_2) + (t^{k_1-1} - t^{2n_2+1})a(n_1) + t^{2n}(1 - t^{k_1-2}). \tag{3.3}$$

Suppose $k_1 \equiv k_2 \equiv 0 (\text{mod } 2)$. Dividing both sides of (3.2), (3.3) by $1 + t$ and putting $t = -1$, we see $P(G/K_1; t)$ and $P(G/K_2; t)$ are even functions by $k_s \geq 2$. So $k_1 = 2n_2 + 2$ by comparing the odd degree terms in (3.3). Hence $n_2 = 0$ by comparing the maximal degree terms in (3.3). So $n_1 = 2n - 1$ and $k_1 = 2$. From (3.2), we see $k_2 = 2n$. Consequently $G/K_1 \sim P_{2n-1}(\mathbb{C})$ and $G/K_2 \sim S^{2n}$. This result is Theorem 3.1 (2).

Suppose k_1 is even and k_2 is odd and put $t = -1$ in (3.2). Then we see $P(G/K_1; t)$ is an even function. So we get

$$P(G/K_1; t) = a(n_1) + t^{k_2-1}a(n_2) + t^{2n-1+k_2}. \tag{3.4}$$

Hence $\dim G/K_1 = \max\{2n_1,\ k_2-1+2n_2,\ 2n-1+k_2\}$. If $\dim G/K_1 = 2n_1$ then $k_2-1 = 2n_1-(k_2-1+2n_2)$ or $2n_1-(2n-1+k_2)$ because of the inequality $n \geq 2$, the Poincaré duality about G/K_1 and the equation (3.4). Hence $k_2-1 = n_1-n_2$ or n_1-n. Since $n_1+n_2 = 2n-1$, n_1-n_2 is an odd number. Therefore $k_2-1 = n_1-n = n-n_2-1$ is an even number. But in this case $n = n_2$ from the Poincaré duality. Hence $\dim G/K_1 \neq 2n_1$. If $\dim G/K_1 = k_2-1+2n_2$, then $2(n_2-n) = k_2-1$ and $n_2 = n$ from the Poincaré duality. This is in contradiction to $k_2 \geq 2$. Hence $\dim G/K_1 = 2n-1+k_2$. In this case $k_2-1 \geq 2n+2 = 2n_1+2$ from the Poincaré duality. So we see $\dim G/K_1 \geq 4n+2$. This is a contradiction. Hence the case k_1 is even and k_2 is odd does not occur.

Suppose k_1 is odd and k_2 is even. In this case we get $P(G/K_2;t) = a(n_2)+t^{k_1-1}a(n_1)+t^{2n}$ from (3.3). One can easily show that this case does not occur similarly from the Poincaré duality.

By Lemma 3.5, there does not exist the case that k_1 and k_2 are odd.

3.2. Preparation for non-orientable cases.

In order to prove two non-orientable cases in Theorem 3.1, it is necessary to show the following proposition.

PROPOSITION 3.2. *If* G/K_2 *is non-orientable, then we have*

$$\begin{aligned} P(G/K_2^o;t) &= (1+t^{k_2})P(G/K_2;t), \\ P(G/K^o;t) &= (1+t^{2k_2-1})P(G/K_2;t) - P(n_1,n_2;t) - \epsilon_2(1-\epsilon_1)(1+t^{-1})t^{2n}, \end{aligned}$$

where

$$P(n_1,n_2;t) = \begin{cases} t^{2n_1+1}+t^{2n_1+2}+\cdots+t^{2n_2} & (n_1 < n_2) \\ 0 & (n_1 \geq n_2). \end{cases}$$

The goal of Section 3.2 is to prove Proposition 3.2. Our proof is essentially due to Uchida (**[Uch77]** 2.4, 2.5 and 2.6).

First of all we show the following lemma.

LEMMA 3.7. *If* $k_1 > 2$, *then* G/K_2 *is simply connected, hence* K_2 *is connected.*

PROOF. We see $\pi_1(M) = \pi_1(G/K_2)$ from $k_1 > 2$, the transversality theorem (**[BJ82]** (14.7)) and Theorem 2.1. Hence G/K_2 is simply connected. So $K_2 = K_2^o$ because a canonical map $G/K_2^o \to G/K_2$ is a finite covering. □

Next we show the following two lemmas (Lemma 3.8 and 3.9) which just come from the condition $k_1 = 2$.

LEMMA 3.8. *If* $k_1 = 2$, *then* $R_k^* = \mathrm{id} : H^*(G/K^o;\mathbb{Q}) \to H^*(G/K^o;\mathbb{Q})$ *for all* $k \in K$, *where* $R_k : [g] \to [gk]$ *and* R_k^* *is an induced homomorphism of* R_k.

PROOF. First we assume $k_2 > 2$. Then K_1 is connected from Lemma 3.7. Because $K_1/K \cong S^1$ by $k_1 = 2$, there is a connected central one dimensional subgroup T in K_1 such that

$$K \subset K_1 = T \cdot K^o.$$

We take a continuous mapping $u : [0,1] \to T$ such that $u(0)$ is the identity in T and $u(1) = u \in T$. Because each $u(t) \in T$ commutes with each element in K, a homotopy

$$H_t : G/K^o \to G/K^o$$

can be defined by $H_t(gK^o) = gu(t)K^o$. For each $k \in K$, there are $u \in T$ and $k' \in K^o$ such that $k = uk'$. Hence there is $u \in T$ such that $R_k = R_u$ for each $k \in K$. Since H_0 is the identity and $H_1 = R_u = R_k$ for each $k \in K$, R_k^* is the identity map.

Next we assume $k_2 = 2$. By Theorem 2.1, we can put X_s the invariant tubular neighborhood of G/K_s $(s = 1, 2)$ in M such that $M = X_1 \cup X_2$ and $X_1 \cap X_2 = \partial X_1 = \partial X_2$. Let $i_s : X_1 \cap X_2 \to X_s$ be the inclusion. Then the induced homomorphism $i_{s*} : \pi_1(X_1 \cap X_2) \to \pi_1(X_s)$ is surjective by the transversality theorem (**[BJ82]** (14.7)). Thus there is a natural surjection

$$h_s : \pi_1(X_s) \simeq \pi_1(X_1 \cap X_2)/(\text{Ker } i_{s*}) \to \pi_1(X_1 \cap X_2)/(\text{Ker } i_{1*}) \cdot (\text{Ker } i_{2*})$$

such that the following diagram is commutative.

$$\begin{array}{lcl} \pi_1(X_1 \cap X_2) & \xrightarrow{i_{1*}} & \pi_1(X_1) \\ i_{2*} \downarrow & & \downarrow h_1 \\ \pi_1(X_2) & \xrightarrow{h_2} & \pi_1(X_1 \cap X_2)/(\text{Ker} i_{1*}) \cdot (\text{Ker} i_{2*}) \end{array}$$

Then there is a surjection

$$\pi_1(X_1 \cup X_2) \to \pi_1(X_1 \cap X_2)/(\text{Ker } i_{1*}) \cdot (\text{Ker } i_{2*})$$

by the van Kampen's theorem. But $M = X_1 \cup X_2$ is simply connected. Hence $\pi_1(X_1 \cap X_2) = (\text{Ker } i_{1*})(\text{Ker } i_{2*})$. On the other hand, the inclusion i_s is homotopy equivalent to the projection $p_s : G/K \to G/K_s$. Thus we have

$$\pi_1(G/K) = (\text{Ker } p_{1*}) \cdot (\text{Ker } p_{2*}).$$

From homotopy exact seqences for the principal bundles

$$G \to G/K \quad \text{and} \quad G \to G/K_s,$$

we have a commutative diagram

$$\begin{array}{ccccc} \pi_1(G) & \longrightarrow & \pi_1(G/K) & \xrightarrow{\theta} & K/K^o \\ \downarrow \text{id} & & \downarrow p_{s*} & & \downarrow \iota_s \\ \pi_1(G) & \longrightarrow & \pi_1(G/K_s) & \xrightarrow{\theta_s} & K_s/K_s^o \end{array}$$

where θ and θ_s are surjective homomorphisms because of $\pi_0(G) = \{e\}$. Thus we have from $\pi_1(G/K) = (\mathrm{Ker}\ p_{1*}) \cdot (\mathrm{Ker}\ p_{2*})$,

$$\begin{aligned} K/K^o &= \theta(\pi_1(G/K)) = \theta((\mathrm{Ker}\ p_{1*}) \cdot (\mathrm{Ker}\ p_{2*})) \\ &= \theta((\mathrm{Ker}\ p_{1*})) \cdot \theta((\mathrm{Ker}\ p_{2*})) \subset (\mathrm{Ker}\ \iota_1) \cdot (\mathrm{Ker}\ \iota_2) \subset K/K^o \end{aligned}$$

Therefore

$$K/K^o = ((K_1^o \cap K)/K^o) \cdot ((K_2^o \cap K)/K^o) \subset (K_1^o/K^o) \cdot (K_2^o/K^o),$$

because $\mathrm{Ker}\ \iota_s = (K_s^o \cap K)/K^o$. Moreover we see K is a normal subgroup of K_s by $K_s/K \cong S^1$. Hence there is a connected subgroup $T \subset K_1^o K_2^o$ of G such that $K \subset T \cdot K^o$. So we can prove Lemma 3.8 for $k_2 = 2$ similarly to the case $k_2 > 2$. □

From Lemma 3.8, we can show the following lemma.

LEMMA 3.9. *If* $k_1 = 2$, *then* $H^*(G/K_s^o;\mathbb{Q}) = \mathrm{Im}(q_s^*) + \mathrm{Ker}(p_s^{o*})$ (*possibly non direct sum*), *where* q_s^* *and* p_s^{o*} *are induced from* $q_s : G/K_s^o \to G/K_s$ *and* $p_s^o : G/K^o \to G/K_s^o$.

PROOF. The natural map $K_s^o/K^o \to K_s/K$ is a surjection because $K_s/K \cong K_s^o/K^o$ is a $(k_s - 1)$-sphere. So we see $K_s = K_s^o K$. In particular for each $a \in K_s$ there exists $k \in K$ such that R_a and R_k are homotopic by the connectedness of K_s^o. Hence $R_a^* = R_k^* : H^*(G/K_s^o;\mathbb{Q}) \to H^*(G/K_s^o;\mathbb{Q})$. By Lemma 3.8 we can consider the following commutative diagram,

$$\begin{array}{ccc} H^*(G/K_s^o;\mathbb{Q}) & \xrightarrow{p_s^{o*}} & H^*(G/K^o;\mathbb{Q}) \\ R_a^* = R_k^* \downarrow & & R_k^* = \mathrm{id} \downarrow \\ H^*(G/K_s^o;\mathbb{Q}) & \xrightarrow{p_s^{o*}} & H^*(G/K^o;\mathbb{Q}), \end{array}$$

for all $a \in K_s$. So we have $p_s^{o*}(u) = p_s^{o*}(R_a^*(u))$ for $u \in H^*(G/K_s^o;\mathbb{Q})$ and $a \in K_s$. K_s/K_s^o acts on $H^*(G/K_s^o;\mathbb{Q})$ by R_k^* for $k \in K_s/K_s^o$. Then we easily see $\mathrm{Im}(q_s^*) = H^*(G/K_s^o;\mathbb{Q})^{K_s/K_s^o}$. Hence $R_k^*(v) = v$ for all $k \in K_s/K_s^o$ and $v \in \mathrm{Im}(q_s^*)$. Moreover if we put $K_s/K_s^o = \{k_1, \cdots, k_r\}$ then $R_{k_1}^*(u) + \cdots + R_{k_r}^*(u) \in \mathrm{Im}(q_s^*)$ for all $u \in H^*(G/K_s^o;\mathbb{Q})$. Therefore there is $x \in H^*(G/K_s;\mathbb{Q})$ such that $p_s^{o*} \circ q_s^*(x) = r p_s^{o*}(u)$. So we see $\mathrm{Im}(p_s^{o*}) = \mathrm{Im}(p_s^{o*} \circ q_s^*)$. Consequently we get the equation $H^*(G/K_s^o;\mathbb{Q}) = \mathrm{Im}(q_s^*) + \mathrm{Ker}(p_s^{o*})$. □

Put $J_k = q_2^* H^k(G/K_2;\mathbb{Q})$ and $J = \oplus_k J_k$. Next we show properties about this J in the following two lemmas (Lemma 3.10 and 3.11) by using Lemma 3.9.

LEMMA 3.10. *Let* χ *be the rational Euler class of the oriented* (k_2-1)-*sphere bundle* $G/K^o \to G/K_2^o$. *If* $k_1 = 2$, *then* $\mathrm{Ker}(p_2^{o*}) = J \cdot \chi + J \cdot \chi^2$.

PROOF. From the Thom-Gysin exact sequence about $p_2^o : G/K^o \to G/K_2^o$ that is,

$$\xrightarrow{p_2^{o*}} H^{q+k_2-1}(G/K_2^o) \xrightarrow{\delta^*} H^q(G/K_2^o) \xrightarrow{\cdot\chi} H^{q+k_2}(G/K_2^o) \xrightarrow{p_2^{o*}} H^{q+k_2}(G/K^o) \xrightarrow{\delta^*},$$

we see $\mathrm{Ker}(p_2^{oq}) = H^{q-k_2}(G/K_2^o; \mathbf{Q})\cdot\chi$. By Lemma 3.9 $H^{q-k_2}(G/K_2^o;\mathbb{Q}) = J_{q-k_2} + \mathrm{Ker}(p_2^{oq-k_2})$. So we have $\mathrm{Ker}(p_2^{oq}) = J_{q-k_2}\cdot\chi + J_{q-2k_2}\cdot\chi^2 + \cdots + J_{q-Nk_2}\cdot\chi^N$ for some integer N. Because of the following bundle mapping

$$\begin{array}{ccc} G/K^o & \xrightarrow{R_k} & G/K^o \\ \downarrow p_2^o & & \downarrow p_2^o \\ G/K_2^o & \xrightarrow{R_k} & G/K_2^o, \end{array}$$

we see $R_k^*(\chi) = \chi$ or $-\chi$ for $k \in K$. Hence $R_k^*(\chi^2) = \chi^2$. Since the equation $J = \mathrm{Im}(q_2^*) = H^*(G/K_2^o;\mathbb{Q})^{K_2}$ holds, we have $\chi^2 \in J$. So we get the equation $\mathrm{Ker}(p_2^{o*}) = J\cdot\chi + J\cdot\chi^2$. □

We remark that non-orientability of G/K_2 is not assumed in Lemma 3.7 through 3.10 unlike Proposition 3.2. From now on we assume G/K_2 is non-orientable. Then $k_1 = 2$ from Lemma 3.7.

LEMMA 3.11. *The following two properties hold.*
(1) $\dim(\mathrm{Ker}(p_2^{o*})) = \dim J + \dim(J \cap \mathrm{Ker}(p_2^{o*}))$.
(2) $J\cdot\chi \cap J\cdot\chi^2 = 0$, $J\cdot\chi^2 = J\cap\mathrm{Ker}(p_2^{o*})$ *and the homomorphism* $E : J \to \mathrm{Ker}(p_2^{o*})$ *is injective, where E is defined by* $E(y) = y\cdot\chi$.

PROOF. First we show the property (1) by proving two inequality. From Lemma 3.9 we get $\dim H^*(G/K_2^o;\mathbb{Q}) = \dim J + \dim(\mathrm{Ker}(p_2^{o*})) - \dim(J\cap\mathrm{Ker}(p_2^{o*}))$. Since G/K_2 is non-orientable, there is $k \in K_2$ such that $R_k : G/K_2^o \to G/K_2^o$ reverses an orientation. So we see $2\dim H^*(G/K_2;\mathbb{Q}) \le \dim H^*(G/K_2^o;\mathbb{Q})$. Since $q_2^* : H^*(G/K_2;\mathbb{Q}) \to H^*(G/K_2^o;\mathbb{Q})$ is an injective map, $\dim J = \dim H^*(G/K_2;\mathbb{Q})$. Hence we get

$$\dim J \le \dim(\mathrm{Ker}(p_2^{o*})) - \dim(J\cap\mathrm{Ker}(p_2^{o*})).$$

From Lemma 3.10 we get $\chi^2 \in J$ and $J\chi^2 \subset \mathrm{Ker}(p_2^{o*})$. So $J\cdot\chi^2 \subset J\cap\mathrm{Ker}(p_2^{o*})$. Moreover we easily see $\dim(J\cdot\chi) \le \dim J$. Hence we get

$$\dim(\mathrm{Ker}(p_2^{o*})) \le \dim J + \dim(J\cap\mathrm{Ker}(p_2^{o*})).$$

So we have the property (1) from the two inequalities above .

Next we show the property (2). From the equation (1), we have $\dim(J\cdot\chi) = \dim J$ (so we get the injectivity of E) and $\dim(J\cdot\chi^2) = \dim(J\cap\mathrm{Ker}(p_2^{o*}))$ (so we get $J\cdot\chi^2 = J\cap\mathrm{Ker}(p_2^{o*})$). From Lemma 3.10 $\mathrm{Ker}(p_2^{o*}) = J\cdot\chi + J\cdot\chi^2$ and $J\cap J\cdot\chi = \{0\}$. Hence we get the property (2). □

From Lemma 3.10 and 3.11, we can prove the following equation.

PROPOSITION 3.3. $P(G/K_2^o;t) = (1 + t^{k_2})P(G/K_2;t)$.

PROOF. From Lemma 3.11, we see $\dim J = \dim(\mathrm{Ker}(p_2^{o*})) - \dim(J\cap\mathrm{Ker}(p_2^{o*}))$. Moreover from Lemma 3.10 and 3.11 we have the equation

$$\mathrm{Ker}(p_2^{o*}) = J\cdot\chi \oplus J\cap\mathrm{Ker}(p_2^{o*}).$$

Since $\chi \in H^{k_2}(G/K_2^o;\mathbb{Q})$ and $\dim H^*(G/K_2;\mathbb{Q}) = \dim J$, by the equation above we get

$$P(\mathrm{Ker}(p_2^{o*});t) = t^{k_2}P(G/K_2;t) + P(J \cap \mathrm{Ker}(p_2^{o*});t). \tag{3.5}$$

Comparing the equation (3.5) with $P(G/K_2^o;t) = P(\mathrm{Im}(q_2^*);t) + P(\mathrm{Ker}(p_2^{o*});t) - P(J \cap \mathrm{Ker}(p_2^{o*});t)$, we get $P(G/K_2^o;t) = (1+t^{k_2})P(G/K_2;t)$ from the injectivity of q_2^*. □

This result is a part of Proposition 3.2.

Next we show the following equation.

PROPOSITION 3.4. $P(G/K^o;t) = (1+t^{2k_2-1})P(G/K_2;t) - (1+t^{-1})P(J \cap \mathrm{Ker}(p_2^{o*});t)$.

PROOF. From the Thom-Gysin exact sequence of $p_2^o : G/K^o \to G/K_2^o$ that is

$$\xrightarrow{p_2^{o*}} H^{q+k_2-1}(G/K_2^o) \xrightarrow{\delta^*} H^q(G/K_2^o) \xrightarrow{\cdot\chi} H^{q+k_2}(G/K_2^o) \xrightarrow{p_2^{o*}} H^{q+k_2}(G/K^o) \xrightarrow{\delta^*},$$

we easily get

$$P(\mathrm{Im}(\delta^*);t) = P(G/K_2^o;t) - t^{-k_2}P(\mathrm{Ker}(p_2^{o*});t), \tag{3.6}$$

$$P(G/K^o;t) = t^{k_2-1}P(\mathrm{Im}(\delta^*);t) + P(\mathrm{Im}(p_2^{o*});t). \tag{3.7}$$

From Lemma 3.11 and the injectivity of q_2^*,

$$P(\mathrm{Im}(p_2^{o*});t) = P(G/K_2;t) - P(J \cap \mathrm{Ker}(p_2^{o*});t). \tag{3.8}$$

Substituting (3.7) for (3.6) and (3.8), we obtain the equation

$$\begin{aligned} P(G/K^o;t) &= t^{k_2-1}P(G/K_2^o;t) - t^{-1}P(\mathrm{Ker}(p_2^{o*});t) \\ &+ P(G/K_2;t) - P(J \cap \mathrm{Ker}(p_2^{o*});t). \end{aligned}$$

Moreover substituting the equation above for (3.5) and $P(G/K_2^o;t) = (1+t^{k_2})P(G/K_2;t)$, the identity of the proposition follows. □

Let us concentrate on the term $(1+t^{-1})P(J \cap \mathrm{Ker}(p_2^{o*});t)$. Consider the following commutative diagram

$$\begin{array}{ccc} H^*(G/K_2;\mathbb{Q}) & \xrightarrow{p_2^*} & H^*(G/K;\mathbb{Q}) \\ q_2^* \downarrow & & q^* \downarrow \\ H^*(G/K_2^o;\mathbb{Q}) & \xrightarrow{p_2^{o*}} & H^*(G/K^o;\mathbb{Q}), \end{array}$$

where q^* is the induced homomorphism from the natural covering map $q : G/K^o \to G/K$. Now q_2^* is an injection and moreover we show

LEMMA 3.12. $q^* : H^*(G/K;\mathbb{Q}) \to H^*(G/K^o;\mathbb{Q})$ *is an isomorphism.*

PROOF. Let $q^! : H^*(G/K^o;\mathbb{Q}) \to H^*(G/K;\mathbb{Q})$ be the transfer of the covering map $q : G/K^o \to G/K$. From Lemma 3.8 $R_k^* = id : H^*(G/K^o;\mathbb{Q}) \to H^*(G/K^o;\mathbb{Q})$, so $q^* \circ q^! : H^*(G/K^o;\mathbb{Q}) \to H^*(G/K^o;\mathbb{Q})$ is r times map where r is the covering degree of q. Hence q^* is surjective. The injectivity of q^* is well known. So q^* is an isomorphism. □

Hence we have $\mathrm{Ker}(p_2^*) = \mathrm{Ker}(p_2^{o*} \circ q_2^*) \simeq \mathrm{Im}(q_2^*) \cap \mathrm{Ker}(p_2^{o*}) = J \cap \mathrm{Ker}(p_2^{o*})$. So we see $P(J \cap \mathrm{Ker}(p_2^{o*}); t) = P(\mathrm{Ker}(p_2^*); t)$. The inclusion $i_s : X_1 \cap X_2 \to X_s$ is homotopy equivalent to $p_s : G/K \to G/K_s$, hence $i_s^* = p_s^*$. Considering the following commutative diagram from the cohomology exact sequences of (M, X_1) and $(X_2, X_1 \cap X_2)$ and the excision isomorphism

$$\begin{array}{ccccc} H^*(M, X_1) & \longrightarrow & H^*(M) & \xrightarrow{f_1^*} & H^*(X_1) \\ \simeq\downarrow & & f_2^* \downarrow & & i_1^* \downarrow \\ H^*(X_2, X_1 \cap X_2) & \longrightarrow & H^*(X_2) & \xrightarrow{i_2^*} & H^*(X_1 \cap X_2), \end{array}$$

we get $f_2^*(\mathrm{Ker}(f_1^*)) = \mathrm{Ker}(i_2^*)$ by this diagram. Hence we obtain the following equations from the definition of n_1 and n_2, that is $f_s^*(c^{n_s}) \neq 0$ and $f_s(c^{n_s+1}) = 0$,

$$P(\mathrm{Ker}(i_2^*); t) = t^{2n_1+2} + \cdots + t^{2n_2} + \epsilon_2(1 - \epsilon_1)t^{2n} \quad (n_1 < n_2)$$

and for $n_1 \geq n_2$

$$P(\mathrm{Ker}(i_2^*); t) = \epsilon_2(1 - \epsilon_1)t^{2n}.$$

Because of the two equations above, $P(J \cap \mathrm{Ker}(p_2^{o*}); t) = P(\mathrm{Ker}(i_2^*); t)$ and Proposition **??**, we complete the proof of Proposition 3.2.

3.3. G/K_1 is orientable, G/K_2 is non-orientable.

Let us prove where the case one of singular orbits is orientable and the other is not so in Theorem 3.1. Assume G/K_1 is orientable and G/K_2 is non-orientable.

From Proposition 3.2, we get the following equation.

LEMMA 3.13. $t^{4n} P(G/K_2; t^{-1}) = t^{2k_2} P(G/K_2; t)$.

PROOF. By Proposition 3.2, $P(G/K_2^o; t) = (1 + t^{k_2}) P(G/K_2; t)$. From the Poincaré duality of G/K_2^o, we see $P(G/K_2^o; t^{-1}) = t^{k_2 - 4n} P(G/K_2^o; t)$. ☐

Since G/K_2 is non-orientable, we see $k_1 = 2$ by Lemma 3.7. Hence we can show the following equation.

LEMMA 3.14. $P(G/K_2; t) = tP(G/K_1; t) + a(n_2) - t^{2n_2+1} a(2n - n_2 - 1) + t^{2n-1}(\epsilon_2 + t\epsilon_2 - 1)$.

PROOF. Since $k_1 = 2$, we see $\dim G/K_1 = 4n - 2$. By the Poincaré-Lefschetz duality and X_1 is a deformation retract to G/K_1,

$$H^q(X_1, \partial X_1; \mathbb{Q}) = H_{4n-q}(X_1; \mathbb{Q}) = H_{4n-q}(G/K_1; \mathbb{Q}) = H^{q-2}(G/K_1; \mathbb{Q}).$$

So we get the equality $P(X_1, \partial X_1; t) = t^2 P(G/K_1; t)$.

From Lemma 3.1 and 3.3, we have the equation

$$\begin{aligned} & P(X_1, \partial X_1; t) - tP(X_2; t) \\ = \;& t^{2n_2+2} + \cdots + t^{4n} + (1 - \epsilon_2)t^{2n} - t(1 + t^2 + \cdots + t^{2n_2} + \epsilon_2 t^{2n}) \\ = \;& t^{2n_2+2} a(2n - n_2 - 1) - ta(n_2) + (1 - \epsilon_2 - t\epsilon_2)t^{2n}. \end{aligned}$$

Putting $P(X_1, \partial X_1; t) = t^2 P(G/K_1; t)$ and $P(X_2; t) = P(G/K_2; t)$ in this equation, we get this claim. □

From Lemma 3.13 and 3.14, we can get the following proposition.

PROPOSITION 3.5. $P(G/K_1; t)$ *is an even function.*

PROOF. Multiplying both sides of the identity in Lemma 3.14 by t^{2k_2-1}, we get

$$\begin{aligned} & t^{2k_2-1}P(G/K_2;t) \\ = \ & t^{2k_2}P(G/K_1;t) + t^{2k_2-1}a(n_2) - t^{2k_2+2n_2}a(2n-n_2-1) + t^{2k_2+2n-2}(\epsilon_2 + t\epsilon_2 - 1). \end{aligned}$$

Moreover multiplying both sides of the equation which substitute t^{-1} for t in Lemma 3.13 by t^{4n-1}, we get

$$\begin{aligned} & t^{4n-1}P(G/K_2;t^{-1}) \\ = \ & t^{4n-2}P(G/K_1;t^{-1}) + t^{4n-2n_2-1}a(n_2) - a(2n-n_2-1) + t^{2n}(\epsilon_2 + t^{-1}\epsilon_2 - 1). \end{aligned}$$

By the Poincaré duality of G/K_1, $P(G/K_1; t) = t^{4n-2}P(G/K_1; t^{-1})$. From the two equations above, Lemma 3.13 and the equation $P(G/K_1; t) = t^{4n-2}P(G/K_1; t^{-1})$, we get

$$\begin{aligned} & (1-t^{2k_2})P(G/K_1;t) \\ = \ & (1-\epsilon_2)t^{2n}(1-t^{2k_2-2}) - \epsilon_2 t^{2n-1}(1-t^{2k_2}) \\ + \ & (t^{2k_2-1} - t^{4n-2n_2-1})a(n_2) + (1-t^{2n_2+2k_2})a(2n-n_2-1). \end{aligned} \tag{3.9}$$

So we easily see $\chi(G/K_1) \neq 0$. Hence $P(G/K_1;\ t)$ is an even function. □

Since $P(G/K_1; t)$ is an even function, it follows from (3.9) that

$$(t^{2k_2-1} - t^{4n-2n_2-1})a(n_2) - \epsilon_2 t^{2n-1}(1-t^{2k_2}) = 0, \tag{3.10}$$

$$(1-t^{2k_2})P(G/K_1;t) = (1-\epsilon_2)t^{2n}(1-t^{2k_2-2}) + (1-t^{2n_2+2k_2})a(2n-n_2-1). \tag{3.11}$$

Comparing the minimal degree terms in (3.10), we get $k_2 = \min\{2n - n_2,\ n\}$. If $k_2 = 2n - n_2$, then we see $\epsilon_2 = 0$ from (3.10). However we see easily $\chi(G/K_1) \notin \mathbb{Z}$ from (3.11) and $k_2 \geq 2$. So this case does not occur.

Hence $k_2 = n$. So we see $\epsilon_2 = 1$ from (3.10).

If $n_2 \neq 0$, then we see $n_2 = n-1$ from (3.10). In this case we can also prove $\chi(G/K_1) \equiv -(1/n) \pmod{\mathbb{Z}}$ up to $n = 2$. Hence $\chi(G/K_1) \notin \mathbb{Z}$. This is a contradiction. Put $n = 2$, then we see $G/K_1 \sim P_2(\mathbb{C})$, $n_2 = 1$ and $k_2 = n = 2$. But we see $P(G/K_2; t) = 1 + t + t^2 + t^4 - t^7$ from Lemma 3.14, this contradicts $\dim H^q(X; \mathbb{Q}) \geq 0$.

Hence $k_2 = n$, $\epsilon_2 = 1$, $n_2 = 0$. Consequently $G/K_1 \sim P_{2n-1}(\mathbb{C})$ from (3.11). So we get $P(G/K_2; t) = 1 + t^{2n}$ from Lemma 3.14. By Proposition 3.2, $P(G/K_2^o; t) = (1 + t^n)(1 + t^{2n})$ and $G/K^o \sim S^{4n-1}$. This is the case that G/K_1 is orientable and G/K_2 is non-orientable in Theorem 3.1.

3.4. Both singular orbits are non-orientable.

Suppose G/K_1 and G/K_2 are non-orientable. By Lemma 3.7 and Proposition 3.2, we have $k_1 = k_2 = 2$, and

$$\begin{aligned} (3.12) \qquad P(G/K_s^o;\ t) &= (1+t^2)P(G/K_s;t), \\ (3.13) \qquad P(G/K^o;t) &= (1+t^3)P(G/K_s;t) - P(n_r, n_s;t) - \epsilon_s(1-\epsilon_r)(1+t^{-1})t^{2n} \end{aligned}$$

where

$$P(n_1, n_2; t) = \begin{cases} t^{2n_1+1} + t^{2n_1+2} + \cdots + t^{2n_2} & (n_1 < n_2) \\ 0 & (n_1 \geq n_2). \end{cases}$$

3.4.1. *The case $\epsilon_1 = \epsilon_2$.*

In this case we see $n_1 = n_2$ from Proposition 3.1. So we get the following two equations from (3.12), (3.13),

$$\begin{aligned} P(G/K_1;t) &= P(G/K_2;t), \\ P(G/K^o;t) &= (1+t^3)P(G/K_s;t). \end{aligned}$$

Now we have

$$P(\mathrm{Im}\ f_s^*;t) = 1 + t^2 + \cdots + t^{2n}$$

from Lemma 3.3 and Proposition 3.1. We can get the following lemma.

