ON SL(3,R)-ACTION ON 4-SPHERE

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Abstract. We construct a natural continuous $SL(3,\mathbf{R})$-action on S^4 which is an extension of the $SO(3)$-action ψ in [8]. The construction is based on the Kuiper's theorem in [3] that the quotient space of $\mathbf{C}P(2)$ by complex conjugation is S^4. We also give a new proof to the Kuiper's theorem.

1. Introduction

One of the central problems in transformation groups is to classify actions of a Lie group on a smooth manifold (for example actions of a special orthogonal group $SO(n)$ on a complex projective space and actions of a special linear group $SL(n,\mathbf{R})$ on a sphere). When the Lie group is compact, there are a rich history and an abundant work. However, when the Lie group is not compact, not much is known. We consider non-compact Lie group actions in this paper.

The classification problem of group actions would be beyond our reach unless we put some assumption on the actions. When the Lie group is $SL(n,\mathbf{R})$ and the manifold is an m-dimensional sphere S^m, there is a history in the classification problem. In 1974 C.R. Schneider ([4]) succeeded in classifying real analytic $SL(2,\mathbf{R})$-actions on S^2, more generally on surfaces. F. Uchida classified real analytic $SL(n,\mathbf{R})$-actions on S^n for $n \geq 3$ in 1979 ([6]) and on S^m for $5 \leq n \leq m \leq 2n-2$ in 1981 ([7]). On the other hand, it is well known that an effective continuous $SL(n,\mathbf{R})$-action on S^{n-1} is unique and any $SL(n,\mathbf{R})$-action on S^m is trivial for $m \leq n-2$ ([2, Theorem 3.5]). From these result, the cases which remain unsolved in the range $m \leq 2n-2$ are $(n,m) = (3,4), (4,5)$ and $(4,6)$.

All the solved cases above have the property that the actions of $SL(n,\mathbf{R})$ restricted to the maximal compact subgroup $SO(n)$ are unique, and this property played an important role in the classification. However the retricted actions are not unique in the remaining cases listed above. In these cases, we need to see how many $SO(n)$-actions there are on S^m (up to equivalence) and then check which of them extends to an $SL(n,\mathbf{R})$-action.

In this paper we will treat the case $(n,m) = (3,4)$. An irreducible $SO(3)$-representation is of odd dimension and unique in each odd dimension. Therefore there are exactly two $SO(3)$-representations of dimension 5. One is the direct sum of the standard $SO(3)$-representation of dimension 3 and the trivial representation of dimension 2. The other is the irreducible $SO(3)$-representation of dimension 5. The unit spheres of these representations produce $SO(3)$-actions on S^4 and it is known that any locally smooth $SO(3)$-action on S^4 is isomorphic to one of these two (see [1, Excercises 7 and 8 for Chapter IV]).

The former $SO(3)$-action on S^4 extends to a smooth $SL(3,\mathbf{R})$-action. In fact, the action $SL(3,\mathbf{R}) \times S^4 \to S^4$ defined by $(A,(x,y)) \mapsto \frac{(Ax,y)}{|(Ax,y)|}$, where $(x,y) \in S^4 \subset \mathbf{R}^3 \times \mathbf{R}^2$ and

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$|(Ax,y)|$ denotes the length of (Ax,y), is a desired extension. However it is unknown whether the latter $SO(3)$-action on S^4 called ψ in [8] extends to a smooth $SL(3,\mathbf{R})$-action. In 1985 Uchida ([8]) constructed an $SL(3,\mathbf{R})$-action on S^4 extending the action ψ. His action is continuous but not smooth. So he gave the following problem (see [8] and [9, (P2)]):

"Can the $SO(3)$-action ψ be extended to a smooth $SL(3,\mathbf{R})$-action on S^4?"

We provide a continuous $SL(3,\mathbf{R})$-action on S^4 which is an extension of ψ but different from Uchida's. Our construction is based on the Kuiper's theorem in [3] that the quotient space of $\mathbf{C}P(2)$ by complex conjugation is S^4. Ours is much more natural than Uchida's, but it turns out that our action is not smooth as well. So the Uchida's problem above is still open and the answer to the problem might be 'No'.

This paper is organized as follows. In section 2 we exhibit two examples showing whether an action of a compact Lie group extends to that of a non-compact Lie group. Section 3 gives a new proof to the Kuiper's theorem. In section 4 we construct a continuous $SL(3,\mathbf{R})$-action on S^4 extending the action ψ and prove that the extended action is not smooth in the last section.

2. Examples

In this section we give two examples concerning the extension problem of a group action.

