Part I
Let $X_1, \cdots, X_n$ be independently and identically distributed random variables from the Poisson distribution with unknown mean $\eta > 0$. From this sample, we are interested in estimating $\theta = P(X_i = 0) = e^{-\eta}$. We consider the estimators
\[ \hat{\theta}_1 = \frac{1}{n} \sum_{i=1}^{n} I(X_i = 0) \]
and
\[ \hat{\theta}_2 = e^{-\bar{X}} \]
where $I(X_i = 0)$ takes the value one if $X_i = 0$ and takes the value zero otherwise, and $\bar{X} = n^{-1} \sum_{i=1}^{n} X_i$. Answer the following questions:

1. Find the limiting distribution of $\sqrt{n} \left( \hat{\theta}_1 - \theta \right)$ as $n \to \infty$.
2. Find the limiting distribution of $\sqrt{n} \left( \hat{\theta}_2 - \theta \right)$ as $n \to \infty$.
3. Compute the asymptotic relative efficiency of $\hat{\theta}_1$ with respect to $\hat{\theta}_2$. Which estimator do you prefer?

Part II
Let $X_1, \cdots, X_n$ be independently and identically distributed random variables from a distribution with density
\[ f(x; \theta) = \begin{cases} 
\theta x^{\theta - 1} & \text{if } 0 < x < 1 \\
0 & \text{otherwise.} 
\end{cases} \]
Define $Y = \max \{X_1, \cdots, X_n\}$. We want to test
\[ H_0 : \theta = 1 \text{ versus } H_1 : \theta > 1 \]
and will reject the null hypothesis when $Y > c$.

4. Find the power function for this test.
5. What choice of $c$ will make the size of the test 0.05?
Part III
Let $X_1, \cdots, X_n$ (for $n > 2$) be a random sample from a normal distribution with mean $\mu$ and variance $\sigma^2$. Suppose that we are interested in estimating $\theta = \mu/\sigma$.

6. Find the maximum likelihood estimator of $\theta$.

7. Find the value of $\gamma$ such that $\gamma S^{-1}$ is unbiased for $\sigma^{-1}$, where

$$S^2 = (n - 1)^{-1} \sum_{i=1}^{n} (X_i - \bar{X})^2.$$ 

[Hint: First show that the $k$-th moment of $\chi^2(\nu)$ distribution is $\Gamma(\nu/2 + k)2^k/\Gamma(\nu/2)$, provided that $\nu/2 + k > 0$.]

8. Find the (uniformly) minimum variance unbiased estimator of $\theta$.

Part IV
Let $(X_1, Y_1)', \cdots, (X_n, Y_n)'$ be a random sample from a bivariate normal distribution with mean $(\mu_x, \mu_y)'$ and variance-covariance matrix $\Sigma = \begin{pmatrix} \sigma_x^2 & \sigma_{xy} \\ \sigma_{xy} & \sigma_y^2 \end{pmatrix}$. We are interested in estimating $\theta = \mu_y$.

9. Assuming that parameters $\mu_x, \sigma_x^2, \sigma_{xy}, \sigma_y^2$ are known, find the maximum likelihood estimator (MLE) of $\theta$ and compute its variance.

10. If the other parameters are also unknown, derive the MLE of $\theta$. Is the estimator here or the estimator from question 9 more efficient? Explain.
Ph.D. Qualifying Exam: Algebraic Topology I: (August 2019)

Justify your answers fully. You should state the theorems and the results that you are using exactly. One can obtain partial points for rough ideas but not the full scores. Also, one must write in good English and in a well-organized way for better grades. You must also write the answers in order and not mix up the answers here and there. (Total 100 pts)

1. (20 pts.) Let $S^1 \times S^1 \times S^1 \times S^1$ be a 4-dimensional torus. Find a CW-complex structure on it, and the cellular-chain complex of the CW-complex structure. Describe the boundary maps. Find the cellular homology group for each dimension. (You won’t get a point if you use another methods.)

2. (20 pts.) Let $Y$ be a closed surface of genus 2 and $f : Y \to Y$ be a homeomorphism inducing homomorphisms $f_1^* : H_1(Y, \mathbb{Z}) \to H_1(Y, \mathbb{Z})$ and $f_2^* : H_2(Y, \mathbb{Z}) \to H_2(Y, \mathbb{Z}) = I$. Let $Z$ be a space obtained by $Y \times I/\sim$ where $(x, 0) \sim (f(x), 1)$. Compute $H_\ast(Z, \mathbb{Z})$ from the properties of $A$.