LEMMA 3.15. *If M is a rational cohomology complex quadric and $P(\mathrm{Im}\ f_s^*;t) = 1+t^2+\cdots+t^{2n}$ then we have*

$$P(G/K_1;t) + P(G/K_2;t) = (1-t^{2n-1})(1+t^2+\cdots+t^{2n}) + P(G/K;t).$$

PROOF. By the Mayer-Vietoris exact sequence that is

$$\cdots \longrightarrow H^q(M) \xrightarrow{f_1^* \oplus f_2^*} H^q(X_1) \oplus H^q(X_2) \longrightarrow H^q(X_1 \cap X_2) \longrightarrow H^{q+1}(M) \longrightarrow \cdots$$

and the assumptions in the lemma, we see $P(X_1;t) + P(X_2;t) = (1-t^{2n-1})(1+t^2+\cdots+t^{2n}) + P(X_1 \cap X_2;t)$. Since X_s is a tubular neighborhood of G/K_s, $H^*(X_s) = H^*(G/K_s)$ and $X_1 \cap X_2 = G/K$. So we get this lemma. □

Since $k_s = 2$ $(s = 1,\ 2)$, we have $q^* : H^*(G/K) \to H^*(G/K^o)$ is an isomorphism. Hence $\chi(G/K) = \chi(G/K^o) = 0$. Therefore $\chi(G/K_s) \neq 0$ (that is $P(G/K_s;t)$ is an even function from Lemma 3.6) from $P(G/K_1;t) = P(G/K_2;t)$ and Lemma 3.15. Substituting Lemma 3.15 for $P(G/K;t) = P(G/K^o;t) = (1+t^3)P(G/K_s;t)$ and comparing the degrees, we have $n = 2$, $P(G/K_s;t) = 1+t^2+t^4$, $P(G/K_s^o;t) = (1+t^2)(1+t^2+t^4)$ and $P(G/K;t) = P(G/K^o;t) = (1+t^3)(1+t^2+t^4)$. This is the case where two singular orbits are both non-orientable in Theorem 3.1.

3.4.2. *The case* $\epsilon_1 \neq \epsilon_2$.

In this case we see $n_1 \neq n_2$ because $n_1+n_2+1 = 2n$ (Proposition 3.1). We may assume $\epsilon_1 = 0$ and $\epsilon_2 = 1$. From (3.13), for $s = 1$,

$$P(G/K^o;t) = (1+t^3)P(G/K_1;t) - P(n_2,n_1;t), \tag{3.14}$$

moreover for $s = 2$

$$P(G/K^o;t) = (1+t^3)P(G/K_2;t) - P(n_1,n_2;t) - (1+t^{-1})t^{2n}. \tag{3.15}$$

From the Mayer-Vietoris exact sequence, we have the following lemma.

LEMMA 3.16. *If* M *is a rational cohomology complex quadric, then*

$$\begin{aligned} & P(G/K_1;t) + P(G/K_2;t) \\ = \; & P(G/K;t) - t^{-1}(1+t^{2n})(1+t^2+\cdots+t^{2n}) + P(\mathrm{Im}\, f_1^* \oplus f_2^*)(1+t^{-1}) \end{aligned}$$

From this lemma, we have following two lemmas.

LEMMA 3.17. *If* $n_1 < n_2$*, then we have*

$$\begin{aligned} & P(G/K_1;t) + P(G/K_2;t) \\ = \; & P(G/K;t) + (1-t^{2n+3m-1})(1+t^2+\cdots+t^{2n-3m}) \\ + \; & t^{2n-3m+2}(1+t^2+\cdots+t^{6m-4}) + t^{2n} \end{aligned}$$

where $m = \chi(G/K_1) - \chi(G/K_2)$.

PROOF. Suppose $n_1 < n_2$. Then we have

$$\begin{aligned} & (1+t)(1-t+t^2)\{P(G/K_2;t) - P(G/K_1;t)\} \\ = \; & t^{2n_1+1}(1+t)(1+t^2+\cdots+t^{2(n_2-n_1)-2}) + (1+t)t^{2n-1} \end{aligned} \tag{3.16}$$

from (3.14) and (3.15). From this equation

$$\chi(G/K_1) - \chi(G/K_2) = m = 3^{-1}(n_2 - n_1 + 1) \in \mathbb{Z}. \tag{3.17}$$

Hence $n_2 - n_1 = 3m - 1$. Since $n_2 + n_1 = 2n - 1$ and $n_2 > n_1$, we have

$$\begin{aligned} n_1 &= n - \frac{3}{2}m, \\ n_2 &= n - 1 + \frac{3}{2}m \end{aligned}$$

and $m (\neq 0)$ is even. Also we have

$$\begin{aligned} P(\mathrm{Im}(f_1^* \oplus f_2^*);t) &= 1 + t^2 + \cdots + t^{2n_2} + t^{2n} \\ &= 1 + t^2 + \cdots + t^{2n-2+3m} + t^{2n}. \end{aligned}$$

Hence we can get the equation in this lemma by Lemma 3.16. □

LEMMA 3.18. *If $n_1 > n_2$, then we have*

$$\begin{aligned} & P(G/K_1;t) + P(G/K_2;t) \\ = \; & P(G/K;t) + (1 - t^{2n+3m'+1})(1 + t^2 + \cdots + t^{2n-3m'-2}) \\ + \; & t^{2n-3m'}(1 + t^2 + \cdots + t^{6m'}) + t^{2n} \end{aligned}$$

where $m' = \chi(G/K_2) - \chi(G/K_1)$.

PROOF. Suppose $n_1 > n_2$, we get from (3.14), (3.15)

$$\begin{aligned} & (1+t)(1-t+t^2)(P(G/K_1;t) - P(G/K_2;t)) \\ = \; & t^{2n_2+1}(1+t)(1+t^2+\cdots+t^{2(n_1-n_2)-2}) - (1+t)t^{2n-1}. \end{aligned} \tag{3.18}$$

By this equation

$$\chi(G/K_2) - \chi(G/K_1) = m' = 3^{-1}(n_1 - n_2 - 1) \in \mathbb{Z}. \tag{3.19}$$

Consequently $n_1 - n_2 = 1 + 3m'$. So we have

$$\begin{aligned} n_1 &= n + \frac{3}{2}m', \\ n_2 &= n - 1 - \frac{3}{2}m' \end{aligned}$$

and m' is even, from $n_1 + n_2 = 2n - 1$. Also we have

$$\begin{aligned} P(\mathrm{Im}(f_1^* \oplus f_2^*);\ t) &= 1 + t^2 + \cdots + t^{2n_1} + t^{2n} \\ &= 1 + t^2 + \cdots + t^{2n+3m'} + t^{2n}. \end{aligned}$$

Hence we can get the equation in this lemma by Lemma 3.16. □

Now we see $\chi(G/K) = \chi(G/K^o) = 0$ by Lemma 3.12, (3.14) and (3.15).

Hence we have $\chi(G/K_1) + \chi(G/K_2) = 2n + 2$ by Lemma 3.17 and 3.18. Therefore we can easily show $\chi(G/K_s) \neq 0$ $(s = 1,\ 2)$ by (3.17) and (3.19). So rank (G) = rank (K_s^o) by Lemma 3.6. Hence we have $H^{odd}(G/K_s^o;\ \mathbb{Q}) = 0$. Therefore we see

$$H^{odd}(G/K_s;\mathbb{Q}) = 0$$

because of the equation (3.12). Hence if $n_1 < n_2$ we have from (3.16),

$$\begin{aligned} P(G/K_2;t) - P(G/K_1;t) &= t^{2n-3m+2}a(3m-2) + t^{2n} \\ t^3(P(G/K_2;t) - P(G/K_1;t)) &= t^{2n-3m+1}a(3m-2) + t^{2n-1}. \end{aligned}$$

Moreover if $n_1 > n_2$ we have from (3.18),

$$\begin{aligned} P(G/K_1;\ t) - P(G/K_2;\ t) &= t^{2n-3m'}a(3m') - t^{2n} \\ t^3(P(G/K_1;\ t) - P(G/K_2;\ t)) &= t^{2n-3m'-1}a(3m') - t^{2n-1}. \end{aligned}$$

By comparing the degrees of these equations, we see the case $\epsilon_1 \neq \epsilon_2$ does not occur.

4. First step to the classification

Let G be a compact connected Lie group and U be its maximal rank closed subgroup. The aim of this section is to find the pair (G, U) from Poincaré polynomials $P(G/U; t)$ which appeared in Theorem 3.1 up to local isomorphism.

4.1. Equivalence relation.

In this section we mention some notations. First we define an *essential isomorphism*.

Definition(essential isomorphism) Put $H = \cap_{x \in M} G_x$. If the induced effective actions $(G/H, M)$ and $(G'/H', M')$ are equivariantly diffeomorphic then we call (G, M) and (G', M') *essential isomorphic*.

We will classify (G, M) up to this equivalence relation. Next we define an *essentail direct product*.

Definition(essential direct product) Let $G_1, \cdots, G_s$ be compact Lie groups, and N be a finite normal subgroup of $G^* \simeq G_1 \times \cdots \times G_s$. We say that the factor group $G = G^*/N$ is an *essential direct product* of $G_1, \cdots, G_s$ and denote $G \simeq G_1 \circ \cdots \circ G_s$.

Note that all compact connected Lie groups are constructed by an essential direct product of some simply connected compact Lie groups and torus (see **[TM]** Corollary 5.31 in Chapter V). Because we would like to classify up to essential isomorphism, we can assume that

$$G \simeq G_1 \times \cdots \times G_k \times T$$

for some simply connected simple Lie groups G_i and a torus group T. Moreover we can assume that G acts almost effectively on M where we say that G acts *almost effectively* on M, if $H = \cap_{x \in M} G_x$ is a finite group. In this case G acts almost effectively on the principal orbit G/K, hence we easily see

PROPOSITION 4.1. *K dose not contain any positive dimensional closed normal subgroup of G.*

4.2. Candidates for (G, K_s).

The purpose of this section is to find the pair (G, U) such that G is a simply connected compact simple Lie group and U is its maximal rank subgroup where a rank of Lie group means a dimension of a maximal torus subgroup. In Theorem 3.1 we get some even functions $P(G/K_i; t)$. If $P(G/K_i; t)$ is an even function, then $\mathrm{rank} G = \mathrm{rank} K_i$ from Lemma 3.6. The following lemma is well known.

LEMMA 4.1 (**[TM]** Theorem 7.2 in Chapter V). *If $G \simeq G_1 \times \cdots \times G_k \times T$ then the maximal rank subgroup of G is $G' \simeq G_1' \times \cdots \times G_k' \times T$. Here G_i' is the maximal rank subgroup of G_i.*

Hence we may only find a simply connected compact simple Lie group G and its maximal rank closed subgroup U to get (G, K_i) such that $P(G/K_i; t)$ is even. All such pairs (G, U) are known (e.g. **[TM]**, **[Wan49]**). So we can compute $P(G/U;t)$ by making use of the following lemma (**[TM]** Theorem 3.21 in Chapter VII).

LEMMA 4.2 (Hirsch formula). *Let* G *be a connected compact Lie group and* U *a maximal rank connected closed subgroup of* G*. Suppose* $H^*(G;\mathbb{Q}) \simeq \Lambda(x_{2s_1+1}, \cdots, x_{2s_l+1})$ *and* $H^*(U;\mathbb{Q}) \simeq \Lambda(x_{2r_1+1}, \cdots, x_{2r_l+1})$ *where* $l = \text{rank } G = \text{rank } U$ *and* x_i *is an element of the* i*-th degree cohomology. Then* $P(G/U;t)$ *satisfies the equation*

$$P(G/U;t) = \prod_{i=1}^{l} \frac{1-t^{2s_i}}{1-t^{2r_i}}.$$

From the above argument we get the following propositions. Note that first three propositions also were known by Uchida **[Uch77]** Section 4.2.

PROPOSITION 4.2. *If* $P(G/U;t) = 1 + t^{2a}$*, then* (G, U) *is locally isomorphic to*

$$(SO(2a+1), SO(2a)) \quad \text{or} \quad (G_2, SU(3)), a = 3.$$

PROPOSITION 4.3. *If* $P(G/U;t) = 1 + t^2 + \cdots + t^{2b}$*, then* (G, U) *is locally isomorphic to one of the following.*

$$\begin{aligned}
&(SU(b+1), S(U(b) \times U(1))),\\
&(SO(b+2), SO(b) \times SO(2)), b = 2m+1,\\
&(Sp(\frac{b+1}{2}), Sp(\frac{b-1}{2}) \times U(1)), b = 2m+1,\\
&(G_2, U(2)), b = 5.
\end{aligned}$$

PROPOSITION 4.4. *If* $P(G/U;t) = (1+t^{2a})(1+t^2+\cdots+t^{2b})$*, then* (G,U) *is locally isomorphic to one of the following.*

$$
\begin{aligned}
&(SO(2m+2), SO(2m)\times SO(2)), a=b=m,\\
&(SO(2m+3), SO(2m)\times SO(2)), a=m, b=2m+1,\\
&(SO(7), U(3)), a=b=3,\\
&(SO(9), U(4)), a=3, b=7,\\
&(SU(3), T^2), a=1, b=2,\\
&(SO(10), U(5)), a=3, b=7,\\
&(SU(5), S(U(2)\times U(3))), a=2, b=4,\\
&(Sp(3), Sp(1)\times Sp(1)\times U(1)), a=2, b=5,\\
&(Sp(3), U(3)), a=b=3,\\
&(Sp(4), U(4)), a=3, b=7,\\
&(G_2, T^2), a=1, b=5,\\
&(F_4, Spin(7)\circ T^1), a=4, b=11,\\
&(F_4, Sp(3)\circ T^1), a=4, b=11.
\end{aligned}
$$

PROPOSITION 4.5. *If* $P(G/U;t) = 1+t^4+t^8+t^{12}$*, then* (G,U) *is locally isomorphic to*

$$(Sp(4), Sp(1)\times Sp(3)).$$

By Theorem 3.1, it is enough to consider above four cases. Before we start the classification, we outline the proof of the classification.

4.3. Outline of the proof of the classification.

In this section we state the outline for the classification. To classify (G, M), where G is a compact Lie group and M is a rational cohomology complex quadric, we will consider five cases corresponding to five Poincaré polynomials which appeared in Theorem 3.1. Let us recall the following theorem.

THEOREM 4.1 (differentiable slice theorem). *Let* G *be a compact Lie group and* M *be a smooth* G*-manifold. Then for all* $x \in M$ *there is a closed tubular neighborhood* U *of the orbit* $G(x) \cong G/G_x$ *and a closed disk* D_x*, which has an orthogonal* G_x*-action via the representation* $\sigma_x : G_x \to O(D_x)$*, such that* $G \times_{G_x} D_x \cong U$ *as a* G*-diffeomorphism.*

We call the representation σ_x in this theorem the slice representation of G_x at $x \in M$. Since we get candidates of singular isotropy groups in Section 4.2, we compute the slice representation of the singular isotropy subgroups K_1 and K_2 from the differentiable slice theorem. Then we will decide the transformation group G and two tubular neighborhoods $X_1 \cong G \times_{K_1} D^{k_1}$ and $X_2 \cong G \times_{K_2} D^{k_2}$ of two singular orbits G/K_1 and G/K_2.

Next we construct the G-manifold M up to equivalence by making use of the structure theorem Theorem 2.1 and the following lemma.

LEMMA 4.3 ([**Uch77**] Lemma 5.3.1). *Let* $f, f' : \partial X_1 \to \partial X_2$ *be G-equivariant diffeomorphisms. Then* $M(f)$ *is equivariantly diffeomorphic to* $M(f')$ *as G-manifolds, if one of the following conditions is satisfied* (*where* $M(f) = X_1 \cup_f X_2$)*:*

(1) f *is G-diffeotopic to* f'.
(2) $f^{-1}f'$ *is extendable to a G-equivariant diffeomorphism on* X_1.
(3) $f'f^{-1}$ *is extendable to a G-equivariant diffeomorphism on* X_2.

From Theorem 2.1, we can put $\partial X_s = G/K$. Hence we may assume the gluing map is in $N(K; G)/K$, because the set of all G-equivariant diffeomorphisms of G/K is isomorphic to $N(K; G)/K$ where $N(K; G)$ is a normalizer group of K in G.

Finally we compute the cohomology of the manifold which we constructed. And we decide whether this manifold is a rational cohomology complex quadric or not. This is a story of the classification.

The following two figures are images of classifisation.

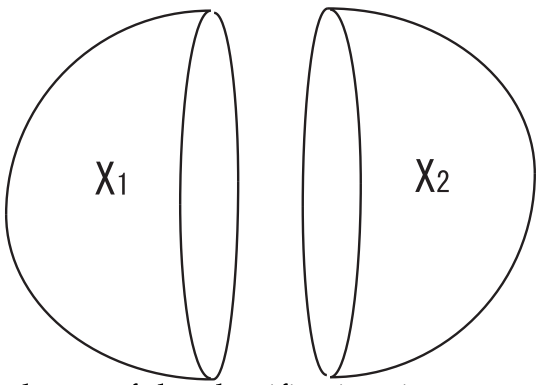

FIGURE 4.1. Second step of the classification, i.e. compute the slice representation and find two tubular neighborhoods X_1 and X_2.

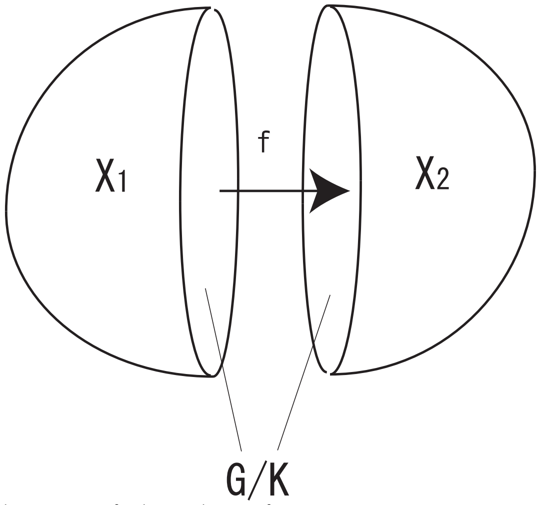

FIGURE 4.2. Third step of the classification, i.e. compute the gluing map $f : G/K \to G/K$.

Let us start to find (G, M) from the next section.

5. The two singular orbits are non-orientable

The goal of this section is to prove this case, that is the two singular orbits are non-orientable, does not occur. By Theorem 3.1, we see $P(G/K_s;t) = 1+t^2+t^4$ and $P(G/K_s^o;t) = (1+t^2)(1+t^2+t^4)$. So rank $G =$ rank K_s^o.

5.1. G/K_s^o is indecomposable.

A manifold is called *decomposable* if it is a product of positive dimensional manifolds. In this section we consider the case where G/K_s^o is indecomposable. By Proposition 4.4 ($a = 1$, $b = 2$), we see $G = SU(3) \times G' \times T^h$ and $K_s^o = T_s^2 \times G' \times T^h$. Here T_s^2 is a maximal torus of $SU(3)$, G' is a product of compact simply connected simple Lie groups and T^h is a torus. First we prove the following lemma.

LEMMA 5.1. $G = SU(3)$, $K_1^o = K_2^o = T^2$ *and* $K_1 = K_2$.

PROOF. Because $k_s = 2$, we see $K_s^o/K^o \cong S^1$. Hence $G' \times T^{h-1} \subset K^o$ from the assumption of G'. Therefore $G' = \{e\}$ and $h = 0$ or 1 from Proposition 4.1.

To show $h = 0$, let us consider the slice representation $\sigma_s : K_s \to O(2)$. Since G/K_s is non-orientable, there is an element $g_s \in K_s - K_s^o$ such that

$$\sigma_s(g_s) = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Since the centralizer of $\sigma_s(g_s)$ in $O(2)$ is a finite group $\mathbb{Z}_2 \times \mathbb{Z}_2$ and the centralizer of g_s in K_s contains $\{e\} \times T^h$, we see $\{e\} \times T^h \subset \mathrm{Ker}(\sigma_s|_{K_s^o}) = K^o$ where $\sigma_s|_{K_s^o}$ is the ristriction to

K_s^o. Hence $h = 0$ from Proposition 4.1. Therefore $K_s^o = T_s^2$ which is the maximal torus of $SU(3)$. Moreover $K_1 = K_2$ because $K \subset K_1 \cap K_2$ and $K_s = KK_s^o$. □

Next we construct the $SU(3)$-manifold. To construct the $SU(3)$-manifold, we will attach two tubular neighborhoods along their boundary. So first we consider two tubular neighborhoods of two singular orbits. Put the slice representation $\sigma_s : K_s \to O(2)$ for $s = 1, 2$. Since we can assume

$$T^2 = K_s^o = \left\{ \begin{pmatrix} u & 0 & 0 \\ 0 & v & 0 \\ 0 & 0 & w \end{pmatrix} = (u, v, w) \in SU(3) \middle| u, v, w \in U(1), uvw = 1 \right\},$$

the restricted slice representation to T^2 is

$$\sigma_s|_{T^2}((u, v, w)) = \phi(u^m)\phi(v^n)\phi(w^l) \tag{5.1}$$

where $\phi : U(1) \to SO(2)$ is a canonical isomorphism and $m, n, l \in \mathbb{Z}$. Now we can easily check $N(T^2; SU(3))/T^2$ is

$$\left\{ I = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, A = \begin{pmatrix} 0 & 0 & -1 \\ 1 & 0 & 0 \\ 0 & -1 & 0 \end{pmatrix}, A^{-1} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & -1 \\ -1 & 0 & 0 \end{pmatrix}, \right.$$
$$\left. \alpha = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \beta = \begin{pmatrix} 0 & -1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}, \gamma = \begin{pmatrix} 0 & 0 & 1 \\ 0 & -1 & 0 \\ 1 & 0 & 0 \end{pmatrix} \right\}.$$

This group is isomorphic to S_3. Hence $N(K_s^o; SU(3))/K_s^o \supset K_s/K_s^o \simeq \mathbb{Z}_2$ or S_3 ($K_s^o = T^2$) by non-orientability of $SU(3)/K_s$. We have following two lemmas.

LEMMA 5.2. *If* $\alpha \in K_s$, *then* $\{(\bar{u}^2, u, u) \in SU(3)\} \subset \mathrm{Ker}(\sigma_s|_{K_s^o})$.
If $\beta \in K_s$, *then* $\{(u, u, \bar{u}^2) \in SU(3)\} \subset \mathrm{Ker}(\sigma_s|_{K_s^o})$.
If $\gamma \in K_s$, *then* $\{(u, \bar{u}^2, u) \in SU(3)\} \subset \mathrm{Ker}(\sigma_s|_{K_s^o})$.

PROOF. Assume $\alpha \in K_s$. The centralizer of α in K_s contains $\{(\bar{u}^2, u, u) | u \in U(1)\}$. Then the slice representation is $\sigma_s(\bar{u}^2, u, u) = \sigma_s(\alpha(\bar{u}^2, u, u)\alpha^{-1}) \in SO(2)$. On the other hand $\sigma_s(\alpha(\bar{u}^2, u, u)\alpha^{-1}) = \sigma_s(\alpha)\sigma_s(\bar{u}^2, u, u)\sigma_s(\alpha)^{-1} = \sigma_s(\bar{u}^2, u, u)^{-1}$ because $\sigma_s(\alpha) \in O(2) - SO(2)$. This means $\sigma_s(\bar{u}^2, u, u) = \{e\}$ for all $u \in U(1)$.

Similarly we can show other cases. □

LEMMA 5.3. $K_s/K_s^o \simeq \mathbb{Z}_2$.

PROOF. If $K_s/K_s^o \simeq S_3$, then $K_s = N(K_s^o; SU(3))$. Hence $\{\alpha, \beta, \gamma, A, A^{-1}\} \subset K_s$. From Lemma 5.2, $\{(\bar{u}^2, u, u), (u, u, \bar{u}^2), (u, \bar{u}^2, u)\} \subset \mathrm{Ker}(\sigma_s|_{K_s^o})$. So we see

$$\{(\bar{u}^2, u, u), (u, u, \bar{u}^2), (u, \bar{u}^2, u)\} \subset K^o.$$

Hence $K^o = T^2$ because K^o is a connected Lie subgroup in $K_s^o = T^2$. This contradicts $K_s^o/K^o \cong S^1$. □

Moreover we can easily see the following lemma from above lemmas and the equation (5.1).

LEMMA 5.4. *For* $m \in \mathbb{N}$*, we have the following properties.*
If $\{I, \alpha\} = K_s/K_s^o$*, then* $K^o = \{(\bar{u}^2, u, u)\}$ *and* $\sigma_s|_{K_s^o}(u\bar{v}, u, v) = \phi(u^m)\phi(v^{-m})$.
If $\{I, \beta\} = K_s/K_s^o$*, then* $K^o = \{(u, u, \bar{u}^2)\}$ *and* $\sigma_s|_{K_s^o}(u, v, u\bar{v}) = \phi(u^m)\phi(v^{-m})$.
If $\{I, \gamma\} = K_s/K_s^o$*, then* $K^o = \{(u, \bar{u}^2, u)\}$ *and* $\sigma_s|_{K_s^o}(u, u\bar{v}, v) = \phi(u^m)\phi(v^{-m})$.

We can easily check $\mathrm{Ker}(\sigma_s|_{K_s^o})/K^o \simeq \mathbb{Z}_m$. Moreover we see $\sigma_1|_{T^2} = \sigma_2|_{T^2}$. Hence we get the tubular neighborhood

$$X_s^{(m)} = SU(3) \times_{K_s} D_m^2$$

where K_s acts on the disk D_m^2 by $\sigma_s : K_s \to O(2)$ such that $\mathrm{Ker}(\sigma_s|_{K_s^o})/K^o \simeq \mathbb{Z}_m$.

Next we consider an attaching map from $X_1^{(m)}$ to $X_2^{(m)}$. Since the attaching map f is equivariantly diffeomorphic to G/K, f is in $N(K;G)/K$. Now the following lemma holds from Lemma 5.4 and $K^o = T^2$.

LEMMA 5.5. $N(K; SU(3)) \simeq U(2)$.

Hence the attaching map is unique up to equivalence by Lemma 4.3 (1.). So we see such an $SU(3)$-manifold exists for each $m \in \mathbb{N}$ and

$$M^{(m)} = SU(3) \times_{K_s} S^2$$

where K_s acts on S^2 via the linear representation $\sigma_s : K_s \to O(2)$ such that $\mathrm{Ker}(\sigma_s|_{K_s^o})/K^o \simeq \mathbb{Z}_m$. From above argument, we have the following proposition.

PROPOSITION 5.1. *Let* M *be an* $SU(3)$*-manifold which has codimension one orbits* $SU(3)/K$ *and two singular orbits* $SU(3)/K_s$ $(s = 1, 2)$*. Then* M *is* $SU(3)$*-equivariant diffeomorphic to* $M^{(m)}$ *for some* $m \in \mathbb{N}$*.*

Finally we show such an $SU(3)$-manifold $M^{(m)}$ is not a rational cohomology complex quadric.

PROPOSITION 5.2. $M^{(m)} = SU(3) \times_{K_s} S^2$ *is not a rational cohomology complex quadric.*

PROOF. If $M^{(m)}$ is a rational cohomology complex quadric, then $M^{(m)}$ is simply connected. The manifold $N = SU(3) \times_{K_s^o} S^2$ is a double covering of $M^{(m)}$. Hence $M^{(m)} \cong N$. Now N is a fiber bundle over $SU(3)/T^2 = SU(3)/K_s^o$ with a fiber S^2 and $SU(3)/T^2$ is simply connected. Hence $H^*(M^{(m)}; \mathbb{Q}) \simeq H^*(N; \mathbb{Q}) \simeq H^*(S^2; \mathbb{Q}) \otimes H^*(SU(3)/T^2; \mathbb{Q})$ because $H^{odd}(S^2; \mathbb{Q}) = H^{odd}(SU(3)/T^2; \mathbb{Q}) = 0$. Hence $H^*(M^{(m)}; \mathbb{Q}) \not\simeq H^*(Q_4; \mathbb{Q})$. This is a contradiction. □

Hence this case does not occur.

5.2. G/K_1^o **is decomposable.**
By Proposition 4.2 ($a = 1$), 4.3 ($b = 2$), we see that

$$G = SU(2) \times SU(3) \times G' \times T^h,$$
$$K_1^o = T^1 \times S(U(2) \times U(1)) \times G' \times T^h.$$

First we prove the following lemma.

LEMMA 5.6. $G = SU(2) \times SU(3)$ *and* $K_1^o = T^1 \times S(U(2) \times U(1)) \simeq K_2^o$.

PROOF. If G/K_2^o is indecomposable, then we see $K_2^o = SU(2) \times T^2 \times G' \times T^h$. Because $K^o \subset K_1^o \cap K_2^o$ and $K_s^o/K^o \cong S^1$ for $s = 1, 2$, this is a contradiction. So G/K_2^o is decomposable. Hence we have $K_1^o \simeq K_2^o$, $G' = \{e\}$ and $h = 0$ or 1. Moreover we can show $h = 0$ like Lemma 5.1. □

Because of the non-orientability of

$$G/K_s,$$
$$N(T^1; SU(2))/T^1 \simeq \mathbb{Z}_2 \text{ and}$$
$$N(S(U(2) \times U(1)); SU(3)) = S(U(2) \times U(1)),$$

we have $K_s = N(T^1; SU(2)) \times S(U(2) \times U(1))$. For the slice representation $\sigma_s : K_s \to O(2)$, there exists $g_s \in K_s - K_s^o$ such that

$$\sigma_s(g_s) = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Here the centeralizer of $\sigma_s(g_s)$ in $O(2)$ is a finite group and the centralizer of g_s in K_s contains $\{e\} \times S(U(2) \times U(1))$. Hence $S(U(2) \times U(1)) \subset \mathrm{Ker}(\sigma_s)$. So the slice representation $\sigma_s : K_s \to O(2)$ has a decomposition $\sigma_s : K_s \to N(T^1; SU(2)) \to O(2)$. Moreover $K^o = \{e\} \times S(U(2) \times U(1))$ by $K_s/K \cong S^1$. Therefore there is an equivariant decomposition

$$M \cong (SU(2) \times_{N(T^1)} D^2) \cup_\partial (SU(2) \times_{N(T^1)} D^2) \times P_2(\mathbb{C})$$

where $N(T^1) = N(T^1; SU(2))$ and as is well known $SU(3)/S(U(2) \times U(1)) \cong P_2(\mathbb{C})$. Hence this case G-manifold is $M \cong N \times P_2(\mathbb{C})$, where N is some $SU(2)$-manifold (In fact we easily see $N = SU(2) \times_{N(T^1)} S^2$). However this contradicts M is indecomposable. So this case does not occur.

6. One singular orbit is orientable, the other is non-orientable

The goal of this section is to prove this case is one of the exotic case in Theorem 1.1.
Assume G/K_1 is orientable, G/K_2 is non-orientable. Then $k_1 = 2$ from Lemma 3.7. Since $k_1 = 2$, we have $K_1/K \cong S^1$. Let us prove the uniqueness of (G, M).

6.1. Uniqueness of (G, M).

By Theorem 3.1, we see $G/K^o \sim S^{4n-1}$, $G/K_1 \sim P_{2n-1}(\mathbb{C})$ (trivially G/K_1 is indecomposable), $P(G/K_2^o; t) = (1+t^n)(1+t^{2n})$ and $P(G/K_2; t) = (1+t^{2n})$. Because of $K_1/K \cong S^1$, we get $G = H \times T^h$, $K_1 = H_1 \times T^h$ ($h = 0$ or 1) where H is a simply connected simple Lie group and H_1 is its closed subgroup. First we show the following lemma.

LEMMA 6.1. $k_2 = n = 2$ *or* 4.