2.1. **$SO(4)$-action on S^6.** Let us consider the $SO(4)$-action $\rho_4 \oplus \pi$ on $S^6 \subset \mathbf{R}^4 \oplus \mathbf{R}^3$ where ρ_4 is a standard action on $\mathbf{R}^4$ and π is a linear action on $\mathbf{R}^3$ induced from a surjective representation from $SO(4)$ to $SO(3)$. Then the following proposition holds.

Proposition 2.1 ([7]). *The $SO(4)$-action $\rho_4 \oplus \pi$ is not extendable to any continuous action of $SL(4,\boldsymbol{R})$ on S^6.*

This proposition was proved by F.Uchida in [7] and is one of the important facts to classify $SL(4,\mathbf{R})$-actions on S^6.

2.2. **$SO(2)$-actions on $\mathbf{R}^2$.** Let us consider the $SO(2)$-action r_p on $\mathbf{R}^2$ defined by

$$r_p : SO(2) \times \mathbf{R}^2 \ni (A,x) \mapsto A^p x \in \mathbf{R}^2$$

where p is a natural number. Then the following proposition holds.

Proposition 2.2. *The $SO(2)$-actions r_1 and r_2 are extendable to $SL(2,\boldsymbol{R})$-actions. However r_p for $p \geq 3$ is not extendable to any $SL(2,\boldsymbol{R})$-action on $\boldsymbol{R}^2$.*

Proof. The $SL(2,\mathbf{R})$-action $R_1 : SL(2,\mathbf{R}) \times \mathbf{R}^2 \to \mathbf{R}^2$ defined by $R_1(A,x) = Ax$, where $A \in SL(2,\mathbf{R})$ and $x \in \mathbf{R}^2$, is an extension of r_1. We identify $\mathbf{R}^2$ with the upper half space $H := \{z \in \mathbf{C} | \mathrm{Im}(z) > 0\}$. Then the Möbius transformation of $SL(2,\mathbf{R})$ on H provides an extension of r_2.

Assume r_p extends to an $SL(2,\mathbf{R})$-action R_p on $\mathbf{R}^2$ for $p \geq 3$. Then the kernel of R_p includes the kernel of r_p where the kernel of an action is the intersection of all isotropy subgroups. In general the kernel of a G-action is a normal subgroup of G. In our case the kernel of r_p is a cyclic group $\mathbf{Z}_p$ of order p. But any proper normal subgroup of $SL(2,\mathbf{R})$ does not contain $\mathbf{Z}_p$ ($p \geq 3$). This is a contradiction. Hence r_p is not extendable to any $SL(2,\mathbf{R})$-action for $p \geq 3$. □

The proposition above can also be derived from the classification result of $SL(2,\mathbf{R})$-actions on $\mathbf{R}^2$ in [4]. But we gave a direct proof.

3. Preparation of construction

We describe the orbit structure of the $SO(3)$-action ψ on S^4 defined in the introduction. Remember that the action ψ was obtained as the unit sphere of the irreducible $SO(3)$-representation of dimension 5. This irreducible representation can be described explicity as the adjoint action of $SO(3)$ on the vector space

$$\mathfrak{sym}(3) = \{A \in M_3(\mathbf{R}) : \mathrm{trace}(A) = 0, A = A^t\}$$

where A^t denotes the transpose of a matrix A. We regard

$$S^4 = \{A \in \mathfrak{sym}(3) : \mathrm{trace}(A^tA) = 1\}.$$

Then the determinant map $\pi\colon S^4 \to \mathbf{R}$ identifies the orbit space of the action ψ with an interval $[-\frac{1}{3\sqrt{6}}, \frac{1}{3\sqrt{6}}]$. We can easily check that $\pi^{-1}(\frac{1}{3\sqrt{6}})$ and $\pi^{-1}(-\frac{1}{3\sqrt{6}})$ are singular orbits isomorphic to $SO(3)/O(2) = \mathbf{R}P(2)$ and other orbits are principal orbits isomorphic to $SO(3)/K$ where $K = \mathbf{Z}_2 \oplus \mathbf{Z}_2$ is

$$\left\{ \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}, \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix} \right\}.$$

We recall a lemma in [5].

Lemma 1 (Lemma 5.3.1 in [5])**.** *Let G be a compact connected Lie group and K_1, K_2 be closed subgroup of G. Assume $X_1 = G \times_{K_1} D^{k_1}$ and $X_2 = G \times_{K_2} D^{k_2}$ where D^{k_1} and D^{k_2} are disks, K_i acts linearly on D^{k_i} $(i = 1, 2)$ and* $\mathrm{dim}X_1 = \mathrm{dim}X_2$. *Let $f, f' : \partial X_1 \to \partial X_2$ be G-equivariant diffeomorphisms. Then $M(f) = X_1 \cup_f X_2$ is equivariantly diffeomorphic to $M(f') = X_1 \cup_{f'} X_2$ as G-manifolds, if one of the following conditions is satisfied.*

(1) *$f^{-1}f'$ is extendable to a G-equivariant diffeomorphism on X_1.*
(2) *$f'f^{-1}$ is extendable to a G-equivariant diffeomorphism on X_2.*

The following proposition gives a characterization of the action ψ. It also gives a criterion of when a smooth 4-manifold is diffeomorphic to S^4.

