Ph.D. Qualifying Exam: Probability
August 2017

Department of Mathematical Sciences, KAIST

Student ID: Name:

Note: Be sure to use English for your answers.

1. **(10 pts)** Suppose $\{A_n\}$ is an infinite family of independent events such that $P(A_n) < 1$ for all n. Show $P(\cup_{n\geq 1} A_n) = 1$ if and only $P(A_n \text{ i.o.}) = 1$.

2. **(15 pts)** Suppose $\{X_n\}$ are independent and $f : \mathbb{R} \to \mathbb{R}$ is continuous. If $P(\lim_{n\to\infty} X_n \text{ exists}) > 0$, what can you say about
$$P(\lim_{n\to\infty} [X_n \cdot f(X_n)] \text{ exists})?$$

3. **(10 pts)** Suppose $\{X_n\}$ converges in distribution to a constant $C > 0$ and $\{Y_n\}$ converges in distribution to Y. Show that $\{X_n Y_n\}$ converges in distribution to $X_n Y_n$. This is a version of the *converging together lemma*.

4. **(10 pts)** Show that if $\{X_n\}$ are uniformly integrable and converge in distribution to X then $X \in L^1$ and $E[X_n] \to E[X]$.

5. **(10 pts)** Let $\{X_n\}$ be iid with mean 0 and variance σ^2. Let $\{R_n\}$ be a sequence of positive random variables such that for some sequence of integers $a_n \to \infty$, $R_n/a_n \to 1$ in distribution. Show that as $n \to \infty$, $\frac{S_{R_n}}{\sigma\sqrt{a_n}}$ converges in distribution to $N(0, 1)$. You may assume the standard CLT.

6. (a) **(8 pts)** Show that the sum of n independent exponential random variables $\{X_k\}$, each with mean $1/\lambda$, is equal in distribution to a gamma random variable which has density of the form $Cx^{n-1}e^{-\lambda x}$ on $(0, \infty)$.

   (b) **(7 pts)** Let $T_k = X_1 + \cdots + X_k$, be the "kth arrival" of a Poisson process with rate λ. Show that the number of arrivals before time t has a Poisson distribution with parameter λt.

   (c) **(5 pts)** Suppose the number of arrivals of the Poisson process with rate λ before time t is equal to 2. Let X denote the first arrival and Y denote the second arrival. Show that (X, Y) has a uniform distribution on the triangle lying above the line $y = x$ and below the line $y = t$ in the first quadrant of the plane.

7. **(10 pts)** If $\{(X_n, \mathcal{F}_n)\}$ is a martingale, show that it converges in L^1 if and only if $X_n = E[X|\mathcal{F}_n]$ for some X and all $n \in \mathbb{N}$.

8. **(15 pts)** If $\{X_n\}$ is a sequence of mean-zero independent random variables with $E(X_n^2) = \sigma_n^2 < \infty$, show that
$$M_n := \left(\sum_{k=1}^{n} X_k\right)^2 - \sum_{k=1}^{n} \sigma_k^2$$
is a martingale with respect to the natural filtration (what is the filtration?).

THE END