Ph.D. Qualifying Exam: Real Analysis

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Student ID: Name:

Note: Be sure to use English for your answers.

1. Prove the following:

   (a) **(10 pts)** There exists a positive continuous function f on $\mathbb{R}$ so that f is integrable on $\mathbb{R}$, but $\limsup_{x\to\infty} f(x) = \infty$.

   (b) **(10 pts)** If f is uniformly continuous on $\mathbb{R}$ and integrable, then $\lim_{|x|\to\infty} f(x) = 0$.

2. **(15 pts)** Let $\{f_n\}$ be a sequence of Lebesgue measurable functions on $E \subset \mathbb{R}$ that converges pointwise to f. Suppose that $\{g_n\}$ is a sequence of nonnegative functions on E that converges pointwise to g and $|f_n| \leq g_n$ on E for all n. Prove that if $\lim_{n\to\infty} \int_E g_n = \int_E g < \infty$, then $\lim_{n\to\infty} \int_E f_n = \int_E f$.

3. **(15 pts)** Suppose that f is absolutely continuous on $[0,1]$. Let T_f be the total variation of f. Prove that
$$\int_0^1 |f'(x)|dx = T_f(0,1)$$

4. **(15 pts)** Let $\mathcal{H}$ be a separable Hilbert space and T be a non-zero linear bounded operator on $\mathcal{H}$. Suppose that T is compact and symmetric. Prove that either $\|T\|$ or $-\|T\|$ is an eigenvalue of T.

5. **(15 pts)** Let μ be a σ-finite measure on a measure space X. Prove that every measurable set of infinite measure in X contains measurable sets of arbitrarily large finite measure.

6. Let F be an increasing function on $[0,1]$ with $F(0) = 0$ and $F(1) = 1$. Let μ be a Borel measure defined by $\mu((a,b)) = F(b^-) - F(a^+)$ and $\mu(\{0\}) = \mu(\{1\}) = 0$. Suppose that the function F satisfies a Lipschitz condition
$$|F(x) - F(y)| \leq A|x-y|$$
for some $A > 0$. Let m be the Lebesgue measure on $[0,1]$.

   (a) **(10 pts)** Prove that $\mu \ll m$.

   (b) **(10 pts)** Prove that $\dfrac{d\mu}{dm} \leq A$ almost everywhere.

THE END