Proposition 3.1. *If a 4-dimensional compact connected smooth $SO(3)$-manifold has two singular orbits $\boldsymbol{R}P(2) \simeq SO(3)/O(2)$ and the principal orbit type is $SO(3)/\boldsymbol{Z}_2 \oplus \boldsymbol{Z}_2$, then the manifold is equivarinatly diffeomorphic to S^4 with the $SO(3)$-action ψ.*

Proof. It follows from [1] and [5] that the manifold in the proposition is of the form $M(f) = X_1 \cup_f X_2$ where $X_i = SO(3) \times_{O(2)} D^2$ $(i = 1,\ 2)$ are tubular neighborhoods of two singular orbits and the attaching map f is an $SO(3)$-equivariant diffeomorphism from $\partial X_1 \simeq SO(3)/\mathbf{Z}_2 \oplus \mathbf{Z}_2$ to $\partial X_2 \simeq SO(3)/\mathbf{Z}_2 \oplus \mathbf{Z}_2$. Because f is an $SO(3)$-map on $SO(3)/\mathbf{Z}_2 \oplus \mathbf{Z}_2$, we can think of f as an element of $N(\mathbf{Z}_2 \oplus \mathbf{Z}_2; SO(3))/\mathbf{Z}_2 \oplus \mathbf{Z}_2$ where $N(H;G)$ is the normalizer of H in G. We can easily check that $N(\mathbf{Z}_2 \oplus \mathbf{Z}_2; SO(3))/\mathbf{Z}_2 \oplus \mathbf{Z}_2$ have the following representatives

$$\left\{ I = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, A = \begin{pmatrix} 0 & 0 & -1 \\ 1 & 0 & 0 \\ 0 & -1 & 0 \end{pmatrix}, A^{-1} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & -1 \\ -1 & 0 & 0 \end{pmatrix}, \right.$$
$$\left. \alpha = \begin{pmatrix} 0 & 0 & 1 \\ 0 & -1 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \beta = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \gamma = \begin{pmatrix} 0 & -1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix} \right\}.$$

Hence the attaching map f can be thought of as one of the above matrices.

Consider the following commutative diagram where $\mathbf{Z}_2 \oplus \mathbf{Z}_2 = K$.

$$\begin{array}{ccc} SO(3) \times_H (H/K) & \xrightarrow{\simeq} & SO(3)/K \\ \downarrow 1 \times R_f & & \downarrow R_f \\ SO(3) \times_H (H/K) & \xrightarrow{\simeq} & SO(3)/K. \end{array}$$

Here $R_f([X]) = [Xf]$ $(X \in SO(3)/K)$, H is a subgroup of $SO(3)$ which is isomorphic to $O(2)$ and contains the element f and the group K, and horizontal arrows are defined by $[g, hK] \mapsto ghK$. It is easy to see $H/K \simeq S^1$ and $SO(3) \times_H (H/K) \simeq SO(3) \times_{O(2)} S^1 = \partial X_1$. If $f = \alpha$, β or γ, then R_f is an antipodal involution of S^1; so R_f is extendable to a diffeomorphism on D^2. Hence $1 \times R_f$ is extendable to an $SO(3)$-equivariant diffeomorphism on $X_1 = SO(3) \times_{O(2)} D^2$ for $f = \alpha$, β or γ. Therefore by Lemma 1 $M(I) \simeq M(f)$ if $f = \alpha$, β or γ. Since $\alpha A = \gamma$ and $\alpha A^{-1} = \beta$, $M(I) \simeq M(\alpha) \simeq M(A) \simeq M(A^{-1})$ by Lemma 1. Therefore all $M(f)$ are $SO(3)$-equivariantly diffeomorphic to S^4 with the action ψ. □

The following theorem was proved by Kuiper in 1974 ([3]), but we shall give a new proof to it.

