1. Introduction

A GKM manifold is a $2m$-dimensional manifold $M^{2m}$ equipped with an effective $T^n$-action whose one and zero dimensional orbits have the structure of a graph, where $n \leq m$. Let $\Gamma$ be the induced $m$-valent graph of the GKM manifold $(M^{2m}, T^n)$. We may identify the fixed points of $(M^{2m}, T^n)$ with the vertices of $\Gamma$. Moreover, we can label each outgoing edge of $\Gamma$ around a vertex $p$ by its tangential representation, called an axial function $\alpha : E \to H^2(BT^n)$ where here $E$ is the set of oriented edges of $\Gamma$. This labeled graph $(\Gamma, \alpha)$ is called a GKM graph induced by the GKM manifold. On the other hand, a GKM graph can be defined abstractly by the labeled graph $(\Gamma, \alpha)$ which satisfies some properties of GKM graphs induced by GKM manifolds (see Section 2 or [3, 6, 7, 8, 13] for detail).

Let $(M^{2m}, T^n)$ be a compact GKM manifold, and $G$ a non-abelian, compact Lie group whose maximal torus is $T^n$. In this article, we study compact GKM manifolds with extended $G$-actions (also see [2, 9, 10, 11, 15, 17, 18, 19] for the related topics). For technical reasons, we assume the followings:

1. A GKM manifold $M^{2m}$ has an almost complex structure which is compatible with the $T^n$-action;
2. $G$ preserves almost complex structure $\mathcal{J}$ on $M$, i.e., $G \subset \text{Diff}(M, \mathcal{J})$;
3. the universal covering $\tilde{G}$ of $G$ has the $SU(2)$-factor;
4. there are codimension two characteristic submanifolds in $(M, T)$; we denote all of them by $\mathcal{F} = \{X_1, \ldots, X_k\}$.

The goal of this article is to introduce a property of GKM graphs induced by GKM manifolds with extended $G$-actions as above (see Theorem 4.1).

The details of this article for more general cases will be appeared in the forthcoming paper [12].

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1In $E$, we distinguish the two same edges $pq$ and $qp$ by regarding that their orientations are different.
2. Definition of GKM graphs

Let $E_p$ be the set of all outgoing edges from the vertex $p$. By the assumption (1) in Section 1, the GKM graph $(\Gamma, \alpha)$ induced by the GKM manifold has the following properties:

- $\Gamma$ is an $m$-valent graph, i.e., $|E_p| = m$ for all vertices $p$;
- $\alpha(e) = -\alpha(\bar{e})$, where $e$ and $\bar{e}$ are the same edge but their orientations are different, e.g., if $e = pq$ then $\bar{e} = qp$;
- $\{\alpha(e_i) | e_i \in E_p\}$ is pairwise linearly independent, i.e., $\alpha(e_i)$ and $\alpha(e_j)$ are linearly independent if $e_i \neq e_j$;
- if two vertices $p$ and $q$ are connected by an edge (called $f$), there is a bijective map $\nabla_f : E_p \to E_q$ such that $\nabla_f = \nabla_f^{-1}$, $\nabla_f(f) = \bar{f}$, and $\alpha(e) - \alpha(\nabla_f(e)) \equiv 0 \pmod{\alpha(f)}$ for $e \in E_p$, (the collection of maps $\nabla = \{\nabla_f | f \in E\}$ is called connection).

On the other hand, if the given labeled graph $(\Gamma, \alpha)$, where $\alpha : E \to H^2(BT^n)$, satisfies the properties above then we call $(\Gamma, \alpha)$ a GKM graph in this article.

3. Basic properties of GKM manifolds with $SU(2)$-symmetries

Assume that the GKM manifold $(M^{2m}, T^n)$ equipped with an extended $G$-action satisfies all the assumptions (1)–(4) mentioned in Section 1.

Let $W$ be the Weyl group of $SU(2)$, i.e., $W \simeq \mathbb{Z}_2$. We let $r \in t^* \simeq H^2(BT^n; \mathbb{R})$ denote a simple root of $SU(2)$. As is well-known, the root system corresponds to the elements in $W$ which act on $t^*$ as the reflections (e.g., see [16, Chapter 5]); so, we let $\sigma \in W$ denote the reflection corresponding to the simple root $r$.

Let $\pi : ET \times_T M \to BT$ be the projection of the Borel construction of $(M, T)$, and $\pi^* : H^*(BT) \to H^*_T(M)$ be the induced homomorphism. The element $\tau_i \in H^2_T(M)$, $i = 1, \ldots, k$, represents the equivariant Thom class of codimension two characteristic submanifold $X_i \in \mathcal{F}$. We denote the set of such equivariant Thom classes by $\mathcal{F}^*$. Then, there is the $W$-action on $\mathcal{F}^*$ induced by the $W$-action on $\mathcal{F}$.

In order to state Theorem 4.1, we first introduce the following lemma.

**Lemma 3.1.** Assume $\sigma(X_s) = X_t$, where $X_s, X_t \in \mathcal{F}$. Then the following equation holds:

$$\pi^*(r) = \tau_s - \tau_t.$$
where $\tau_1, \tau_2 \in H^2_\ell(\Gamma, \alpha)$ are the Thom classes of some $(m - 1)$-valent GKM subgraphs $\Gamma_1$, $\Gamma_2$, respectively.

Now we may state the following theorem.

**Theorem 4.1.** Suppose that there is $r \in H^2(BT^n)$ such that E.q. (4.1) holds for some GKM subgraphs $\Gamma_1$ and $\Gamma_2$. Then, one of the following cases occur:

- **The 1st case:** if $\Gamma_1 \cap \Gamma_2 = \emptyset$, there is the GKM fiber bundle $\rho : (\Gamma, \alpha) \to (I, \alpha_I)$, where $I$ is the compact 1-valent graph (i.e., two vertices $p, q$ and one edge $e = pq$) and $\alpha_I$ satisfies that $\alpha_I(p) = r$ and $\alpha_I(q) = -r$;

- **The 2nd case:** otherwise, there is the GKM blow-up $(\tilde{\Gamma}, \tilde{\alpha}) \to (\Gamma, \alpha)$ along $\Gamma_1 \cap \Gamma_2$ such that $(\tilde{\Gamma}, \tilde{\alpha})$ satisfies the 1st case.

The geometric interpretation of this theorem is as follows (also see [1]). The 1st case corresponds to that $M$ is $T^n$-equivariantly diffeomorphic to the crossed product $SU(2) \times_{S^1} N$ for some $(2m - 2)$-dimensional GKM manifold $N$. The 2nd case is otherwise, i.e., $M$ does not decompose into the crossed product; however, there is the codimension-4 GKM submanifold $X$ such that there is the blow up $\tilde{M} \to M$ along $X$ and $\tilde{M}$ is equivariantly diffeomorphic to the crossed product $SU(2) \times_{S^1} N$ for some $N$.

See [12] for more general results (in particular, for GKM manifolds with larger symmetries) and further studies.

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**References**


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