3. (20 pts.) We have a commutative diagram of two long exact sequences for all $n \geq 0$

\[
\begin{array}{ccccccc}
\rightarrow & C_{n+1} & \rightarrow & A_n & \rightarrow & B_n & \rightarrow & C_n & \rightarrow & A_{n-1} & \rightarrow & B_{n-1} & \rightarrow \\
\downarrow & \downarrow & = & \downarrow & \downarrow & \downarrow & = & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow \\
\rightarrow & E_{n+1} & \rightarrow & A_n & \rightarrow & D_n & \rightarrow & E_n & \rightarrow & A_{n-1} & \rightarrow & D_{n-1} & \rightarrow \\
\end{array}
\]

Assume $A_i = 0, B_i = 0, C_i = 0, D_i = 0, E_i = 0$ for $i < 0$. Prove that there is a following long exact sequence

\[
\rightarrow E_{n+1} \rightarrow B_n \rightarrow C_n \oplus D_n \rightarrow E_n \rightarrow B_{n-1} \rightarrow .
\]

Also, describe each map by the maps in the above diagram. (You need to show the steps and details)

4. (20 pts.) Let $S_g$ be a closed surface of genus $g, g \geq 2$. Let $M_1 = S_{g_1} \times S_{g_2}$ and $M_2 = S_{g_3} \times S_{g_4}$. Suppose that $g_1 + g_2 = g_3 + g_4$ and $g_1 \neq g_3$. Show that $M_1$ and $M_2$ are not homotopy equivalent.

5. (20 pts.) Let $X$ be the join $(S^1 \times S^1) \ast (S^1 \times S^1)$. Compute the fundamental group and the homology groups of $X$ with coefficient $\mathbb{Z}$ for all dimensions.
1. Let $G$ be a nilpotent group.
   (a) (10 pts) Prove that every subgroup of $G$ is solvable.
   (b) (10 pts) Prove that if $H$ is a proper subgroup of $G$, then $H$ is a proper subgroup of the normalizer of $H$ in $G$.

2. Let $G$ be a group of order $p^nq$, where $p$, $q$ are distinct prime numbers and $n$ is a positive integer. Assume that $G$ has at least two distinct Sylow $p$-subgroups $H'$ and $K'$ such that $H' \cap K' \neq \{e\}$.
   (a) (10 pts) Let $H$ and $K$ be two distinct Sylow $p$-subgroups of $G$ such that the order of $H \cap K$ is maximal among all possible intersections of two distinct Sylow $p$-subgroups of $G$. Show that the order of the normalizer of $H \cap K$ in $G$ is divisible by $q$.
   (b) (15 pts) Show that the intersection of all Sylow $p$-subgroups of $G$ is a nontrivial proper normal subgroup of $G$.

3. (10 pts) Let $R = \mathbb{Z}[\sqrt{-21}] = \{a + b\sqrt{-21} \mid a, b \in \mathbb{Z}\}$. Show that the ideal $I$ of $R$ generated by 5 and $2 + \sqrt{-21}$ is a projective $R$-module but not a free $R$-module.

4. (15 pts) Show that a free group of rank 2 contains as a subgroup a free group of any finite rank.

5. (10 pts) Let $K$ be a field and let $M_n(K)$ be the ring of all $n \times n$ matrices with entries in $K$. Find all the two-sided ideals of $M_n(K)$.

6. Let $R$ be a Noetherian domain.
   (a) (10 pts) Prove that every nonzero, nonunit element is a product of irreducible elements.
   (b) (10 pts) Prove that the localization $R_P$ of $R$ at any prime ideal $P$ is a Noetherian domain.

THE END
Each problem is worth 10 points.

1. Prove or disprove the following statement: If \(|a_n|^{1/n} < 1\) for every \(n\), then \(\sum_{n=1}^{\infty} a_n\) converges. (Justify your answer.)

2. Find all possible values of \(\log(-1)\).

3. Let \(f(z)\), \(z \in \mathbb{C}\), be an entire function such that \(|f(x+iy)| \leq e^x\) for all \(z = x + iy, x, y \in \mathbb{R}\). Find all such \(f(z)\).

4. Compute the area \(A_n\) enclosed by the line segments connecting the neighboring solutions of \(z^n = 1, n \geq 3\). Find the limit of \(A_n\) as \(n \to \infty\).

5. Let \(b > 1\) and \(f(z) = z + e^{-z}\). Show that there exists a unique solution of \(f(z) = b\) in the region \(\text{Re } z > 0\).