Theorem 1. *Consider the $\boldsymbol{Z}_2$-action on the complex projective space $\boldsymbol{C}P(2) = (\boldsymbol{C}^3 - \{0\})/\boldsymbol{C}^*$ by a complex conjugation. Then the orbit space is 4-dimensional sphere, that is*

$$\boldsymbol{C}P(2)/\boldsymbol{Z}_2 \simeq S^4.$$

Proof. The natural $SO(3)$-action ϕ on $\mathbf{C}^3 - \{0\}/\mathbf{C}^* = \mathbf{C}P(2)$ has two singular orbits; one is the real projective plane $\mathbf{R}P(2)$ sitting naturally in $\mathbf{C}P(2)$ and the other is diffeomorphic to S^2. The principal orbits of ϕ are isomorphic to $SO(3)/\mathbf{Z}_2$. Since the action ϕ commutes with the complex conjugation, it induces an $SO(3)$-action $\bar{\phi}$ on $\mathbf{C}P(2)/\mathbf{Z}_2$. Because one singular orbit $\mathbf{R}P(2)$ of ϕ is the fixed point set of the complex conjugation on $\mathbf{C}P(2)$ and the other one is S^2, the induced $SO(3)$-action $\bar{\phi}$ on $\mathbf{C}P(2)/\mathbf{Z}_2$ has two singular orbits both of which are diffeomorphic to $\mathbf{R}P(2)$ and principal orbits of the induced action $\bar{\phi}$ are isomorphic to $SO(3)/\mathbf{Z}_2 \oplus \mathbf{Z}_2$. Therefore $\mathbf{C}P(2)/\mathbf{Z}_2$ is diffeomorphic to S^4 (and moreover $\bar{\phi}$ agrees with the action ψ) by Proposition 3.1. □

4. Construction of the $SL(3, \mathbf{R})$-action on S^4

Consider the natural $SL(3, \mathbf{R})$-action Φ on $\mathbf{C}^3 - \{0\}/\mathbf{C}^* = \mathbf{C}P(2)$. Since Φ commutes with complex conjugation on $\mathbf{C}P(2)$, it induces an $SL(3, \mathbf{R})$-action Ψ on S^4 by Theorem 1. The restricted action of Ψ to $SO(3)$ agrees with the $SO(3)$-action ψ as observed in the proof of Theorem 1. Therefore our $SL(3, \mathbf{R})$-action Ψ is an extension of ψ.

Remark . *The action Ψ has one singular orbit $\boldsymbol{R}P(2)$ and one open orbit $S^4 - \boldsymbol{R}P(2)$. Because the continuous $SL(3, \boldsymbol{R})$-action on S^4 in* [8] *has two open orbits and one singular orbit $SO(3)/\boldsymbol{Z}_2 \oplus \boldsymbol{Z}_2$, the action Ψ is different from the action in* [8]. *These two actions look very different but there might be some relation between them.*

5. Non-differentiability of Ψ

We shall observe that our action Ψ is not smooth. Let $p = [1 : 0 : 0] \in (\mathbf{C}^3 - \{0\})/(\mathbf{C}^* \times \mathbf{Z}_2) = S^4$. Take a small neighborhood U_p of p and a local chart $\varphi \colon U_p \to \mathbf{R}^4$ defined by

$$\varphi([1 : x_1 + y_1\mathbf{i} : x_2 + y_2\mathbf{i}]) = (x_1, x_2, y_1^2 - y_2^2, y_1y_2).$$

The point p corresponds to the origin of $\mathbf{R}^4$ through φ. We consider a transformation Ψ_t induced by the action Ψ of a matrix

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & t \\ 0 & 0 & 1 \end{pmatrix}.$$

By definition $\Psi_t([1 : z_1 : z_2]) = [1 : z_1 + tz_2 : z_2]$ for $[1 : z_1 : z_2] \in S^4$. For each t, there is a sufficiently small neighborhood $U_p' \subset U_p$ such that $\Psi_t(U_p') \subset U_p$. Therefore the map Ψ_t around the point p can be described through the local chart φ. In fact, if we denote $\varphi \circ \Psi_t \circ \varphi^{-1}$ by F_t, then we have

$$F_t((0, 0, y_1^2 - y_2^2, y_1y_2)) = (0, 0, y_1^2 - y_2^2 + 2ty_1y_2 + t^2y_2^2, y_1y_2 + ty_2^2).$$

If we set $Y_1 = y_1^2 - y_2^2$ and $Y_2 = y_1y_2$, then $y_2^2 = (\sqrt{Y_1^2 + 4Y_2^2} - Y_1)/2$; so we have

$$F_t((0, 0, Y_1, Y_2)) = \Big(0, 0, Y_1 + 2tY_2 + t^2(\sqrt{Y_1^2 + 4Y_2^2} - Y_1)/2, Y_2 + t(\sqrt{Y_1^2 + 4Y_2^2} - Y_1)/2\Big).$$

This shows that F_t with $t \neq 0$ is not differentiable at the origin, proving that Ψ is not smooth.

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