PROOF. We see $n = k_2$ from Theorem 3.1. Assume $k_2 = n$ is an odd number.
Now we have, from Proposition 4.3,

$$\begin{aligned}(H, H_1) \ \simeq\ & (SU(2n), S(U(2n-1) \times U(1))),\\ & (SO(2n+1), SO(2n-1) \times SO(2)),\\ & (Sp(n), Sp(n-1) \times U(1)) \text{ or}\\ & (G_2, U(2)),\ n = 7.\end{aligned}$$

If $(H, H_1) = (SU(2n), S(U(2n-1) \times U(1)))$, then the slice representatoion $\sigma_1 : K_1 \xrightarrow{\rho} U(1) \xrightarrow{\cong} SO(2)$ is as follows;

$$\rho\left(\begin{pmatrix} A & 0 \\ 0 & \det(A^{-1}) \end{pmatrix}, x\right) = \det(A^{-1})^l x^m \in U(1)$$

where $(l, m) \in \mathbf{Z}^2 - \{(0,0)\}$. Moreover we see $\mathrm{Ker}(\rho) = K$. Hence we have

$$\begin{aligned}K^o &\simeq SU(2n-1) \text{ if } h = 0 \text{ or}\\ K^o &\simeq U(2n-1) \text{ if } h = 1.\end{aligned}$$

Since $k_2 = n$ is an odd number, $K_2^o/K^o (\cong S^{n-1})$ is an even dimensional sphere. So we see $\mathrm{rank}\, K_2^o = \mathrm{rank}\, K^o$ by $\chi(K_2^o/K^o) \neq 0$ and Lemma 3.6. Hence (K_2^o, K^o) is locally isomorphic to one of the following pair

$$\begin{aligned}&(SO(n), SO(n-1)),\\ &(G_2, SU(3)) \text{ if } n = 7\end{aligned}$$

from Proposition 4.2. However this contradicts $K^o \simeq SU(2n-1)$ or $U(2n-1)$. Hence we see $k_2 = n$ is an even number for the case $(H, H_1) = (SU(2n), S(U(2n-1) \times U(1)))$. Also for other cases we see $k_2 = n$ is an even number by the similar argument. Therefroe $k_2 = n$ is an even number.

Hence we see $k_2 = n = 2$ or 4 from propositions in Section 4.2. □

We already have $G = H \times T^h$, $K_1 = H_1 \times T^h$. Moreover we have $K_2^o = H_2 \times T^h$ ($h = 0$ or 1) from Lemma 6.1, where H is a simply connected simple Lie group and H_s is its closed subgroup. By Proposition 4.3, 4.4 and 4.5,

$$\begin{aligned}(H, H_s^o) &\approx (SU(4), S(U(3)\times U(1))\ (n=2),\\ &\quad (Sp(2), Sp(1)\times U(1))\ (n=2) \text{ or}\\ &\quad (SO(5), SO(3)\times SO(2)) \sim (Sp(2), U(2))\ (n=2),\\ (H, H_1, H_2^o) &\approx (Sp(4), Sp(3)\times U(1), Sp(1)\times Sp(3))\ (n=4).\end{aligned}$$

where $(A_1, B_1) \approx (A_2, B_2)$ means (A_1, B_1) and (A_2, B_2) are locally isomorphic. Since G/K_2 is non-orientable, $N(K_2^o; G) \neq K_2^o$. Hence $H = Sp(2)$ and $n = 2$. Moreover we see $h = 0$ by the similar proof to Lemma 5.1.

Therefore this case has just the following three pairs (G, K_1^o, K_2^o).

$$\begin{aligned}(G, K_s^o) &\simeq (Sp(2), Sp(1)\times U(1)),\\ (G, K_s^o) &\simeq (Sp(2), U(2)),\\ (G, K_s^o, K_r^o) &\simeq (Sp(2), U(2), Sp(1)\times U(1))\end{aligned}$$

for $s + r = 3$. Let us prove the following lemma.

LEMMA 6.2. *In this case* $G = Sp(2)$, $K_1 = Sp(1)\times U(1)$, $K_2 \simeq Sp(1)\times U(1)_j \cup U(1)_j\mathbf{i}$ *and* $K \simeq Sp(1)\times\{1, -1, \mathbf{i}, -\mathbf{i}\}$ *where* $\{1, \mathbf{i}, \mathbf{j}, \mathbf{k}\}$ *is the basis of* $\mathbf{H}$ *and* $U(1)_j = \{a + b\mathbf{j} \mid a^2 + b^2 = 1\}$.

PROOF. Suppose $(G, K_s^o) \simeq (Sp(2), U(2))$. Since G/K_2 is non-orientable, we have $K_2 \simeq N(U(2); Sp(2))$ (K_2 has two components). We can put $K_1 = U(2)$. So $K^o = SU(2)$ since $K_1/K \cong S^1$. Since $K_1 \cap K_2 \supset K$ and $K_2 = N(U(2); Sp(2))$, we get $K_2/K \cong S^1 \times S^1$. This contradicts $K_2/K \cong S^1$. So this case does not occur.

Next put $(G, K_s^o, K_r^o) \simeq (Sp(2), U(2), Sp(1)\times U(1))$ $(s + r = 3)$. Because $K_s^o \supset K^o \supset SU(2)$ is not conjugate to $K_r^o \supset K^o \supset Sp(1)$ by $k_s = k_r = 2$, we have $K_1^o \cap K_2^o = U(2) \cap (Sp(1)\times U(1)) = U(1)\times U(1) \supset K^o$. Hence $K_s^o/K^o \cong S^2$, this contradicts $K_s/K \cong S^1$. So this case does not occur.

Therefore $(G, K_s^o) \simeq (Sp(2), Sp(1)\times U(1))$. Since G/K_1 is orientable and G/K_2 is non-orientable, we have $K_1 = Sp(1)\times U(1) = K_1^o$ and $K_2 = N(K_2^o; G)$. Since $K_s/K \cong S^1$, we have $K = Sp(1)\times F$ (where F is a finite subgroup of $U(1)$). If $K_2^o = K_1 = Sp(1)\times U(1)$, then $K_2/K \cong N(U(1); Sp(1))/F \cong S^1 \times S^1$. This contradicts $K_2/K \cong S^1$. So we can put $K_2^o = Sp(1)\times U(1)_j$ without loss of generality. Then $K_2 = Sp(1)\times(U(1)_j \cup U(1)_j\mathbf{i})$ and $K_1 \cap K_2 = Sp(1)\times\{1, -1, \mathbf{i}, -\mathbf{i}\}$. Since $K \subset K_2 \cap K_1$, we have $F = \{1, -1, \mathbf{i}, -\mathbf{i}\}$. □

Next we prove the following lemma.

LEMMA 6.3. *Let* $(Sp(2), M)$ *be an* $Sp(2)$*-manifold which has codimension one principal orbits* $Sp(2)/Sp(1)\times\{1, -1, \mathbf{i}, -\mathbf{i}\}$, *two singular orbits* $Sp(2)/Sp(1)\times U(1)$ *and* $Sp(2)/Sp(1)\times(U(1)_j \cup U(1)_j\mathbf{i})$. *Then this* $(Sp(2), M)$ *is unique up to essential isomorphism.*

PROOF. The slice representations of $K_1 = Sp(1)\times U(1)$ and $K_2 = Sp(1)\times(U(1)_j \cup U(1)_j\mathbf{i})$ decompose into the factor as follows:

$$\sigma_1 : K_1 \to U(1) \overset{\rho_1}{\to} O(2),$$
$$\sigma_2 : K_2 \to N(U(1)_j; Sp(1)) = U(1)_j \cup U(1)_j\mathbf{i} \overset{\rho_2}{\to} O(2).$$

Since $\mathrm{Ker}(\rho_1) = F$, we can assume

$$\rho_1(\exp(\mathbf{i}\theta)) = \begin{pmatrix} \cos(4\theta) & -\sin(4\theta) \\ \sin(4\theta) & \cos(4\theta) \end{pmatrix}$$

up to equivalence. So the slice representation σ_1 is unique up to equivalence. Since $K_2/K \cong S^1$ and $\mathrm{Ker}(\rho_2|_{U(1)_j}) = \{1, -1\}$, we can put

$$\rho_2(\mathbf{i}) = \rho_2(-\mathbf{i}) = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Therefore the slice representation σ_2 is unique up to equivalence. Moreover $N(K; G)/K \simeq U(1)/F$ has only one connected component. In this case the action is unique by Lemma 4.3. □

Consequently the following proposition holds.

PROPOSITION 6.1. *Let* M *be an* $Sp(2)$*-manifold which satisfies the conditions of Lemma 17.3. Then* $M \cong S^7 \times_{Sp(1)} P_2(\mathbb{C})$.

PROOF. If $M = S^7 \times_{Sp(1)} P_2(\mathbb{C})$ where $S^7 \cong Sp(2)/Sp(1)$, $Sp(2)$ acts naturally on S^7 and $Sp(1)$ acts on $P_2(\mathbb{C}) = P(\mathbb{R}^3 \otimes_{\mathbb{R}} \mathbb{C})$ through the double covering $Sp(1) \to SO(3)$ (see **[Uch77]** Example 3.2). Then we can easily check this manifold satisfies the conditions of Lemma 17.3. From Lemma 17.3, we get this proposition. □

Hence this case has a unique (G, M) up to essential isomorphism.

6.2. Topology of $M = S^7 \times_{Sp(1)} P_2(\mathbb{C})$.

In this section, we study the topology of M.

First we show M is a rational cohomology complex quadric. This manifold M is a $P_2(\mathbb{C})$-bundle over $S^7/Sp(1) \cong S^4$. Since $H^{odd}(S^4) = H^{odd}(P_2(\mathbb{C})) = 0$ and S^4 is simply connected, the induced map $p^* : H^*(S^4) \to H^*(M)$ is injective where $p : M \to S^4$ is a projection and $i^* : H^*(M) \to H^*(P_2(\mathbb{C}))$ is surjective where $i : P_2(\mathbf{C}) \cong p^{-1}(w) \to M$ for fixed $w \in S^4$ by **[TM]** Theorem 4.2 in Chapter III. Hence there exists a generator $x \in H^4(M)$ such that $x^2 = 0 \in H^8(M)$ and $c \in H^2(M)$ such that $i^*(c) \in H^2(P_2(\mathbb{C}))$ is a generator of $H^*(P_2(\mathbb{C}))$. Because $i^*(x) = 0$, we see $c^2 \neq x$ in $H^4(M) \simeq \mathbb{Q} \oplus \mathbb{Q}$. Next we assume $S^7 \times P_2(\mathbb{C})$ a $Sp(1)$-bundle over M. From the Thom-Gysin exact sequence, $H^6(M) \simeq \mathbb{Q}$ is generated by xc and $H^8(M) \simeq \mathbb{Q}$ is generated by xc^2.

Let us show $0 \neq c^3 \in H^6(M)$. This manifold M has an $Sp(2)$-action and this action has codimension one principal orbits from Section 6.1. Therefore we can use the

Mayer-Vietoris exact sequence from Theorem 2.1. If we put the principal orbit G/K, the orientable singular orbit G/K_1 and the non-orientable singular orbit G/K_2, then we have $H^*(G/K) \simeq H^*(S^7)$ and $H^*(G/K_2) \simeq H^*(S^4)$ from Theorem 3.1. Moreover we see, from Section 6.1, the orientable singular orbit G/K_1 is diffeomorphic to $P_3(\mathbb{C})$. Hence the induced homomorphism $j^* : H^2(M) \to H^2(G/K_1)$ is isomorphic. Therefore $j^*(c)$ is a generator in $H^2(G/K_1)$ and $j^*(c^3) = j^*(c)^3 \neq 0$ because $H^*(P_3(\mathbb{C})) \simeq \mathbb{Q}[c]/(c^4)$. Hence this manifold M is a rational cohomology complex quadric.

Next we show the tangent bundle of M does not have a spin structure, we call such a manifold non-spin. It is easy to show if a fiber is non-spin then its total space is also non-spin. Hence M is non-spin because $P_2(\mathbb{C})$ is non-spin, that is, the second Stiefel-Whiteny class $w_2(P_2(\mathbb{C})) \neq 0$. By definition, Q_4 is a degree 2 non-singular algebraic hypersurface in $P_5(\mathbb{C})$. So Q_4 is a spin manifold (see Section 16.5 in **[BH58]** or **[MS74]**). Therefore M is not diffeomorphic to Q_4.

Hence we get the following proposition.

PROPOSITION 6.2. *The 8-dimensional manifold* $S^7 \times_{Sp(1)} P_2(\mathbb{C})$ *is not diffeomorphic to* Q_4, *but a rational cohomology complex quadric .*

7. $G/K_1 \sim P_{2n-1}(\mathbb{C}), G/K_2 \sim S^{2n}$

The goal of this section is to prove there are three cases (G, M) up to essential isomorphism. In this case G/K_1, G/K_2 are indecomposable. Since $k_1 = 2$, $k_2 = 2n$, $n \geq 2$ and Lemma 3.7, $G = H \times T^h$ and $K_1^o = K_1 = H_1 \times T^h$ ($h = 0$ or 1). By Proposition 4.3,

$$\begin{aligned}(H, H_1) \ \approx\ & (SU(2n), S(U(2n-1) \times U(1))) \text{ or} \\ & (SO(2n+1), SO(2n-1) \times SO(2)) \text{ or} \\ & (Sp(n), Sp(n-1) \times U(1)) \text{ or} \\ & (G_2, U(2)),\ n = 3.\end{aligned}$$

Since $k_1 = 2$, we can use Lemma 3.9 and Lemma 3.10. So we have

$$H^*(G/K_2^o; \mathbb{Q}) = \mathrm{Im}(q_2^*) + J \cdot \chi + J \cdot \chi^2 \text{ (possibly non direct sum)}$$

where $q_2^* : H^*(G/K_2; \mathbb{Q})(\simeq H^*(S^{2n}; \mathbb{Q})) \to H^*(G/K_2^o; \mathbb{Q})$ is the injective induced homomorphism, $J_k = q_2^* H^k(G/K_2; \mathbb{Q})$ and $J = \oplus_k J_k$. Since $\chi \in H^{2n}(G/K_2^o; \mathbb{Q})$ by $k_2 = 2n$, we see $J \cdot \chi^2 = 0$ and $J \cdot \chi = H^{2n}(G/K_2^o; \mathbb{Q})$. Hence $P(G/K_2^o; t) = P(G/K_2; t) = 1 + t^{2n}$.

Therefore we see $(G, K_2^o) \approx (SO(2n+1), SO(2n))$ or $(G_2, SU(3))$ and $n = 3$ by Proposition 4.2. So we have that

$$\begin{aligned}(H, H_1, H_2) \ =\ & (Spin(2n+1), Spin(2n-1) \circ T^1, Spin(2n)) \text{ or} \\ & (G_2, U(2), SU(3)) \text{ and } n = 3\end{aligned}$$

where $K_2^o = H_2 \times T^h$.

7.1. $G = \mathrm{Spin}(2n+1) \times T^h$.

First we show the following lemma.

LEMMA 7.1. $h = 0$.

PROOF. If $h = 1$, then $K_2^o = \mathrm{Spin}(2n) \times T^1$. Because G/K_2 is orientable, we get $K_2 = K_2^o$. Since $k_2 = 2n$, we have the slice representation $\sigma_2 : K_2 \to SO(2n)$. From $n \geq 2$, we see the restricted representation $\sigma_2|_{\mathrm{Spin}(2n)}$ is a natural projection from $\mathrm{Spin}(2n)$ on $SO(2n)$. Hence $\sigma_2(\{e\} \times T^1) \subset C(SO(2n))$ where $C(SO(2n))$ is the center of $SO(2n)$ that is $C(SO(2n)) = \{I_{2n}, -I_{2n}\}$. Hence $\{e\} \times T^1 \subset \mathrm{Ker}(\sigma_2) \subset K$. This contradicts Proposition 4.1. So we have $h = 0$. □

Hence we have $K_1 = \mathrm{Spin}(2n-1) \circ T^1, K_2 = \mathrm{Spin}(2n), K^o = \mathrm{Spin}(2n-1)$. We see $K = K^o$ from $K_2/K \cong S^{2n-1}$. Let us prove the following lemma.

LEMMA 7.2. *Let* (G, M) *be a* G*-manifold which has codimension one orbits* G/K*, two singular orbits* Q_{2n-1} *and* S^{2n} *where* $G = \mathrm{Spin}(2n+1)$, $K = \mathrm{Spin}(2n-1)$. *Then this* (G, M) *is unique up to essential isomorphism.*

PROOF. Because $n \geq 2$, we can decompose the slice representation $\sigma_1 : K_1 \to O(2)$ into $\sigma_1 : K_1 = \mathrm{Spin}(2n-1) \circ T^1 \overset{proj}{\to} T^1 \overset{\rho}{\to} O(2)$. Since $\mathrm{Ker}(\sigma_1) \subset K$, ρ is an injection. So the slice representation σ_1 is unique up to equivalence. Next we consider the slice representation $\sigma_2 : K_2 \to SO(2n) \subset O(2n)$. Since $\mathbb{Z}_2 \subset \mathrm{Ker}(\sigma_2) \subset \sigma_2^{-1}(SO(2n-1)) = K$, σ_2 decomposes into $\sigma_2 : K_2 = \mathrm{Spin}(2n) \overset{proj}{\to} SO(2n) \overset{\rho}{\to} SO(2n)$. Because $SO(2n)$ acts transitively on S^{2n-1}, we see that ρ is an isomorphism by **[HH65]** Section I and $n \geq 2$. Hence the slice representation σ_2 is unique up to equivalence.

Since $N(K, G)$ has two components, we can assume

$$p(y) = \begin{pmatrix} -I_{2n} & 0 \\ 0 & 1 \end{pmatrix}$$

where $p : \mathrm{Spin}(2n+1) \to SO(2n+1)$ is the natural projection, $[y] \in N(K, G)/N(K, G)^o$ ($y \in G = \mathrm{Spin}(2n+1)$). It suffices to prove that the right translation R_y on G/K is extendable to a G-diffeomorphic map on X_2 from Lemma 4.3 (3.). Because y is in the center of $K_2 = \mathrm{Spin}(2n)$, we have the following commutative diagram

$$\begin{array}{ccc} G \times_{K_2} K_2/K & \longrightarrow & G/K \\ \downarrow R_y \times 1 & & \downarrow R_y \\ G \times_{K_2} K_2/K & \longrightarrow & G/K. \end{array}$$

Here $G \times_{K_2} K_2/K = \partial(G \times_{K_2} D^{2m}) = \partial X_2$. It is clear that $R_y \times 1$ is extendable to a G-diffeomorhpic map on X_2. □

Consequently (G, M) is unique up to essential isomorphism. Such an example of (G, M) will be constructed in Section 11.1. This is one of the results in Theorem 1.1.

7.2. $G = G_2 \times T^h$.

The exceptional Lie group G_2 is defined by Aut($\mathbb{O}$). Here $\mathbb{O}$ is the Cayley numbers generated by $\mathbb{R}$-basis $\{1, e_1, \cdots, e_7\}$. It is well known that $G_2 \subset SO(7)$ and $SU(3) \simeq \{A \in G_2 | A(e_1) = e_1\}$.

Let us consider the case $h = 0$ and 1.

7.2.1. $h = 0$.

In this case $K_1 \simeq U(2), K_2^o \simeq SU(3), K^o \simeq SU(2)$. We can put $K_2^o = \{A \in G_2 | A(e_1) = e_1\}$. Then $N(K_2^o, G)$ has two components. Since G/K_2 is orientable and $G_2/SU(3) \cong S^6$, $K_2 = K_2^o$ and $K = K^o$. Also in this case (G, M) is unique by the following lemma.

LEMMA 7.3. *Let (G_2, M) be a G_2-manifold which has codimension one orbits $G_2/SU(2)$, two singular orbits $G_2/U(2)$ and S^6. Then (G_2, M) is unique up to essential isomorphism.*

PROOF. Because K_2 acts transitively on $K_2/K \cong S^5$, the slice representation σ_2 is unique up to equivalence by **[HH65]** Section I. Then we see that $\sigma_2^{-1}(SO(5)) = \{B \in K_2 | B(e_2) = e_2\} = K \simeq SU(2)$.

The slice representation σ_1 decomposes into $\sigma_1 : K_1 \to U(1) \xrightarrow{\rho} O(2)$, because $\mathrm{Ker}(\sigma_1) \subset K$. Here ρ is an injection to $SO(2)$. So the slice representation σ_1 is unique up to equivalence.

Now $N(K; G)/K \simeq SO(3)$ is known (Section 7.4 in **[Uch77]**). Consequently (G, M) is unique up to essential isomorphism by Lemma 4.3 (1.). □

Hence, in this case, (G, M) is unique up to essential isomorphism. Such an example of (G, M) will be constructed in Section 11.5. This is one of the results in Theorem 1.1.

7.2.2. $h = 1$.

In this case we have $G = G_2 \times T^1$, $K_1 = U(2) \times T^1$, $K_2 = SU(3) \times T^1$ and $K^o \simeq SU(2) \times T^1$, from the same argument as Section 7.2.1. First we show the following lemma.

LEMMA 7.4. *For each natural number m, the pair $(G_2 \times T^1, M^{(m)})$, which has codimension one orbits $(G_2 \times T^1)/K$ and two singular orbits $G_2/U(2)$ and S^6, is unique up to equivalence.*

PROOF. First we consider the slice representations. Because $K_2/K \simeq S^5$ and $\sigma_2(\{e\} \times T^1) \subset C(\sigma_2(SU(3) \times \{e\}); SO(6))$, where $C(X; Y) = \{b \in Y | ab = ba \text{ for all } a \in X\}$ for $X \subset Y$, the slice representation $\sigma_2 : K_2 = SU(3) \times T^1 \to O(6)$ is as follows

$$\sigma_2(A + \mathbf{i}B, \cos(\theta) + \mathbf{i}\sin(\theta)) = \begin{pmatrix} A & -B \\ B & A \end{pmatrix} \begin{pmatrix} \cos(m\theta)I_3 & -\sin(m\theta)I_3 \\ \sin(m\theta)I_3 & \cos(m\theta)I_3 \end{pmatrix}$$

for some $m \in \mathbb{N}$ up to equivalence. Hence

$$\begin{aligned} K &= \sigma_2^{-1}(SO(5)) \\ &= \left\{ \left(\begin{pmatrix} e^{mi\theta} & 0 \\ 0 & X \end{pmatrix}, e^{i\theta} \right) \middle| \det(X) = e^{-mi\theta} \right\}. \end{aligned}$$

From this equation, we have

$$\begin{aligned} K_1 &= U(2)\times T^1 \\ &= \left\{ \left(\begin{pmatrix} e^{i\theta} & 0 \\ 0 & X \end{pmatrix}, e^{i\phi} \right) \middle| \, 0 \le \theta, \phi \le 2\pi, \ \det(X) = e^{-i\theta} \right\}. \end{aligned}$$

Moreover we see the slice representation $\sigma_1 : K_1 = U(2)\times T^1 \overset{\rho}{\to} U(1) \overset{\simeq}{\to} SO(2)$ is as follows

$$\rho\left(\begin{pmatrix} e^{i\theta} & 0 \\ 0 & X \end{pmatrix}, e^{i\phi} \right) = e^{i\theta} e^{-mi\phi}$$

because $\mathrm{Ker}(\rho) = K$. Therefore there is a unique pair (σ_1, σ_2) for each $m \in \mathbb{N}$.

Next we consider the gluing map. Now we can assume $K = SU(2)\times T^1 \subset SO(7)\times T^1$ as follows:

$$\left\{ \left(\begin{pmatrix} I_3 & 0 \\ 0 & X \end{pmatrix}, r \right) \middle| \, X \in SU(2) \subset SO(4), r \in T^1 \right\}.$$

Because $N(K;G) = N(K; SO(7)\times T^1) \cap (G_2 \times T^1)$, we have

$$N(K;G)/K \simeq SO(3).$$

Consequently $(G, M^{(m)})$ is unique up to equivalence for each $m \in \mathbb{N}$ by Lemma 4.3 (1.). Hence we have this lemma. □

Next we prepare some notations. Let $G_{\mathbb{R}}(2, \mathbb{O})$ be the set of oriented 2-dimensional real linear subspace of $\mathbb{O}$. We identify an oriented 2-dimensional real linear subspaces of $\mathbb{O}$ with an element $\xi = u \wedge v \in \Lambda^2\mathbb{O}$ where $u,\ v \in \mathbb{O}$ is an oriented orthonormal basis of the 2-dimension subspace. Thus,

$$G_{\mathbb{R}}(2,\mathbb{O}) = \{\xi \in \Lambda^2\mathbb{O} |\ \xi = u \wedge v \text{ for some } u,\ v \text{ orthonormal in } \mathbb{O}\}$$

denotes the grassmannian of oriented 2-dimensional suspaces of $\mathbb{O}$. Then this manifold is diffeomorphic to Q_6 (see Section 14 in **[Har90]**).

Moreover we can show the following proposition.

PROPOSITION 7.1. *Let $M^{(m)}$ be a $G_2 \times T^1$-manifold which satisfies the conditions of Lemma 7.4. Then $M^{(m)} \cong G_{\mathbb{R}}(2,\mathbb{O})$ for all $m \in \mathbb{N}$ and $(G_2 \times T^1, M^{(m)})$ is essential isomorphic to $(G_2 \times T^1, M^{(1)})$ for all $m \in \mathbb{N}$.*

PROOF. Put $M = G_{\mathbb{R}}(2,\mathbb{O})$. Assume $(G_2 \times T^1, G_{\mathbb{R}}(2,\mathbb{O}))_{(m)}$ is a pair such that $g \in G_2$ acts on $u\wedge v \in M$ by $g\cdot u\wedge v = g(u)\wedge g(v)$ and $e^{i\theta} = \cos(\theta) + \mathbf{i}\sin(\theta) \in T^1$ acts on $u \wedge v \in M$ by $e^{i\theta}\cdot u\wedge v = (\cos(m\theta)u - \sin(m\theta)v)\wedge(\sin(m\theta)u + \cos(m\theta)v)$. Then we can easily check this action is well defined and this pair satisfies the condition of Lemma 7.4. Hence this pair is essentially isomorphic to $(G, M^{(m)})$. So we can assume $M^{(m)} = G_{\mathbb{R}}(2,\mathbb{O})$.

Let the action of the pair $(G, M^{(m)})$ be $\phi_{(m)}$. Then Ker $\phi_{(m)} = \{e\} \times \mathbb{Z}_m \subset G_2 \times T^1$. Hence we see (G, M^1) and $(G, M^{(m)})$ are essentially isomorphic for all $m \in \mathbb{N}$. □

Hence this case has a unique (G, M) up to essential isomorphism and such action will be constructed in Section 11.8 again.

8. $G/K_s \sim P_n(\mathbb{C})$

In this case $K_s = K_s^o$ because $k_s = 2n$ ($n \geq 2$) and Lemma 3.7. First we assume that $G = H_1 \times H_2 \times G' \times T^h$, $K_1 = H_{(1)} \times H_2 \times G' \times T^h$, $K_2 = H_1 \times H_{(2)} \times G' \times T^h$ where H_s is a simply connected simple Lie group, $H_{(s)}$ is its closed subgroup, G' is a product of simply connected simple Lie groups and T^h is a torus. Then $K_1 \cap K_2 = H_{(1)} \times H_{(2)} \times G' \times T^h$. So $\dim(G/K_1 \cap K_2) = 4n \leq \dim(G/K)$ because $K \subset K_1 \cap K_2$. This contradicts $\dim G/K = 4n - 1$. Hence we can put

$$\begin{aligned} G &= H \times G' \times T^h, \\ K_s &= H_{(s)} \times G' \times T^h. \end{aligned}$$

where H is a simply connected simple Lie group and $H_{(s)}$ is its closed subgroup. By Proposition 4.3,

$$\begin{aligned} (H, H_{(s)}) \approx\ & (SU(n+1), S(U(n) \times U(1))) \text{ or} \\ & (SO(n+2), SO(n) \times SO(2)),\ n = 2m+1 \text{ or} \\ & (Sp(\frac{n+1}{2}), Sp(\frac{n-1}{2})),\ n = 2m+1 \text{ or} \\ & (G_2, U(2)),\ n = 5. \end{aligned}$$

Next we show the following lemma.

LEMMA 8.1. *If M is a rational cohomology complex quadric, then $H = SU(n+1)$ and $H_{(s)} \simeq S(U(n) \times U(1))$.*

PROOF. If $H_{(1)}$ acts non-transitively on $K_1/K \cong S^{2n-1}$, then $V = G' \times T^h$ acts transitively on K_1/K by **[MS43]** Theorem I′ and $K_1/K \cong V/V'$ where $V' = K \cap V$. So we see $p_1(K) = H_{(1)} = p_1(K_1)$ where $p_1 : G \to H$ from $\{pt\} = V\backslash K_1/K \cong p_1(K_1)/p_1(K)$. Hence $V\backslash M$ is a mapping cylinder of $V\backslash G/K_1 = H/H_{(1)} \cong V\backslash G/K \to V\backslash G/K_2 = H/H_{(2)}$. From the following commutative diagram

$$\begin{array}{ccc} G/K_2 & \longrightarrow & M \\ \downarrow = & & \downarrow p \\ V\backslash G/K_2 = H/H_{(2)} & \xrightarrow{i} & V\backslash M \end{array}$$

where i is a homotopy equivalent map, we get the induced diagram

$$\begin{array}{ccc} H^*(V\backslash M) & \xrightarrow{i^*} & H^*(V\backslash G/K_2) \simeq H^*(H/H_{(2)}) \\ \downarrow p^* & & \downarrow = \\ H^*(M) & \longrightarrow & H^*(G/K_2). \end{array}$$

From this diagram we see p^* is an injective map. Put the generator $c \in H^2(V\backslash M) \simeq H^2(H/H_{(2)})$. Then $p^*(c) = u \in H^2(M)$ is a generator. Since $c^{n+1} = 0$, we see $p^*(c)^{n+1} = u^{n+1} = 0$. This is a contradiction to $u^{n+1} \neq 0$ from $H^*(M) = H^*(Q_{2n})$.

So $H_{(s)}$ acts transitively on $K_s/K \simeq S^{2n-1}$. By making use of **[HH65]** Section I, we get $(H, H_{(s)}) \simeq (SU(n+1), S(U(n) \times U(1)))$. Hence we can put $G = SU(n+1) \times G' \times T^h$ and $K_s \simeq S(U(n) \times U(1)) \times G' \times T^h$. □

Consider the slice representation $\sigma_s : S(U(n) \times U(1)) \times G' \times T^h \to O(2n)$. Because $SU(n)$ acts transitively on $K_s/K \cong S^{2n-1}$, we can assume that $\sigma_s|_{SU(n)}$ is a natural inclusion up to equivalence. Hence we can assume $\sigma_s(K_s) \subset U(n)$ and $\sigma_s(\{e\} \times G' \times T^h)$ is in the center of $U(n)$. This implies $G' \subset \mathrm{Ker}(\sigma_s) \subset K$. Hence $G' = \{e\}$ from Proposition 4.1. So we can assume the slice representation decomposes into $S(U(n) \times U(1)) \times T^h \xrightarrow{\rho_s} U(n) \xrightarrow{c} O(2n)$ where c is a canonical injective representation. Then we see $\rho_s|_{S(U(n)\times U(1))\times\{e\}} = \tau_{x_s}$ for some integer x_s where $\tau_{x_s} : S(U(n) \times U(1)) \to U(n)$ is

$$\tau_{x_s}\begin{pmatrix} A & 0 \\ 0 & \det(A^{-1}) \end{pmatrix} = (\det(A^{-1}))^{x_s} A \quad \text{for} \quad A \in U(n).$$

Moreover we get $K \simeq (SU(n-1) \times \{e\}) \circ T^{h+1}$ by $K_s/K \cong S^{2n-1}$. From Proposition 4.1, we see $h \leq 1$.

8.1. $h = 0$.

Assume $h = 0$, then the following lemma holds.

LEMMA 8.2. *If* $h = 0$, *then* $G = SU(n+1)$, $K_s \simeq S(U(n) \times U(1))$ *and* $K = S(U(n-1) \times U(1))$. *Moreover we have* $x_s = 0$.

