Ph.D. Qualifying Exam: Advanced Statistics
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Department of Mathematical Sciences, KAIST

Student ID: Name:

Note: Be sure to use English for your answers.

1. **(10 pts)** Let X_1, X_2, X_3 be binary variables taking on 0 or 1. Let X_1 and X_2 be independent each other and $P(X_i = 1) = p_i$ for $i = 1, 2$. Also let $P(X_3 = 1 \mid X_1 = X_2) = 1$ and $P(X_3 = 1 \mid X_1 \neq X_2) = 0$.

   (a) Check independence among the three X variables when $p_i = 0.5$, $i = 1, 2$.

   (b) Check independence among the three X variables when $p_i \neq 0.5$, $i = 1, 2$.

2. Let $\{(x_{1i}, x_{2i}, y_i),\ i = 1, 2, \dots, n\}$ be a data set from a linear regression model

$$Y = \beta_0 + \beta_1 x_1 + \beta_2 x_2 + \epsilon.$$

   Suppose $\epsilon_1, \dots, \epsilon_n$ are iid from Normal$(0, \sigma^2)$.

   (a) **(8 pts)** Find the MLE's of $\beta_0, \beta_1, \beta_2$, and σ^2.

   (b) **(8 pts)** Check if the above four MLE's are unbiased.

   (c) **(6 pts)** Suppose we fit the model given by $Y = \alpha_0 + \alpha_1 x_1 + \epsilon$ to the data $\{(x_{1i}, y_i) : i = 1, 2, \dots, n\}$. Find the MLE's of α_0, α_1, and σ^2.

   (d) **(6 pts)** Check if the MLE's of α_0 and α_1 are unbiased for β_0 and β_1, respectively.

3. Suppose that signal data are received at a sensor from a target in polar coordinates as (d_i, θ_i), $i = 1, 2, \dots, n$ where d_i and θ_i are the measurements of sensor-target distance and angle, respectively. The angle is measured counter-clock-wise with the X-axis as the reference line. The sensor is at the origin (0.0). Denote by D and Θ respectively the random variables of the sensor-target distance and angle. Let the joint distribution of (D, Θ) be bivariate Normal(μ, Σ) with

$$\mu = \begin{pmatrix} \mu_d \\ \mu_\theta \end{pmatrix} \quad \text{and} \quad \Sigma = \begin{pmatrix} \sigma_{dd} & 0 \\ 0 & \sigma_{\theta\theta} \end{pmatrix}.$$

   (a) **(8 pts)** Find the MLE's of μ and Σ.

   (b) **(4 pts)** Denote the target location in the Cartesian coordinate system as $\mu^c = (\mu_x, \mu_y)'$. In other words, $\mu_x = \mu_d \cos(\mu_\theta)$ and $\mu_y = \mu_d \sin(\mu_\theta)$. Find the MLE $\widehat{\mu^c}$ of μ^c.

   (c) **(8 pts)** Find the mean squared error (MSE) of $\widehat{\mu^c}$.

4. **(10 pts)** Let $X_1, \dots, X_n$ be iid with pdf $f(x; \mu) = e^{-(x-\mu)}$, $x \geq \mu$. Find $1 - \alpha$ confidence intervals for μ by applying the CDF method and the pivotal method.

5. Let $\mathbf{X} = (X_1, \dots, X_p)'$ be a p-dimensional Normal random vector with mean $\mu = (\mu_1, \dots, \mu_p)'$ and variance-covariance $\sigma^2 \mathbf{I}$, where σ is known and $\mathbf{I}$ is a $p \times p$ identity matrix.

   (a) **(4 pts)** Find the MLE $\hat{\mu}$ of μ.

   (b) **(3 pts)** Find the MSE of $\hat{\mu}$, i.e., $MSE(\hat{\mu}) = E[\sum_{i=1}^p (\hat{\mu}_i - \mu_i)^2]$.

   (c) **(10 pts)** Assume that $\mu_1, \dots, \mu_p$ are iid random variables from Normal(μ_0, σ_0^2) where μ_0 and σ_0 are both known. Find a Bayes estimate $\tilde{\mu}$ of μ based on the observed sample, $x_1, \dots, x_p$.

   (d) **(5 pts)** Find $MSE(\tilde{\mu})$ and compare this with $MSE(\hat{\mu})$.

6. **(10 pts)** Consider a sequence of random variables. Prove equivalence of convergence of the sequence in distribution to a constant a and its convergence in probability to a.

THE END