Ph.D. Qualifying Exam: Algebra I
February 2017

Student ID: Name:

Note: Be sure to use English for your answers.

1. Answer the following questions.
   (a) [10 pts] Let F be a field and x be an indeterminate. Prove that $F[x]$ is a Euclidean domain.
   (b) [10 pts] Let R be a Euclidean domain. Prove that R is a PID.
2. [10 pts] Let G be a finite group acting on itself by conjugation. Using the orbits of the action, prove the class equation.
3. [10 pts] For a prime number p, consider the set G of equivalence classes of integers modulo p that is not equivalent to 0, i.e., $G = (\mathbb{Z}/p)^\times = \{\overline{1}, \cdots, \overline{p-1}\}$. Prove that G is a group with respect to the operation induced by the multiplication of the integers.
4. Answer the following questions.
   (a) [10 pts] Let G be a simple group of order n acting nontrivially on a finite set of size r. Prove that $n|r!$.
   (b) [10 pts] Show that when G is a simple group of order 60, any proper subgroup $H < G$ has the cardinality at most 12.
5. Let $U \subset \mathbb{C}$ be a nonempty open subset of the complex plane and let $\mathcal{H}(U)$ be the set of holomorphic functions $U \to \mathbb{C}$. Answer the following questions:
   (a) [10 pts] Prove that $\mathcal{H}(U)$ is an integral domain if and only if U is connected.
   (b) [10 pts] Is $\mathcal{H}(U)$ always a unique factorization domain for a nonempty open subset $U \subset \mathbb{C}$? If so, give a proof. Otherwise, give a counterexample.
   (c) [10 pts] For $p \in \mathbb{C}$, consider $\mathcal{H}_p$ defined to be the equivalence classes (f, U) with $f \in \mathcal{H}(U)$, with $p \in U$, where we define the equivalence relation $(f, U) \sim (g, V)$ for two open sets $U, V \subset \mathbb{C}$ satisfying $p \in U$ and $p \in V$, if there exists an open subset $W \subset U \cap V$ with $p \in W$ such that $f|_W = g|_W$. Prove that $\mathcal{H}_p$ is a unique factorization domain.
6. [10 pts] Let R be a nonzero commutative ring with unity. Prove that R is an integral domain if and only if the zero ideal (0) is a prime ideal of R.

THE END

Ph.D. Qualifying Exam: Algebra II
February 2017

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Note: Be sure to use English for your answers.

1. [25 pts] Prove or disprove the following:
   (a) The rings $\mathbb{Q}[x,y]/(y^2-x)$ and $\mathbb{Q}[x,y]/(xy-1)$ are not isomorphic.
   (b) $\mathbb{Q}$ is a projective $\mathbb{Z}$-module.
   (c) $\mathbb{Q}$ is not a flat $\mathbb{Z}$-module.
   (d) $f(x)^q = f(x^q)$ for any polynomial $f(x) \in \mathbb{F}_q[x]$.
   (e) A splitting field K of a polynomial of degree n over F has degree $[K:F]$ which divides $n!$.
2. [15 pts] Let R be a subring of the commutative ring T with identity $1 \in R$.
   (a) If T is a finitely generated R-module, then T is integral over R.
   (b) If R is a U.F.D. then R is integrally closed.
3. [10 pts] Let (R,m) be a Noetherian local ring and M be a finitely generated R-module. Show that if $M \otimes_R R/m = 0$ then $M = 0$.
4. [20 pts] Let R be a commutative ring with identity 1_R.
   (a) If M is a projective R-module, then M is flat as an R-module.
   (b) If R is an integral domain with its quotient field K. Let V be the K-vector space, then $K \otimes_R V \simeq V$.
5. [30 pts] Let F be a field and K be the splitting field of the polynomial x^4-2 over F. Find $[K:F]$, a primitive element of K and the Galois group $\mathrm{Gal}(K/F)$ where F is one of the following fields:
   (a) $F = \mathbb{R}$.
   (b) $F = \mathbb{Q}$.
   (c) $F = \mathbb{Z}_3$.

THE END

Ph.D. Qualifying Exam: Differential Geometry
February 2017

Student ID: Name:

Note: Be sure to use English for your answers.

1. [15 pts] Show in detail that RP^n is a smooth manifold.

2. [15 pts] Let M denote a smooth manifold. Suppose K to be a compact subset of M and $O \subset M$ to be an open set containing K. Show there exists a smooth function $\beta : M \to [0, 1]$, that is identically equal to 1 on K and its compact support is contained in O.

3. [15 pts] State and give a proof of the Inverse Mapping Theorem for differentiable manifolds.

4. [15 pts] Show that the set on $n \times n$ orthogonal real matrices $O(n, R)$ is a submanifold of the manifold of all square real matrices $M(n, R)$.