PROOF. Because $h = 0$, we see $G = SU(n+1)$, $K_s \simeq S(U(n) \times U(1))$ and $K \simeq (SU(n-1) \times \{e\}) \circ T^1$. Put the slice representation $\sigma_s = c \circ \tau_{x_s}$ and $\sigma'_s = c \circ \tau_{-x_s}$ where c is a canonical injection $c : U(n) \to O(2n)$. Then $\sigma_s, \sigma'_s : S(U(n) \times U(1)) \to O(2n)$ are equivalent representations. So $\mathrm{Ker}(\sigma_s) \simeq \mathrm{Ker}(\sigma'_s)$. Since the canonical representation c is injective, $\mathrm{Ker}(\tau_{x_s}) \simeq \mathrm{Ker}(\tau_{-x_s})$. However if $x_s \neq 0$

$$\begin{aligned} \mathrm{Ker}(\tau_{x_s}) &= \left\{ \begin{pmatrix} A & 0 \\ 0 & \det(A^{-1}) \end{pmatrix} \middle| (\det(A^{-1}))^{x_s} A = I_n \right\} \\ &= \left\{ \begin{pmatrix} \alpha I_n & 0 \\ 0 & \alpha^{-n} \end{pmatrix} \middle| \alpha \in U(1), \alpha^{-nx_s+1} = 1 \right\}. \end{aligned}$$

This is a contradiction to $\mathrm{Ker}(\tau_{x_s}) \simeq \mathrm{Ker}(\tau_{-x_s})$. Hence $x_s = 0$. From $\tau_0^{-1}(U(n-1)) \simeq K$, we can put $K = S(U(n-1) \times U(1))$. □

From this lemma, the slice representation σ_s is unique. Since $N(K;G)/K$ is connected, the attaching map from X_1 to X_2 is unique up to equivalence by Lemma 4.3 (1.). Hence, in this case, $(SU(n+1), M)$ is unique. Such a pair will be constructed in Section 11.2.

8.2. $h = 1$.

Next we put $h = 1$. In this case the slice representation is

$$\sigma_s : S(U(n) \times U(1)) \times T^1 \overset{\rho_s}{\to} U(n) \overset{c}{\to} O(2n).$$

Consider the restricted representation of σ_s to $S(U(n) \times U(1))$. By using the same argument as in Lemma 8.2, we see $x_s = 0$. Hence the representation $\rho_s = \rho_s^{(m)}$ is

$$\rho_s^{(m)} : \left(\begin{pmatrix} A & 0 \\ 0 & \det(A^{-1}) \end{pmatrix}, z \right) \mapsto z^m A$$

for some integer m where $z \in T^1$. From Proposition 4.1, we have $m \neq 0$. Moreover we can take $m > 0$ because two slice representations $\sigma_s^{(m)} = c \circ \rho_s^{(m)}$ and $\sigma_s^{(-m)} = c \circ \rho_s^{(-m)}$ are equivalent representations.

Since $(\rho_s^{(m)})^{-1}(U(n-1)) = K$, we have

$$K = \left\{ \left(\begin{pmatrix} z^{-m} & 0 & 0 \\ 0 & X & 0 \\ 0 & 0 & z^m \det(X^{-1}) \end{pmatrix}, z \right) \middle| z \in T^1, X \in U(n-1) \right\}$$

Because $N(K,G)/K$ is connected, the attaching map is unique. Hence we get the unique pair $(SU(n+1) \times T^1, M^{(m)})$ for each $m \in \mathbb{N}$, because of Lemma 4.3 (1.). Therefore we get the following lemma.

LEMMA 8.3. *For each natural number m, the pair $(SU(n+1) \times T^1, M^{(m)})$, which has two singular orbits $(SU(n+1) \times T^1)/K_s$ and principal orbits $(SU(n+1) \times T^1)/K$, is unique up to equivalence.*

Let us construct such a pair $(SU(n+1) \times T^1, M^{(m)})$. Take $M^{(m)} = Q_{2n}$ and the $SU(n+1) \times T^1$-action on Q_{2n} by the representation $\sigma^{(m)} : SU(n+1) \times T^1 \to SO(2n+2)$ which is defined by

$$\sigma^{(m)} : (A, z) \mapsto c(z^m A).$$

Here $c : U(n+1) \to SO(2n+2)$ is a canonical representation and $A \in SU(n+1)$, $z \in T^1$. We can easily check this pair $(SU(n+1) \times T^1, M^{(m)})$ has orbits which are the same orbits in Lemma 8.3. However the following proposition holds.

PROPOSITION 8.1. *For all $m \in \mathbb{N}$, the pair $(SU(n+1) \times T^1, M^{(m)})$ is essentially isomorphic to $(U(n+1), Q_{2n})$ where $U(n+1)$ acts on Q_{2n} by canonical representation.*

PROOF. First we put a subgroup

$$\mathbb{Z}_{n+1} = \{(zI_{n+1}, z^{-1}) | z \in \mathbb{Z}_{n+1}\}$$

which is the center of $G = SU(n+1) \times T^1$ and the following holds

$$SU(n+1) \times_{\mathbb{Z}_{n+1}} T^1 \simeq U(n+1).$$

Next we consider a kernel of the G-action on $M^{(m)}$ where the kernel of G-action means $\cap_{x \in M^{(m)}} G_x$. Then we have

$$\begin{aligned} \cap_{x \in M^{(m)}} G_x &= \mathrm{Ker}(\sigma^{(m)}) \\ &= \{(X, z) | z^m X = I_{n+1}\} \\ &= \{(z^{-m} I_{n+1}, z) | z^{m(n+1)} = 1\}. \end{aligned}$$

Hence $\mathbb{Z}_m \subset \cap_{x \in M^{(m)}} G_x$. So we see $(G, M^{(m)})$ is essentially isomorphic to $(G, M^{(1)})$ for all $m \in \mathbb{N}$. Moreover we see $\mathbb{Z}_{n+1} = \cap_{x \in M^{(1)}} G_x$. Therefore the pair $(G, M^{(1)})$ is essentially isomorphic to $(U(n+1), Q_{2n})$. □

Hence this case is unique.

9. $P(G/K_1; t) = (1 + t^{k_2 - 1}) a(n)$, k_2 is odd: No.1, G/K_1 is decomposable.

In this case we have $K_1 = K_1^o$ because $k_2 > 2$ and Lemma 3.7. Because G/K_1 is decomposable, we can put $G = H_1 \times H_2 \times G''$ and $K_1 = H_{(1)} \times H_{(2)} \times G''$ where $H_1/H_{(1)} \sim S^{k_2-1}$, $H_2/H_{(2)} \sim P_n(\mathbb{C})$. Then $G/K_1 = H_1/H_{(1)} \times H_2/H_{(2)}$. So by Propositions 4.2 and 4.3,

$$\begin{aligned} (H_1, H_{(1)}) &= (Spin(k_2), Spin(k_2 - 1)) \text{ or} \\ &\quad (G_2, SU(3)) \ (k_2 = 7). \\ (H_2, H_{(2)}) &= (SU(n+1), S(U(n) \times U(1))) \text{ or} \\ &\quad (Spin(n+2), Spin(n) \circ T^1) \ (n \text{ is odd}) \text{ or} \\ &\quad (Sp(\frac{n+1}{2}), Sp(\frac{n-1}{2}) \times U(1)) \ (n \text{ is odd}) \text{ or} \\ &\quad (G_2, U(2)) \ (n = 5). \end{aligned}$$

9.1. Preliminary.

The goal of this section is to prove the following proposition.

PROPOSITION 9.1. *If M is a rational cohomology complex quadric, then $H_{(2)}$ acts transitively on K_1/K.*

In the beginning, we prepare the following lemmas.

LEMMA 9.1 (Theorem I' in **[MS43]**). *If $K \times H$ acts transitively on M, then K or H acts transitively on M.*

From Lemma 9.1, we have the following lemma.

LEMMA 9.2. *Let* H *be a subgroup of* $G = G_1 \times G_2$ *and* $p : G \to G_2$ *be a projection. Then the following two conditions are equivalent.*

(1) G_1 *acts transitively on* G/H.
(2) $p(H) = G_2$.

Next we prove the following technical lemma.

LEMMA 9.3. *Let* $V \subset G$ *be a subgroup such that*

$$\begin{aligned}&\pi^* : H^*(V\backslash G/K_s) \longrightarrow H^*(V\backslash G/K) \text{ is injective},\\ &p^* : H^*(V\backslash G/K_r) \longrightarrow H^*(G/K_r) \text{ is injective},\\ &V\backslash G/K_r \cong V\backslash G/K\end{aligned}$$

where $s + r = 3$ *and* $\pi : V\backslash G/K \to V\backslash G/K_s$ *and* $p : G/K_r \to V\backslash G/K_r$ *are projections. Then* $f^* : H^*(V\backslash M) \to H^*(M)$ *is injective where* $f : M \to V\backslash M$ *is a projection. If* M *is a rational cohomology complex quadric, then* $H^2(V\backslash G/K_s;\mathbb{Q}) = 0$.

PROOF. Assume $H^2(V\backslash G/K_s;\mathbb{Q}) \neq 0$. Now $V\backslash M$ is a mapping cylinder of

$$\pi : V\backslash G/K_r \cong V\backslash G/K \to V\backslash G/K_s = G/K_s.$$

Consider a diagram

$$\begin{array}{ccccc} G/K_s & \xrightarrow{i_s} & M & \xleftarrow{i_r} & G/K_r \\ \downarrow & & f\downarrow & & p\downarrow \\ V\backslash G/K_s & \xrightarrow{j_s} & V\backslash M & \xleftarrow{j_r} & V\backslash G/K_r \\ \downarrow & & & & \downarrow \\ V\backslash G/K_s & & \xleftarrow{\pi} & & V\backslash G/K_r. \end{array}$$

where i_s, i_r, j_s, j_r are natural inclusions. Now j_s is a homotopy equivalece. This diagram induces a commutative diagram

$$\begin{array}{ccccc} H^*(G/K_s) & \xleftarrow{i_s^*} & H^*(M) & \xrightarrow{i_r^*} & H^*(G/K_r) \\ \uparrow & & f^*\uparrow & & p^*\uparrow \\ H^*(V\backslash G/K_s) & \xleftarrow{j_s^*} & H^*(V\backslash M) & \xrightarrow{j_r^*} & H^*(V\backslash G/K_r) \\ \uparrow & & & & \uparrow \\ H^*(V\backslash G/K_s) & & \xrightarrow{\pi^*} & & H^*(V\backslash G/K_r). \end{array}$$

From the assumptions, f^* is an injection.

Since we assume $H^2(V\backslash G/K_s) \neq 0$, we can take $c \in H^2(V\backslash M) \simeq H^2(V\backslash G/K_s)$. Hence $f^*(c^{2n}) = f^*(c)^{2n} \neq 0$ because $H^*(M) \simeq H^*(Q_{2n})$ where $n \geq 2$. Therefore $0 \neq c^{2n} \in H^{4n}(V\backslash G/K_s)$. This contradicts $\dim(V\backslash G/K_s) \leq \dim(G/K_s) \leq \dim(M) - 2 = 4n - 2$. □

Before we prove Proposition 9.1, we show the following lemma.

LEMMA 9.4. *If* M *is a rational cohomology complex quadric, then* $H_{(1)} \times H_{(2)}$ *acts transitively on* K_1/K.

PROOF. If $H_{(1)} \times H_{(2)}$ acts non-transitively on K_1/K then G'' acts transitively on K_1/K by Lemma 9.1. Hence $p(K) = H_{(1)} \times H_{(2)} = p(K_1)$ by Lemma 9.2 where $p : G \to H_1 \times H_2$ is the natural projection. Put $H_1 \times H_2 = G'$, $H_{(1)} \times H_{(2)} = K_1'$ and $p(K_2) = K_2'$. Then K_2'/K_1' is connected, because the induced map $p' : S^{k_2-1} \cong K_2/K \to K_2'/K_1'$ from $p : G \to H_1 \times H_2$ is continuous. Hence we see K_2' is connected from the fibre bundle $K_1' \to K_2' \to K_2'/K_1'$ and the connectedness of K_1'. Now $K_1' = p(K) \subset p(K_2) = K_2' \subset G'$. Therefore rank $K_1' =$ rank $G' =$ rank K_2'. So we get

$$P(G/K_1; t) = (1 + t^{k_2-1})a(n) = P(G'/K_1'; t) = P(K_2'/K_1'; t)P(G'/K_2'; t) \tag{9.1}$$

by the fibration $K_2'/K_1' \to G'/K_1' \to G'/K_2'$.

Since $K_2/K \cong K_2^o/K^o$ is an even dimensional sphere S^{k_2-1}, we see rank $K_2^o =$ rank K^o. So rank$(K_1 \cap K_2^o) =$ rank K^o. Hence $H^{odd}((K_1 \cap K_2^o)/K) = H^{odd}(K_2'/K_1') = 0$. Because of the fibration $(K_1 \cap K_2^o)/K^o \to K_2^o/K^o \xrightarrow{p''} K_2'/K_1'$ where p'' is the induced map from p and the simply connectedness of $K_2^o/K^o \cong S^{k_2-1}$, we see K_2'/K_1' is simply connected. Hence we have

$$P(K_2^o/K^o; t) = 1 + t^{k_2-1} = P(K_2'/K_1'; t)P((K_1 \cap K_2^o)/K^o; t). \tag{9.2}$$

From equations (9.1) and (9.2), we see $H^2(G'/K_2') = H^2(G''\backslash G/K_2) \neq 0$. Now we have $G''\backslash G/K = G''\backslash G/K_1 = G/K_1$. Moreover we see $\pi^* : H^*(G''\backslash G/K) = H^*(G'/K_1') \to H^*(G'/K_2')$ is injective by the fibration $K_2'/K_1' \to G'/K_1' \xrightarrow{\pi} G'/K_2'$. This contradicts Lemma 9.3. Therefore $H_{(1)} \times H_{(2)}$ acts transitively on K_1/K. □

To show Proposition 9.1.1, we prepare some notations.

Let $p_t : G \to H_t$, $p_t' : G \to H_t \times G''$ be the natural projection, and let $h_t : H_t \to G$, $h_t' : H_t \times G'' \to G$ be the natural inclusion. Put

$$L_{st} = p_t(K_s),\ L_t = p_t(K),\ L_{st}' = p_t'(K_s),\ L_t' = p_t'(K),$$
$$N_{st} = h_t^{-1}(K_s),\ N_t = h_t^{-1}(K),\ N_{st}' = (h_t')^{-1}(K_s),\ N_t' = (h_t')^{-1}(K).$$

Then $N_{st} \triangleleft L_{st}$, $N_t \triangleleft L_t$, $N_{st}' \triangleleft L_{st}'$ and $N_t' \triangleleft L_t'$ where $A \triangleleft B$ means a group A is a normal subgroup of B. In particular $L_{1t} = N_{1t} = H_{(t)}$ and $L_{1t}' = N_{1t}' = H_{(t)} \times G''$ by the equality $K_1 = H_{(1)} \times H_{(2)} \times G''$.

Let us prove Proposition 9.1.

Proof of Proposition 9.1. If $H_{(2)}$ does not act transitively on K_1/K, then $H_{(1)}$ acts transitively on K_1/K by Lemma 9.1 and 9.4. Hence $L_2 = H_{(2)} = L_{12}$ by Lemma 9.2. Then f^* is an injective homomorphism from Lemma 9.3, where $f^* : H^*((H_1 \times G'')\backslash M) \to H^*(M)$ is an induced homomorphism from the natural projection $f : M \to (H_1 \times G'')\backslash M$.

Now $L_{22}/H_{(2)}$ is connected because the induced map $p_2' : K_2/K \to L_{22}/H_{(2)}$ is continuous. Hence L_{22} is connected by the fibration $H_{(2)} \to L_{22} \to L_{22}/H_{(2)}$.

Since $L_2 = H_{(2)} \subset L_{22} \subset H_2$, we have rank $H_{(2)} =$ rank $L_{22} =$ rank H_2 and $H^{odd}(L_{22}/H_{(2)}) = H^{odd}(H_2/L_{22}) = 0$. Because L_{22} is connected, H_2/L_{22} is simply connected. Hence we have an isomorphism $H^*(P_n(\mathbb{C})) \simeq H^*(H_2/H_{(2)}) \simeq H^*(L_{22}/H_{(2)}) \otimes H^*(H_2/L_{22})$ from the fibration $L_{22}/H_{(2)} \to H_2/H_{(2)} \xrightarrow{\pi} H_2/L_{22}$.

Assume we can take $a \in H^{2m}((H_1 \times G")\backslash M) \simeq H^{2m}(H_2/L_{22}) \neq 0$ for some $0 \neq m \leq n$.

If $m \neq n$, then we can put $f^*(a) = c^m$ for $0 < m < n$ where $c \in H^2(M)$ is a generator. However there is an l such that $n < l + m < 2n$ and $f^*(a^l) = c^{l+m} \neq 0$. This contradicts $\dim H_2/L_{22} \leq 2n$.

Hence $m = n$. Then we have $H^*((H_1 \times G")\backslash M) \simeq H^*(H_2/L_{22}) \simeq H^*(S^{2n})$ and we also have $\dim H_2/L_{22} = 2n$. Therefore $H_{(2)} = L_{22}$ from the fibration $L_{22}/H_{(2)} \to H_2/H_{(2)} \to H_2/L_{22}$. Hence we have $H_2/H_{(2)} \cong H_2/L_{22}$. This contradicts $H_2/H_{(2)} \sim P_n(\mathbb{C})$.

Therefore $H^{2m}(H_2/L_{22}) = 0$ for $m \neq 0$. Hence we see $L_{22} = H_2$. Therfore $\dim(L_{22}/L_2) = 2n$. From the fibration $(K_1 \cap K_2^o)/K^o \to K_2^o/K^o \cong S^{k_2-1} \to L_{22}/L_2$, we see $k_2 - 1 \geq 2n$. This contradicts $k_1 + k_2 = 2n + 1$ and $k_1 \geq 2$.

□

9.2. Candidates for (G, K_1).

The goal of this section is to prove $k_1 = 2n-2$, $k_2 = 3$ and the pair (G, K_1) is one of the following

$$(G, K_1) = \left(Sp(1) \times Sp(\frac{n+1}{2}) \times G", T^1 \times Sp(\frac{n-1}{2}) \times U(1) \times G"\right)$$

or $n = 9$,

$$(G, K_1) = \left(Sp(1) \times Spin(11) \times G", T^1 \times Spin(9) \circ T^1 \times G"\right).$$

From Proposition 9.1, $H_{(2)}$ acts transitively on K_1/K. Then $H_{(2)}/N_2 \cong K_1/K \cong S^{k_1-1}$. Since $\{pt\} = H_{(2)}\backslash K_1/K \cong (H_{(1)} \times G")/L_1'$, we have the following lemma.

LEMMA 9.5. $L_1' = H_{(1)} \times G"$ *and* $L_1 = H_{(1)} = L_{11}$.

Moreover we can easily show the natural homomorphisms $K/(N_1' \times N_2) \to L_1'/N_1'$ and $K/(N_1' \times N_2) \to L_2/N_2$ are isomorphic. Hence $L_1'/N_1' \cong L_2/N_2$. Since L_2/N_2 acts freely on $H_{(2)}/N_2 \cong S^{k_1-1}$, we have the following lemma by **[Bre72]** 6.2. Theoreom in Chapter IV.

LEMMA 9.6. $\dim L_1'/N_1' = \dim L_2/N_2 \leq 3$.

Let us prove the following lemma.

LEMMA 9.7. *If M is a rational cohomology complex quadric, then* $L_{21} = H_1$.

PROOF. First we have L_{21} is connected because K_2/K is connected, $H_{(1)} = L_1$ is connected and the map $\bar{p_1} : K_2/K \to L_{21}/L_1 = L_{21}/H_{(1)}$ which induced by $p_1 : G \to H_1$ is continuous. Consider the fibration

$$L_{21}/H_{(1)} \longrightarrow H_1/H_{(1)} \longrightarrow H_1/L_{21}.$$

Then rank $H_{(1)}$ = rank L_{21} = rank H_1 by $H_{(1)} = L_1 \subset L_{21} \subset H_1$. So we have $H^*(H_1/H_{(1)}) \simeq H^*(S^{k_2-1}) \simeq H^*(H_1/L_{21}) \otimes H^*(L_{21}/H_{(1)})$. Therefore we see $L_{21} = H_{(1)}$ or H_1.

If we put $L_{21} = H_{(1)} = L_1$, then $(H_2 \times G")\backslash M \cong [0,1] \times H_1/H_{(1)}$. Now we consider the following commutative diagram

$$\begin{array}{ccc} H_1/H_{(1)} \times H_2/H_{(2)} \cong G/K_1 & \xrightarrow{i_1} & M \\ \downarrow q_1 & & \downarrow f \\ H_1/H_{(1)} \cong (H_2 \times G")\backslash G/K_1 & \xrightarrow{j_1} & (H_2 \times G")\backslash M. \end{array}$$

Here j_1 is a homotopy equivalence. Hence $q_1^* \circ j_1^*$ is injective. Therefore $f^* : H^*((H_2 \times G")\backslash M) \simeq H^*(S^{k_2-1}) \to H^*(M) \simeq H^*(Q_{2n})$ is injective. Hence $k_2 \geq 2n+1$. But this contradicts $k_1 + k_2 = 2n+1$ and $k_1 \geq 2$. Hence we see $L_{21} = H_1$. □

Hence we can prove the following lemma.

LEMMA 9.8. *If* M *is a rational cohomology complex quadric, then* $N_1 \neq H_{(1)}$.

PROOF. Suppose $N_1 = H_{(1)}$, then $H_{(1)} \subset N_{21} \triangleleft L_{21} = H_1$ by Lemma 9.7. Since H_1 is a simple Lie group, we see $N_{21} = H_1$. Hence we can put $K_2 = H_1 \times X$ and $K = H_{(1)} \times X$ where $X < H_2 \times G"$. Therefore $H_1\backslash M$ is a mapping cylinder of $H_1\backslash G/K = (H_2 \times G")/X \to H_1\backslash G/K_1 = H_2/H_{(2)}$. From the following commutative diagram

$$\begin{array}{ccc} H_1/H_{(1)} \times H_2/H_{(2)} \cong G/K_1 & \longrightarrow & M \\ \downarrow q_2 & & \downarrow p \\ H_2/H_{(2)} \cong H_1\backslash G/K_1 & \xrightarrow{i} & H_1\backslash M \end{array}$$

where i is a homotopy equivalent map, we have the following diagram

$$\begin{array}{ccc} H^*(H_1\backslash M) & \xrightarrow{i^*} & H^*(H_2/H_{(2)}) \\ \downarrow p^* & & \downarrow q_2^* \\ H^*(M) & \longrightarrow & H^*(H_1/H_{(1)}) \otimes H^*(H_2/H_{(2)}). \end{array}$$

Hence p^* is an injection. This contradicts $H^*(M) \simeq H^*(Q_{2n})$, $H^*(H_1\backslash M) \simeq H^*(P_n(\mathbb{C}))$. □

Next we show the following proposition.

PROPOSITION 9.2. $k_1 = 2n-2$, $k_2 = 3$ *and* $(H_1, H_{(1)}) = (Sp(1), T^1)$.

PROOF. Let us recall,

$$(H_1, H_{(1)}) = (Spin(k_2), Spin(k_2-1))) \text{ or } (G_2, SU(3)) : k_2 = 7.$$

Because $N_1' \subset N_1 \times G"$, we have $3 \geq \dim L_1'/N_1' \geq \dim H_{(1)} - \dim N_1$ by Lemma 9.5 and Lemma 9.6. So we have $\dim N_1 \neq 0$ if $k_2 \neq 3$ because k_2 is odd.

If $k_2 > 6$, then $H_{(1)}$ is a simple Lie group. Hence $N_1 = H_{(1)}$ from $N_1 \triangleleft H_{(1)} = L_1$ and $\dim N_1 \neq 0$. This contradicts Lemma 9.8. Hence $k_2 = 3$ or 5.

If $k_2 = 5$, then $(H_1, H_{(1)}) = (Sp(2), Sp(1) \times Sp(1))$. Then $\dim N_{21} \geq \dim N_1 > 0$ from $\dim N_1 \neq 0$. Now H_1 is a simple Lie group and $N_{21} \lhd L_{21} = H_1$ from Lemma 9.7. Hence $N_{21} = H_1$. This implies $K_2 = H_1 \times X$ where X is a subgroup of $H_2 \times G''$. Because $K_1 = H_{(1)} \times H_{(2)} \times G''$, we see $K \subset K_1 \cap K_2 = H_{(1)} \times (X \cap (H_{(2)} \times G'')) \subset K_2$. Consider the fibration

$$(H_{(1)} \times (X \cap (H_{(2)} \times G'')))/K \to K_2/K \to K_2/(H_{(1)} \times (X \cap (H_{(2)} \times G''))).$$

Because $K_2/K \simeq S^{k_2-1}$, $K_2 = H_1 \times X$ and $\dim H_1/H_{(1)} = k_2 - 1$, we have $\dim X \cap (H_{(2)} \times G'') = \dim X$ and $K = H_{(1)} \times Y$ where $\dim X/Y = 0$. Hence $N_1 = H_{(1)}$. This contradicts Lemma 9.8. Consequently $k_2 = 3$. Hence $k_1 = 2n - 2$ and $(H_1, H_{(1)}) = (Sp(1), T^1)$. □

So $H_{(2)}$ acts transitively on S^{2n-3} from Proposition 9.1 and 9.2. Hence by Proposition 4.3 and **[HH65]** Section I, we have the following two cases where $k_1 = 2n - 2$, $k_2 = 3$,

$$\begin{aligned} G &= Sp(1) \times Sp(\frac{n+1}{2}) \times G'', \\ K_1 &= T^1 \times Sp(\frac{n-1}{2}) \times U(1) \times G'', \end{aligned}$$

and $n = 9$,

$$\begin{aligned} G &= Sp(1) \times Spin(11) \times G'', \\ K_1 &= T^1 \times Spin(9) \circ T^1 \times G''. \end{aligned}$$

In these cases $K_2 = K_2^o$ because n is an odd number and Lemma 3.7 and $K = K^o$ because $K_2/K \cong S^2$ is simply connected.

In next two sections we will discuss slice reprepsentations and attaching maps in each case.

9.3. $G = Sp(1) \times Sp(\frac{n+1}{2}) \times G''$**.**

If $G = Sp(1) \times Sp(\frac{n+1}{2}) \times G''$, then $K_1 = T^1 \times Sp(\frac{n-1}{2}) \times U(1) \times G''$. Now $Sp(\frac{n-1}{2}) \times U(1)$ acts transitively on $K_1/K \cong S^{2n-3}$ because of Proposition 9.1. So we can assume the restricted slice representation $\sigma_1|_{Sp(\frac{n-1}{2})}$ is a natural inclusion to $SO(2n-2)$. Hence $\sigma_1(T^1 \times \{e\} \times U(1) \times G'') \subset C(\sigma_1(Sp(\frac{n-1}{2})); SO(2n-2))$ where $C(K; G) = \{g \in G \mid gk = kg$ for all $k \in K\}$. We put the natural inclusion $\sigma_1|_{Sp(\frac{n-1}{2})} = i : Sp(\frac{n-1}{2}) \to SO(2n-2)$ as follows:

$$i(X + Y\mathbf{i} + Z\mathbf{j} + W\mathbf{k}) = \begin{pmatrix} X & -Y & Z & -W \\ Y & X & -W & -Z \\ -Z & W & X & -Y \\ W & Z & Y & X \end{pmatrix}. \tag{9.3}$$

Then

$$(9.4)\quad C(\sigma_1(Sp(\frac{n-1}{2})); SO(2n-2)) = \left\{ \begin{pmatrix} h_1 I_m & h_3 I_m & -h_2 I_m & h_4 I_m \\ -h_3 I_m & h_1 I_m & -h_4 I_m & -h_2 I_m \\ h_2 I_m & h_4 I_m & h_1 I_m & -h_3 I_m \\ -h_4 I_m & h_2 I_m & h_3 I_m & h_1 I_m \end{pmatrix} \right\}$$

where $h_1^2 + h_2^2 + h_3^2 + h_4^2 = 1$ and $m = \frac{n-1}{2}$. Hence we have

$$G" \subset Sp(1) \times T^h$$

where $h \leq 1$ by Proposition 4.1 and we can assume the slice representation as $\sigma_1 : K_1 \to Sp(1) \times Sp(\frac{n-1}{2})$ such that $\sigma_1|_{Sp(\frac{n-1}{2})} : Sp(\frac{n-1}{2}) \to \{e\} \times Sp(\frac{n-1}{2})$ is isomorphic and $\sigma_1(T^1 \times U(1) \times G") \subset Sp(1) \times \{e\}$ by (9.3) and (9.4).

Moreover we have the following lemma.

LEMMA 9.9. $G" = \{e\}$ *or* T^1 *and we can assume the slice representation as* $\sigma_1 : K_1 \to U(1) \times Sp(\frac{n-1}{2})$.

PROOF. Suppose $G" = Sp(1) \times T^h$. Then the restricted slice representation $\sigma_1|_{T^1 \times U(1) \times G"}$ is $r : T^1 \times \{e\} \times U(1) \times G" \to Sp(1)$. Because $Sp(1)$ is a simple Lie group, $r|_{Sp(1)}$ is isomorphic or trivial representation. If $r|_{Sp(1)}$ is isomorphic, then we have $Ker(r) = T^1 \times \{e\} \times U(1) \times \{e\} \times T^h$ because $C(r(Sp(1)); Sp(1)) = \{1, -1\}$. Since $Ker(r) \subset K$, we have $H_{(1)} = T^1 \subset K$. This contradicts the fact $H_{(1)} = T^1 \not\subset K$ from Lemma 9.8. So we see $r|_{Sp(1)}$ is trivial and $Sp(1) \subset Ker(r) \subset K$. But this contradicts Proposition 4.1. Hence $G" = T^h$ for $h \leq 1$. Moreover we see easily the slice representation $\sigma_1 : K_1 \to U(1) \times Sp(\frac{n-1}{2})$. So we get this lemma. □

Assume $h = 1$. Then we can put the slice representation $\sigma_1 : K_1 = T^1 \times Sp(\frac{n-1}{2}) \times U(1) \times T^1 \to U(1) \times Sp(\frac{n-1}{2})$ as follows;

$$\sigma_1 \left(x, \begin{pmatrix} A & 0 \\ 0 & y \end{pmatrix}, z \right) \mapsto (x^l y^m z^n, A)$$

where l, m, n are in $\mathbb{Z}$. Now we can assume the $U(1) \times Sp(\frac{n-1}{2})$-action ρ (via σ_1) on $S^{2n-3} \subset \mathbb{H}^{\frac{n-1}{2}}$ as $\rho((t, X), \mathbf{h}) = X\mathbf{h}\bar{t}$. Hence we have

$$K = \left\{ \left(x, \begin{pmatrix} x^l y^m z^n & 0 & 0 \\ 0 & B & 0 \\ 0 & 0 & y \end{pmatrix}, z \right) \middle| B \in Sp(\frac{n-3}{2}), x, y, z \in T^1 \right\}$$

where $l \neq 0$ by Lemma 9.8.

Since $K_2/K \cong S^2$, $l \neq 0$ and $L_{21} = Sp(1)$ by Lemma 9.7, we have

$$K_2 = \left\{ \left(h, \begin{pmatrix} h & 0 & 0 \\ 0 & B & 0 \\ 0 & 0 & y \end{pmatrix}, z \right) \middle| B \in Sp(\frac{n-3}{2}), h \in Sp(1), y, z \in T^1 \right\}.$$

Therefore we have $G'' = T^1 \subset \mathrm{Ker}(\sigma_2) \subset K$ by the slice representation $\sigma_2 : K_2 \to SO(3)$. This contradicts Proposition 4.1. Hence we have $G'' = \{e\}$ that is $h = 0$.

Moreover, from the same argument, we have

$$\begin{aligned}
K_1 &= T^1 \times Sp(\frac{n-1}{2}) \times U(1) \\
K_2 &= \left\{ \left(h, \begin{pmatrix} h & 0 & 0 \\ 0 & B & 0 \\ 0 & 0 & y \end{pmatrix} \right) \middle| B \in Sp(\frac{n-3}{2}), h \in Sp(1), y \in T^1 \right\}, \\
K &= \left\{ \left(x, \begin{pmatrix} x & 0 & 0 \\ 0 & B & 0 \\ 0 & 0 & y \end{pmatrix} \right) \middle| B \in Sp(\frac{n-3}{2}), x, y \in T^1 \right\}
\end{aligned}$$

and

$$\sigma_1 \left(x, \begin{pmatrix} A & 0 \\ 0 & y \end{pmatrix} \right) \mapsto (x, A)$$

up to equivalence. We also see the slice representation $\sigma_2 : K_2 \to SO(3)$ is unique up to equivalence.

