Ph.D. Qualifying Exam: Algebraic Topology I
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Note: Be sure to use English for your answers.

If you are using certain theorems or facts while you are solving the following problems, you should specify exactly which theorems or facts you are using by either mentioning the name of the theorems or by stating the theorems or the facts. You don't have to prove these theorems or facts.

1. **(16 pts)** Let X be a compact Hausdorff space, and $A \subset X$ be a closed subset. For a continuous map $f : A \to A$, let $f^n(A) := f \circ f \circ \cdots \circ f(A)$ be the image of the set A by the nth composition of f, and let $B := \bigcap_{n=1}^{\infty} f^n(A)$. Suppose a loop $\alpha : S^1 \to X \setminus B$ is null-homotopic. Prove that there exists a positive integer N such that the induced map $\overline{\alpha} : S^1 \to X \setminus f^N(A)$ defined by $\overline{\alpha}(t) = \alpha(t)$ for any $t \in S^1$ is also null-homotopic.

2. Prove or disprove the following statements.

   (a) **(8 pts)** There exists a continuous map $f : S^n \to S^1$ which is not null-homotopic for $n \geq 2$.

   (b) **(8 pts)** There exists a continuous map $g : \mathbb{R}P^n \to S^1$ which is not null-homotopic for $n \geq 2$.

   (c) **(8 pts)** Let $T = S^1 \times S^1$ be a torus. There exists a continuous map $h : T \to S^1$ which is not null-homotopic.

3. Let $X = A \cup B \cup C$ and $Y = A \cup C$ where $A = \{(x, y, z) \in \mathbb{R}^3 \mid x^2 + y^2 + z^2 = 1\}$, $B = \{(x, y, 0) \in \mathbb{R}^3 \mid x^2 + y^2 \leq 1\}$ and $C = \{(0, 0, z) \in \mathbb{R}^3 \mid -1 \leq z \leq 1\}$.

   (a) **(10 pts)** Find the fundamental group of Y and X.

   (b) **(10 pts)** Find the integral homology groups of Y and X.

4. Let $X := \{(x, y, z) \in \mathbb{R}^3 \mid xyz = 0\}$.

   (a) **(12 pts)** Compute the relative homology groups $H_*(X, X \setminus \{x\} : \mathbb{Z})$ for every $x \in X$. You should consider the cases when $x = 0$, $x \neq 0$ but in the coordinate axis, and x is not in the coordinate axises sperately.

   (b) **(8 pts)** Using the result above, show that any homeomorphism $f : X \to X$ must preserve the origin, i.e., f should send $0 \in X$ to $0 \in X$.

5. Let X be a connected CW-complex with two 0-cells, four 1-cells, three 2-cells, and no higher cells. Assume that $H_1(X) \cong \mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z}/3$.

   (a) **(7 pts)** Give the definition of Euler characteristic of an arbitrary topological space Y, and compute the Euler characteristic $\chi(X)$ of the above CW-complex X.

   (b) **(13 pts)** Using the Euler characteristic of X, determine all possible $H_2(X : \mathbb{Z})$.

THE END