5. [15 pts] Find the integral curves in R^2 of the vector field $X = e^{-x}\frac{\partial}{\partial x} + \frac{\partial}{\partial y}$ and determine if X is complete or not.

6. [10 pts] Determine if the differential 1-form

$$\alpha := \frac{xdy - ydx}{x^2 + y^2}$$

is globally conservative, locally conservative, exact in $R^2 \setminus \{(0,0)\}$.

7. [15 pts] Explain in detail the definition of the integral of a differential n-form with compact support on an oriented smooth n-manifold (without boundary and with boundary).

THE END

Ph.D. Qualifying Exam: Complex Analysis
February 2017

Student ID: Name:

Note: Be sure to use English for your answers.

1. [10 pts] Let f be a function on a domain $U \subset \mathbb{C}$. Suppose that both f and $\overline{f}$ are holomorphic. Prove that f is a constant.

2. [10 pts] Let f be a continuous function on the unit disc $D(0,1)$. Suppose that for any closed curve γ in $D(0,1)$, $\oint_\gamma f(z)dx = 0$. Prove that f is holomorphic.

3. [15 pts] Let u be a harmonic function in $D(0,1) \setminus \{0\}$. Prove that if

$$\lim_{|(x,y)|\to 1} u(x,y) = 0 \text{ and } \lim_{|(x,y)|\to 0} u(x,y)/\log(x^2+y^2) = 0,$$

   u is identically zero on $D(0,1)$.

4. [15 pts] Prove or disprove that there exists an entire function f satisfying $\lim_{|z|\to\infty} |f(z)|/|z|^{3/2} = 1$.

5. [15 pts] Prove that for any compact subset K of the unit disc $D(0,1)$, there exists a constant $C > 0$ such that for any holomorphic function f on $D(0,1)$,

$$\max_{z\in K} |f(z)| \leq C\Big(\int_{D(0,1)} |f(z)|^2 dxdy\Big)^{1/2}.$$

6. [15 pts] Compute $\int_{-\infty}^{\infty} \frac{\cos x}{1+x^2} dx$.

7. [20 pts] Find all conformal mappings from $\mathbb{C}$ to $\mathbb{C}$.

THE END

Ph.D. Qualifying Exam: Real Analysis
February 2017

Student ID: Name:

Note: Be sure to use English for your answers.

1. [10 pts] Suppose that μ is a semifinite measure and $\mu(E) = \infty$. Prove that for any $C > 0$ there exists $F \subset E$ such that $C < \mu(F) < \infty$.

2. [10 pts] Interpret Fatou's lemma, the monotone convergence theorem and the dominated convergence theorem when μ is the counting measure on $\mathbb{N}$.

3. [15 pts] Let $f : [a, b] \to \mathbb{R}$ be Lebesgue measurable and $\epsilon > 0$. Prove that there is a compact set $E \subset [a, b]$ such that

$$\mu(E^c) < \epsilon \text{ and } f\big|_E \text{ is continuous.}$$

4. [15pts] Suppose that μ, ν are σ-finite measures on $(X, \mathcal{M})$ with $\nu \ll \mu$, and let $\lambda = \mu + \nu$ and $f = \frac{d\nu}{d\lambda}$. Prove that $0 \leq f < 1$ μ-a.e. and $\frac{d\nu}{d\mu} = \frac{f}{1-f}$.

5. [15 pts] Let f be a measurable function on $(X, \mathcal{M}, \mu)$ and define $\lambda_f : (0, \infty) \to [0, \infty]$ as $\lambda_f(\alpha) = \mu(\{x : |f(x)| > \alpha\})$. Prove that

$$f \in L^1 \text{ if and only if } \sum_{k=-\infty}^{\infty} 2^k \lambda_f(2^k) < \infty.$$

6. [10 pts] Prove that every nonempty closed convex set K in a Hilbert space has a unique element of minimal norm.

7. Let $(X, \mathcal{M}, \mu)$ be a measure space.
   (a) [15 pts] Prove that $L^p(X)$ is complete for $1 \leq p < \infty$.
   (b) [10 pts] Prove that the set of simple functions

$$f = \sum_{j=1}^{n} a_j \chi_{E_j}, \quad \text{where } \mu(E_j) < \infty \text{ for all } j,$$

   is dense in $L^p(X)$ for $1 \leq p < \infty$.

THE END

Ph.D. Qualifying Exam: Combinatorics
February 2017

Student ID: Name:

Note: Be sure to use English for your answers.

1. [20 pts] Let $\mathcal{F}$ be a family of subsets of a finite set, where each member of $\mathcal{F}$ has size at least 2. Let A and B be two blocking sets of $\mathcal{F}$ of minimal size, that is $|A| = |B| = \tau(\mathcal{F})$. Consider a bipartite graph G with parts A and B, where $a \in A$ is connected to $b \in B$ if there is an $F \in \mathcal{F}$ containing both a and b. Prove that G has a perfect matching.

