Ph.D. Qualifying Exam: Algebra I
February 2018

Department of Mathematical Sciences, KAIST

Student ID: Name:

Note: Be sure to use English for your answers.

1. Let $n \geq 2$ be an integer and let E_{ij} be the $n \times n$ matrix such that the (i, j) entry is 1, but all the other entries are 0.

   (a) (10 pts) Prove that the group $\mathrm{SL}_n(\mathbb{C})$ is generated by $\{I_n + aE_{ij} : a \in \mathbb{C} \text{ and } i \neq j\}$, where I_n denotes the identity matrix.

   (b) (10 pts) Prove that neither $\mathrm{SL}_n(\mathbb{C})$ nor $\mathrm{GL}_n(\mathbb{C})$ is solvable.

2. (10 pts) Let H be a nontrivial normal subgroup of a finite p-group G. Show that the intersection $H \cap Z(G)$ is nontrivial, where $Z(G)$ denotes the center of G.

3. (20 pts) Let G be a finite group and let p be a prime. We shall denote by $\mathrm{Syl}_p(G)$ the set of Sylow p-subgroups of G. Show that if N is a normal subgroup of G, then $|\mathrm{Syl}_p(G/N)| \leq |\mathrm{Syl}_p(G)|$. Show also that if H is a subgroup of G, then $|\mathrm{Syl}_p(H)| \leq |\mathrm{Syl}_p(G)|$.

4. Let R be a principal ideal domain and let I be a nonzero prime ideal of the polynomial ring $R[x]$ such that $R \cap I = \{0\}$.

   (a) (10 pts) Show that $I = (f)$ for some irreducible $f \in R[x]$.

   (b) (10 pts) Assume that R has infinitely many prime elements. Show that I is not maximal.

5. (10 pts) Let M be an R-module, where R denotes a commutative ring with identity. Let N be a Noetherian submodule of M and let P be a submodule of M with $N \subseteq P$. Prove that if the quotient modules M/P and P/N are Noetherian, then M is Noetherian.

6. (a) (10 pts) Classify all groups of order 8 up to isomorphism.

   (b) (10 pts) For each group G in (a), calculate $|\mathrm{Aut}(G)|$, where $\mathrm{Aut}(G)$ denotes the automorphism group of G.

THE END

Ph.D. Qualifying Exam: Algebra II
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1. (20 pts) Suppose that R is a commutative Noetherian ring with 1. Show that $R[X]$ is Noetherian.

2. (20 pts) Let K be an extension field of a field k and $\mathrm{Aut}_k(K)$ be the set of all the automorphisms of K fixing k. Show that K is a Galois extension of k if and only if k is the fixed field of $\mathrm{Aut}_k(K)$.

3. (20 pts) Let $K = \mathbb{Q}(\sqrt{-7})$.

   (a) Find the integral closure $\mathcal{O}_K$ of $\mathbb{Z}$ in K.

   (b) Show that $\mathcal{O}_K$ is a Euclidean domain.

4. (20 pts) Let R be a local ring with maximal ideal $\mathfrak{m}$ and M, N be finitely generated R-modules. Show that if $M \otimes N = 0$, then $M = 0$ or $N = 0$.

5. (20 pts) Given two morphisms $\phi : A \to X$ and $\psi : B \to X$ in a category $\mathcal{C}$, a *pull-back* of (ϕ, ψ) is defined to be a triple (Y, α, β), where $\alpha : Y \to A$ and $\beta : Y \to B$ are morphisms in $\mathcal{C}$ such that $\phi \circ \alpha = \psi \circ \beta$ with the following universal property;

   Given two morphisms $\gamma : Z \to A$ and $\delta : Z \to B$ with $\phi \circ \gamma = \psi \circ \delta$, there exists a unique morphism $\eta : Z \to Y$ such that $\gamma = \alpha \circ \eta$ and $\delta = \beta \circ \eta$.

   Show that there always exists a pull-back in the category of R-modules, where R is a commutative ring with 1 and it is unique up to isomorphism.

THE END

Ph.D. Qualifying Exam: Complex Analysis
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1. (10 pts) Find a bijective conformal mapping $f : D_1 \to D_2$ where
$$D_1 = \{z \in \mathbb{C} : 0 < |z| < 1, 0 < \operatorname{Arg} z < \frac{\pi}{2}\} \text{ and } D_2 = \{z \in \mathbb{C} : |z| < 1\}.$$

2. (10 pts)
   (a) Find all values of $(1+i)^{1+i}$.
   (b) Solve $\cos z = \frac{1}{2}$, $z \in \mathbb{C}$.

