

# Ph.D. Qualifying Exam: Numerical Analysis

## August 2017

Department of Mathematical Sciences, KAIST

Student ID:

Name:

Note: Be sure to use English for your answers.

1. (30 pts)

- (a) Define a Lagrange interpolation polynomial with data  $\{(x_i, f(x_i))\}_{i=0}^n$ .
- (b) What is the error form in the above? Derive it.
- (c) Define a Newton's form of interpolation polynomial using the same data.
- (d) Explain how to compute the coefficients in (c).

2. (10 pts) Consider the following integral equation:

$$x(t) = \int_a^b K(t, s)x(s) ds + g(t),$$

where  $K(t, s) : \mathbb{R}^2 \rightarrow \mathbb{R}$  and  $g : \mathbb{R} \rightarrow \mathbb{R}$  are continuous functions.

- (a) Define a Picard method to find the solution  $x(t)$ .
- (b) Prove the solution exists in  $C[a, b]$ , the space of continuous functions under the maximum norm  $\|\cdot\|_\infty = \max_{[a, b]} |x(t)|$ .

3. (10 pts) Describe the Newton's method to solve a system of nonlinear equations of the form

$$\mathbf{F}(\mathbf{x}) := A\mathbf{x} + g(\mathbf{x})\mathbf{x} = 0,$$

starting from some initial point  $\mathbf{x}_0$ . Here  $\mathbf{x} = (x_1, \dots, x_n)$ ,  $A$  is an  $n \times n$ , nonsingular constant matrix and  $g(\mathbf{x})$  is a  $C^1$  scalar function of  $\mathbf{x}$ .

4. (25 pts)

- (a) Let  $\mathbf{u}, \mathbf{v}$  are any vectors in  $\mathbb{R}^n$ . Find an  $n \times n$  matrix of the form  $H_{\mathbf{w}} = I - 2\mathbf{w}\mathbf{w}^*$ , for some unit vector  $\mathbf{w} \in \mathbb{R}^n$  such that  $H_{\mathbf{w}}\mathbf{u} = \mathbf{v}$ .
- (b) Explain how to obtain an upper Hessenberg form of a given  $n \times n$  matrix  $A$  by a similarity transformations using  $H_{\mathbf{w}}$ . (In your explanation, mention how to avoid the instability)
- (c) Explain how to find all the eigenvalues of  $A$  using a transformations like above. Discuss detailed algorithm, computational complexity, and convergence, etc.

5. (10 pts) Describe a predictor corrector method to solve an ODE:

$$\dot{x} = f(t, x(t)), \quad x(0) = x_0.$$

Give at least one specific example.

6. (15 pts) State Gaussian quadrature using  $n$  points for computing the integral  $\int_a^b f(x)dx$  and prove that it is exact for polynomials of degree  $2n - 1$ . What is the error form?

THE END