Next we see

$$N(K; G)/K \simeq (N(T^1; Sp(1))/T^1) \times (N(U(1); Sp(1))/U(1)).$$

If we denote by a the generator of $N(T^1; Sp(1))/T^1 \simeq \mathbb{Z}_2$, then $xa = a\bar{x}$ for all $x \in T^1$. Hence we can consider the following diagram

$$\begin{array}{ccc}
G \times_{K_2} K_2/K & \xrightarrow{f} & G/K \\
\downarrow 1 \times R_\alpha & & \downarrow R_\alpha \\
G \times_{K_2} K_2/K & \xrightarrow{f} & G/K.
\end{array}$$

Here $f([g, kK]) = gkK$ and $\alpha = (a, e, e) \in N(K; K_2)$. We have $gkK\alpha = gk\alpha K$ for all $g \in G$ and $k \in K_2$. So this diagram is commutative. In this case R_α is the antipodal involution on $K_2/K \cong S^2$. Hence R_α is extendable to a K_2-equivariant diffeomorphism on D^3. Hence $M(R_\alpha) \cong M(id)$ from Lemma 4.3 (3.). Since $N(U(1); Sp(1))/U(1) \simeq \mathbb{Z}_2$, there are just two manifolds up to essential isomorphism. Hence we get the following proposition.

PROPOSITION 9.3. *Let* (G, M) *be a* G*-manifold which has codimension one orbit* G/K *and two singular orbit* G/K_1 *and* G/K_2 *where* $G = Sp(1) \times Sp(\frac{n+1}{2})$, $K_1 = T^1 \times Sp(\frac{n-1}{2}) \times U(1)$,

$$\begin{aligned}
K_2 &= \left\{ \left(h, \begin{pmatrix} h & 0 & 0 \\ 0 & B & 0 \\ 0 & 0 & y \end{pmatrix} \right) \middle| B \in Sp(\frac{n-3}{2}), h \in Sp(1), y \in T^1 \right\} \text{ and} \\
K &= \left\{ \left(x, \begin{pmatrix} x & 0 & 0 \\ 0 & B & 0 \\ 0 & 0 & y \end{pmatrix} \right) \middle| B \in Sp(\frac{n-3}{2}), x, y \in T^1 \right\}.
\end{aligned}$$

Then there are just two such (G, M) *up to essential isomorphism which are* $M = Q_{2n}$ *and* $M = (Sp(1) \times Sp(k+1)) \times_{Sp(1)\times Sp(k)\times U(1)} S^{4k+2}$ *where* $k = \frac{n-1}{2}$.

PROOF. By the above argument, this case has just two types up to essential isomorphism. If $M = Q_{2n}$, then this case will be realized in Section 11.3. If $M = (Sp(1) \times Sp(k+1)) \times_{Sp(1)\times Sp(k)\times U(1)} S^{4k+2}$ such that $k = \frac{n-1}{2}$ and $S^{4k+2} \subset \mathbb{R}^3 \times \mathbb{H}^k$ has the trivial $U(1)$-action, the canonical $Sp(1)$-action on $\mathbb{R}^3$ and the canonical $Sp(1) \times Sp(k)$-action on $\mathbb{H}^k$. Then this manifold has the $Sp(1) \times Sp(k+1)$-action. We can easily check this manifold satisfies the assumption of this proposition. □

$M = (Sp(1) \times Sp(k+1)) \times_{Sp(1)\times Sp(k)\times U(1)} S^{4k+2}$ is the fibre bundle over $Sp(k+1)/U(1) \times Sp(k) \cong P_{2k+1}(\mathbb{C})$ with the fibre S^{4k+2}. We see easily check $H^{odd}(P_{2k+1}(\mathbb{C})) = H^{odd}(S^{4k+2}) = 0$ and $P_{2k+1}(\mathbb{C})$ is simply connected. Hence $p^* : H^*(P_{2k+1}(\mathbb{C})) \to H^*(M)$ is injective where $p : M \to P_{2k+1}(\mathbb{C})$ is a projection. Hence the $2k+2$ times cup product of $c \in H^2(M)$ is vanishing in $H^{4k+4}(M)$. Hence this is not a rational cohomology complex quadric. So this case is unique up to essential isomorphism and such (G, M) will be constructed in Section 11.3.

9.4. $G = Sp(1) \times Spin(11) \times G''$.

If $G = Sp(1) \times Spin(11) \times G''$, then we have $K_1 = T^1 \times Spin(9) \circ T^1 \times G''$ and $G'' = \{e\}$ or T^1. Put the slice representation $\sigma_1 : K_1 \to O(16)$. Then the restricted representation $\sigma_1|_{Spin(9)}$ is the spin representation to $SO(16)$ and we can easily show $C(\sigma_1(Spin(9)); SO(16))$ is a finite group. So we have $\sigma_1(T^1 \times \{e\}) = \{I_{16}\}$ because $T^1 \times \{e\} \subset C(Spin(9); K_1)$, where $e \in Spin(9) \circ T^1 \times G''$ and $I_{16} \in O(16)$ are identity elements. Therefore we see $K \supset Ker(\sigma_1) \supset T^1 \times \{e\}$. So $N_1 = h_1^{-1}(K) = T^1 = H_{(1)}$, recall h_1 denotes the natural inclusion $H_1 \to G$. This contradicts Lemma 9.8. Hece this case does not occur.

10. $P(G/K_1; t) = (1 + t^{k_2-1})a(n)$, k_2 is odd: No.2, G/K_1 is indecomposable.

In this case $K_1 = K_1^o$ by $k_2 > 2$ and Lemma 3.7. Because G/K_1 is indecomposable, we can put $G = G' \times G''$ and $K_1 = K_1' \times G''$ where G' is a simple Lie group and G'' is a direct product of some simple Lie groups and a toral group. By Proposition 4.4, (G', K_1')-pair which satisfies

$$P(G/K_1; t) = P(G'/K_1'; t) = (1 + t^{2a})(1 + t^2 + \cdots + t^{2b})$$

where $2a = k_2 - 1$ and $b = n$ is one of the following thirteen pairs

$$\begin{aligned}
&(SO(2n+2), SO(2n) \times SO(2)), a = b = n,\\
&(SO(k_2+2), SO(k_2-1) \times SO(2)), a = (k_2-1)/2, b = k_2,\\
&(SO(7), U(3)), a = b = 3,\\
&(SO(9), U(4)), a = 3, b = 7,\\
&(SU(3), T^2), a = 1, b = 2,\\
&(SO(10), U(5)), a = 3, b = 7,\\
&(SU(5), S(U(2) \times U(3))), a = 2, b = 4,\\
&(Sp(3), Sp(1) \times Sp(1) \times U(1)), a = 2, b = 5,\\
&(Sp(3), U(3)), a = b = 3,\\
&(Sp(4), U(4)), a = 3, b = 7,\\
&(G_2, T^2), a = 1, b = 5,\\
&(F_4, Spin(7) \circ T^1), a = 4, b = 11,\\
&(F_4, Sp(3) \circ T^1), a = 4, b = 11.
\end{aligned}$$

In the beginning, we will find the candidates for (G', K_1').

10.1. Candidates for (G', K_1').

The goal of this section is to prove the pair (G', K_1') is one of the following

$$\begin{aligned}
&(Spin(9), Spin(6) \circ T^1)\ (k_1 = 8, k_2 = n = 7) \text{ or}\\
&(SU(3), T^2)\ (k_1 = 2, k_2 = 3, n = 2).
\end{aligned}$$

Now $k_1 \geq 2$ and $k_1 + k_2 = 2n + 1$. So we can easily see the following three cases do not satisfy $k_1 \geq 2$. (Recall $a = \frac{k_2-1}{2}$ and $b = n$.)

$$\begin{aligned}
&(SO(2n+2), SO(2n) \times SO(2)), a = b = n,\\
&(SO(7), U(3)), a = b = 3,\\
&(Sp(3), U(3)), a = b = 3.
\end{aligned}$$

We can show the following proposition similarly to Proposition 9.1.

PROPOSITION 10.1. K_1' *acts transitively on* $K_1/K \cong S^{k_1-1}$.

Hence we see the following six cases contradict Proposition 10.1 by the paper **[HH65]** Section I.

$$(SO(k_2+2), SO(k_2-1)\times SO(2)), a=(k_2-1)/2, b=k_2,$$
$$(SO(10), U(5)), a=3, b=7,$$
$$(Sp(3), Sp(1)\times Sp(1)\times U(1)), a=2, b=5,$$
$$(G_2, T^2), a=1, b=5,$$
$$(F_4, Spin(7)\circ T^1), a=4, b=11,$$
$$(F_4, Sp(3)\circ T^1), a=4, b=11.$$

Therefore in this case we have that

$$\begin{aligned}(G', K_1') &= (Spin(9), Spin(6)\circ T^1)\ (k_1=8, k_2=n=7) \text{ or}\\ &\quad (SU(3), T^2)\ (k_1=2, k_2=3, n=2) \text{ or}\\ &\quad (SU(5), S(U(3)\times U(2)))\ (k_1=4, k_2=5, n=4) \text{ or}\\ &\quad (Sp(4), U(4))\ (k_1=8, k_2=n=7).\end{aligned}$$

If $(G', K_1') = (SU(5), S(U(3)\times U(2)))$, then $k_1=4$. Hence $K_1/K \cong S^3$. Since $U(2)$ $(\subset K_1')$ acts transitively on K_1/K by Proposition 10.1, we can assume the slice reprepsentation as $\sigma_1 : K_1 \to U(2)$. Therefore we see $G" = T^h$ $(h\le 1)$ and $K \simeq S(U(3)\times\{e\})\circ T^{1+h}$ by Proposition 4.1 and Proposition 10.1. In particular we see $K_2 \supset K \supset SU(3)$. Since $K_2/K \cong S^4$, $(K_2, K) = (A\circ N, B\circ N)$ where $(A, B)\approx (SO(5), SO(4))$ by Proposition 4.2. Now we easily see $N \supset SU(3)$. So $K_2 \supset A\circ SU(3)$. That is $\dim K_2 \ge \dim(A\circ SU(3)) = 18$. However we have $\dim(K_2) = 13$ or 14 by $\dim(K) = \dim(S(U(3)\times\{e\})\circ T^{1+h}) = 9+h$ $(h\le 1)$ and $K_2/K \cong S^4$. This is a contradiction. Hence this case does not occur.

If $(G', K_1') = (Sp(4), U(4))$, then $k_1 = 8$ and $K_1/K \cong S^7$. From Proposition 10.1, we can assume the slice reprepsentation as $\sigma_1 : K_1 \to U(4)$. So $G" = \{e\}$ or T^1 by Proposition 4.1. Since $K_2/K \cong S^6$ and $K_1 = U(4)$ or $U(4)\times T^1$, we have $(K_2, K)\approx (G_2\circ T^1, SU(3)\circ T^1)$ or $(G_2\circ T^2, SU(3)\circ T^2)$ by Proposition 4.2. Therefore we get $Sp(4)\supset G_2$. However the following proposition holds.

PROPOSITION 10.2. $Sp(4) \not\supset G_2$.

PROOF. Assume $Sp(4)\supset G_2$. Let V be the $Sp(4)$-$\mathbb{C}$ irreducible 8-dimensional representation space (complex dimensional). Then we can consider $Sp(4)$ acts effectively on V by the natural representation $\rho : Sp(4)\to U(8)$. Since $Sp(4)\supset G_2$ and $\mathrm{Ker}(\rho) = \{e\}$, we see the restricted reprepsentation to G_2 $\rho|_{G_2}$ is not trivial. Because the least dimension of non-trivial complex representation of G_2 is 7 and V is an 8-dimensional space, there is an irreducible decomposition $V = V^7 \oplus W$ where V^7 is a complex seven dimensional G_2-space which has a representation $\rho|_{G_2}$ and W is a complex one dimensional space which has trivial G_2-action. Then V has the structure map $J : V\to V$ such that J is a $Sp(4)$ map,

$J^2(v) = -v$ and $J(zv) = \bar{z}J(v)$ for $z \in \mathbb{C}$ and $v \in V$ (see **[Ada69]** 3.2). Moreover $J(w) \in W$ for $w \in W$ because J is a $G_2(\subset Sp(4))$ map. However W is a complex one dimensional space, so this contradicts W does not have such map. Therefore we see $Sp(4) \not\supset G_2$. □

Hence the following two cases remain.

10.2. $(G', K_1') = (Spin(9), Spin(6) \circ T^1)$**.**
If $(G', K_1') = (Spin(9), Spin(6) \circ T^1)$, then $k_1 = 8$. So $K_1/K \cong S^7$, hence $G'' = T^h$ $(h \leq 1)$ from Proposition 4.1 and Proposition 10.1.

Assume $h = 1$. Since $K_2/K \cong S^6$, we see $(K_2, K) = (G_2 \circ T^2, SU(3) \circ T^2)$. Consider the slice representation $\sigma_2 : G_2 \circ T^2 \to SO(7)$. Because K_2 acts transitively on $K_2/K \cong S^6$, the restricted representation $\sigma_2|_{G_2}$ is a natural inclusion. So $C(\sigma_2(G_2); SO(7)) = \{e\}$ where $C(K; G) = \{g \in G | \, gk = kg \text{ for all } k \in K\}$. Therefore $G'' \subset Ker(\sigma_2) = T^2 \subset K$. Now $G'' = T^1$ is a normal subgroup of G. This contradicts Proposition 4.1. Hence $h = 0$.

We get $G'' = \{e\}$ and $(G, K_1) = (Spin(9), Spin(6) \circ T^1)$. Since $h = 0$ and $K_2/K \cong S^6$, we see $(K_2, K) = (G_2 \circ T^1, SU(3) \circ T^1)$. Hence we can easily show that slice representations $\sigma_1 : K_1 \to SO(8)$ and $\sigma_2 : K_2 \to SO(7)$ are unique up to equivalence (σ_1 through $Spin(6) \simeq SU(4)$). Moreover we see $N(K; G)/N(K; G)^o = \mathbb{Z}_2$. Hence in this case there are just two G-manifolds M up to essential isomorphism. Hence the following proposition holds.

PROPOSITION 10.3. *Let* $(Spin(9), M)$ *be a* $Spin(9)$*-manifold which has codimension one orbits* $Spin(9)/SU(3) \circ T^1$ *and two singular orbits* $Spin(9)/K_1$ *and* $Spin(9)/K_2$ *where* $K_1 = Spin(6) \circ T^1$ *and* $K_2 = G_2 \circ T^1$*. Then there are just two such* $(Spin(9), M)$ *up to essential isomorphism, that is,* $M = Q_{14}$ *and* $M = Spin(9) \times_{Spin(7) \circ T^1} S^{14}$.

PROOF. From the above argument this case has just two such $(Spin(9), M)$ up to essential isomorphism. If $M = Q_{14}$, then we will be constructed in Section 11.4. Put $M = Spin(9) \times_{Spin(7) \circ T^1} S^{14}$ such that T^1 acts $S^{14} \subset \mathbb{R}^8 \times \mathbb{R}^7$ trivially and $Spin(7)$ acts canonically on $\mathbb{R}^7$ and acts on $\mathbb{R}^8$ through the spin representation $Spin(7) \to SO(8)$. Then this manifold has a canonical $Spin(9)$ action and satisfies the assumption of this case. □

But $M = Spin(9) \times_{Spin(7) \circ T^1} S^{14}$ is the fibre bundle over $Spin(9)/Spin(7) \circ T^1 \cong P_{14}(\mathbb{C})$ with the fibre S^{14}. Hence this is not a rational cohomology complex quadric. So this case is unique up to essential isomorphism and such (G, M) will be constructed in Section 11.4.

10.3. $(G', K_1') = (SU(3), T^2)$**.**
If $(G', K_1') = (SU(3), T^2)$, then $k_1 = 2$. Hence $G'' = T^h$ and $h \leq 1$. From $K_2/K \cong S^2$ and Proposition 4.2, we have $K_2^o \simeq SU(2) \circ N$ and $K^o \simeq T^1 \circ N$.

If $h = 0$ then we have $N = \{e\}$ because $K_1/K^o \cong S^1$. Then the slice reprepsentation $\sigma_1 : K_1 = T^2 \to U(1)(\to O(2))$ is

$$\sigma_1(x, y) = x^m y^n$$

where $m, n \in \mathbb{Z}$ and $(m, n) \neq 0$. We see $\mathrm{Ker}(\sigma_1) = K$ and we can put $T^2 = \{(x^{-1}y^{-1}, x, y) \in SU(3)\}$ and $m \geq n$ without loss of generality where (x, y, z) is a diagonal matrix in $SU(3)$, then

$$K = \{(x^{-1}y^{-1}, x, y) |\ x, y \in T^1 \text{ and } x^m y^n = 1\}.$$

Hence we have $K \simeq T^1 \times F$ where F is a finite group and then $K_2 \simeq SU(2) \times F$. Moreover we see the slice representation $\sigma_2 : K_2 \to SO(3)$ is unique up to equivalence because the restricted representation $\sigma_2|_{SU(2)}$ is a canonical double covering and $\mathbb{Z}_2 \circ F = \mathrm{Ker}(\sigma_2)$.

Next we discuss $N(K; G)/N(K; G)^o$. First of all we define the following notations.

$$\alpha = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \beta = \begin{pmatrix} 0 & -1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}, \gamma = \begin{pmatrix} 0 & 0 & 1 \\ 0 & -1 & 0 \\ 1 & 0 & 0 \end{pmatrix}.$$

We can easily show the next four statements.

(1) If $m = n (\neq 0)$, we have $N(K; G)/N(K; G)^o = \{I_3, \alpha\}$.
(2) If $m = 0 > n$, we have $N(K; G)/N(K; G)^o = \{I_3, \beta\}$.
(3) If $m > n = 0$, we have $N(K; G)/N(K; G)^o = \{I_3, \gamma\}$.
(4) If $m > n$ and $mn \neq 0$, we have $N(K; G)/N(K; G)^o = \{I_3\}$.

We also see K^o is conjugate to the following subgroup of G except the last case above.

$$\left\{ \begin{pmatrix} 1 & 0 & 0 \\ 0 & x & 0 \\ 0 & 0 & x^{-1} \end{pmatrix} \middle|\ x \in T^1 \right\}.$$

Hence $P(G/K^o; t) = (1 + t^2)(1 + t^5)$ from the fibration $SU(2)/K^o \cong S^2 \to G/K^o \to G/SU(2) \cong S^5$. Therefore we have, from the fibration $K_2^o/K^o \cong S^2 \to G/K^o \to G/K_2^o$, the Poincaré polynomial of G/K_2^o is $P(G/K_2^o; t) = 1 + t^5$. This contradicts $P(G/K_2; t) = (1 + t)(1 + t^2 + t^4)$ and an injectivity of $p^* : H^*(G/K_2; \mathbb{Q}) \to H^*(G/K_2^o; \mathbb{Q})$.

Therefore $N(K; G)/N(K; G)^o = \{I_3\}$. Hence $N(K; G)/K$ is connected and the attaching map is unique up to equivalence by Lemma 4.3 (1.). So we can put such $SU(3)$-manifold as $M = SU(3) \times_{S(U(2)\times U(1))} S^4$ where $S(U(2) \times U(1))$ acts on $S^4 \subset \mathbb{R}^3 \times \mathbb{R}^2$ by $S(U(2) \times U(1)) \xrightarrow{p_1} SU(2) \xrightarrow{c} SO(3)$ and $S(U(2) \times U(1) \xrightarrow{p_2} T^1 \xrightarrow{\tau_F} SO(2)$ where p_1, p_2 are projections, c is a canonical double covering and $\mathrm{Ker}(\tau_F) = F$. The manifold M is a fibre bundle over $SU(3)/S(U(2) \times U(1)) \cong P_4(\mathbb{C})$ with fibre S^4. This is not a rational cohomology complex quadric. Hence this case does not occur.

Hence $h = 1$, $G = SU(3) \times T^1$ and $K_1 = T^2 \times T^1$. Moreover we see $N = T^1$ because $K_1/K \cong S^1$. Hence $K_2 \simeq SU(2) \circ T^1 \times F$ and $K \simeq T^2 \times F$ where $F \subset T^1$ is a finite subgroup. Then we can show easily the slice representation σ_1 decomposes into $K_1 \to T^1 \xrightarrow{\rho} O(2)$ such that $\mathrm{Ker}(\rho) = F$. Moreover σ_2 decomposes into $\sigma_2 : K_2 \to SU(2) \xrightarrow{\tau} SO(3)$ where τ is a canonical double covering. Hence σ_2 is unique up to equivalence.

Because $N(K;G)/N(K;G)^o = N(K_1;G)/K_1 = \mathbb{Z}_2 \times \mathbb{Z}_3$ and $\mathbb{Z}_3 \subset K_2$, this case has just two (G, M) up to essential isomorphism for the fixed positive integer $m = |F|$. Hence the following proposition holds.

PROPOSITION 10.4. *Let* (G, M) *be a* G*-manifold which has codimension one orbit* G/K*, and two singular orbits* G/K_1 *and* G/K_2 *where* $G \simeq SU(3) \times U(1)$*. Then this* (G, M) *has just two types up to essential isomorphism, that is,* $M = Q_4$ *and* $M = SU(3) \times_{S(U(2)\times U(1))} S^4$.

PROOF. There are two types (G, M) from above argument for the positive integer m. Put $M = Q_4$, and the representation $r_m : SU(3) \times T^1 \to S(U(3) \times U(1))$ such that

$$r_m(A, x) = \begin{pmatrix} x^{-m/6}A & 0 \\ 0 & x^{m/2} \end{pmatrix}.$$

As in Section 11.6, there is a representation $\rho : S(U(3) \times U(1)) \to SO(6)$ from the natural double covering surjection $SU(4) \to SO(6)$. Then $\mathrm{Ker}(\rho \circ r_m) = \{I_3\} \times F$. Hence the $SU(3) \times T^1$ acts on Q_4 by $\rho \circ r_m$ such that $K = T^2 \times F$. Moreover we see, for all $m = |F|$, the induced effective actions are equivariantly diffeomorphic. Therefore such action is unique up to essential isomorphism.

Put $M = SU(3) \times_{S(U(2)\times U(1))} S^4$ such that $S(U(2) \times U(1))$ acts on $S^4 \subset \mathbb{R}^3 \times \mathbb{R}^2$ through the representation $S(U(2) \times U(1)) \to SO(3)$. This manifold has the action of $SU(3) \times T^1$, that is, $SU(3)$ acts on $SU(3)$ canonically and T^1 acts on $S^4 \cap \mathbb{R}^2$ by m-fold. Then this $SU(3) \times T^1$-manifold M satisfies the assumption of this case. We can assume T^1-action on $S^4 \cap \mathbb{R}^2$ is canonical because all m-fold actions are essentially isomorphic. □

The manifold $M = SU(3) \times_{S(U(2)\times U(1))} S^4$ is an S^4-bundle over $P_4(\mathbb{C}) \cong SU(3)/S(U(2) \times U(1))$. Hence this is not a rational cohomology complex quadric. So this case is unique up to essential isomorphism and such (G, M) will be constructed in Section 11.6.

11. Compact transformation groups on rational cohomology complex quadrics with codimension one orbits.

All the pair (G, M) which has codimension one principal orbits are exhibited in this last section.

11.1. $(SO(2n + 1), Q_{2n})$**.**
In this case $M = Q_{2n}$ and $SO(2n + 1)$ acts on M through the canonical representation to $SO(2n + 2)$. Then there are two singular orbits S^{2n} and Q_{2n-1}. The principal orbit type is $\mathbb{R}V_{2n+1,2} \cong SO(2n + 1)/SO(2n - 1)$.

Put $\mathbb{Z}_2 = \left\{ I_{n+2}, \begin{pmatrix} -1 & 0 \\ 0 & I_{n+1} \end{pmatrix} \right\}$. This group canonically acts on Q_n and commutes with the action of $SO(n+1)$. $(SO(n+1), Q_n/\mathbb{Z}_2)$ has two singular orbits $P_{2n}(\mathbb{R})$ and Q_{n-1} and the principal orbit is $\mathbb{R}V_{n+1,2}/\mathbb{Z}_2$. From **[Uch77]** Section 9.6, such manifold $(SO(n +$

1), M) is unique up to essential isomorphism that is $(SO(n+1), M) \simeq (SO(n+1), P_n(\mathbb{C}))$. Hence we get the following proposition

PROPOSITION 11.1. *For* $n \geq 3$, $Q_n/\mathbb{Z}_2 \cong P_n(\mathbb{C})$.

11.2. $(SU(n+1), Q_{2n})$**.**
In this case $M = Q_{2n}$ and $SU(n+1)$ acts by the natural representation of $SO(2n+2)$ that is

$$SU(n+1) \ni A + B\mathbf{i} \mapsto \begin{pmatrix} A & -B \\ B & A \end{pmatrix} \in SO(2n+2).$$

Then there are two singular orbits, both orbit types are $P_n(\mathbb{C})$. The principal orbit type is $SU(n+1)/(SO(2) \times SU(n-1))$.

For $G = U(n+1)$ we get a similar result.

11.3. $(Sp(1) \times Sp(m), Q_{4m-2})$**,** $m \geq 1$**.**
In this case $M = Q_{4m-2}$ $(n = 2m-1)$ and the action of $Sp(1) \times Sp(m)$ on $\mathbb{H}^m$ is defined by $A\mathbf{x}\bar{h}$ where $(h, A) \in Sp(1) \times Sp(m)$ and $\mathbf{x} \in \mathbb{H}^m$. Then there is a representation $\rho : Sp(1) \times Sp(m) \to SO(4m)$ that is

$$\rho(h, A) = \begin{pmatrix} h_1 I_m & h_3 I_m & -h_2 I_m & h_4 I_m \\ -h_3 I_m & h_1 I_m & -h_4 I_m & -h_2 I_m \\ h_2 I_m & h_4 I_m & h_1 I_m & -h_3 I_m \\ -h_4 I_m & h_2 I_m & h_3 I_m & h_1 I_m \end{pmatrix} \begin{pmatrix} X & -Y & Z & -W \\ Y & X & -W & -Z \\ -Z & W & X & -Y \\ W & Z & Y & X \end{pmatrix}$$

where $h = h_1 + h_2\mathbf{i} + h_3\mathbf{j} + h_4\mathbf{k} \in Sp(1)$ and $A = X + Y\mathbf{i} + Z\mathbf{j} + W\mathbf{k} \in Sp(m)$.

Hence there is an action of $Sp(1) \times Sp(m)$ on Q_{4m-2} through the representation ρ. Then there are two singular orbits $S^2 \times P_m(\mathbb{C})$ and $Sp(m)/(Sp(m-2) \times U(1))$. The principal orbit type is $Sp(1) \times_{T^1} Sp(m)/(Sp(m-2) \times U(1))$.

11.4. $(Spin(9), Q_{14})$**.**
In this case $M = Q_{14}$. It is well known that $Spin(9)$ acts on S^{15} transitively by the spin representation $\rho : Spin(9) \to SO(16)$ (**[Yok73]**). Hence $Spin(9)$ acts on Q_{14} through this representation. Then the principal orbit type is $Spin(9)/SU(3) \circ T^1$ and two singular orbits are $Spin(9)/Spin(6) \circ T^1$ and $Spin(9)/G_2 \circ T^1$.

11.5. (G_2, Q_6)**.**
In this case $M = Q_6$ and the exceptional Lie group G_2 acts through the canonical representation to $SO(7)$. Then there are two singular orbits S^6 and $G_2/S(U(1) \times U(2))$. The principal orbit type is $\mathbb{R}V_{7,2} \cong G_2/SU(2)$.

11.6. $(S(U(3) \times U(1)), Q_4)$**.**
In this case $M = Q_4$. It is well known that there is the double covering representation $\rho : SU(4) \to SO(6)$ (**[Har90]**, **[Yok73]**) because of $SU(4) \simeq Spin(6)$. Hence $S(U(3) \times U(1))$ acts on this manifold through the restricted representation of ρ, and the principal isotropy

group is $S(U(1)\times U(1))\times S(U(1)\times U(1))$. Consequently principal orbits are of codimension one.

11.7. $(Sp(2), S^7 \times_{Sp(1)} P_2(\mathbb{C}))$.

In this case $M = S^7 \times_{Sp(1)} P_2(\mathbb{C})$ and $Sp(2)$ canonical acts on $S^7 \cong Sp(2)/Sp(1)$. The manifold M is a quotient manifold of $S^7 \times P_2(\mathbb{C})$ by the action $Sp(1)$ where $Sp(1)$ acts on $S^7 \cong Sp(2)/Sp(1)$ canonically and on $P_2(\mathbb{C})$ by the double covering $Sp(1) \to SO(3)$. Then the $Sp(1)$ action on $P_2(\mathbb{C})$ has codimension one principal orbits $Sp(1)/\{1, -1, \mathbf{i}, -\mathbf{i}\}$ and two singular orbits $Sp(1)/U(1)$ and $Sp(1)/U(1)_j \cup U(1)_j\mathbf{i}$ where $U(1)_j = \{a + b\mathbf{j} | \ a^2 + b^2 = 1\}$. Hence the $Sp(2)$ action on M has codimension one principal orbits $Sp(2)/Sp(1) \times \{1, -1, \mathbf{i}, -\mathbf{i}\}$ and two singular orbits $Sp(2)/Sp(1)\times U(1)$ and $Sp(2)/Sp(1)\times(U(1)_j\cup U(1)_j\mathbf{i})$.

11.8. $(G_2 \times T^1, G_{\mathbb{R}}(2, \mathbb{O}))$.

In this case $M = G_{\mathbb{R}}(2, \mathbb{O})$. Then $g \in G_2$ acts $u \wedge v \in M$ by $g \cdot u \wedge v = g(u) \wedge g(v)$. We see $g(u)$, $g(v)$ is an oriented orthonrmal basis because of $G_2 \subset SO(7)$. Hence this action is well defined on M. Moreover T^1 acts on M by the induced action from the canonical $SO(2)$-action on $\mathbb{O}^2$. These two actions are commutative. Therefore we have the $G_2 \times T^1$-action on M.

Put $G = G_2 \times T^1$. Then the isotropy subgroup $G_{1\wedge \mathbf{i}}$ is $SU(3) \times T^1$, $G_{\mathbf{i}\wedge \mathbf{j}}$ is $U(2) \times T^1$ and $G_{1\wedge 1/\sqrt{2}(\mathbf{i}+\mathbf{j})}$ is $SU(2)\circ T^1$. Hence this action has codimension one orbit $(G_2\times T^1)/SU(2)\circ T^1$ and two singular orbits $(G_2 \times T^1)/(SU(3) \times T^1) \cong S^6$ and $(G_2 \times T^1)/(U(2) \times T^1)$.

Part 2

Equivarinat Graph Cohomology of Hypertorus graph and $(n+1)$-dimensional Torus action on $4n$-dimensional manifold

12. Introduction of Part 2

A research in Part 2 is motivated by two problems about GKM-graphs and hypertoric manifolds. First we mention a GKM-graph.

Let M^{2m} be a $2m$-dimensional manifold which has an n-dimensional torus action. We denote it by (M^{2m}, T^n). This pair (M^{2m}, T^n) is called a *GKM-manifold* if it satisfies the following three conditions (*GKM-condition*);

- Its fixed point set M^T is finite.
- (M^{2m}, T^n) is an *equivariantly formal space*.
- (M^{2m}, T^n) satisfies a *pairwise linearly independence* around its fixed point.

Here an equivariantly formal space (M^{2m}, T^n) means the spectral sequence of the fibre bundle

$$M \to ET \times_T M \to BT$$

collapses (see **[GKM98]**), and a pairwise linearly independence means the induced T^n-action on the tangent space of a fixed point $T_p(M)$ is equivariantly decompose into $V(\alpha_1) \times \cdots \times V(\alpha_m)$ such that the weights $\{\alpha_1, \ldots, \alpha_m\}$ are pairwise linearly independent in $\mathfrak{t}^*$, where $\mathfrak{t}^*$ is a dual Lie algebra of the torus T.

