Ph.D. Qualifying Exam: Algebra I
February 2018
Department of Mathematical Sciences, KAIST

Student ID: Name:

Note: Be sure to use English for your answers.

1. Let \( n \geq 2 \) be an integer and let \( E_{ij} \) be the \( n \times n \) matrix such that the \((i, j)\) entry is 1, but all the other entries are 0.
   
   (a) (10 pts) Prove that the group \( \text{SL}_n(\mathbb{C}) \) is generated by \( \{ I_n + aE_{ij} : a \in \mathbb{C} \text{ and } i \neq j \} \), where \( I_n \) denotes the identity matrix.
   
   (b) (10 pts) Prove that neither \( \text{SL}_n(\mathbb{C}) \) nor \( \text{GL}_n(\mathbb{C}) \) is solvable.

2. (10 pts) Let \( H \) be a nontrivial normal subgroup of a finite \( p \)-group \( G \). Show that the intersection \( H \cap Z(G) \) is nontrivial, where \( Z(G) \) denotes the center of \( G \).

3. (20 pts) Let \( G \) be a finite group and let \( p \) be a prime. We shall denote by \( \text{Syl}_p(G) \) the set of Sylow \( p \)-subgroups of \( G \). Show that if \( N \) is a normal subgroup of \( G \), then \( |\text{Syl}_p(G/N)| \leq |\text{Syl}_p(G)| \). Show also that if \( H \) is a subgroup of \( G \), then \( |\text{Syl}_p(H)| \leq |\text{Syl}_p(G)| \).

4. Let \( R \) be a principal ideal domain and let \( I \) be a nonzero prime ideal of the polynomial ring \( R[x] \) such that \( R \cap I = \{0\} \).
   
   (a) (10 pts) Show that \( I = (f) \) for some irreducible \( f \in R[x] \).
   
   (b) (10 pts) Assume that \( R \) has infinitely many prime elements. Show that \( I \) is not maximal.

5. (10 pts) Let \( M \) be an \( R \)-module, where \( R \) denotes a commutative ring with identity. Let \( N \) be a Noetherian submodule of \( M \) and let \( P \) be a submodule of \( M \) with \( N \subseteq P \). Prove that if the quotient modules \( M/P \) and \( P/N \) are Noetherian, then \( M \) is Noetherian.

6. (a) (10 pts) Classify all groups of order 8 up to isomorphism.
   
   (b) (10 pts) For each group \( G \) in (a), calculate \( |\text{Aut}(G)| \), where \( \text{Aut}(G) \) denotes the automorphism group of \( G \).

THE END
1. (20 pts) Suppose that $R$ is a commutative Noetherian ring with 1. Show that $R[X]$ is Noetherian.

2. (20 pts) Let $K$ be an extension field of a field $k$ and $\text{Aut}_k(K)$ be the set of all the automorphisms of $K$ fixing $k$. Show that $K$ is a Galois extension of $k$ if and only if $k$ is the fixed field of $\text{Aut}_k(K)$.

3. (20 pts) Let $K = \mathbb{Q}(\sqrt{-7})$.
   
   (a) Find the integral closure $\mathcal{O}_K$ of $\mathbb{Z}$ in $K$.
   
   (b) Show that $\mathcal{O}_K$ is a Euclidean domain.

4. (20 pts) Let $R$ be a local ring with maximal ideal $m$ and $M, N$ be finitely generated $R$-modules. Show that if $M \otimes N = 0$, then $M = 0$ or $N = 0$.

5. (20 pts) Given two morphisms $\phi : A \rightarrow X$ and $\psi : B \rightarrow X$ in a category $\mathcal{C}$, a pull-back of $(\phi, \psi)$ is defined to be a triple $(Y, \alpha, \beta)$, where $\alpha : Y \rightarrow A$ and $\beta : Y \rightarrow B$ are morphisms in $\mathcal{C}$ such that $\phi \circ \alpha = \psi \circ \beta$ with the following universal property:

   Given two morphisms $\gamma : Z \rightarrow A$ and $\delta : Z \rightarrow B$ with $\phi \circ \gamma = \psi \circ \delta$, there exists a unique morphism $\eta : Z \rightarrow Y$ such that $\gamma = \alpha \circ \eta$ and $\delta = \beta \circ \eta$.

   Show that there always exists a pull-back in the category of $R$-modules, where $R$ is a commutative ring with 1 and it is unique up to isomorphism.

THE END
Ph.D. Qualifying Exam: Complex Analysis
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1. (10 pts) Find a bijective conformal mapping $f : D_1 \to D_2$ where

$D_1 = \{ z \in \mathbb{C} : 0 < |z| < 1, 0 < \text{Arg} z < \frac{\pi}{2} \}$ and $D_2 = \{ z \in \mathbb{C} : |z| < 1 \}$.