6. Show that the zeros of \(\tan z\) are all simple.

7. Let \(f(z) = \sum_{n=0}^{\infty} a_n z^n\) be analytic on \(D = \{z \in \mathbb{C} : |z| \leq 1\}\). Assume that \(f : D \to \mathbb{C}\) is one-to-one. Prove that the area of \(f(D)\) is equal to \(\pi \sum_{n=1}^{\infty} n|a_n|^2\). (Hint: For \(f(z) = u + iv\), define a mapping \(g : D \subset \mathbb{R}^2 \to \mathbb{R}^2\) by \(g = (u, v)\).)

8. Let \(P(z) = z^3 + z^2 + 2z + 1\). Find the number of zeros of \(P(z)\) in \(|z| < 3\).

9. (i) Find the Taylor expansion of the function \(g(z)\) around \(z = 0\) where \(g(z)\) is defined by \(g(z) = \frac{\ln z}{z}\) for \(z \neq 0\) and \(g(0) = \lim_{z \to 0} \frac{\ln z}{z}\). (ii) Let \(C\) be a contour \(|z - \frac{1}{2}| = \frac{3}{2}\) with positive orientation. Evaluate \(\int_C \frac{\tan z}{z} \, dz\).

10. (i) Let \(f(z) = 1/(z^4 + 1)\). Find the principal part of the Laurent expansion in a deleted neighborhood of \(z = \text{e}^{\pi i/4}\). (ii) Using the contour integral, find \(\int_0^{\infty} \frac{1}{1 + x^4} \, dx\).
1. (20 pts)
   (a) Define a Lagrange interpolation polynomial with data \( \{(x_i, f(x_i))\}_{i=0}^{n} \) \( x_i \) all distinct.
   (b) Derive the error form in the above.
   (c) Define a Newton’s form of interpolation polynomial using the same data.
   (d) Explain what happens if some \( x_i \) are repeated, and in this case what is the correct data corresponding to the repeated points?

2. (15 pts) Explain two popular methods to find the roots of a system of equations \( f(x) = 0 \) in \( \mathbb{R}^n \). Here \( f(x) = (f_1(x), \cdots, f_n(x))^T \) and \( x \in \mathbb{R}^n \). (State proper conditions to guarantee the convergence)
Prove the convergence of one of the methods (specify which one you would prove) when \( n = 1 \).

3. (15pts) Explain the Runge phenomena to approximate a continuous function on some interval \([a, b]\) with polynomials and show how one can avoid it.

4. (20 pts)
   (a) Let \( \mathbf{u}, \mathbf{v} \) are any vectors in \( \mathbb{R}^n \) having the same size. Find an \( n \times n \) matrix of the form \( H_w = I - 2\mathbf{w}\mathbf{w}^* \), for some unit vector \( \mathbf{w} \in \mathbb{R}^n \) such that \( H_w \mathbf{u} = \mathbf{v} \).
   (b) Show that \( H_w \) is symmetric and orthogonal.
   (c) Explain how to transform \( A \) into an upper triangular matrix \( R \) using above transformations. (Including how to avoid the instability)
   (d) Discuss at least two ways to orthonormalize given set of \( k \leq n \) vectors in \( \mathbb{R}^n \). (Advantages/disadvantages)

5. (10 pts) Explain the Givens rotation and Givens algorithm to tridiagonalize a symmetric matrix.

6. (10 pts) Describe Euler’s method(explicit and implicit) to solve an ODE. \( \dot{x} = f(t, x(t)) \), \( x(0) = x_0 \). Discuss advantages and disadvantages.

7. (10 pts) State Gaussian quadrature using \( n \) points for the approximation of integral \( \int_{a}^{b} f(x)dx \) and prove it is exact for polynomials of degree \( 2n - 1 \).
PH.D QUALIFYING EXAM:
REAL ANALYSIS
3 HOURS

Problem 1. (20 pt) Let $1 \leq p_0 < p_1 \leq \infty$ be fixed.

1. Assume $f \in L^{p_0} (\mathbb{R}^d)$ with $\|f\|_{p_0} = A_0$ and $\|f\|_{p_1} = A_1$. Show that $f \in L^p (\mathbb{R}^d)$ for $p_0 \leq p \leq p_1$. Find the optimal (largest) constant of $\|f\|_p$ and a maximizer.

2. Assume $f$ belongs to weak $L^{p_i}$ spaces, i.e.
   \[
   \sup_{t>0} t \cdot m(\{ x \in \mathbb{R}^d : |f(x)| \geq t \})^{1/p_i} = B_i < \infty, \quad i = 0, 1.
   \]
   Show that $f \in L^p (\mathbb{R}^d)$ for $p_0 < p < p_1$. i.e. $\|f\|_p \leq C(p, B_1, B_2)$ for some $C(p, B_1, B_2)$.