A regular m-valent graph $\Gamma(M)(= \Gamma) = (V^\Gamma, E^\Gamma)$ can be defined by the above GKM-manifold (M^{2m}, T^n), regarding the fixed point in M^T as a vertex in V^Γ and the connected component in the orbit space of one-dimensional orbits as an oriented edges E^Γ. Moreover "labels" on the oriented edges E^Γ are defined by its isotropy weight representations (in the dual Lie algebra $(\mathfrak{t}^n)^*_{\mathbb{Z}}$). We denote it $\alpha : E^\Gamma \to (\mathfrak{t}^n)^*_{\mathbb{Z}}$ and call α an *axial function* on Γ. The important fact in **[CS74]** and **[GKM98]** is that the equivariant cohomology ring of (M^{2m}, T^n) is isomorphic to an equivariant graph cohomology of $\Gamma(M)$ defined by (M^{2m}, T^n) (see Section 13), that is the following equation holds;

$$H_T^*(M; \mathbb{Z}) \simeq H_T^*(\Gamma(M),\ \alpha),$$

where $H_T^*(M; \mathbb{Z})$ is the equivariant cohomology of (M, T) and $H_T^*(\Gamma(M), \alpha)$ is the equivariant graph cohomology of the GKM-graph $\Gamma(M)$.

Now a GKM-manifold (M^{2m}, T^n) is a geometrical object, on the other hand, a GKM-graph Γ can be assumed to be a combinatorial object. So we are naturally led to study to compute the equivariant cohomology ring $H_T^*(M; \mathbb{Z})(\simeq H_T^*(\Gamma,\ \alpha))$ from combinatorial strucures of Γ. In fact, it has already succeeded in some cases. In 2001, Guillemin and Zara initiated to study a class of the GKM-graph Γ (a *toric graph*) which contains the GKM-graph defined by the toric manifold. They give generators of $H_T^*(\Gamma)$ as $H^*(BT)$-module by combinatorial structures of Γ and they compute the Betti number of $H_T^*(\Gamma)$ in **[GZ01]**. In 2003, Masuda and Panov study on a *torus manifold* which is more general than the toric manifold in **[MP03]**. They compute its equivariant cohomology ring and describe it by combinatorial structures of their characteristic manifolds. Maeda and they also define a class of GKM-graphs (a *torus graph*) in **[MMP05]** which contains the GKM-graph defined by the torus manifold and describe its graph cohomology ring by combinatorial structures (by a *connection* θ of Γ which will be defined in Section 13). The torus manifold M (contains the toric manifold) is a $2n$-dimensional manifold which has a T^n-action and it satisfies a GKM condition if M holds $H^{odd}(M) = 0$. So, in the above cases, the GKM-graph Γ is an n-valent graph and has an axial function $\alpha : E^\Gamma \to (t^n)^*_{\mathbb{Z}}$. However little is known about the GKM-graph Γ which is an m-valent and has an axial function $\alpha : E^\Gamma \to (t^n)^*_{\mathbb{Z}}$ such that $m > n$, we call such GKM-graph an (m, n)-*type* GKM-*graph*. Therefore the first motivation of Part 2 is as follows.

PROBLEM 1. *Let* (Γ, α, θ) *be an* (m, n)*-type* GKM*-graph, where* $m > n$ *. Describe the graph cohomology ring* $H_T^*(\Gamma,\ \alpha)$ *by its combinatorial structures* (a *connection* θ)*.*

Next we mention a hypertoric manifold.

In 2000 [**BD00**], Bielowski and Dancer define $4n$-dimensional variety, the *hypertoric variety* M^{4n}, by the hyperKäler quotient of a torus action on the quaternionic spaces. Remark the *hypertoric variety* and the *toric hyperKäler* are same things, we use *hypertoric variety* in Part 2. The hypertoric variety corresponds to a hyperplane arrangement. In the same year, Konno computed the equivariant cohomology ring (and the ordinary cohomology ring) of (M^{4n}, T^n) by the combinatorial structure of hyperplane arrangements [**Kon00**] and [**Kon03**]. The hypertoric variety (M^{4n}, T^n) satisfies the upper two conditions in GKM-condition, that is M^T is finite and it is the equivariantly formal (because of $H^{odd}(M) = 0$). However it does not satisfy the pairwise linearly independentness on M^T, that is, it does not satisfy the GKM-condition. Hence we can not define the GKM-graph from the hypertoric variety (M^{4n}, T^n). In 2004 [**HH04**], Harada and Holm found a hypertoric variety (M^{4n}, T^n) extends to a transformation group (M^{4n}, T^{n+1}) and it satisfies the GKM-condition. Therefore we can define the $(2n, n+1)$-type GKM-graph from $(M^{4n}, T^n \times S^1)$. Moreover they describe the eqivariant cohomology ring of $(M^{4n}, T^n \times S^1)$ in terms of its hyperplane arrangement and they coorespond its generator to an element of its equivariant graph cohomology $H_T^*(\Gamma)$. So we are naturally led to think there might be a class of GKM-graphs which contains the GKM-graph defined by the hypertoric variety like the Masuda-Maeda-Panov's torus graph in [**MMP05**]. From the above researches, we can denote the second motivation as follows;

PROBLEM 2. *Define the class of GKM-graphs which contains the GKM-graph coming from the hypertoric variety and compute its equivariant graph cohomology.*

The aim of Part 2 is to solve the above Problem 2. We define a *hypertorus graph* as a new class of GKM-graphs, which contains the GKM-graph defined by the hypertoric variety or the cotangentbundle of the torus manifold and describe its equivariant graph cohomology in terms of its combinatorial structure for some cases. The hypertorus graph is $(2n, n+1)$-type GKM-graph, so to describe its equivariant graph cohomology is to solve the Problem 1 partially. A main result of Part 2 is a generalization of the main result in [**HH04**] from the other aspect, which is not the hyperplane arrangement but the hypertorus graph. We also define a *quaternionic torus graph* as generalization of hypertorus graph. This graph contains the GKM-graph coming from a *complex quadric* or a *quaternion projective space*. We do not compute its equivariant graph cohomology in Part 2, but to compute it might be an important problem.

The main result of Part 2 is as follows.

MAIN THEOREM 2. *Assume for each codimension two hypertorus subgraph* L *there is a unique hyperfacet* H *and its opposite side* $\bar{H}$ *such that* $\partial H = L$, *and* $H \cap G = \emptyset$ *or connected for all hyperfacets* H *and* G. *Then there is the following isomorphism*:

$$H_T^*(\Gamma,\ \alpha) \simeq \mathbb{Z}[\Gamma,\ \theta].$$

The organization of Part 2 is as follows. First we recall a GKM-graph and its equivariant graph cohomology in Section 13. Because we would like to define a quaternionic torus graph which contains the GKM-graph coming from T^{n+1}-action on $\mathbb{H}P^n$, we define the GKM-graph more general than the *GKM-graph* which is defined in other papers, for example [**GZ01**] or [**MMP05**]. In Section 14 we define a hypertorus graph and quaternionic torus graph, before to define we recall a hypertoric variety. Next we exhibit three examples of hypertorus graph and quaternionic torus graph in Section 15. To state our main theorem, we have to prepare some notations and propositions in Section 16. Finally we prove the main theorem in Section 17. To prove the main theorem, we consider three cases as follows:

(1) Γ has only one vertex;

(2) Γ is a *minimal* hypertorus graph;

(3) Γ is general hypertorus graph.

13. GKM-graph and equivariant graph cohomology

Guillemin and Zara [**GZ01**] introduce a *GKM-graph* to study equivariant cohomology rings of GKM-manifolds from combinatorial aspects. They succeed to translate the notations from toric manifolds into *toric graphs*, where a toric graph contains a GKM-graph which is defined by toric manifolds. For instance they define a Betti number of toric graphs by its combinatorial information and show it accords with a Betti number of toric manifolds.

Maeda, Masuda and Panov introduced the *torus graph* in [**MMP05**] as a GKM-graph which is more general than a toric graph. They succeed to describe its equivariant graph cohomology rings.

In this section we state a definition of a GKM-graph which has more general condition (of an axial function) than the above two papers [**GZ01**] and [**MMP05**] and also define its equivariant graph cohomology.

Remark. A toric graphs called GKM-graphs in **[GZ01]**, but we use the terminology *GKM-graph* is more general meaning in Part 2.

First we state some notations. Let $\Gamma = (V^\Gamma, \mathcal{E}^\Gamma)$ be a connected regular m-valent graph which is possible to have a *leg*, where V^Γ is a set of finite vertices of Γ. Here $\mathcal{E}^\Gamma$ is a set which consists of two parts as follows:

$$\mathcal{E}^\Gamma = E^\Gamma \cup L^\Gamma.$$

Here E^Γ is a set of all oriented edges, so each edge $e \in E^\Gamma$ has two possible orientations, and L^Γ is a set of legs where a leg $l \in L^\Gamma$ is an out going half line from the vertex, so each leg l has an only one orientation. The following two figures are examples of our graphs.

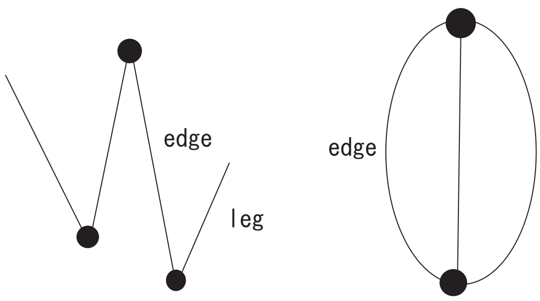

FIGURE 13.1. Examples.

The above left example is a 2-valent graph which has two legs and edges and the right one is 3-valent graph which has no legs.

An opposite orientation of the edge $e = pq$ is denoted by $\bar{e} = qp$, we also denote the initial vertex of $e = pq$ by $i(e)(= p)$ and the terminal vertex of e by $t(e)(= q)$. So a leg l does not have terminal vertex but it has an initial vertex $i(l)$, hence we can state the leg l is an out going half line from $i(l)$. The leg has only one orientation.

Next we prepare two important notations, a *connection* and an *axial function*. We can regard a connection as a combinatorial structure on the graph Γ, on the other hand an axial function as an algebraic structure on it.

For $p \in V^\Gamma$ we put

$$\mathcal{E}^\Gamma_p = \{e \in \mathcal{E}^\Gamma \mid i(e) = p\},$$

and for an edge $e = pq \in E^\Gamma$ we denote a collection of bijections

$$\theta_e : \mathcal{E}^\Gamma_p \to \mathcal{E}^\Gamma_q$$

by $\theta = \{\theta_e\}$. Now we denote the number of all edges and legs which have a same initial vertex p by $|\mathcal{E}^\Gamma_p|$. In our case $|\mathcal{E}^\Gamma_p| = m$ holds for all $p \in V^\Gamma$ because the graph Γ is an m-valent graph. Hence the bijective map θ_e always exits on all edges E^Γ. Let us state a definition of a connection.

Definition[connection]. A *connection* on Γ is a collection $\theta = \{\theta_e\}$ which satisfies the following two conditions:

(1) $\theta_{\bar{e}} = \theta_e^{-1}$;

(2) $\theta_e(e) = \bar{e}$.

We can easily show an m-valent graph Γ admits different $((m-1)!)^g$ connections, where g is the number of (non- oriented) edges E^Γ.

Next we define an axial function which is more general than the definition of axial functions in **[GZ01]** and **[MMP05]**.

Definition[axial function]. We call a map $\alpha : \mathcal{E}^\Gamma \to \mathrm{Hom}(T, S^1) = H^2(BT) = \mathfrak{t}_{\mathbb{Z}}$ an *axial function* (associated with the connection θ) if it satisfies the following three conditions:

(1) $m_{i(\bar{e})}\alpha(\bar{e}) = m_{i(e)}\alpha(e)$ for some $m_{i(\bar{e})},\ m_{i(e)} \in \mathbb{Z} - \{0\}$;

(2) Elements of $\alpha(\mathcal{E}_p^\Gamma)$ are pairwise linearly independence for each $p \in V^\Gamma$;

(3) $m'_{e'}\alpha(\theta_e(e')) - m_{e'}\alpha(e') \equiv 0 \pmod{\alpha(e)}$ for any $e \in \mathcal{E}^\Gamma$, $e' \in \mathcal{E}^\Gamma_{i(e)}$ and some non-zero integer $m'_{e'}$, $m_{e'}$ which depend on e'.

We call the above third relation a *congruence relation*.

Remark. The GKM-graph which defines in **[GZ01]** (resp. in **[MMP05]**) is the first condition of the axial function was $m_{i(\bar{e})} = -1 = -m_{i(e)}$ (resp. $m_{i(\bar{e})} = \pm 1$ and $m_{i(e)} = 1$) and the congruence relation was both of them were $m'_{e'} = m_e$. Because we would like to define a *quaternionic torus manifold* as a class of GKM-graphs which contains the GKM-graph defined by T^{n+1}-action on $\mathbb{H}P^n$, we need to define the axial function which has more general condition than **[GZ01]** and **[MMP05]**.

Let us define a *GKM-graph*.

Definition[GKM-graph]. Assume a connection θ defines on an m-valent graph Γ and Γ is labeled by an axial function α whose terget is $\mathfrak{t}_{\mathbb{Z}}^n$. Then we call (Γ, α, θ) a (m, n)-*type GKM-graph*.

The GKM-graph is defined by a GKM-manifold as follows. Put vertices V^Γ by M^T, edges and legs $\mathcal{E}^\Gamma$ by the set of connected components of $\mathfrak{s}$, where $\mathfrak{s} = \{x \in M \mid \dim T(x) = 1\}/T$. Remark $\mathfrak{s}$ is a one dimensional open manifold from the GKM-condition. Set the graph $\Gamma(M)$ by $\bar{\mathfrak{s}}$, where $\bar{\mathfrak{s}} = \{x \in M \mid \dim T(x) = 1\}^c/T = \mathfrak{s} \cup V^\Gamma$. Then we call $\bar{\mathfrak{s}}$ a *one skelton* of T-action on M (X^c means a closure of X). Since GKM-manifold satisfies the pairwise linerly independence around its fixed points, there is an isotropy weight decomposition on the tangent space of $p \in M^T = V^\Gamma$ as

$$T_p(M) \simeq V(\alpha_1) \oplus \cdots \oplus V(\alpha_n),$$

where $\alpha_i \in (\mathfrak{t}^m)^*$ is a weight of an isotorpy group representation on $T_p(M)$ and the representation space of α_i is denoted by $V(\alpha_i) \simeq \mathbb{C}$ for all $i = 1, \cdots n$. Now each α_i corresponding to some $e_i \in \mathcal{E}^\Gamma$ which has an initial point p.

Remark. We can assume e_i as the T^{m-1}-invariant manifold in M, that is $(T\bar{e}_i)^c \simeq \mathbb{C}P(1)$ or $\mathbb{C}$ which contains $p \in M^T$.

So an axial function α_M on $\Gamma(M)$ is the map which satisfies $\alpha_M(e_i) = \alpha_i$. Finally we define the connection θ from the above axial function α_M (possibly not unique). Therefore we get an (m, n)-type GKM-graph $(\Gamma(M), \alpha_M, \theta)$ from a GKM-manifold (M^{2n}, T^m).

Guillemin and Zara define a *toric graph* in **[GZ01]**. The toric graph is an (n, n)-type GKM-graph (without legs) and it satisfies $m_{i(e)} = -m_{i(\bar{e})} = 1$ on the first condition and $m'_{e'} = m_{e'} = 1$ on the third condition of the axial function. We call such axial function a *toric axial function*. Moreover $\alpha(E_p^\Gamma)$ forms a basis of $\mathfrak{t}_\mathbb{Z}^n$.

Maeda, Masuda and Panov define a *torus graph* in **[MMP05]**. The torus graph is an (n, n)-type GKM-graph (without legs) and it satisfies $m_{i(e)}, m_{i(\bar{e})} = \pm 1$ on the first condition and $m'_{e'} = m_{e'} = 1$ on the third condition of the axial function. We call such axial function a *torus axial function*. In this case $\alpha(E_p^\Gamma)$ also forms basis of $\mathfrak{t}_\mathbb{Z}^n$. The torus graph contains the toric graph and Maeda, Masuda and Panov show its equivariant graph cohomology is isomorphic to some ring which is defined by its combinatorial information. The following Figure 13.2 is an example of torus graph and the next one Figure 13.3 is an example of *generalized* torus graph which is possible to have legs.

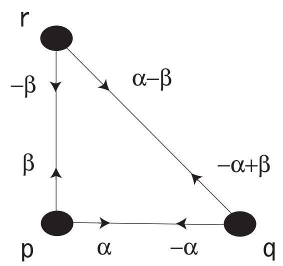

FIGURE 13.2. The GKM-graph associated with T^2-action on $\mathbb{C}P(2)$.

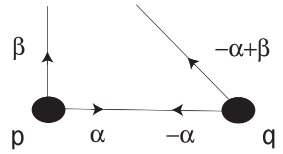

FIGURE 13.3. The GKM-graph associated with T^2-action on $\mathbb{C}P(2) - \{r\}$.

Give a GKM-graph (Γ, α, θ). Then we can define a ring $H^*_{T^n}(\Gamma, \alpha)$ which is called an *equivariant graph cohomlogy*.

Definition[equivariant graph cohomology]. Let (Γ, α, θ) be an (m, n)-type GKM-graph. Then we set an *equivariant graph cohomology* $H^*_{T^n}(\Gamma, \alpha)$ as follows:

$$H^*_{T^n}(\Gamma, \alpha) = \{f : V^\Gamma \to H^*_{T^n}(pt) \mid f(p) - f(q) \equiv 0 \pmod{\alpha(pq)}\}$$

where $pq \in E^\Gamma$ is an edge.

As is well known $H^*_{T^n}(pt)$ is a polynomial ring $\mathbb{Z}[x_1, \cdots, x_n]$ where $x_i \in H^2_{T^n}(pt) = \mathfrak{t}^n_{\mathbb{Z}}$. Now we exhibit an example of an element of $H^*_T(\Gamma, \alpha)$.

Examples. Let Γ be a GKM-graph as the above Figure 13.2. Put $f : V^\Gamma \to H^*_T(pt)$ as follows:

$$f(p) = \alpha(2\alpha + \beta),\ f(q) = 2\alpha\beta,\ f(r) = 2\alpha^2 + \beta(\alpha - \beta),$$

where $\mathfrak{t}^2_{\mathbb{Z}} \simeq \langle \alpha,\ \beta \rangle$. Then we have $f(p) - f(q) = 2\alpha^2 - \alpha\beta \equiv 0 \pmod{\alpha}$, $f(q) - f(r) = 2\alpha(\beta - \alpha) - \beta(\alpha - \beta) \equiv 0 \pmod{\beta - \alpha}$ and $f(r) - f(p) = -\beta^2 \equiv 0 \pmod{-\beta}$. Hence the map f is an element of $H^*_T(\Gamma)$.

14. Hypertoric variety and hypertorus graph

A hyperkähler quotient is defined by Hitchin, Karlhede, Lindström and Roček in **[HKLR87]** as a quotient which constructs an hyperkähler manifold. In 2000, Bielowsky and Dancer study a special case of a hyperkähler quotient that is a hypertoric variety in **[BD00]**. A hypertoric variety is constructed by a hyperkähler quotient of torus action on $T^*\mathbb{C}^N \simeq \mathbb{H}^N$ (see Section 14.1) like a toric variety, which is defined by a kähler quotient of torus action on $\mathbb{C}^N$. In same year **[Kon00]**, H. Konno studies its cohomology ring structure in detail. In 2004, Harada and Holm relates the hypertoric variety with the GKM theory in **[HH04]**. In this section, we recall a hypertoric variety and define a hypertorus graph.

14.1. hypertoric variety.

A hypertoric variety is motivated to define a hypertorus graph in this thesis, note that in the paper **[Kon00]** a hypertoric variety called a *toric hyperkähler* but we use the name *hypertoric variety* as **[HH04]**. Let us recall a hypertoric variety. Consider the natural torus group $K(\subset T^N)$-action on $T^*\mathbb{C}^N$. Then we can define a *hyperkähler moment map* as follows:

$$\mu_{HK} : T^*\mathbb{C}^N \xrightarrow{\mu} \mathfrak{t}^* \oplus \mathfrak{t}^*_{\mathbb{C}} \xrightarrow{i^*} \mathfrak{k}^* \oplus \mathfrak{k}^*_{\mathbb{C}}.$$

Here $\mathfrak{t}^*$ and $\mathfrak{k}^*$ are dual Lie algebras of T^N and K, i^* is an induced homomorphism from the inclusion $i : \mathfrak{k} \to \mathfrak{t}$, and a map μ is defined by

$$\mu(z, w) = \frac{1}{2} \sum_{i=1}^{N} (|z_i|^2 - |w_i|^2)\partial_i \oplus \sum_{j=1}^{N} z_j w_j \partial_j,$$

where $z = (z_1, \cdots, z_N)$ is a point of a base space $\mathbb{C}^N$, $w = (w_1, \cdots, w_N)$ is a point of a fibre space and $\partial_1, \cdots, \partial_N$ are canonical basis of $\mathfrak{t}^*$. Take a regular value of $(\nu, 0) \in \mathfrak{k}^* \oplus \mathfrak{k}^*_{\mathbb{C}}$, then a manifold $\mu_{HK}^{-1}(\nu, 0)(\subset T^*\mathbb{C}^N)$ has an almost free $K(\subset T^N)$-action. Hence the quotient space $\mu_{HK}^{-1}(\nu, 0)/K = M^{4n}$ is an orbifold and it has a $(T^N/K =)T^n$-action. This orbifold is called a *hypertoric variety*. Moreover M^{4n} has an induced residual S^1-action from a scaler multiplication on fibres of $T^*\mathbb{C}^N$. Therefore M^{4n} has a $T^n \times S^1$-action and it satisfies a GKM-condition. The tangent space of its fixed point p has isotropy weight decomposition as follows:

$$T_p(M) = V(\alpha_1) \oplus \cdots \oplus V(\alpha_n) \oplus V(-\alpha_1 + x) \oplus \cdots \oplus V(-\alpha_n + x),$$

where $\mathfrak{t}^* \simeq \langle \alpha_1, \cdots, \alpha_n \rangle$ and $\mathfrak{s} \simeq \langle x \rangle$.

Remark. T^n-action on M^{4n} does not satisfy a pairwise linerly independentness.

Hence there exists a GKM-graph coming from a hypertoric variety M^{4n} with $(T^n \times S^1)$-action which has a properties, that the edges or legs consist n-pairs $\{e_i^+, e_i^-\}$ $(i = 1, \cdots, n)$ in $\mathcal{E}_p^\Gamma$ and their axial function holds $\alpha(e_i^+) + \alpha(e_i^-) = x$. From these properties, we will define a hypertorus graph as in next section.

The most essential example of the hypertoric variety is a cotangent bundle of complex projective space $T^*\mathbb{C}P(n)$ which has a natural T^n-action and S^1-action on fibres. The following Figure 14.1 is the GKM-graph coming from $T^2 \times S^1$-action on $T^*\mathbb{C}P(2)$.

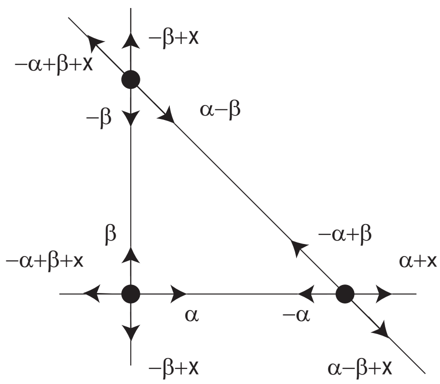

FIGURE 14.1. The GKM graph associated with $T^2 \times S^1$-action on $T^*\mathbb{C}P^2$.

14.2. Hypertorus graph.

In the begining, we define a *quaternionic torus graph* as generalization of *hypertorus graph*.

Definition[quaternionic torus graph]. Let $\Gamma = (V^\Gamma, \mathcal{E}^\Gamma)$ be a regular $2n$-valent graph, possibly with legs. Let (Γ, α, θ) be a $(2n, n+1)$-type GKM-graph. Its axial function α satisfies two conditions such that

(1) $\alpha(e) = \pm\alpha(\bar{e})$

(2) $\alpha(e') \equiv \epsilon_{e'}\alpha(\theta_e(e'))$ (mod $\alpha(e)$) for any $e \in \mathcal{E}^\Gamma$, $e' \in \mathcal{E}^\Gamma_{i(e)}$ and $\epsilon_{e'} = 1$ or -1.

For each $p \in V^\Gamma$, we can put $\mathcal{E}^\Gamma_p = \{e^+_1(p), \cdots, e^+_n(p), e^-_1(p), \cdots, e^-_n(p)\}$ and the pair $(e^+_i(p), e^-_i(p))$ satisfies

$$\alpha(e^+_i(p)) + \alpha(e^-_i(p)) = x(p) \tag{14.1}$$

for all $i = 1, \cdots, n$ where an element $x(p) \in (t^{n+1})^*$ depends on $p \in V_\Gamma$. Moreover the set

$$\{\alpha(e^+_1(p)), \cdots, \alpha(e^+_n(p)), x(p)\}$$

is a basis of t^{n+1} for all $p \in V^\Gamma$. Then we call such GKM-graph a *quaternionic torus graph*.

The following proposition can be proved by easy calculating.

PROPOSITION 14.1. *Let* (Γ, α, θ) *be a quaternionic torus graph and* $x(p) \in t^*$ *be a value of* $\alpha(e^+_i(p)) + \alpha(e^-_i(p))$ *for each* $p \in V^\Gamma$, *where* $\{e^+_i(p),\ e^-_i(p)\}$ *is a pair of* $\mathcal{E}^\Gamma_p$. *Then the following two statements are equivalent.*

(1) *The edge* $pq \in \mathcal{E}^\Gamma$ *satisfies* $\theta_{pq}(e^+_i) = h^+_i$, $\theta_{pq}(e^-_i) = h^-_i$ *for all* $i = 1, \cdots, n$ *and* $\alpha(e) \equiv \alpha(\theta_{pq}(e))$ (mod $\alpha(pq)$) *for all* $e \in \mathcal{E}^\Gamma_p$, *that is* $\epsilon_e = 1$ *for all* $e \in \mathcal{E}^\Gamma_p$.

(2) *The equation* $x(p) - x(q) \equiv 0$ mod $\alpha(pq)$ *holds for the edge* pq.

PROOF. First we show $(1 \Rightarrow 2)$. Because Γ is a quaternionic torus graph, we have

$$\begin{aligned} \alpha(e^+_i(p)) + \alpha(e^-_i(p)) &= x(p) \text{ and} \\ \alpha(e^+_j(q)) + \alpha(e^-_j(q)) &= x(q), \end{aligned}$$

for all $i, j = 1, \cdots, n$. Now we can put $\theta_{pq}(e^+(p)) = e^+(q)$ and $\theta_{pq}(e^-(p)) = e^-(q)$, and then we have $\alpha(e^+(p)) - \alpha(e^+(q)) \equiv 0$ and $\alpha(e^-(p)) - \alpha(e^-(q)) \equiv 0$ (mod $\alpha(pq)$) by the assumption 1. From the above equations, we have

$$(\alpha(e^+(p)) - \alpha(e^+(q))) + (\alpha(e^-(p) - \alpha(e^-(q))) = x(p) - x(q) \equiv 0 \ (\text{mod } \alpha(pq)).$$

So we get $(1 \Rightarrow 2)$.

Next we show $(1 \Leftarrow 2)$. Put the edge pq by $e(= e^+)$ and qp by $h(= h^+)$. First we begin to show $\theta_e(e^-) = h^-$. Now we have the following equations by the definition of the quaternionic torus graph:

$$\begin{aligned} &\alpha(e) + \alpha(e^-) = x(p); \\ &\alpha(e^-) - \epsilon_e\alpha(\theta_e(e^-)) \equiv 0 \text{ mod } \alpha(e). \end{aligned}$$

Hence we have

$$\begin{aligned} \alpha(e^-) - \epsilon_e\alpha(\theta_e(e^-)) &= x(p) - \alpha(e) - \epsilon_e\alpha(\theta_e(e^-)) \\ &\equiv x(p) - \epsilon_e\alpha(\theta_e(e^-)) \equiv 0 \ (\text{mod } \alpha(h)), \end{aligned}$$

by $\alpha(e) = \pm\alpha(h)$. Therefore we get $\epsilon_e\alpha(\theta_e(e^-)) - x(q) \equiv 0 \pmod{\alpha(h)}$ by the assumption 2. So we get

$$\begin{aligned} \epsilon_e\alpha(\theta_e(e^-)) - r\alpha(h) &= x(q) \\ &= \alpha(h) + \alpha(h^-) \end{aligned}$$

for some $r \in \mathbb{Z}$. Because of the definition of quaternionic torus graph, we have $\epsilon_e = 1$, $r = -1$ and $\theta_e(e^-) = h^-$. Therefore for $(\mathcal{E}^\Gamma_p \ni) e_i^+, e_i^- \neq pq$, we see $\theta_{pq}(e_i^+) = h_1$, $\theta_{pq}(e_i^-) = h_2 \in \mathcal{E}^\Gamma_q$ are not equal to the pair of qp.

Next we show $\alpha(h_1) + \alpha(h_2) = x(q)$. Now we have

$$\begin{aligned} &\alpha(e_i^+) + \alpha(e_i^-) = x(p), \\ &x(p) - x(q) \equiv 0 \bmod \alpha(pq), \\ &\alpha(e_i^+) - \epsilon_1\alpha(h_1) \equiv 0 \bmod \alpha(pq), \\ &\alpha(e_i^-) - \epsilon_2\alpha(h_2) \equiv 0 \bmod \alpha(pq) \end{aligned}$$

for some $\epsilon_1, \epsilon_2 = 1$ or -1. Therefore we have the following equation:

$$\begin{aligned} & x(p) - \epsilon_1\alpha(h_1) - \epsilon_2\alpha(h_2) \\ \equiv\ & x(q) - \epsilon_1\alpha(h_1) - \epsilon_2\alpha(h_2) \equiv 0 \bmod \alpha(pq). \end{aligned}$$

So we have $\epsilon_1 = \epsilon_2 = 1$ and $\alpha(h_1) + \alpha(h_2) = x(q)$ because of the definition of the quaternionic torus graph. □

The quaternionic torus graph contains the GKM-graph coming from a T^{n+1}-action on a quarternionic projective space $\mathbb{H}P(n)$ (see next section). Let us define a *hypertorus graph*.

Definition[hypertorus graph]. Let $\Gamma = (V^\Gamma, \mathcal{E}^\Gamma)$ be a regular $2n$-valent graph, possibly with legs. We say $(2n, n+1)$-type GKM-graph (Γ, α, θ) is a *hypertorus graph* if α is a torus axial function, that is $m_{i(e)}, m_{i(\bar{e})} = \pm 1$ and $m'_{e'} = m_{e'} = 1$ on the definition of the axial function, and it satisfies

$$\alpha : \mathcal{E}^\Gamma \to (\mathfrak{t}^n \times \mathfrak{s}^1)^*_{\mathbb{Z}} = \langle \alpha_1, \cdots, \alpha_n \rangle_{\mathbb{Z}} \times \langle x \rangle_{\mathbb{Z}}$$

where the α_i $(i = 1, \cdots, n)$ is a basis of $(\mathfrak{t}^n)^*$ and x is a basis of $(\mathfrak{s}^1)^*$ and $(\mathfrak{t}^n \times \mathfrak{s}^1)^*_{\mathbb{Z}}$ is the weight lattice in the dual Lie algebra of $T^n \times S^1$. Moreover for all $p \in V^\Gamma$ we can put

$$\mathcal{E}^\Gamma_p = \{e_1^+(p), \cdots, e_n^+(p), e_1^-(p), \cdots, e_n^-(p)\}$$

and its axial function satisfies

$$\alpha(e_i^+(p)) + \alpha(e_i^-(p)) = x, \tag{14.2}$$

$$\langle \alpha(e_1^+(p)), \cdots, \alpha(e_n^+(p)), x \rangle_{\mathbb{Z}} = \langle \alpha_1, \cdots, \alpha_n \rangle_{\mathbb{Z}} \times \langle x \rangle_{\mathbb{Z}} \tag{14.3}$$

for all $i = 1, \cdots, n$ and vertices.