2. (10 pts)
(a) Find all values of $(1 + i)^{1+i}$.
(b) Solve $\cos z = \frac{1}{2}$, $z \in \mathbb{C}$.

3. (10 pts) Let $f(z) = u(x, y) + iv(x, y)$ be an entire function where $z = x + iy$ and $u$, $v$ are real-valued functions. Suppose that $u_y - v_x = -2$ for all $z \in \mathbb{C}$. Find all such $f(z)$.

4. (10 pts) Evaluate the following integrals:

(a) $\int_{|z|=1} \frac{dz}{z}$
(b) $\int_{|z|=1} \frac{dz}{|z|}$
(c) $\int_{|z|=1} \frac{|dz|}{z}$
(d) $\int_{|z|=1} \frac{dz}{|z|}$

5. (10 pts) Assume that $f(z)$ is analytic for all $|z| < r$ with some $r > 0$ and $f$ satisfies the equation $f(2z) = f(z)^2$ for all $z$ sufficiently close to 0.

(a) Show that $f(z)$ can be extended to an entire function.
(b) Find all such entire functions $f(z)$ explicitly.

6. (10 pts) Evaluate

$$\int_{|z|=1} \frac{z^2 + e^z}{z(z - 3)} \, dz.$$

7. (10 pts) Evaluate the integral

$$\int_{\gamma} \frac{\sqrt{z^2 - 1}}{dz}$$

where $\gamma$ is the circle $|z| = \frac{1}{2}$.

8. (10 pts) Suppose that $f(z)$ is entire and that

$$\lim_{|z| \to \infty} \frac{f(z)}{z} = 0.$$

Prove that $f'(z) = 0$ for every $z \in \mathbb{C}$.

9. (10 pts) Consider the function $f(z) = e^{z^2}$ on the unit disk $|z| \leq 1$.

(a) What is the maximum of $f(z)$ on the unit disk? At which points does the function have the maximum?
(b) What is the minimum of $f(z)$ on the unit disk? At which points does the function have the minimum?

10. (10 pts) Using the contour integral in complex analysis, evaluate

$$\int_{-\infty}^{\infty} \frac{dx}{x^4 + 1}.$$

THE END
1. (20 pts)
   (a) Define a Lagrange interpolation polynomial with data \( \{(x_i, f(x_i))\}_{i=0}^n \) where all \( x_i \)'s are distinct.
   (b) What is the error form in the above? Derive it.
   (c) Define Newton’s form of interpolation polynomial using the same data.
   (d) Explain what happens if some \( x_i \)'s are repeated, and in this case what is the correct data corresponding to the repeated points?

2. (15 pts) Describe Newton’s method to solve a system of nonlinear equations

   \[ \mathbf{F}(\mathbf{x}) := A\mathbf{x} + g(\mathbf{x})\mathbf{x} = 0 \]

   starting from some initial points \( \mathbf{x}_0 \). Here \( \mathbf{x} = (x_1, \ldots, x_n) \), \( A \) is an \( n \times n \), nonsingular constant matrix and \( g(\mathbf{x}) \) is a scalar \( C^1 \)-function of \( \mathbf{x} \).

3. (10 pts) Suggest at least one more method to solve above system (Problem 2) and provide a sufficient condition for the convergence.

4. (10 pts) Define Chebyshev points on \([-1, 1]\) and explain a usage for them.

5. (10 pts) Define a Jacobi iteration to solve the linear \( n \times n \) system \( A\mathbf{x} = \mathbf{b} \). Prove the Jacobi iteration converges if \( A \) is diagonal dominant, i.e., \( |a_{ii}| \geq \sum_{j \neq i} |a_{ij}| \) for all \( i = 1, \ldots, n \).

6. (15 pts)
   (a) Let \( \mathbf{u}, \mathbf{v} \) are any vectors in \( \mathbb{R}^n \). Find an \( n \times n \) matrix of the form \( H_w = I - 2w w^* \), for some unit vector \( w \in \mathbb{R}^n \) such that \( H_w \mathbf{u} = \mathbf{v} \).
   (b) Show that \( H_w \) is symmetric and orthogonal
   (c) Explain how to transform \( A \) into an upper triangular matrix \( R \) using above transformations (including how to avoid instability).

7. (10 pts) Describe Euler’s method (explicit and implicit) to solve an ODE

   \[ \dot{x} = f(t, x(t)), \quad x(0) = x_0. \]

   Discuss advantages and disadvantages.

8. (10 pts) State the Gaussian quadrature using \( n \) points for the approximation of integral \( \int_a^b f(x) \, dx \) and prove it is exact if \( f \) is a polynomial of degree \( 2n - 1 \).

