Ph.D. Qualifying Exam: Complex Analysis

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Department of Mathematical Sciences, KAIST

Student ID: Name:

Note: Be sure to use English for your answers.

1. **(10 pts)** Let $f(z) = \displaystyle\sum_{n=0}^{\infty} \frac{z^{2n}}{(n!)^2}$. Prove that
$$z^2 f''(z) + z f'(z) = 4z^2 f(z).$$

2. **(10 pts)** Compute the following integral:
$$\int_0^{\infty} \frac{v^{a-1}}{1+v}\, dv \quad \text{for } 0 < a < 1.$$

3. **(15 pts)** State the open mapping theorem for holomorphic functions and prove it.

4. **(15 pts)** Prove that the meromorphic functions in the extended complex plane are the rational functions.

5. **(10 pts)** Let $s > 0$. Characterize an entire function g satisfying the following: there exist an increasing sequence $\{r_n\}$ of positive reals and a constant $C > 0$ (independent of n) such that $\displaystyle\lim_{n\to\infty} r_n = \infty$ and
$$|g(z)| \leq C r_n^s \quad \text{whenever } |z| = r_n \text{ for some } n.$$

6. **(20 pts)** Let $\{\alpha_n\}$ be a sequence in the unit disc D such that $\alpha_n \neq 0$ for all n and $\sum_{n=1}^{\infty}(1-|\alpha_n|)$ converges. Show that for each $0 < r < 1$, the product
$$f(z) = \prod_{n=1}^{\infty} \frac{\alpha_n - z}{1 - \overline{\alpha}_n z} \frac{|\alpha_n|}{\alpha_n}$$
converges uniformly for $|z| \leq r$.

7. **(20 pts)** Let S be a sector given by
$$S = \left\{ z : -\frac{\pi}{4} < \arg z < \frac{\pi}{4} \right\}.$$
Let F be a holomorphic function in S that is continuous on the closure of S. Assume that $|F(z)| \leq 1$ on ∂S and that there are constants $C, c > 0$ such that
$$|F(z)| \leq C e^{c|z|} \quad \text{for all } z \in S.$$
Prove that $|F(z)| \leq 1$ for all $z \in S$.

THE END