We see easily $\langle \alpha(e_1^-(p)), \cdots, \alpha(e_n^-(p)), x \rangle_{\mathbb{Z}} = \langle \alpha_1, \cdots, \alpha_n \rangle_{\mathbb{Z}} \times \langle x \rangle_{\mathbb{Z}}$.

The following corollary is easy to show by Proposition 14.1.

COROLLARY 14.1. *Let* (Γ, α, θ) *be a hypertorus graph and* $\theta_{pq}(e^+) = h_1$ *and* $\theta_{pq}(e^-) = h_2$, *where* e^+, e^- *are pair of* $\mathcal{E}_p^\Gamma$ *and* h_1, h_2 *are elements in* $\mathcal{E}_q^\Gamma$. *Then we have* $h_1 = h^+$ *and* $h_2 = h^-$, *that is* h_1 *and* h_2 *consist the pair in* $\mathcal{E}_q^\Gamma$.

We will exhibit examples in the next section.

15. Typical examples

In this chapter, we exhibit some examples which define a hypertorus graph and a quaternionic torus graph as a GKM-graph. First example is about a hypertorus graph and second and third examples are about a quaternionic torus graph.

15.1. cotangent bundle of torus manifold.

The *torus manifold* is a $2n$-dimensional compact smooth manifold M with an effective action of an n-dimensional torus T whose fixed point set (finite) is non-empty. A *characteristic submanifold* of M is a codimension-two connected component of the fixed pointwise by a circle subgroup of T. An *omniorientation* of M consists of a choice of orientation for M and for each characteristic submanifold. The torus manifold defined by Masuda in **[Mas99]** and **[HM03]**. It contans the toric manifolds and it satisfies a GKM-condition if $H^{odd}(M) = 0$. So we get a GKM-graph as a torus graph from torus manifold.

Denote a cotangent bundle of a torus manifold M by T^*M. Then T^*M has a canonical T^n-action and the scaler S^1-action on fibres. Of course this case also satisfies a GKM-condition, so we have a GKM-graph. Moreover we can easily show this graph is a hypertorus graph.

One of the example of torus manifolds (but not toric manifolds) is $2n$-dimensional sphere S^{2n} $(n \geq 2)$. This manifold $S^{2n}(\subset \mathbb{C}^n \times \mathbb{R})$ has a T^n-action ρ coming from the canonical T^n-action on $\mathbb{C}^n$. Define $T^n \times S^1$-action on T^*S^{2n} by the T^n-action induced from the above action ρ and the scaler S^1-action on fibres. Let $\Gamma = (V^\Gamma, \mathcal{E}^\Gamma)$ be the graph given by finite fixed points and the one skelton of the orbit space. In this case there are just two fixed points that is $V^\Gamma = \{N, S\}$, n edges connecting two vertices N, S and each vertex has n legs, that is Γ has $2n$ legs. Moreover we can get the axial function α by the isotropy weight representation on fixed points and the connection θ is defined by this function α. Then (Γ, α, θ) is a hypertorus graph. The following Figure 15.1 is a hypertorus graph associated with $T^2 \times S^1$-action on T^*S^4.

15.2. quaternionic projective space.

The quaternionic projective space $\mathbb{H}P(n)$ is a $4n$-dimensional projective space over the quaternionic numbers which defines as follows:

$$\mathbb{H}P(n) = (\mathbb{H}^{n+1} - \{0\})/\mathbb{H}^*,$$

where $\mathbb{H}$ is the quaternionic numbers and $\mathbb{H}^*$ is $\mathbb{H} - \{0\}$. Remark the scaler multiplication of $\mathbb{H}^*$ on $\mathbb{H}^{n+1} - \{0\}$ by the right side.

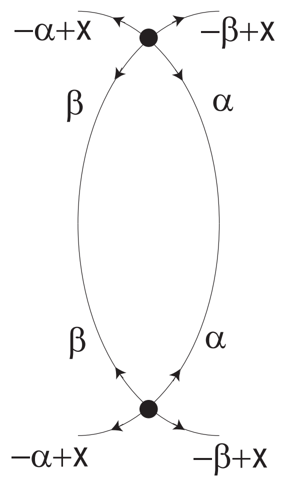

FIGURE 15.1. The hypertorus graph associated with $T^2 \times S^1$-action on T^*S^4.

Then $(n+1)$-dimensional torus T^{n+1} acts $\mathbb{H}P(n)$ as follows:

$$(t_1, \cdots, t_n, t_{n+1}) \cdot [h_0 : h_1 : \ldots : h_n] = [t_{n+1}^{1/2} h_0 : t_{n+1}^{1/2} t_1 h_1 : \ldots : t_{n+1}^{1/2} t_n h_n],$$

where $(t_1, \cdots, t_n, t_{n+1}) \in T^{n+1}$ and $[h_0 : h_1 : \ldots : h_n] \in \mathbb{H}P(n)$. Note that the left diagonal action of $t_{n+1} \in S^1$ on $\mathbb{H}P(n)$ is not trivial because the scaler $\mathbb{H}^*$ acts from right side.

Then this action defines a GKM-graph Γ. This graph Γ is not a hypertorus graph but a quaternionic torus graph.

The following Figure 15.2 is a quaternionic torus graph coming from T^3-action on $\mathbb{H}P(2)$.

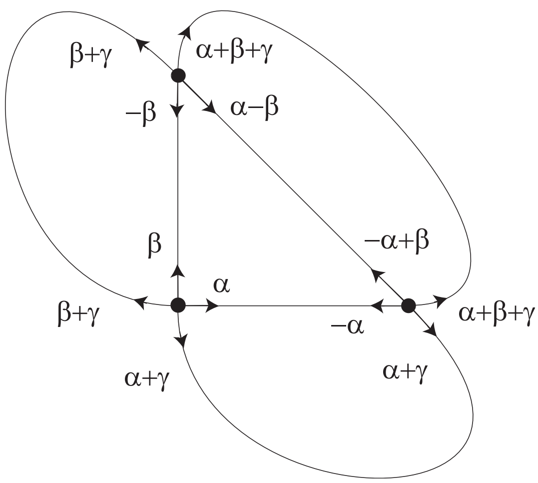

FIGURE 15.2. The quaternionic torus graph associated with T^3-action on $\mathbb{H}P^2$.

15.3. complex quadric.

The complex quadric Q_{2n} is a non-degenarate degree two homogeneous space in the $(2n+1)$-dimensional complex projective space $P_{2n+1}(\mathbb{C})$ which is defined as

$$Q_{2n} = \{z \in P_{2n+1}(\mathbb{C}) \mid z_1 z_2 + \cdots + z_{2n+1} z_{2n+2} = 0\}$$

where $z = [z_1 : \ldots : z_{2n+2}] \in P_{2n+1}(\mathbb{C})$. This manifold Q_{2n} has an $(n+1)$-dimensional torus T^{n+1} action as follows:

$$(t_1, \cdots, t_{n+1}) \circ [z_1 : \ldots : z_{2n+2}] = [t_1 z_1 : t_1^{-1} z_2 : \ldots : t_{n+1} z_{2n+1} : t_{n+1}^{-1} z_{2n+2}],$$

where $(t_1, \cdots, t_{n+1}) \in T^{n+1}$. Then this action satisfies the GKM-condition. It has $2n+2$ fixed points and the axial function $\alpha(pq) = -\alpha(qp)$. The shape of graph is the complete graph except all diagonal edges. The following Figure 15.3 is the quaternionic torus graph coming from T^3-action on Q_4.

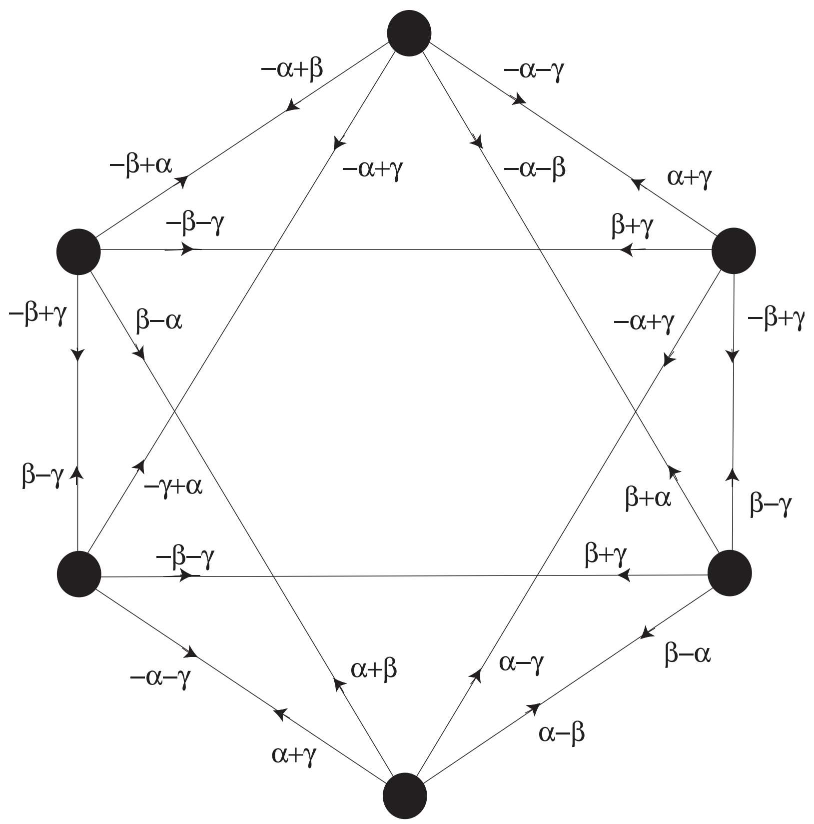

FIGURE 15.3. The quaternionic torus graph associated with T^3-action on Q_4

In next section we will state a main theorem.

16. Equivariant graph cohomology of hypertorus graph
—Main theorem and Preparation—

In this section we state a main theorem of Part 2. To state a theorem, we prepare some terminologies. From now on (Γ, α, θ) means a hypertorus graph. First we give a definitoin of a *pre-hyperfacet*.

Definition[pre-hyperfacet]. Put a subgraph $H = (V^H, \mathcal{E}^H) \subset \Gamma = (V^\Gamma, \mathcal{E}^\Gamma)$ such that $|\mathcal{E}_p^H| = 2n - 1$ or $2n$ for all $p \in V^H$, where $\mathcal{E}_p^H = \mathcal{E}_p^\Gamma \cap \mathcal{E}^H$ and $|\mathcal{E}_p^\Gamma|$ means a number of out going edges and legs on $p \in V^H$ in H. Moreover this H is closed by a connection on Γ, that is $\theta_{pq}|_H : \mathcal{E}_p^H \to \mathcal{E}_q^H$ is bijective (if $|\mathcal{E}_p^H| = |\mathcal{E}_q^H|$) or injective (if $|\mathcal{E}_p^H| < |\mathcal{E}_q^H|$) and if $|\mathcal{E}_p^H| < |\mathcal{E}_q^H|$ then $\theta_{pq}|_H$ satisfies

$$\alpha(e) - \alpha(\theta_{pq}(e)) = 0 \ (\mathrm{mod}\ \alpha(pq)) \text{ and}$$
$$\alpha(h) - x = 0 \ (\mathrm{mod}\ \alpha(pq))$$

where $h \in \mathcal{E}_q^\Gamma$ is an element which is not in $\mathrm{Im}\theta_{pq}|_H$ ($h \notin \mathrm{Im}\theta_{pq}|_H$). We call H a *pre-hyperfacet*.

If the edge $pq \in \mathcal{E}^H$ satisfies $|\mathcal{E}_p^H| < |\mathcal{E}_q^H|$, then the element of $\mathcal{E}_p^\Gamma - \mathcal{E}_p^H$ denotes by $n_H(p)$ and we call it a *normal edge* or *leg* of H on p . The following proposition is easy to show by the definition.

PROPOSITION 16.1. *The normal edge (or leg) $n_H(p)$ stisfies $\theta_{pq}(n_H(p)) \notin \mathrm{Im}\ \theta_{pq}|_H$ for $pq \in \mathcal{E}_p^H$ which satisfies a number of edges (or legs) $|\mathcal{E}_q^H| = 2n$.*

PROPOSITION 16.2. *Let $n_H(p) = e^+$ be a normal edge (or leg) of a pre-hyperfacet H. If the vertex $q \in V^H$ satisfies the assumption of Proposition 16.1, then we have $pq = e^-$.*

PROOF. From Proposition 16.1, we see $\theta_{pq}(e^+) = h^+ \notin \mathrm{Im}\theta_{pq}|_H$. Hence we have

$$\alpha(h^+) - x = k\alpha(pq)$$

for some integer k by the definition of pre-hyperfacet. Since our GKM-graph is a pairwise linearly independent on q and the equations $\alpha(h^+) + \alpha(h^-) = x$ and $\alpha(pq) = \pm\alpha(qp)$, we get $qp = h^-$ and $k = 1$ or -1. By the equation $\theta_{pq}(e^+) = h^+$ and the congruence relation on pq, the following equation holds for some integer k':

$$\alpha(e^+) - \alpha(h^+) = k'\alpha(pq).$$

Hence we get $\alpha(e^+) - (k + k')\alpha(pq) = x$ from the above equations. Because our GKM-graph is a pairwise linearly independent on p and $\alpha(e^+) + \alpha(e^-) = x$, we have

$$\alpha(pq) = \alpha(e^-) \text{ and } k + k' = -1.$$

Hence we have $pq = e^-$. □

Next we start to mention generators of the equivariant graph cohomology $H_T^*(\Gamma, \alpha)$.

Definition[Thom class of pre-hyperfacet]. Define $\tau_H : V^\Gamma \to H^2(BT)$ by

$$\tau_H(p) = \begin{cases} 0 & p \notin V^H \\ x & |\mathcal{E}_p^H| = 2n \\ \alpha(n_H(p)) & |\mathcal{E}_p^H| = 2n-1. \end{cases}$$

We call τ_H a *Thom class* of the pre-hyperfacet H.

Then we have $\tau_H \in H_T^*(\Gamma, \alpha)$ from the following proposition.

PROPOSITION 16.3. *A Thom class τ_H is an element of an equivariant graph cohomology $H_T^*(\Gamma, \alpha)$.*

PROOF. A Thom class is a map $\tau_H : V^\Gamma \to H^2(BT)$, so we check this map satisfies the conditions of an element in $H_T^*(\Gamma, \alpha)$ (the congruence relation). In the case $|\mathcal{E}_p^H| = |\mathcal{E}_q^H| = 2n$ (pq is an edge), we have $\tau_H(p) - \tau_H(q) = x - x = 0$. So this map satisfies the congruence relation on pq if $|\mathcal{E}_p^H| = |\mathcal{E}_q^H| = 2n$, that is $\tau_H(p) - \tau_H(q) \equiv 0(\mathrm{mod}\ \alpha(pq))$. Because the pre-hyperfacet H is closed by connection θ of Γ, we have the congruence relation even if $|\mathcal{E}_p^H| = |\mathcal{E}_q^H| = 2n-1$.

If $|E_p^H| < |E_q^H|$ (resp. $|E_p^H| > |E_q^H|$), then we have $\tau_H(p) - \tau_H(q) = \alpha(e') - x$ (resp. $x - \alpha(e')$). From Proposition 16.2, the equation $\alpha(e') - x = -\alpha(pq)$ holds. Hence a Thom class τ_H satisfies the congruence relation for all edges. Therefore $\tau_H \in H_T^*(\Gamma)$. □

Thom classes will be generators of $H_T^*(\Gamma, \alpha)$.

Next we define an *opposite side* of pre-hyperfacet to except a Thom class associated with a disconnected pre-hyperfacet from genarators of $H_T^*(\Gamma, \alpha)$.

Definition[opposite side of pre-hyperfacet]. If a pre-hyperfacet $\bar{H}$ satisfies the following:

$$\tau_{\bar{H}} + \tau_H = x$$

for a pre-hyperfacet H, then we call $\bar{H}$ an *opposite side* of H.

The following proposition holds for the opposite side of the pre-hyperfacet.

PROPOSITION 16.4. *For all pre-hyperfacet* H *in the hypertorus graph* (Γ, α, θ)*, there is a unique opposite side* $\bar{H}$ *and the opposite side* $\bar{H}$ *is a pre-hyperfacet.*

PROOF. Take a pre-hyperfacet $H = (V^H, \mathcal{E}^H)$ in Γ which have $p \in V^H$ such that $|\mathcal{E}_p^H| = 2n-1$. We construct $\bar{H}$ as follows. If the vertex $q \in V^H$ has $2n$ out going edges (or legs) in H that is $\mathcal{E}_p^H = \mathcal{E}_p^\Gamma$, then we put $p \notin V^{\bar{H}}$ and $\mathcal{E}_p^{\bar{H}} = \emptyset$. If the vertex q is not in V^H ($q \notin V^H$), then we put $q \in V^{\bar{H}}$ and $\mathcal{E}_q^{\bar{H}} = \mathcal{E}_q^\Gamma$. If the vertex $r \in V^H$ has $2n-1$ edges or legs in H and $\mathcal{E}_r^H = \{e_1^+, \cdots, e_n^+, e_1^-, \cdots, e_{n-1}^-\}$, then we set $r \in V^{\bar{H}}$ and $\mathcal{E}_r^{\bar{H}} = \{e_1^+, \cdots, e_{n-1}^+, e_1^-, \cdots, e_n^-\}$. The above $\bar{H} = (V^{\bar{H}}, \mathcal{E}^{\bar{H}})$ is closed under the connection $\theta|_{\bar{E}}$ from Proposition 16.2. So this pre-hyperfacet $\bar{H}$ is an opposite side of H, that is $\tau_H + \tau_{\bar{H}} = x$, from the above construction and uniqueness is easy to show. □

The following Figure 16.1 is one of the pre-hyperfacet and its opposite side in the example of Figure 14.1, the value on each vertex is the value of its Thom class.

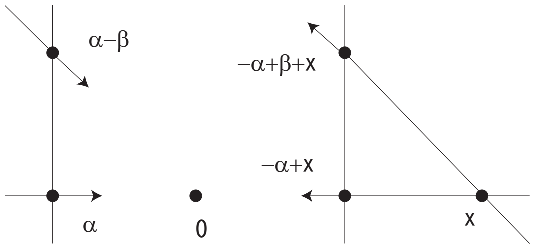

FIGURE 16.1. Thom class of pre-hyperfacet and its opposite side.

Here we state a generator of the equivariant graph cohomology of Γ.

Definition[hyperfacet]. We call a connected pre-hyperfacet a *hyperfacet* if its opposite side is connected.

The following Figure 16.2 is an example which is not a hyperfacet but a pre-hyperfacet.

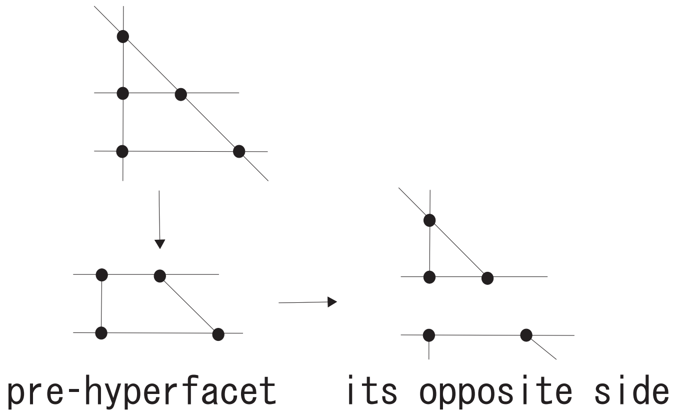

FIGURE 16.2

By the definition of hyperfacet and Proposition 16.4, we have the following proposition.

PROPOSITION 16.5. *For the hyperfacet* H, *its opposite side* $\bar{H}$ *is the hyperfacet.*

We prepare the following notation.

Definition[boundary of hyperfacet]. We denote $\partial H = H \cap \bar{H}$ where H is a hyperfacet and we call ∂H a *boundary* of hyperfacet.

From the connection of hyperfacet, we have the following proposition.

PROPOSITION 16.6. $\partial H = (V^{\partial H}, \mathcal{E}^{H} \cap \mathcal{E}^{\bar{H}})$ *is a codimension two* ($(2n-2)$*-valent*) *hypertorus subgraph.*

For the codimension two hypertorus subgraph, the following proposition holds.

PROPOSITION 16.7. *For the vertex* $p \in V^{\Gamma}$ *and the edge (or leg)* $e \in \mathcal{E}_p^{\Gamma}$*, there is a unique codimension two hypertorus subgraph* $\Gamma' = (V^{\Gamma'}, \mathcal{E}^{\Gamma'})$ *whose normal edge (or leg) on* p *is* e*.*

PROOF. Put $e = e^+ \in \mathcal{E}_p^{\Gamma}$. Let $\{e^+,\ e^-\}$ be a pair in $\mathcal{E}_p^{\Gamma}$. Then we can construct a $(2n-2)$-valent hypertorus subgraph Γ' such that $\mathcal{E}_p^{\Gamma'} = \mathcal{E}_p^{\Gamma} - \{e^+,\ e^-\}$ as follows.

First we take $n-1$ lines via p which contain $\mathcal{E}^{\Gamma} - \{e^+,\ e^-\} = \mathcal{E}_p^{\Gamma'}$. We denote an abstruct *graph* which is defined by these lines by $\mathcal{L}_p$. Take $q \in \mathcal{L}_p$ such that $pq \in \mathcal{E}^{\Gamma}$. Then we can take $\mathcal{E}_q^{\Gamma} - \{\theta_{pq}(e^+),\ \theta_{pq}(e^-)\} = \mathcal{E}_q^{\Gamma'}$. From Corollary 14.1, $\mathcal{E}_q^{\Gamma'}$ consists of $n-1$ pairs in $\mathcal{E}_q^{\Gamma}$ and the restricted bijection $\theta_{pq}|_{\mathcal{E}_p^{\Gamma'}} : \mathcal{E}_p^{\Gamma'} \to \mathcal{E}_q^{\Gamma'}$ is well-defined. Next we take $n-1$ lines via q which contain $\mathcal{E}_q^{\Gamma'}$ (and denote it by $\mathcal{L}_q$). Similarly we can get a graph $\Gamma_1 = \cup_{q \in V^{\mathcal{L}_p}} \mathcal{L}_q$.

If this graph Γ_1 is $(2n-2)$-valent graph then we get a codimension two hypertorus subgraph that we want. Assume this graph Γ_1 has a vertex r which is not $(2n-2)$-valent. Then there is a path l from p to r. Denote the edge (or leg) in $\mathcal{E}_r^{\Gamma}$ which corresponds to e^+ by $\theta_l(e^+)$. If we can take two diffenrent paths $l_1,\ l_2$ from p to r. Then we have $\theta_{l_1}(e^+) = \theta_{l_2}(e^+)$ or $\{\theta_{l_1}(e^+),\ \theta_{l_2}(e^+)\}$ is a pair in $\mathcal{E}_r^{\Gamma}$, because of Corollary 14.1, the congruence relation and the definition of the hypertorus graph that if we put $\{e_1^+,\ \cdots,\ e_n^+\} \subset \mathcal{E}_p^{\Gamma}$, then $\langle x,\ \alpha(e_1^+),\ \cdots,\ \alpha(e_n^+)\rangle \simeq t_{\mathbb{Z}}$ for all p. Hence we get a $(2n-2)$-valent hypertorus subgraph $\Gamma' \subset \Gamma$ to apply the similar argument. □

Before to state a main theorem, we prepare a notation.

Notation. Let (Γ, α, θ) be a hypertorus graph. Denote the set of all hyperfacet of Γ by $\mathcal{H}$. The algebra $\mathbb{Z}[\Gamma, \theta]$ is as follows:

$$\mathbb{Z}[\Gamma,\ \theta] = \mathbb{Z}[x,\ H \mid H \in \mathcal{H}]/\mathcal{I},$$

where $\mathbb{Z}[x,\ H \mid H \in \mathcal{H}]$ is a polynomial ring generated by all hyperfacets of Γ, x, and the ideal $\mathcal{I}$ is generated by

$$H + \bar{H} - x \text{ for all } H \in \mathcal{H} \text{ and}$$
$$\prod_{H \in \mathcal{H}'} H \text{ where } \mathcal{H}' \subset \mathcal{H} \text{ is the set } \bigcap_{H \in \mathcal{H}'} H = \emptyset.$$

Let us state a main theorem.

MAIN THEOREM 2. *Assume for each codimension two hypertorus subgraph* L *there is a unique hyperfacet* H *and its opposite side* $\bar{H}$ *such that* $\partial H = L$*, and* $H \cap G = \emptyset$ *or connected for all hyper facets* H *and* G*. Then there is the following isomorphism:*

$$H_T^*(\Gamma, \alpha) \simeq \mathbb{Z}[\Gamma, \theta].$$

Before to show the above theorem we prepare notation.

Definition[neighborhood of subgraph]. Let H be a subgraph of Γ. Put $N(H)$ be a $2n$-valent graph in Γ which satisfies the following properties:

$$V^{N(H)} = V^{H};$$
$$\mathcal{E}_p^{N(H)} = \mathcal{E}_p^{H} \text{ if } |\mathcal{E}_p^{H}| = 2n;$$
$$\mathcal{E}_q^{N(H)} = \mathcal{E}_q^{H} \cup \{l(n(q)_1), \cdots, l(n(q)_k)\} \text{ if } |\mathcal{E}_q^{H}| = 2n - k,$$

where $\{n(q)_1, \cdots, n(q)_k\} = \mathcal{E}_q^{\Gamma} - \mathcal{E}_q^{H}$. Here if $n(q)$ is a leg then $l(n(q)) = n(q)$, if not so then we regard the edge $n(q)$ as a leg whose initial vertex is q (denote it by $l(n(q))$). We call $N(H)$ a *neighborhood* of the subgraph H in Γ.

Remark. We *do not* call a neighborhood $N(H)$ a subgraph of Γ if $N(H)$ has a leg $l(n(q))$ such that $n(q)$ is an edge in Γ. Of course the neighborhoods $N(H)$ is a hypertorus graph for every hyperfacet H.

The following figure is an image of the neighborhood of pre-hyperfacet in Figure 14.1. The upper image is an example whose neighborhood is not a subgraph in Γ.

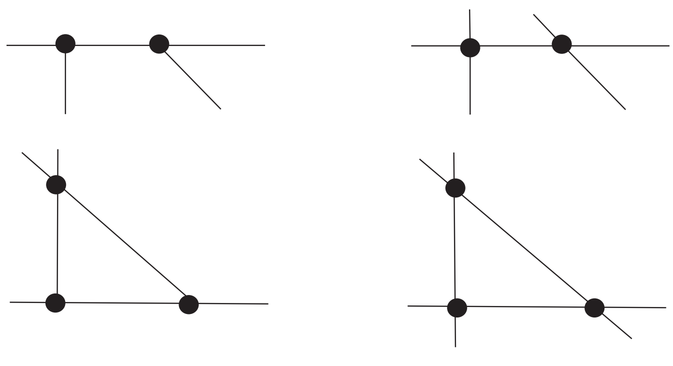

FIGURE 16.3. Hyperfacet and its neighborhood.

From the next section we will prove the main theorem.

17. Proof of the main theorem

In this section we show the main theorem. The program of proof is first we will prove the case $|V^{\Gamma}| = 1$, and next we will prove about a *minimal* hypertorus graph by the inductive argument for $|V^{\Gamma}|$, finally we will prove the general hypertorus graph by the inductive argument and the Mayer-Vietoris analogue dividing a non minimal hypertorus graph into two hypertorus graphs.

17.1. The case $|V^\Gamma| = 1$.

First of all we prove Theorem 17.1 about the easiest hypertorus graph, that is

$$\Gamma = (\{p\}, \{e_1^+, \cdots, e_n^+, e_1^-, \cdots, e_n^-\}),$$

where e_i^+ and e_i^- are legs for all $i = 1, \cdots, n$.

Remark. If the *quaternionic torus graph* Γ has only one vertex ($|V^\Gamma| = 1$), then Γ is always the above graph. So the following theorem also holds on the quaternionic torus graph.

THEOREM 17.1. *Set the hypertorus graph $\Gamma = (\{p\}, \mathcal{E}^\Gamma)$, that is the graph consists of only one vertex and $2n$ legs. Then we have $\mathbb{Z}[\Gamma, \theta] \simeq H_T^*(\Gamma, \alpha)$.*

PROOF. We can put $\mathcal{E}^\Gamma = \{e_1^+, \cdots, e_n^+, e_1^-, \cdots, e_n^-\}$. By the definition of hypertorus graph, we have $\langle \alpha(e_1^+), \cdots, \alpha(e_n^+)\rangle \simeq \mathfrak{t}_{\mathbb{Z}}^n$ and $\alpha(e_i^-) = x - \alpha(e_i^+)$ for all $i = 1, \cdots, n$. Put $\alpha_i = \alpha(e_i^+)$. Then we have

$$H_T^*(\Gamma, \alpha) = \{f : \{p\} \to H_T^*(pt)\} \simeq H_T^*(pt) = H^*(BT) = \mathbb{Z}[x, \alpha_1, \cdots, \alpha_n]$$

where $\langle x, \alpha_1, \cdots, \alpha_n\rangle = \mathfrak{t}_{\mathbb{Z}}$.

Let $\mathcal{H}$ be all hyperfacets in Γ. Then we can put $\mathcal{H} = \{H_1, \cdots, H_n, \bar{H}_1, \cdots, \bar{H}_n\}$ such that $\tau_{H_i}(p) = \alpha_i$ (that is $n_{H_i}(p) = e_i^+$ and $n_{\bar{H}_i}(p) = e_i^-$) from the definition of the hyperfacet and $|V^\Gamma| = 1$. Since the intersection of all hyperfacets is $\cap_{H \in \mathcal{H}} = \{p\}$, we have

$$\mathbb{Z}[\Gamma, \theta] = \mathbb{Z}[x, H \mid H \in \mathcal{H}]/\langle H + \bar{H} - x \mid H \in \mathcal{H}\rangle.$$

By the above ideal $\langle H + \bar{H} - x \mid H \in \mathcal{H}\rangle = \langle H_i + \bar{H}_i - x \mid i = 1, \cdots, n\rangle = \mathcal{I}$, we can assume $[\bar{H}_i] = [x - H_i]$ in $\mathbb{Z}[\Gamma, \theta]$, so we get the natural surjective homomorphism

$$\varphi : \mathbb{Z}[x, H_1, \cdots, H_n] \to \mathbb{Z}[\Gamma, \theta].$$

Next we put

$$\rho : \mathbb{Z}[\Gamma, \theta] \to \mathbb{Z}[x, H_1, \cdots, H_n]$$

by $\rho([x]) = x$, $\rho([H_i]) = H_i$ and $\rho([\bar{H}_i]) = x - H_i$. If $[X] = [Y] \in \mathbb{Z}[\Gamma, \theta]$ then $X - Y \in \mathcal{I} \subset \mathbb{Z}[x, H \mid H \in \mathcal{H}]$, that is $X - Y = Z(H_i + \bar{H}_i - x)$ for some i and $Z \in \mathbb{Z}[x, H \mid H \in \mathcal{H}]$. Hence we have $\rho([X]) = \rho([Y])$ from $\rho([X - Y]) = \rho([Z])(H_i + \rho([\bar{H}_i]) - x) = 0$. So this map ρ is a well-defined homomorphism. Because the composite map is $\rho \circ \varphi = id$ from the definition of ρ, we have φ is an isomorphism and $\rho = \varphi^{-1}$. So we get

$$\mathbb{Z}[\Gamma, \theta] \simeq \mathbb{Z}[x, H_1, \cdots, H_n].$$

Hence the following map is isomorphic:

$$\Psi : \mathbb{Z}[\Gamma, \theta] \xrightarrow{\rho} \mathbb{Z}[x, H_1, \cdots, H_n] \xrightarrow{\psi} \mathbb{Z}[x, \alpha_1, \cdots, \alpha_n] \simeq H_T^*(\Gamma, \alpha)$$

such that $\psi(x) = x$ and $\psi(H_i) = \tau_{H_i}(p) = \alpha_i$. Therefore we have $\Psi([x]) = x$, $\Psi([H_i]) = \tau_{H_i}$ and $\Psi([\bar{H}_i]) = x - \tau_{H_i} = \tau_{\bar{H}_i}$. □

Next we show our main theorem about the subclass (*minimal hypertorus graphs*) of the hypretorus graphs.

