Ph.D. Qualifying Exam: Algebra I
February 2017

Student ID: Name:

Note: Be sure to use English for your answers.

1. Answer the following questions.
   (a) [10 pts] Let $F$ be a field and $x$ be an indeterminate. Prove that $F[x]$ is a
       Euclidean domain.
   (b) [10 pts] Let $R$ be a Euclidean domain. Prove that $R$ is a PID.

2. [10 pts] Let $G$ be a finite group acting on itself by conjugation. Using the
   orbits of the action, prove the class equation.

3. [10 pts] For a prime number $p$, consider the set $G$ of equivalence classes of integers
   modulo $p$ that is not equivalent to 0, i.e., $G = (\mathbb{Z}/p)^\times = \{1, \ldots, p-1\}$.
   Prove that $G$ is a group with respect to the operation induced by the multi-
   plication of the integers.

4. Answer the following questions.
   (a) [10 pts] Let $G$ be a simple group of order $n$ acting nontrivially on a finite
       set of size $r$. Prove that $n|r!$.
   (b) [10 pts] Show that when $G$ is a simple group of order 60, any proper
       subgroup $H < G$ has the cardinality at most 12.

5. Let $U \subset \mathbb{C}$ be a nonempty open subset of the complex plane and let $\mathcal{H}(U)$ be
   the set of holomorphic functions $U \to \mathbb{C}$. Answer the following questions:
   (a) [10 pts] Prove that $\mathcal{H}(U)$ is an integral domain if and only if $U$ is con-
       nected.
   (b) [10 pts] Is $\mathcal{H}(U)$ always a unique factorization domain for a nonempty
       open subset $U \subset \mathbb{C}$? If so, give a proof. Otherwise, give a counterexample.
   (c) [10 pts] For $p \in \mathbb{C}$, consider $\mathcal{H}_p$ defined to be the equivalence classes
       $(f, U)$ with $f \in \mathcal{H}(U)$, with $p \in U$, where we define the equivalence
       relation $(f, U) \sim (g, V)$ for two open sets $U, V \subset \mathbb{C}$ satisfying $p \in U$ and
       $p \in V$, if there exists an open subset $W \subset U \cap V$ with $p \in W$ such that
       $f|_W = g|_W$. Prove that $\mathcal{H}_p$ is a unique factorization domain.

6. [10 pts] Let $R$ be a nonzero commutative ring with unity. Prove that $R$ is an
   integral domain if and only if the zero ideal $(0)$ is a prime ideal of $R.$
Ph.D. Qualifying Exam: Algebra II
February 2017

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Note: Be sure to use English for your answers.

1. [25 pts] Prove or disprove the following:
   (a) The rings $\mathbb{Q}[x, y]/(y^2 - x)$ and $\mathbb{Q}[x, y]/(xy - 1)$ are not isomorphic.
   (b) $\mathbb{Q}$ is a projective $\mathbb{Z}$-module.
   (c) $\mathbb{Q}$ is not a flat $\mathbb{Z}$-module.
   (d) $f(x^n) = f(x^n)$ for any polynomial $f(x) \in \mathbb{F}_p[x]$.
   (e) A splitting field $K$ of a polynomial of degree $n$ over $F$ has degree $[K : F]$ which divides $n!$.

2. [15 pts] Let $R$ be a subring of the commutative ring $T$ with identity $1 \in R$.
   (a) If $T$ is a finitely generated $R$-module, then $T$ is integral over $R$.
   (b) If $R$ is a U.F.D. then $R$ is integrally closed.

3. [10 pts] Let $(R, m)$ be a Noetherian local ring and $M$ be a finitely generated $R$-module. Show that if $M \otimes_R R/m = 0$ then $M = 0$.

4. [20 pts] Let $R$ be a commutative ring with identity $1_R$.
   (a) If $M$ is a projective $R$-module, then $M$ is flat as an $R$-module.
   (b) If $R$ is an integral domain with its quotient field $K$. Let $V$ be the $K$-vector space, then $K \otimes_R V \simeq V$.

5. [30 pts] Let $F$ be a field and $K$ be the splitting field of the polynomial $x^4 - 2$ over $F$. Find $[K : F]$, a primitive element of $K$ and the Galois group $\text{Gal}(K/F)$ where $F$ is one of the following fields:
   (a) $F = \mathbb{R}$.
   (b) $F = \mathbb{Q}$.
   (c) $F = \mathbb{Z}_3$.

THE END
Ph.D. Qualifying Exam: Differential Geometry  
February 2017  

Student ID:  
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Note: Be sure to use English for your answers.

1. [15 pts] Show in detail that $\mathbb{R}P^n$ is a smooth manifold.

2. [15 pts] Let $M$ denote a smooth manifold. Suppose $K$ to be a compact subset of $M$ and $O \subset M$ to be an open set containing $K$. Show there exists a smooth function $\beta : M \to [0, 1]$, that is identically equal to 1 on $K$ and its compact support is contained in $O$.

3. [15 pts] State and give a proof of the Inverse Mapping Theorem for differentiable manifolds.