3. (10 pts) Let $f(z) = u(x,y) + iv(x,y)$ be an entire function where $z = x + iy$ and u, v are real-valued functions. Suppose that $u_y - v_x = -2$ for all $z \in \mathbb{C}$. Find all such $f(z)$.

4. (10 pts) Evaluate the following integrals:
$$\text{(a)} \int_{|z|=1} \frac{dz}{z} \qquad \text{(b)} \int_{|z|=1} \frac{dz}{|z|} \qquad \text{(c)} \int_{|z|=1} \frac{|dz|}{z} \qquad \text{(d)} \int_{|z|=1} \left|\frac{dz}{z}\right|$$

5. (10 pts) Assume that $f(z)$ is analytic for all $|z| < r$ with some $r > 0$ and f satisfies the equation $f(2z) = f(z)^2$ for all z sufficiently close to 0.
   (a) Show that $f(z)$ can be extended to an entire function.
   (b) Find all such entire functions $f(z)$ explicitly.

6. (10 pts) Evaluate
$$\int_{|z|=1} \frac{z^2 + e^z}{z(z-3)}\, dz.$$

7. (10 pts) Evaluate the integral
$$\int_\gamma \sqrt{z^2 - 1}\, dz$$
where γ is the circle $|z| = \frac{1}{2}$.

8. (10 pts) Suppose that $f(z)$ is entire and that
$$\lim_{|z|\to\infty} \frac{f(z)}{z} = 0.$$
Prove that $f'(z) = 0$ for every $z \in \mathbb{C}$.

9. (10 pts) Consider the function $f(z) = e^{z^2}$ on the unit disk $|z| \le 1$.
   (a) What is the maximum of $f(z)$ on the unit disk? At which points does the function have the maximum?
   (b) What is the minimum of $f(z)$ on the unit disk? At which points does the function have the minimum?

10. (10 pts) Using the contour integral in complex analysis, evaluate
$$\int_{-\infty}^{\infty} \frac{dx}{x^4 + 1}.$$

THE END

Ph.D. Qualifying Exam: Numerical Analysis
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1. (20 pts)
   (a) Define a Lagrange interpolation polynomial with data $\{(x_i, f(x_i))\}_{i=0}^n$ where all x_i's are distinct.
   (b) What is the error form in the above? Derive it.
   (c) Define Newton's form of interpolation polynomial using the same data.
   (d) Explain what happens if some x_i's are repeated, and in this case what is the correct data corresponding to the repeated points?

2. (15 pts) Describe Newton's method to solve a system of nonlinear equations

$$\mathbf{F}(\mathbf{x}) := A\mathbf{x} + g(\mathbf{x})\mathbf{x} = 0$$

   starting from some initial points $\mathbf{x}_0$. Here $\mathbf{x} = (x_1, \ldots, x_n)$, A is an $n \times n$, nonsingular constant matrix and $g(\mathbf{x})$ is a scalar C^1-function of $\mathbf{x}$.

3. (10 pts) Suggest at least one more method to solve above system (Problem 2) and provide a sufficient condition for the convergence.

4. (10 pts) Define Chebysheff points on $[-1, 1]$ and explain a usage for them.

5. (10 pts) Define a Jacobi iteration to solve the linear $n \times n$ system $A\mathbf{x} = \mathbf{b}$. Prove the Jacobi iteration converges if A is *diagonal dominant*, i.e., $|a_{ii}| \geq \sum_{j \neq i} |a_{ij}|$ for all $i = 1, \ldots, n$.

6. (15 pts)
   (a) Let $\mathbf{u}, \mathbf{v}$ are any vectors in $\mathbb{R}^n$. Find an $n \times n$ matrix of the form $H_\mathbf{w} = I - 2\mathbf{w}\mathbf{w}^*$, for some unit vector $\mathbf{w} \in \mathbb{R}^n$ such that $H_\mathbf{w}\mathbf{u} = \mathbf{v}$.
   (b) Show that $H_\mathbf{w}$ is symmetric and orthogonal
   (c) Explain how to transform A into an upper triangular matrix R using above transformations (including how to avoid instability).

7. (10 pts) Describe Euler's method (explicit and implicit) to solve an ODE

$$\dot{x} = f(t, x(t)), \quad x(0) = x_0.$$

   Discuss advantages and disadvantages.

8. (10 pts) State the Gaussian quadrature using n points for the approximation of integral $\int_a^b f(x)\,dx$ and prove it is exact if f is a polynomial of degree $2n - 1$.

THE END

Ph.D. Qualifying Exam: Real Analysis
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Note: Be sure to use English for your answers.

