Ph.D. Qualifying Exam: Algebra I

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Student ID: Name:

Note: Be sure to use English for your answers.

1. Let F be a field. For any commutative ring R with unity, let $R^\times$ denote the multiplicative group of the units of R. Answer the following questions.
   (a) **(10 pts)** For the polynomial ring $F[x]$, what is $F[x]^\times$? Prove your assertion.
   (b) **(10 pts)** For the formal power series ring $F[[x]]$, what is $F[[x]]^\times$? Prove your assertion.
2. **(10 pts)** Let R be a commutative ring with unity. Recall that an R-module M is called *simple* if it has no proper nontrivial R-submodule. Prove that if R is noetherian and M is simple, then there exists a prime ideal $I \subset R$ such that $R/I \simeq M$.
3. **(10 pts)** Let G be a finite group and suppose for a prime number p, we have $p \mid |G|$. Let P be a Sylow p-subgroup of G. Then prove that $N_G(N_G(P)) = N_G(P)$.
4. **(10 pts)** Give an integer d such that the ring $\mathbb{Z}[\sqrt{d}]$ is not a unique factorization domain, and prove your assertion.
5. Let G be a finite group. Answer the following questions.
   (a) **(10 pts)** State and prove the class equation for the group G.
   (b) **(10 pts)** For a fixed prime number p, prove that all groups of order p^2 are abelian and classify *all groups* of order p^2 up to isomorphism.
6. Let F be a field. Answer the following questions.
   (a) **(10 pts)** Let $p(x) \in F[x]$ be an irreducible polynomial. Prove that $F[x]/(p(x))$ is a field, and as an F-vector space, its dimension is equal to $\deg p(x)$.
   (b) **(10 pts)** Now take $F = \mathbb{Q}$. For each integer $n \geq 2$, decide whether there exists an irreducible polynomial of degree n over $\mathbb{Q}$. When there is for n, give an example of such a polynomial $p(x)$. Whether there exists or not, you should justify your answer.
7. **(10 pts)** Prove that the ring $\mathbb{Z}[x]$ of polynomials in x with the coefficients in $\mathbb{Z}$ is a unique factorization domain, but it is neither a principal ideal domain, nor a Euclidean domain.

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