Problem 2. (20pt) Let $(X, \mathcal{M})$ be a measurable space. State the Lebesgue-Radon-Nikodym theorem for $\sigma$-finite (signed) measures $\mu$ and $\nu$ on $(X, \mathcal{M})$. Assuming the Lebesgue-Radon-Nikodym theorem for finite positive measures, finish the proof for general cases.

Problem 3. (20 pt) Let $S$ be the Schwarz class. We define the Fourier transform by
   \[
   \hat{f}(\xi) = \int_{\mathbb{R}^d} e^{-i 2\pi x \cdot \xi} f(x) \, dx, \quad f \in S.
   \]

1. Find the Fourier transform of Gaussian $e^{-a|x|^2}$ for $a > 0$.

2. Using the above, prove the Fourier inversion formula. i.e.
   \[
   f(x) = \int_{\mathbb{R}^d} e^{i 2\pi x \cdot \xi} \hat{f}(\xi) \, d\xi.
   \]

Problem 4. (20 pt)

1. Recall that any regular Borel measure on $\mathbb{R}$ can be uniquely decomposed as
   \[
   \mu = \mu_d + \mu_{ac} + \mu_{sc}.
   \]
   Find an example of singularly continuous measures. (Verify your answer.)

2. Assume $F, G : \mathbb{R} \to \mathbb{R}$ are increasing and $F$ is continuous. Show that
   \[
   \int_{(a,b)} F \, dG + \int_{(a,b)} G \, dF = F(b)G(b) - F(a)G(a).
   \]

Problem 5. (20 pt) Let $B$ be a Banach space.

1. Let $x \neq 0 \in B$. For $l \in B^*$ we know $\|l\| = \sup_{\|x\|=1} |l(x)|$. On the other hand, show that for $x \in B$, $\|x\| = \sup_{\|l\|=1} |l(x)|$. Also, show that the supremum is attained. i.e. there exists $l \in B^*$ such that $\|l\| = 1$ and $l(x) = \|x\|$.

2. Show that $B$ is isometrically embedded in $B^{**}$.

Date: Aug 2019.
1. [15 pts] Show that if $X_1 \in L^1$ and $\{X_i\}$ are iid, then the family $\{\frac{S_n}{n}, n \in \mathbb{N}\}$ is uniformly integrable.

2. [14 pts] Prove Kolmogorov’s Maximal Inequality for independent (not necessarily identically distributed) zero-mean random variables $\{X_k\}$ which have finite variances: If $S_n = X_1 + \cdots + X_n$ then

$$P(\max_{1 \leq k \leq n} |S_k| \geq \epsilon) \leq \epsilon^{-2}\text{Var}(S_n)$$

for all $\epsilon > 0$.

3. [14 pts] Suppose that the sequence $(X_n/a_n, n \in \mathbb{N})$ converges in distribution where $a_n \uparrow \infty$. If $(b_n)$ is a sequence of numbers such that $a_n/b_n \to 0$ as $n \to \infty$, show that $(X_n/b_n, n \in \mathbb{N})$ converges in probability to 0. (Remark: this shows that the CLT implies the WLLN under finite second moments.)

4. [14 pts] Let $(X_k, k \in \mathbb{N})$ be an i.i.d. sequence such that $E|X_1| < \infty$. If $\tau$ is a stopping time with respect to the filtration given by $\mathcal{F}_n = \sigma(X_1, \ldots, X_n)$, and $E\tau < \infty$, then

$$E(X_1 + \cdots + X_\tau) = E(\tau)E(X_1).$$

(Remark: this is called Wald’s equation, and it is not true when $E\tau = \infty$.)

5. [15 pts] If $\{(X_n, \mathcal{F}_n)\}$ is a martingale, show that it converges in $L^1$ if and only if $X_n = E[X|\mathcal{F}_n]$ for some $X$ and all $n \in \mathbb{N}$.

6. [14 pts] If $\{X_n\}$ is a sequence of mean-zero independent random variables with $E(X_n^2) = \sigma_n^2 < \infty$, show that

$$M_n := \left(\sum_{k=1}^{n} X_k\right)^2 - \sum_{k=1}^{n} \sigma_k^2$$

is a martingale with respect to the natural filtration (what is the filtration?).

7. [14 pts] If $X$ is equal in distribution to $E(X|\mathcal{F})$ then, in fact, they are equal a.s.

(for partial credit, you may assume $E(X^2) < \infty$)