GKM GRAPHS INDUCED BY GKM MANIFOLDS WITH $SU(2)$-SYMMETRIES

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1. Introduction

A *GKM manifold* is a $2m$-dimensional manifold M^{2m} equipped with an effective T^n-action whose one and zero dimensional orbits have the structure of a graph, where $n \leq m$. Let Γ be the induced m-valent graph of the GKM manifold (M^{2m}, T^n). We may identify the fixed points of (M^{2m}, T^n) with the vertices of Γ. Moreover, we can label each outgoing edge of Γ around a vertex p by its tangential representation, called an *axial function* $\alpha : E \to H^2(BT^n)$ where here E is the set of oriented edges[1] of Γ. This labeled graph (Γ, α) is called a *GKM graph* induced by the GKM manifold. On the other hand, a *GKM graph* can be defined abstractly by the labeled graph (Γ, α) which satisfies some properties of GKM graphs induced by GKM manifolds (see Section 2 or [3, 6, 7, 8, 13] for detail).

Let (M^{2m}, T^n) be a compact GKM manifold, and G a non-abelian, compact Lie group whose maximal torus is T^n. In this article, we study compact GKM manifolds with extended G-actions (also see [2, 9, 10, 11, 15, 17, 18, 19] for the related topics). For technical reasons, we assume the followings:

1. a GKM manifold M^{2m} has an almost complex structure which is compatible with the T^n-action;
2. G preserves almost complex structure $\mathcal{J}$ on M, i.e., $G \subset \mathrm{Diff}(M, \mathcal{J})$;
3. the universal covering $\widetilde{G}$ of G has the $SU(2)$-factor;
4. there are codimension two characteristic submanifolds in (M, T); we denote all of them by $\mathcal{F} = \{X_1, \ldots, X_k\}$.

The goal of this article is to introduce a property of GKM graphs induced by GKM manifolds with extended G-actions as above (see Theorem 4.1).

The details of this article for more general cases will be appeared in the forthcoming paper [12].

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[1]In E, we distinguish the two same edges pq and qp by regarding that their orientations are different.

2. Definition of GKM graphs

Let E_p be the set of all out going edges from the vertex p. By the assumption (1) in Section 1, the GKM graph (Γ, α) induced by the GKM manifold has the following properties:

- Γ is an m-valent graph, i.e., $|E_p| = m$ for all vertices p;
- $\alpha(e) = -\alpha(\bar{e})$, where e and $\bar{e}$ are the same edge but their orientations are different, e.g., if $e = pq$ then $\bar{e} = qp$;
- $\{\alpha(e_i) \mid e_i \in E_p\}$ is pairwise linearly independent, i.e., $\alpha(e_i)$ and $\alpha(e_j)$ are linearly independent if $e_i \neq e_j$;
- if two vertices p and q are connected by an edge (called f), there is a bijective map $\nabla_f : E_p \to E_q$ such that $\nabla_{\bar{f}} = \nabla_f^{-1}$, $\nabla_f(f) = \bar{f}$, and $\alpha(e) - \alpha(\nabla_f(e)) \equiv 0 \pmod{\alpha(f)}$ for $e \in E_p$, (the collection of maps $\nabla = \{\nabla_f \mid f \in E\}$ is called *connection*).

On the other hand, if the given labeled graph (Γ, α), where $\alpha : E \to H^2(BT^n)$, satisfies the properties above then we call (Γ, α) a *GKM graph* in this article.

3. Basic properties of GKM manifolds with $SU(2)$-symmetries

Assume that the GKM manifold (M^{2m}, T^n) equipped with an extended G-action satisfies all the assumptions (1)–(4) mentioned in Section 1.

Let $\mathcal{W}$ be the Weyl group of $SU(2)$, i.e., $\mathcal{W} \simeq \mathbb{Z}_2$. We let $r \in \mathfrak{t}^* \simeq H^2(BT; \mathbb{R})$ denote a simple root of $SU(2)$. As is well-known, the root system corresponds to the elements in $\mathcal{W}$ which act on $\mathfrak{t}^*$ as the reflections (e.g. see [16, Chapter 5]); so, we let $\sigma \in \mathcal{W}$ denote the reflection corresponding to the simple root r.

Let $\pi : ET \times_T M \to BT$ be the projection of the Borel construction of (M, T), and $\pi^* : H^*(BT) \to H_T^*(M)$ be the induced homomorphism. The element $\tau_i \in H_T^2(M)$, $i = 1, \ldots, k$, represents the equivariant Thom class of codimension two characteristic submanifold $X_i \in \mathcal{F}$. We denote the set of such equivariant Thom classes by $\mathcal{F}^*$. Then, there is the $\mathcal{W}$-action on $\mathcal{F}^*$ induced by the $\mathcal{W}$-action on $\mathcal{F}$.

In order to state Theorem 4.1, we first introduce the following lemma.

Lemma 3.1. *Assume $\sigma(X_s) = X_t$, where $X_s, X_t \in \mathcal{F}$. Then the following equation holds:*

$$\pi^*(r) = \tau_s - \tau_t.$$

4. Main theorem

Let (Γ, α) be an abstract GKM graph, and $H_T^*(\Gamma, \alpha)$ its graph equivariant cohomology. Now $\pi^* : H^*(BT^n) \to H_T^*(\Gamma, \alpha)$ is defined by $\pi^*(x) = x$ ($x \in H^*(BT^n)$), i.e., the constant function $x : p \mapsto x$ for all vertices p.

Abstractly, we assume that (Γ, α) satisfies the property in Lemma 3.1. That is, there is the element $r \in H^2(BT^n)$ such that the following E.q. (4.1) holds:

$$\pi^* : r \mapsto \tau_1 - \tau_2, \tag{4.1}$$

where τ_1, $\tau_2 \in H^2_T(\Gamma, \alpha)$ are the Thom classes of some $(m-1)$-valent GKM subgraphs Γ_1, Γ_2, respectively.

Now we may state the following theorem.

Theorem 4.1. *Suppose that there is $r \in H^2(BT^n)$ such that E.q. (4.1) holds for some GKM subgraphs Γ_1 and Γ_2. Then, one of the following cases occur:*

- **The 1^{st} case:** *if $\Gamma_1 \cap \Gamma_2 = \emptyset$, there is the GKM fiber bundle $\rho : (\Gamma, \alpha) \to (I, \alpha_I)$, where I is the compact 1-valent graph (i.e., two vertices p, q and one edge $e = pq$) and α_I satisfies that $\alpha_I(p) = r$ and $\alpha_I(q) = -r$;*
- **The 2^{nd} case:** *otherwise, there is the GKM blow-up $(\widetilde{\Gamma}, \widetilde{\alpha}) \to (\Gamma, \alpha)$ along $\Gamma_1 \cap \Gamma_2$ such that $(\widetilde{\Gamma}, \widetilde{\alpha})$ satisfies the 1^{st} case.*

The geometric interpretation of this theorem is as follows (also see [1]). The 1^{st} case corresponds to that M is T^n-equivariantly diffeomorphic to the crossed product $SU(2) \times_{S^1} N$ for some $(2m-2)$-dimensional GKM manifold N. The 2^{nd} case is otherwise, i.e., M does not decompose into the crossed product; however, there is the codimension-4 GKM submanifold X such that there is the blow up $\widetilde{M} \to M$ along X and $\widetilde{M}$ is equivariantly diffeomorphic to the crossed product $SU(2) \times_{S^1} N$ for some N.

See [12] for more general results (in particular, for GKM manifolds with larger symmetries) and further studies.

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