17.2. The case where Γ is the minimal hypertorus graph.

A minimal hypertorus graph is a hypertorus graph which can not divide into two hypertorus graphs. The rigorous definition is as follows.

Definition[minimal hypertorus graph]. Let Γ be a hypertorus graph. We put a set of all hyperfacets in Γ as follows:

$$\mathcal{H} = \{H_1, \cdots, H_m, \bar{H}_1, \cdots, \bar{H}_m\}.$$

Then we call Γ is a *minimal* if it satisfies the neighborhood of $\bar{H}_i$ coincides with Γ, that is $N(\bar{H}_i) = \Gamma$ for all $i = 1, \cdots, m$.

In this section, we show the main theorem on all minimal hypertorus graphs.

17.2.1. *Injectivity.*

First we show the following lemma.

LEMMA 17.1. *Let Γ be a minimal hypertorus graph and $\mathcal{H} = \{H_1, \cdots, H_m, \bar{H}_1, \cdots, \bar{H}_m\}$ is a set of all hyperfacets in Γ such that $N(\bar{H}_i) = \Gamma$ for all $i = 1, \cdots, m$. Then there is the following isomorphism*:

$$\begin{aligned} \mathbb{Z}[\Gamma, \theta] &= \mathbb{Z}[x, H_1, \cdots, H_m, \bar{H}_1, \cdots, \bar{H}_m]/\mathcal{I} \\ &\simeq \mathbb{Z}[x, H_1, \cdots, H_m]/\langle \prod_{H \in \mathcal{H}'} H \mid \mathcal{H}' \subset \{H_1, \cdots, H_m\}\rangle, \end{aligned}$$

where $\mathcal{H}'$ is a set of disjoint hyperfacets in $\{H_1, \cdots, H_m\}$.

PROOF. Because the relation $N(\bar{H}_i) = \Gamma$ holds, we have $V^{\Gamma} = V^{\bar{H}_i}$ for all $i = 1, \cdots, m$. So we see that if $\bar{H}_i \cap (\cap_{j=1}^{l} H_j) = \emptyset$, then $\cap_{j=1}^{l} H_j = \emptyset$. Hence we can put

$$\mathcal{I} = \langle \prod_{H \in \mathcal{H}'} H, \ H + \bar{H} - x\rangle$$

such that $\mathcal{H}' \subset \{H_1, \cdots, H_m\}$.

Put the ideal $\mathcal{I}'$ as follws:

$$\mathcal{I}' = \langle \prod_{H \in \mathcal{H}'} H\rangle.$$

Because we have $[\bar{H}_i] = [x - H_i]$ for all $i = 1, \cdots, m$ in $\mathbb{Z}[\Gamma, \theta]$, the following map is well-defined and surjective:

$$\varphi : \mathbb{Z}[x, H_1, \cdots, H_m]/\mathcal{I}' \to \mathbb{Z}[x, H_1, \cdots, H_m, \bar{H}_1, \cdots, \bar{H}_m]/\mathcal{I}$$

such that $\varphi((x)) = [x], \ \varphi((H_i)) = [H_i]$.

Moreover the following map ρ is an inverse map of φ:

$$\rho : \mathbb{Z}[x, H_1, \cdots, H_m, \bar{H}_1, \cdots, \bar{H}_m]/\mathcal{I} \to \mathbb{Z}[x, H_1, \cdots, H_m]/\mathcal{I}'$$

such that $\rho([x]) = (x), \ \rho([H_i]) = (H_i), \ \rho([\bar{H}_i]) = (x - H_i)$. Hence φ is an isomorphism. Therefore we get $\mathbb{Z}[\Gamma, \theta] \simeq \mathbb{Z}[x, H_1, \cdots, H_n]/\mathcal{I}'$. □

Next we put the homomorphism $\Psi : \mathbb{Z}[\Gamma, \theta] \to H_T^*(\Gamma, \alpha)$ as follows:

$$\Psi([x]) = x;$$
$$\Psi([H]) = \tau_H.$$

Then this map is well-defined by the following equations:

$$\tau_H + \tau_{\bar{H}} = x;$$
$$\prod_{H \in \mathcal{H}'} \tau_H = 0,$$

where $\mathcal{H}'$ is a set in $\mathcal{H}$ such that the intersection of all elements is $\cap\{H \in \mathcal{H}'\} = \emptyset$. The following lemma holds for this map Ψ.

LEMMA 17.2 (Injectivity). *Let Γ be a minimal hypertorus graph. If there exists unique hyperfacet H and its opposite side $\bar{H}$ such that $\partial H = L$ for every codimension two hypertorus subgraph L, then Ψ is injective.*

PROOF. Define $\psi : \mathbb{Z}[x,\ H_1,\ \cdots,\ H_m]/\mathcal{I}' \to H_T^*(\Gamma, \alpha)$ as follows:

$$\psi([x]) = x$$
$$\psi(H_i) = \tau_{H_i}.$$

Then we have $\psi \circ \rho = \Psi$. So the injectivity of Ψ is equivalent to the injectivity of ψ. We will prove the injectivity of ψ.

Put $\mathbb{Z}[\Gamma]_p = \mathbb{Z}[x,\ H_1,\ \cdots,\ H_m]/\langle H \mid p \notin V^H \rangle$. Then we have

$$\mathbb{Z}[\Gamma]_p \simeq \mathbb{Z}[x,\ H \mid p \in V^H].$$

Put the homomorphism $\psi_p : \mathbb{Z}[x,\ H \mid p \in V^H] \to H_T^*(pt)$ as follows:

$$\psi_p(x) = x(p),$$
$$\psi_p(H) = \tau_H(p).$$

Now the hypertorus graph Γ is minimal and the opposite side of the generator H of $\mathbb{Z}[x,\ H \mid p \in V^H]$ is $N(\bar{H}) = \Gamma$. So we have if $p \in V^H$ then $p \in V^{\partial H}$. Since the axial functions around of the vertex p and x span $\mathfrak{t}$ and we have Proposition 16.7 and the assumption of this lemma, we have this map ψ_p is isomorphic that is there is the following isomorphism:

$$\mathbb{Z}[\Gamma]_p \simeq \mathbb{Z}[x,\ H \mid p \in V^{\partial H}] \simeq H_T^*(pt).$$

We can put $\chi_p : \mathbb{Z}[x, H_1, \cdots, H_m]/\mathcal{I}' \to \mathbb{Z}[\Gamma]_p$ by the canonical surjection because of $\mathcal{I}' \subset \langle H \mid p \notin V^H \rangle$. So we have the following commutative diagram:

$$\begin{array}{ccc} \mathbb{Z}[x,\ H_1,\ \cdots,\ H_m]/\mathcal{I}' & \xrightarrow{\chi} & \oplus_{p \in V^\Gamma} \mathbb{Z}[\Gamma]_p \\ \psi \downarrow & & \downarrow\simeq \\ H_T^*(\Gamma, \alpha) & \xrightarrow{\phi} & \oplus_{p \in V^\Gamma} H_T^*(pt), \end{array}$$

where $\chi = \oplus_{p\in V^\Gamma}\chi_p$ and $\phi(f) = \oplus_{p\in V^\Gamma} f(p)$. Because ϕ is injective, if the injectivity of χ is known then we have the injectivity of ψ.

Since we have $\mathrm{Ker}(\chi_p) = \langle H \mid p \notin V^H\rangle/\mathcal{I}'$, the followings hold:

$$\begin{aligned}\mathrm{Ker}(\chi) &= \mathrm{Ker}(\oplus_{p\in V^\Gamma}\chi_p)\\ &= \cap_{p\in V^\Gamma}\mathrm{Ker}\chi_p\\ &= \cap_{p\in V^\Gamma}(\langle H \mid p \notin V^H\rangle/\mathcal{I}')\\ &= (\cap_{p\in V^\Gamma}\langle H \mid p \notin V^H\rangle)/\mathcal{I}'.\end{aligned}$$

Hence we assume $\chi([X]) = 0$ for some element $[X] \in \mathbb{Z}[x,\ H_1,\ \cdots,\ H_m]/\mathcal{I}'$, then we have

$$[X] \in (\cap_{p\in V^\Gamma}\langle H \mid p \notin V^H\rangle)/\mathcal{I}'.$$

Assume $X \in \cap_{p\in V^\Gamma}\langle H \mid p \notin V^H\rangle \subset \mathbb{Z}[x,\ H_1,\ \cdots,\ H_m]$. Then we can denote uniquely by

$$X = \sum_{a,\ a_1,\ \cdots,\ a_m} k_{(a,\ a_1,\ \cdots,\ a_m)} x^a H_1^{a_1}\cdots H_m^{a_m}$$

for some $k_{(a,\ a_1,\ \cdots,\ a_m)} \in \mathbb{Z}$, because there is no relation on $x^a H_1^{a_1}\cdots H_m^{a_m}$ and $x^{a'} H_1^{a_1'}\cdots H_m^{a_m'}$ in $\mathbb{Z}[x,\ H_1,\ \cdots,\ H_m]$ if $(a,\ a_1,\ \cdots,\ a_m) \neq (a',\ a_1',\ \cdots,\ a_m')$. Because of $X \in \langle H \mid p \notin V^H\rangle$, we have there is a hyperfacet H_j such that $p \notin V^{H_j}$ and $a_j \neq 0$ for each term $k_{(a,\ a_1,\ \cdots,\ a_m)} x^a H_1^{a_1}\cdots H_m^{a_m}$ of X. This fact holds for all $p \in V^\Gamma$ because $X \in \cap_{p\in V^\Gamma}\langle H \mid p \notin V^H\rangle$. Hence there are $j_1,\ \cdots,\ j_r$ such that $\cap_{s=1}^r H_{j_s} = \emptyset$ and $a_{j_1},\ \cdots,\ a_{j_s} \neq 0$ for each term of X. This means $X \in \mathcal{I}'$. Hence we assume $\chi([X]) = 0$ then $[X] = 0$ in $\mathbb{Z}[x,\ H_1,\ \cdots,\ H_m]/\mathcal{I}'$. So we have the injectivity of χ. □

17.2.2. *Surjectivity.*

Next we prepare the following concept of the hypertorus graph.

Definition[line and end point]. Let Γ be a hypertorus graph. We call the 2-valent hypertorus subgraph l in Γ a *line*. We call an *end point of the line* l such that $\mathcal{E}_p^l$ has a leg. We also call an *end point of* Γ such that p is an end point of all line which through on p.

For the end point of the minimal hypertorus graph Γ, we have the following properties.

LEMMA 17.3. *Let Γ be a minimal hypertorus graph which satisfies the condition there is a unique pair hyperfacet H and its opposite side $\bar{H}$ for all codimension two hypertorus subgraph in Γ, then all the elementes of V^Γ are end points of Γ.*

PROOF. From Proposition 16.7 and an assumption of this Lemma, we can take H_i $(i = 1, \cdots, n)$ such that $N(H_i) = \Gamma$ for a vertex $p \in V^\Gamma$ as it has a normal edge $e \in \mathcal{E}_p^\Gamma$. Now the normal edge e on p of H_i is a leg because of $V^{H_i} = V^\Gamma$. Hence p is an end point of all lines through on p. So the vertex p is an end point of Γ. □

Moreover we have the following proposition for the hypertorus graph which is in Lemma 17.3

PROPOSITION 17.1. *Let* Γ *be a hypertorus graph. If all vertices are end points of* Γ;, *then the number of vertices on* l *is* $|V^l| \leq 2$ *for all lines* $l \subset \Gamma$.

PROOF. Assume there is a line l such that $|V^l| \geq 3$. Then there is a vertex $p \in V^l$ which is not an end point for this line l. Hence this is a contradiction. □

The following lemma will be used to prove the surjectivity of $\Psi : \mathbb{Z}[\Gamma, \theta] \to H_T^*(\Gamma, \alpha)$ for the minimal hypertorus graph Γ. First we prove the following proposition.

PROPOSITION 17.2. *Let* Γ *be a graph. Then there exists a vertex* $p \in V^\Gamma$ *such that* $\Gamma - N(p)$ *is connected.*

PROOF. We show the statement by the inductive argument. If the number of vertices $|V^\Gamma| = 2$, then we easily see this proposition. Assume the statement of this proposition holds for all Γ such that $|V^\Gamma| < k$. When the number of vertices $|V^\Gamma| = k$, take a vertex $p \in V^\Gamma$. If $\Gamma - N(p)$ is not connected, then we can put $\Gamma - N(p) = \Gamma_1 \cup \Gamma_2$. Because we have $|V^{\Gamma_1}| < |V^\Gamma| = k$ and the assumption of the induction, there exists $q \in V^{\Gamma_1} \subset V^\Gamma$ such that $\Gamma_1 - N(q)$ is connected. Hence $\Gamma - N(q)$ is connected. □

LEMMA 17.4. *Let* Γ *be a hypertorus graph. There is a vertex* $p \in V^\Gamma$ *such that* $L - N(p) \cap L$ *is connected for all codimension two hypertorus subgraph* L.

PROOF. If $|V^\Gamma| = 2$, then we can easily show this lemma. Assume this statement holds all hypertorus graph Γ such that $|V^\Gamma| \leq k - 1$. If we put $|V^\Gamma| = k$, we can take $p \in V^\Gamma$ such that $\Gamma' = \Gamma - N(p)$ is connected from Proposition 17.2. Then Γ' is a hypertorus graph which has $k - 1$vertices. So from our assumption, there is a vertex $q \in V^{\Gamma'} \subset V^\Gamma$ such that $\Gamma' - N(q)$ satisfies the statement of this lemma.

Now we denote the codimension two hypertorus subgraph in Γ' by L'. Then there is a codimension two hypertorus subgraph L of Γ such that $L' \subset L$. If $p \notin V^L$, then $L = L'$. If not so then there are two cases where

(1) $L' = L - L \cap N(p)$ is connected

(2) $L - L \cap N(p)$ is a disjoint union $L' \cup L''$.

In each above case, $(L' - N(q) \cap L') \cup (L \cap N(p))$ is connected. Because $N(q) \cap L' = N(q) \cap L$ and $p \neq q$, we have $(L' - N(q) \cap L') \cup (L \cap N(p)) = L - N(q) \cap L$. Hence we have $L - N(q) \cap L$ is connected for all codimension two hypertorus graph L in Γ. □

Let us prove the surjectivity.

LEMMA 17.5 (Surjectivity). *Let* Γ *be a minimal hypertorus graph. If it holds* $H_1 \cap H_2 = \emptyset$ *or connected for all hyperfacets* H_1 *and* H_2 *in* Γ *and there is a unique pair* $\{H, \bar{H}\}$ *such taht* $\partial H = L$ *for every codimension two hypertorus subgraph* L, *then* Ψ *is surjective.*

PROOF. We only show the surjectivity of ψ because of $\Psi = \psi \circ \rho$. There is the following commutative diagram:

$$\begin{array}{ccc} \mathbb{Z}[x, H_1, \cdots, H_m] & \stackrel{p}{\longrightarrow} & \mathbb{Z}[x, H_1, \cdots, H_m]/\mathcal{I}' \\ \pi \downarrow & & \downarrow \psi \\ H_T^*(\Gamma, \alpha) & = & H_T^*(\Gamma, \alpha) \end{array}$$

where the natural projection p is surjective. So we will show that π is surjective by the inductive argument for $|V^\Gamma|$.

If $|V^\Gamma| = 1$, then we have this lemma by Theorem 17.1, hence π is surjective. Assume the surjectivity of π holds for all minimal hypertorus graphs Γ such that $|V^\Gamma| < k$ and take the minimal hypertorus graph Γ such that $|V^\Gamma| = k$. From Lemma 17.4, we can take $p \in V^\Gamma$ such taht $\Gamma' = \Gamma - N(p)$ is connected and $L - L \cap N(p)$ is connected for all codimension two hypertorus subgraphs L. Now we have, for the edge (or leg) e in Γ', there is a codimension two hypertorus graph L' of Γ' from Proposition 16.7 such taht e is a normal edge (leg) of L'. Moreover for this e we can take codimension two hypertorus subgraph L of Γ such taht e is a normal edge (leg) of L. Then we see $L' = L - L \cap N(p)$. Because of our assumption, we can take hyperfacets H and $\bar{H}$ such that $\partial H = L$. We also have $H' = H - H \cap N(p)$ is a hyperfacet of Γ' such that $\partial H' = L'$. Hence Γ' is a minimal hypertorus graph and it satsfies our assumptions. Moreover we have $|V^{\Gamma'}| = k - 1$, so we have the following map $\pi' \circ r$ is surjective from the assumption of the induction:

$$\begin{array}{ccc} \mathbb{Z}[x,\ H_1,\ \cdots,\ H_m] & \stackrel{\pi}{\longrightarrow} & H_T(\Gamma, \alpha) \\ r \downarrow & & \downarrow r' \\ \mathbb{Z}[x,\ H_1',\ \cdots,\ H_l'] & \stackrel{\pi'}{\longrightarrow} & H_T(\Gamma', \alpha|_{\mathcal{E}^{\Gamma'}}), \end{array}$$

where $r(H) = H \cap \Gamma'$, $r(x) = x$ and $r'(f) = f|_{\Gamma'}$. Hence we have $r' \circ \pi = \pi' \circ r$ is surjective. So all $f|_{\Gamma'}$ are denoted by $\mathbb{Z}[x, H_1, \cdots, H_m]$ as identified $H'(= H \cap \Gamma')$ and H.

Because we see $g = f - f|_{\Gamma'}$ is in $H_T^*(\Gamma, \alpha)$ and $g(q) = 0$ for all $q \neq p$, the following equation holds from the definition of $H_T^*(\Gamma, \alpha)$ and the definition of the hyperfacet:

$$g(p) = k \prod_{pq \in E_p^\Gamma} \alpha(pq) = k \prod_{pq \in E_p^\Gamma} \tau_{H_q}(p),$$

where H_q is the hyperfacet whose normal edge on p is pq and some element $k \in H_T^*(pt)$. Moreover we have the vertices of $X = \cap_{pq \in E_p^\Gamma} H_q$ is only one point p that is

$$\{p\} = V^X$$

from the assumption that $H \cap H' = \emptyset$ or coonected for all hyper facets H and H'. Therefore we see

$$g = k \prod_{pq \in E_p^\Gamma} H_q.$$

Hence we have $f = f|_{\Gamma'} + g \in \mathrm{Im}(\pi)$. □

Hence we have the following theorem from Lemma 17.2 and 17.5.

THEOREM 17.2. *Let* Γ *be a minimal hypertorus graph. If it holds* $H_1 \cap H_2 = \emptyset$ *or connected for every hyperfacet* H_1 *and* H_2 *in* Γ *and there is a unique pair* $\{H, \bar{H}\}$ *such that* $H \cap \bar{H} = L$ *for all codimension two hypertorus subgraphs* L, *then we have* $\mathbb{Z}[\Gamma, \theta] \simeq H_T^*(\Gamma, \alpha)$.

17.3. Proof of the main theorem.

In this section we will prove the main theorem. To prove it, we will use an inductive argument for $|V^\Gamma|$ and the Mayer-Vietoris analogue.

First of all we can assume the statement of Main Theorem 2 holds for all hypertorus graphs Γ such that $|V^\Gamma| < k-1$ because we have already known Main Theorem 2 holds for $|V^\Gamma| = 1$ by Theorem 17.1. We also have already known the statement of Main Theorem 2 holds for the minimal hypertorus graph by Theorem 17.2. So there is the codimension two hypertorus subgraph $L \subset \Gamma$ which has the unique hyperfacet H and $\bar{H}$ such that $H \cap \bar{H} = L$ and $N(H)$, $N(\bar{H}) \neq \Gamma$. Put these neighborhood $N(H) = \Gamma_1$, $N(\bar{H}) = \Gamma_2$ and $\Gamma_1 \cap \Gamma_2 = N(L) = \Gamma_3$. Then the graph $\Gamma_i = (V^{\Gamma_i}, \mathcal{E}^{\Gamma_i})$ is the hypertorus graph which has a restricted connection $\theta|_{\Gamma_i}$ and a restricted axial function $\alpha|_{\Gamma_i}$ of the hypertorus graph (Γ, α, θ) and all $i = 1, 2, 3$. Since we assume the condition $H \cap G = \emptyset$ or connected for all hyperfacets H and G of (Γ, α, θ), $H \cap \Gamma_i$ is connected for every hyperfacet H of (Γ, α, θ). Hence all hyperfacets of Γ_i are inherited from hyperfacets of (Γ, α, θ), that is the set of all hyperfacets of Γ_i is a set $\mathcal{H}_i = \{\Gamma_i \cap H \mid H \in \mathcal{H}\}$ for $i = 1, 2, 3$. So we also have these hypertorus graph satisfies conditions as follows:

1. There is a unique hyperfacet H and its opposite side $\bar{H}$ for every codimension two hyperfacet L in $(\Gamma_i, \alpha|_{\Gamma_i}, \theta|_{\Gamma_i})$ such that $\partial H = L$.

2. For all hyperfacets H and G, these intersection $H \cap G = \emptyset$ or connected.

From the assumption of the induction, we also have $\Psi_i : \mathbb{Z}[\Gamma_i, \theta|_{\Gamma_i}] \to H_T^*(\Gamma_i, \alpha|_{\Gamma_i})$ is isomorphic for each $i = 1, 2, 3$.

Put the homomorphism $\rho_1 : \mathbb{Z}[\Gamma, \theta] \to \mathbb{Z}[\Gamma_1, \theta|_{\Gamma_1}] \oplus \mathbb{Z}[\Gamma_2, \theta|_{\Gamma_2}]$ such that

$$\rho_1(H) = \Gamma_1 \cap H \oplus \Gamma_2 \cap H,$$
$$\rho_1(x) = x \oplus x,$$

and $\rho_2 : \mathbb{Z}[\Gamma_1, \theta|_{\Gamma_1}] \oplus \mathbb{Z}[\Gamma_2, \theta|_{\Gamma_2}] \to \mathbb{Z}[\Gamma_3, \theta|_{\Gamma_3}]$ such that

$$\rho_2(H_1 \oplus H_2) = \Gamma_3 \cap H_1 - \Gamma_3 \cap H_2,$$
$$\rho_2(x \oplus H_2) = x - \Gamma_3 \cap H_2,$$
$$\rho_2(H_1 \oplus x) = \Gamma_3 \cap H_1 - x,$$

where H (resp. H_i) is a hyperfacet of Γ (resp. Γ_i) and assume if $\Gamma_i \cap H = \emptyset$ then $\Gamma_i \cap H = 0$ in $\mathbb{Z}[\Gamma_i, \theta|_{\Gamma_i}]$. Because all hyperfacets in Γ_i are inherited from Γ, these maps are well-defined.

Then ρ_1 is injective because we have $\mathrm{Ker}(p_1 \circ \rho_1) \cap \mathrm{Ker}(p_2 \circ \rho_1) = \{0\} = \mathrm{Ker}(\rho_1)$ by the definition of the assumption if $\Gamma_i \cap H = \emptyset$ then $\Gamma_i \cap H = 0$ in $\mathbb{Z}[\Gamma_i, \theta|_{\Gamma_i}]$ and $\Gamma_1 \cup \Gamma_2 = \Gamma$, where $p_i = \mathbb{Z}[\Gamma_1, \theta|_{\Gamma_1}] \oplus \mathbb{Z}[\Gamma_2, \theta|_{\Gamma_2}] \to \mathbb{Z}[\Gamma_i, \theta|_{\Gamma_i}]$.

Because all hyperfacets of Γ_3 are inherited from Γ_1, we can get all generators G of $\mathbb{Z}[\Gamma_3, \theta|_{\Gamma_3}]$ by $\rho_2(H \oplus 0) = \Gamma_3 \cap H$ for some generator $H \in \mathbb{Z}[\Gamma_1, \theta|_{\Gamma_1}]$. So we see ρ_2 is surjective.

Moreover we have the following lemma.

LEMMA 17.6. *The following sequence is exact*:

$$\{0\} \longrightarrow \mathbb{Z}[\Gamma, \theta] \xrightarrow{\rho_1} \mathbb{Z}[\Gamma_1, \theta|_{\Gamma_1}] \oplus \mathbb{Z}[\Gamma_2, \theta|_{\Gamma_2}] \xrightarrow{\rho_2} \mathbb{Z}[\Gamma_3, \theta|_{\Gamma_3}] \longrightarrow \{0\}.$$

PROOF. We may only show $\mathrm{Im}(\rho_1) = \mathrm{Ker}(\rho_2)$. First we can get $\mathrm{Im}(\rho_1) \subset \mathrm{Ker}(\rho_2)$ from the following equation:

$$\begin{aligned} & \rho_2 \circ \rho_1(X) \\ = \ & \rho_2(X \cap \Gamma_1 \oplus X \cap \Gamma_2) \\ = \ & X \cap \Gamma_3 - X \cap \Gamma_3 \\ = \ & 0. \end{aligned}$$

Next we assume generators $H_1 \in \mathbb{Z}[\Gamma_1, \theta|_{\Gamma_1}]$ and $H_2 \in \mathbb{Z}[\Gamma_2, \theta|_{\Gamma_2}]$ satisfy $\Gamma_3 \cap H_1 - \Gamma_3 \cap H_2 = 0$. Then this means the hyperfacet H_1 of Γ_1 coinsides with the hyperfacet H_2 of Γ_2 on the hypertorus graph Γ_3. So $H_1 \cup H_2 = H$ is a hyperfacet of Γ. Hence we have $\mathrm{Im}(\rho_1) \supset \mathrm{Ker}(\rho_2)$. □

Next we consider the equivariant graph cohomologies $H_T^*(\Gamma, \alpha)$ and $H_T^*(\Gamma_i, \alpha|_{\Gamma_i})$ ($i = 1, 2, 3$).

Put the homomorphism $\rho_1' : H_T^*(\Gamma, \alpha) \to H_T^*(\Gamma_1, \alpha|_{\Gamma_1}) \oplus H_T^*(\Gamma_2, \alpha|_{\Gamma_2})$ such that

$$\rho_1'(f) = f|_{\Gamma_1} \oplus f|_{\Gamma_2}$$

and $\rho_2' : H_T^*(\Gamma_1, \alpha|_{\Gamma_1}) \oplus H_T^*(\Gamma_2, \alpha|_{\Gamma_2}) \to H_T^*(\Gamma_3, \alpha|_{\Gamma_3})$ such that

$$\rho_2'(g \oplus h) = g|_{\Gamma_3} - h|_{\Gamma_3}.$$

Now ρ_1' is injective because we see if $\rho_1'(f) = 0$ then $f(p) = 0$ for all $p \in V^{\Gamma_1} \cup V^{\Gamma_2} = V^{\Gamma}$.

Moreover we have the following lemma.

LEMMA 17.7. *The following sequence is exact*:

$$\{0\} \longrightarrow H_T^*(\Gamma, \alpha) \xrightarrow{\rho_1'} H_T^*(\Gamma_1, \alpha|_{\Gamma_1}) \oplus H_T^*(\Gamma_2, \alpha|_{\Gamma_2}) \xrightarrow{\rho_2'} H_T^*(\Gamma_3, \alpha|_{\Gamma_3}).$$

PROOF. First we have $\rho_2' \circ \rho_1'(f) = f|_{\Gamma_3} - f|_{\Gamma_3} = 0$, so $\mathrm{Im}(\rho_1') \subset \mathrm{Ker}(\rho_2')$ holds. Next we take $g \oplus h \in \mathrm{Ker}(\rho_2')$, then $g|_{\Gamma_3} = h|_{\Gamma_3}$. Hence the following map $f : V^{\Gamma} \to H_T^*(pt)$ is well-defined and in $H_T^*(\Gamma, \alpha)$:

$$\begin{aligned} f(p) &= g(p) \text{ if } p \in V^{\Gamma_1}, \\ f(q) &= h(q) \text{ if } q \in V^{\Gamma_2}. \end{aligned}$$

So we have $\mathrm{Im}(\rho_1') \supset \mathrm{Ker}(\rho_2')$. □

LEMMA 17.8. *The following diagram is commutative*:

$$\begin{array}{ccccccc} \{0\} & \longrightarrow & \mathbb{Z}[\Gamma,\theta] & \xrightarrow{\rho_1} & \mathbb{Z}[\Gamma_1,\theta|_{\Gamma_1}]\oplus\mathbb{Z}[N(\Gamma_2),\theta|_{\Gamma_2}] & \xrightarrow{\rho_2} & \mathbb{Z}[\Gamma_3,\theta|_{\Gamma_3}] \\ \downarrow & & \Psi\downarrow & & \Psi_1\oplus\Psi_2\downarrow & & \Psi_3\downarrow \\ \{0\} & \longrightarrow & H_T^*(\Gamma,\alpha) & \xrightarrow{\rho_1'} & H_T^*(\Gamma_1,\alpha|_{\Gamma_1})\oplus H_T^*(\Gamma_1),\alpha|_{\Gamma_2}) & \xrightarrow{\rho_2'} & H_T^*(\Gamma_3,\alpha|_{\Gamma_3}) \end{array}$$

PROOF. Now we see $\rho_1'(\tau_H) = \tau_{H\cap\Gamma_1} \oplus \tau_{H\cap\Gamma_2}$, $\rho_1'(x) = x\oplus x$ by the definition of ρ_1'. Hence we see the left square is commute from the definitions of ρ_1, Ψ, Ψ_1 and Ψ_2. Similarly we have the right square is commute by definitions of ρ_2 and ρ_2'. □

From the assumption of the induction, we have $\Psi_1 \oplus \Psi_2$ and Ψ_3 are isomorphic. Hence we have Ψ is isomorphic from the above Lemma 17.6 through 17.8 and the five lemma. Therefore we have the Main Theorem 2.

Finally we exhibit two examples which does not satisfies two assumptions of Main Theorem 2 that is

(1) There is a unique hyperfacet H and its opposite side $\bar{H}$ for every codimension two hyperfacet L in (Γ, α, θ) such that $\partial H = L$.

(2) For all hyperfacets H and G, these intersection $H \cap G = \emptyset$ or connected.

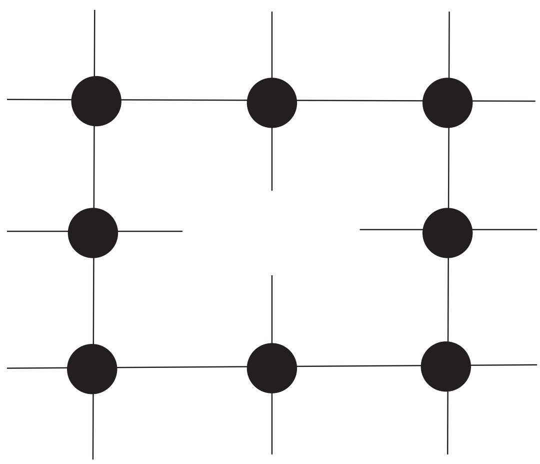

FIGURE 17.1. The figure which does not satisfy the assumption 1.

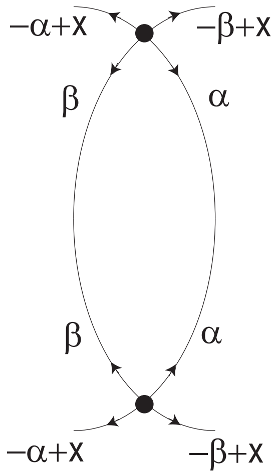

FIGURE 17.2. The figure which does not satisfy the assumption 2.

On the above two cases Main Theoreom 2 deos not hold, that is

$$\mathbb{Z}[\Gamma, \theta] \not\simeq H_T^*(\Gamma, \alpha).$$

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