4. [15 pts] Show that the set on $\mathbb{R}^{n \times n}$ orthogonal real matrices $O(n, \mathbb{R})$ is a submanifold of the manifold of all square real matrices $M(n, \mathbb{R})$.

5. [15 pts] Find the integral curves in $\mathbb{R}^2$ of the vector field $X = e^{-x} \frac{\partial}{\partial z} + \frac{\partial}{\partial y}$ and determine if $X$ is complete or not.

6. [10 pts] Determine if the differential 1-form  

$$\alpha := \frac{xdy - ydx}{x^2 + y^2}$$

is globally conservative, locally conservative, exact in $\mathbb{R}^2 \setminus \{(0, 0)\}$.

7. [15 pts] Explain in detail the definition of the integral of a differential $n$-form with compact support on an oriented smooth $n$-manifold (without boundary and with boundary).

THE END
1. [10 pts] Let $f$ be a function on a domain $U \subset \mathbb{C}$. Suppose that both $f$ and $\bar{f}$ are holomorphic. Prove that $f$ is a constant.

2. [10 pts] Let $f$ be a continuous function on the unit disc $D(0,1)$. Suppose that for any closed curve $\gamma$ in $D(0,1)$, $\oint_{\gamma} f(z) dx = 0$. Prove that $f$ is holomorphic.

3. [15 pts] Let $u$ be a harmonic function in $D(0,1) \setminus \{0\}$. Prove that if
\[
\lim_{|(x,y)| \to 1} u(x,y) = 0 \quad \text{and} \quad \lim_{|(x,y)| \to 0} u(x,y) / \log(x^2 + y^2) = 0,
\]
u is identically zero on $D(0,1)$.

4. [15 pts] Prove or disprove that there exists an entire function $f$ satisfying $\lim_{|z| \to \infty} |f(z)|/|z|^{3/2} = 1$.

5. [15 pts] Prove that for any compact subset $K$ of the unit disc $D(0,1)$, there exists a constant $C > 0$ such that for any holomorphic function $f$ on $D(0,1)$,
\[
\max_{z \in K} |f(z)| \leq C \left( \int_{D(0,1)} |f(z)|^2 dxdy \right)^{1/2}.
\]

6. [15 pts] Compute $\int_{-\infty}^{\infty} \frac{\cos x}{1+x^2} dx$.

7. [20 pts] Find all conformal mappings from $\mathbb{C}$ to $\mathbb{C}$.

THE END
Ph.D. Qualifying Exam: Real Analysis
February 2017

Student ID: Name:

Note: Be sure to use English for your answers.

1. [10 pts] Suppose that $\mu$ is a semifinite measure and $\mu(E) = \infty$. Prove that for any $C > 0$ there exists $F \subset E$ such that $C < \mu(F) < \infty$.

2. [10 pts] Interpret Fatou's lemma, the monotone convergence theorem and the dominated convergence theorem when $\mu$ is the counting measure on $\mathbb{N}$.

3. [15 pts] Let $f : [a, b] \to \mathbb{R}$ be Lebesgue measurable and $\epsilon > 0$. Prove that there is a compact set $E \subset [a, b]$ such that
\[ \mu(E^c) < \epsilon \] and $f|_E$ is continuous.

4. [15 pts] Suppose that $\mu, \nu$ are $\sigma$-finite measures on $(X, \mathcal{M})$ with $\nu \ll \mu$, and let $\lambda = \mu + \nu$ and $f = \frac{d\nu}{d\lambda}$. Prove that $0 \leq f < 1$ $\mu$-a.e. and $\frac{d\nu}{d\mu} = \frac{1}{f}$.

5. [15 pts] Let $f$ be a measurable function on $(X, \mathcal{M}, \mu)$ and define $\lambda_f : (0, \infty) \to [0, \infty]$ as $\lambda_f(\alpha) = \mu(\{x : |f(x)| > \alpha\})$. Prove that
\[ f \in L^1 \text{ if and only if } \sum_{k=-\infty}^{\infty} 2^k \lambda_f(2^k) < \infty. \]

6. [10 pts] Prove that every nonempty closed convex set $K$ in a Hilbert space has a unique element of minimal norm.

7. Let $(X, \mathcal{M}, \mu)$ be a measure space.
   
   (a) [15 pts] Prove that $L^p(X)$ is complete for $1 \leq p < \infty$.
   
   (b) [10 pts] Prove that the set of simple functions
   \[ f = \sum_{j=1}^{n} \alpha_j \chi_{E_j}, \text{ where } \mu(E_j) < \infty \text{ for all } j, \]
   is dense in $L^p(X)$ for $1 \leq p < \infty$.

THE END
1. [20 pts] Let $\mathcal{F}$ be a family of subsets of a finite set, where each member of $\mathcal{F}$ has size at least 2. Let $A$ and $B$ be two blocking sets of $\mathcal{F}$ of minimal size, that is $|A| = |B| = \tau(\mathcal{F})$. Consider a bipartite graph $G$ with parts $A$ and $B$, where $a \in A$ is connected to $b \in B$ if there is an $F \in \mathcal{F}$ containing both $a$ and $b$. Prove that $G$ has a perfect matching.