1. (15 pts) Let A be the set of all $x \in \mathbb{R}$ such that there exist infinitely many fractions p/q, with relatively prime integers p and q such that
$$\left| x - \frac{p}{q} \right| \leq \frac{1}{q^3}.$$
Prove that A is a set of (Lebesgue) measure zero.

2. (15 pts) Evaluate the following limit with justification:
$$\lim_{n\to\infty} \int_0^\infty ne^{-nx} \sin\left(\frac{1}{x}\right) dx.$$

3. Let $F : \mathbb{R} \to \mathbb{R}$ be an increasing, continuous function.
   (a) (10 pts) Prove that $\int_a^b F'(x)\, dx \leq F(b) - F(a)$ for any $a < b$.
   (b) (10 pts) Give an example of an increasing, continuous function $F : \mathbb{R} \to \mathbb{R}$ and $a, b \in \mathbb{R}$ such that $\int_a^b F'(x)\, dx \neq F(b) - F(a)$.

4. An operator T on a Hilbert space $\mathcal{H}$ is called an *isometry* if $\|Tf\| = \|f\|$ for all $f \in \mathcal{H}$.
   (a) (10 pts) Prove that $T^*T = I$ if T is an isometry.
   (b) (10 pts) Give an example of an isometry that is not unitary.

5. (15 pts) Given an exterior measure μ_* on a set X, prove that the collection of (Carathéodory) measurable sets forms a σ-algebra.

6. (15 pts) Let ν be a signed measure on a measure space X and μ be a positive measure on X. Consider the following conditions:
   (a) ν is absolutely continuous with respect to μ.
   (b) For any $\varepsilon > 0$, there exists $\delta > 0$ such that $|\nu(E)| < \varepsilon$ whenever $\mu(E) < \delta$.

   Prove that (b) implies (a). Prove also that, if $|\nu|$ is a finite measure, then (a) implies (b).

THE END

Ph.D. Qualifying Exam: Probability
February 2018

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1. (a) (2 pts) State Dynkin's $\pi - \lambda$ lemma.

   (b) (8 pts) Let $(\Omega, \mathcal{F}, P)$ be a probability space and $X \in L^1$. Show that if $\mathcal{P} \subset \mathcal{F}$ is a π-system generating $\mathcal{F}$ and containing Ω, then for another random variable Y,

   $$\int_A X\, dP = \int_A Y\, dP$$

   holds for all $A \in \mathcal{F}$ if and only if it holds for all $A \in \mathcal{P}$. (You are allowed to use the $\pi - \lambda$ lemma without proving it).

2. (15 pts) Show that if $\{X_n\}$ are independent, then the event

   $$\left\{\omega : \lim_{n\to\infty} \frac{X_1(\omega) + \cdots + X_n(\omega)}{n} = \sqrt{2}\right\}$$

   has probability either 0 or 1.

3. (15 pts) Suppose $\{X_n\}$ converges in distribution to X_∞ and that $\sup_n E|X_n|^{2+\delta} < \infty$ for some $\delta > 0$. Show that X_∞ has finite second moment and that $EX_n^p \to EX_\infty^p$ for $p = 1, 2$.

4. (15 pts) Prove Kolmogorov's Maximal Inequality for independent (not necessarily indentically disributed) zero-mean random variables $\{X_k\}$ which have finite variances: If $S_n = X_1 + \cdots + X_n$ then

   $$P(\max_{1\le k\le n} |S_k| \ge \varepsilon) \le \varepsilon^{-2}\operatorname{Var}(S_n)$$

   for all $\varepsilon > 0$.

5. (15 pts) Let $\{X_n\}$ be iid with mean 0 and variance σ^2. Let $\{R_n\}$ be a sequence of positive random variables such that for some sequence of integers $a_n \to \infty$, $R_n/a_n \to 1$ in distribution. Show that as $n \to \infty$, $\frac{S_{R_n}}{\sigma\sqrt{a_n}}$ converges in distribution to $N(0,1)$. You may assume the standard CLT. (Hint: be careful on this problem!)

6. (15 pts) Let $(X_n, \mathcal{F}_n)$ be a supermartingale. If $0 \le H_n < B$ for some positive $B \in \mathbb{R}$ and H_n is *predictable*, i.e., $H_n \in \mathcal{F}_{n-1}$, then $(Y_n, \mathcal{F}_n)$ is a supermartingale where

   $$Y_n := \sum_{k=1}^{n} H_k(X_k - X_{k-1}).$$

7. (15 pts) Let $(X_n, \mathcal{F}_n)$ be a martingale. Show that (X_n) converges in L^1 if and only if there exists $X \in L^1$ such that $E(X|\mathcal{F}_n) = X_n, n \ge 0$.

THE END