THE END
1. (15 pts) Let $A$ be the set of all $x \in \mathbb{R}$ such that there exist infinitely many fractions $p/q$, with relatively prime integers $p$ and $q$ such that

$$|x - \frac{p}{q}| \leq \frac{1}{q^3}.$$ 

Prove that $A$ is a set of (Lebesgue) measure zero.

2. (15 pts) Evaluate the following limit with justification:

$$\lim_{n \to \infty} \int_0^\infty ne^{-nx} \sin \left( \frac{1}{x} \right) \, dx.$$ 

3. Let $F : \mathbb{R} \to \mathbb{R}$ be an increasing, continuous function.

   (a) (10 pts) Prove that $\int_a^b F'(x) \, dx \leq F(b) - F(a)$ for any $a < b$.

   (b) (10 pts) Give an example of an increasing, continuous function $F : \mathbb{R} \to \mathbb{R}$ and $a, b \in \mathbb{R}$ such that $\int_a^b F'(x) \, dx \neq F(b) - F(a)$.

4. An operator $T$ on a Hilbert space $\mathcal{H}$ is called an isometry if $\|Tf\| = \|f\|$ for all $f \in \mathcal{H}$.

   (a) (10 pts) Prove that $T^*T = I$ if $T$ is an isometry.

   (b) (10 pts) Give an example of an isometry that is not unitary.

5. (15 pts) Given an exterior measure $\mu_*$ on a set $X$, prove that the collection of (Carathéodory) measurable sets forms a $\sigma$-algebra.

6. (15 pts) Let $\nu$ be a signed measure on a measure space $X$ and $\mu$ be a positive measure on $X$. Consider the following conditions:

   (a) $\nu$ is absolutely continuous with respect to $\mu$.

   (b) For any $\varepsilon > 0$, there exists $\delta > 0$ such that $|\nu(E)| < \varepsilon$ whenever $\mu(E) < \delta$.

Prove that (b) implies (a). Prove also that, if $|\nu|$ is a finite measure, then (a) implies (b).

THE END
Ph.D. Qualifying Exam: Probability
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1. (a) (2 pts) State Dynkin’s $\pi - \lambda$ lemma.
   
   (b) (8 pts) Let $(\Omega, \mathcal{F}, P)$ be a probability space and $X \in L^1$. Show that if $\mathcal{P} \subset \mathcal{F}$ is a $\pi$-system generating $\mathcal{F}$ and containing $\Omega$, then for another random variable $Y$,
   $$\int_A X \, dP = \int_A Y \, dP$$
   
   holds for all $A \in \mathcal{F}$ if and only if it holds for all $A \in \mathcal{P}$. (You are allowed to use the $\pi - \lambda$ lemma without proving it).

2. (15 pts) Show that if $\{X_n\}$ are independent, then the event
   $$\left\{ \omega : \lim_{n \to \infty} \frac{X_1(\omega) + \cdots + X_n(\omega)}{n} = \sqrt{2} \right\}$$
   
   has probability either 0 or 1.

3. (15 pts) Suppose $\{X_n\}$ converges in distribution to $X_\infty$ and that $\sup_n E|X_n|^{2+\delta} < \infty$ for some $\delta > 0$. Show that $X_\infty$ has finite second moment and that $EX_n^p \to EX_\infty^p$ for $p = 1, 2$.

4. (15 pts) Prove Kolmogorov’s Maximal Inequality for independent (not necessarily identically distributed) zero-mean random variables $\{X_k\}$ which have finite variances: If $S_n = X_1 + \cdots + X_n$ then
   $$P\left( \max_{1 \leq k \leq n} |S_k| \geq \varepsilon \right) \leq \varepsilon^{-2} Var(S_n)$$
   
   for all $\varepsilon > 0$.

5. (15 pts) Let $\{X_n\}$ be iid with mean 0 and variance $\sigma^2$. Let $\{R_n\}$ be a sequence of positive random variables such that for some sequence of integers $a_n \to \infty$, $R_n/a_n \to 1$ in distribution. Show that as $n \to \infty$, $\frac{S_n}{\sigma \sqrt{a_n}}$ converges in distribution to $N(0, 1)$. You may assume the standard CLT. (Hint: be careful on this problem!)

6. (15 pts) Let $(X_n, \mathcal{F}_n)$ be a supermartingale. If $0 \leq H_n < B$ for some positive $B \in \mathbb{R}$ and $H_n$ is predictable, i.e., $H_n \in \mathcal{F}_{n-1}$, then $(Y_n, \mathcal{F}_n)$ is a supermartingale where
   $$Y_n := \sum_{k=1}^{n} H_k (X_k - X_{k-1}).$$

7. (15 pts) Let $(X_n, \mathcal{F}_n)$ be a martingale. Show that $(X_n)$ converges in $L^1$ if and only if there exists $X \in L^1$ such that $E(X|\mathcal{F}_n) = X_n, n \geq 0$.

THE END