2. [20 pts] Prove that for every integer $k$ there exists a graph $G$ with girth $g(G) > k$ and chromatic number $\chi(G) > k$.

3. [20 pts] Let $(\mathcal{P}, \mathcal{L})$ be a finite projective plane of order $q \geq 3$, and let $B \subset \mathcal{P}$ be a non-trivial blocking set. Prove that
   (i) $|B| \leq q^2 - \sqrt{q}$.
   (ii) no line in $\mathcal{L}$ contains more than $|B| - q$ points from $B$.

4. [20 pts] Let $L \subset \{0, 1, 2, \ldots \}$ be a finite set of integers and let $\mathcal{F}$ be a family of subsets of $\{1, 2, \ldots, n\}$ such that $|A \cap B| \in L$ for every pair $A, B$ of distinct members of $\mathcal{F}$. Prove that
   $$|\mathcal{F}| \leq \sum_{i=0}^{\lfloor L/2 \rfloor} \binom{n}{i}.$$

5. [20 pts] Let $\mathbb{F}$ be a finite field with $|\mathbb{F}| = q$. Prove that every non-zero polynomial $f(x_1, \ldots, x_n) \in \mathbb{F}[x_1, \ldots, x_n]$ of degree $d$ ($1 \leq d \leq q$) has at most $d q^{n-1}$ roots.

THE END
Ph.D. Qualifying Exam: Probability Theory
February 2017

Student ID: 
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Note: Be sure to use English for your answers.

1. [15 pts] For \( n \in \mathbb{N} \), consider the probability space \((\Omega_n, \mathcal{F}_n, P_n)\) where \( \Omega_n = \{1, 2, \ldots, n\} \), \( \mathcal{F}_n \) is the set of all subsets of \( \Omega_n \), and \( P_n \) is the uniform probability measure, i.e., \( P_n(\{k\}) = 1/n \) for all \( k \in \Omega_n \). Find the largest positive integer \( N \) for which the following statement holds for all \( n \leq N \):
   If \( X \) and \( Y \) are independent random variables on \((\Omega_n, \mathcal{F}_n, P_n)\), then either \( X \) or \( Y \) must be constant.

2. [10 pts] Two sequences of random variables, \((X_n)\) and \((Y_n)\), are tail equivalent if
   \[
   \sum_{n \geq 1} P(X_n \neq Y_n) < \infty.
   \]
   Suppose \( \Omega_X \subset \Omega \) is the set on which \( \sum_n X_n \) converges, and \( \Omega_Y \subset \Omega \) is the set on which \( \sum_n Y_n \) converges. Show that the symmetric difference of \( \Omega_X \) and \( \Omega_Y \) has probability zero, i.e., \( P(\Omega_X \Delta \Omega_Y) = 0 \).

3. [15 pts] Prove Kolmogorov's Maximal Inequality for independent zero-mean random variables \( \{X_k\} \) which have finite variances: If \( S_n = X_1 + \cdots + X_n \) then
   \[
   P \left( \max_{1 \leq k \leq n} |S_k| \geq \varepsilon \right) \leq \frac{\varepsilon^{-2} \text{Var}(S_n)}{n}
   \]
   for all \( \varepsilon > 0 \).

4. [10 pts] Suppose that the sequence \((X_n/a_n, n \in \mathbb{N})\) converges in distribution where \( a_n \uparrow \infty \). If \((b_n)\) is a sequence of numbers such that \( a_n/b_n \to 0 \) as \( n \to \infty \), show that \((X_n/b_n, n \in \mathbb{N})\) converges in probability to 0. (Remark: this shows that the CLT implies the WLLN under finite second moments.)

5. [15 pts] Suppose \( \sup_n E|X|^p < \infty \) for some \( p > 0 \). Show that there is a subsequence \( \{X_{n_k}\} \) such that \( \{X_{n_k}, k \in \mathbb{N}\} \) converges in distribution.

6. (a) [10 pts] Consider \( \sigma \)-fields \( \mathcal{F}_1 \subset \mathcal{F}_2 \). Show \( E(E(X|\mathcal{F}_1)|\mathcal{F}_2) = E(X|\mathcal{F}_1) \) and
   \[
   E(E(X|\mathcal{F}_2)|\mathcal{F}_1) = E(X|\mathcal{F}_1).
   \]
   (b) [10 pts] If \( EX^2 < \infty \), then \( E(X|\mathcal{F}) \) is (a version of) the random variable \( Y \) which is \( \mathcal{F} \)-measurable and minimizes \( E(Y - X)^2 \).

7. [15 pts] Let \( \{X_k, k \in \mathbb{N}\} \) be an i.i.d. sequence such that \( E|X_1| < \infty \). If \( \tau \) is a stopping time with respect to the filtration given by \( \mathcal{F}_n = \sigma(X_1, \ldots, X_n) \), and \( E\tau < \infty \), then
   \[
   E(X_1 + \cdots + X_\tau) = E(\tau)E(X_1).
   \]
   (Remark: this is called Wald's equation, and it is not true when \( E\tau = \infty \).

THE END