2. [20 pts] Prove that for every integer k there exists a graph G with girth $g(G) > k$ and chromatic number $\chi(G) > k$.

3. [20 pts] Let $(\mathcal{P}, \mathcal{L})$ be a finite projective plane of order $q \geq 3$, and let $B \subset \mathcal{P}$ be a non-trivial blocking set. Prove that
   (i) $|B| \leq q^2 - \sqrt{q}$.
   (ii) no line in $\mathcal{L}$ contains more than $|B| - q$ points from B.

4. [20 pts] Let $L \subset \{0, 1, 2, \dots\}$ be a finite set of integers and let $\mathcal{F}$ be a family of subsets of $\{1, 2, \dots, n\}$ such that $|A \cap B| \in L$ for every pair A, B of distinct members of $\mathcal{F}$. Prove that
$$|\mathcal{F}| \leq \sum_{i=0}^{|L|} \binom{n}{i}.$$

5. [20 pts] Let $\mathbb{F}$ be a finite field with $|\mathbb{F}| = q$. Prove that every non-zero polynomial $f(x_1, \dots, x_n) \in \mathbb{F}[x_1, \dots, x_n]$ of degree d $(1 \leq d \leq q)$ has at most dq^{n-1} roots.

THE END

Ph.D. Qualifying Exam: Probability Theory
February 2017

Student ID: Name:

Note: Be sure to use English for your answers.

1. [15 pts] For $n \in \mathbb{N}$, consider the probability space $(\Omega_n, \mathcal{F}_n, P_n)$ where $\Omega_n = \{1, 2, \dots, n\}$, $\mathcal{F}_n$ is the set of all subsets of Ω_n, and P_n is the uniform probability measure, i.e., $P_n(\{k\}) = 1/n$ for all $k \in \Omega_n$. Find the largest positive integer N for which the following statement holds for all $n \le N$:

   If X and Y are independent random variables on $(\Omega_n, \mathcal{F}_n, P_n)$, then either X or Y must be constant.

2. [10 pts] Two sequences of random variables, (X_n) and (Y_n), are *tail equivalent* if
$$\sum_{n \ge 1} P(X_n \neq Y_n) < \infty.$$
   Suppose $\Omega_X \subset \Omega$ is the set on which $\sum_n X_n$ converges, and $\Omega_Y \subset \Omega$ is the set on which $\sum_n Y_n$ converges. Show that the symmetric difference of Ω_X and Ω_Y has probability zero, i.e., $P(\Omega_X \,\Delta\, \Omega_Y) = 0$.

3. [15 pts] Prove Kolmogorov's Maximal Inequality for independent zero-mean random variables $\{X_k\}$ which have finite variances: If $S_n = X_1 + \cdots + X_n$ then
$$P\left(\max_{1 \le k \le n} |S_k| \ge \epsilon\right) \le \epsilon^{-2} \mathrm{Var}(S_n)$$
   for all $\epsilon > 0$.

4. [10 pts] Suppose that the sequence $(X_n/a_n, n \in \mathbb{N})$ converges in distribution where $a_n \uparrow \infty$. If (b_n) is a sequence of numbers such that $a_n/b_n \to 0$ as $n \to \infty$, show that $(X_n/b_n, n \in \mathbb{N})$ converges in probability to 0. (Remark: this shows that the CLT implies the WLLN under finite second moments.)

5. [15 pts] Suppose $\sup_n E|X_n|^p < \infty$ for some $p > 0$. Show that there is a subsequence (n_k) such that $(X_{n_k}, k \in \mathbb{N})$ converges in distribution.

6. (a) [10 pts] Consider σ-fields $\mathcal{F}_1 \subset \mathcal{F}_2$. Show $E(E(X|\mathcal{F}_1)|\mathcal{F}_2) = E(X|\mathcal{F}_1)$ and
$$E(E(X|\mathcal{F}_2)|\mathcal{F}_1) = E(X|\mathcal{F}_1).$$
   (b) [10 pts] If $EX^2 < \infty$, then $E(X|\mathcal{F})$ is (a version of) the random variable Y which is $\mathcal{F}$-measurable and minimizes $E(X - Y)^2$.

7. [15 pts] Let $(X_k, k \in \mathbb{N})$ be an i.i.d. sequence such that $E|X_1| < \infty$. If τ is a stopping time with respect to the filtration given by $\mathcal{F}_n = \sigma(X_1, \dots, X_n)$, and $E\tau < \infty$, then
$$E(X_1 + \cdots + X_\tau) = E(\tau)E(X_1).$$
   (Remark: this is called Wald's equation, and it is not true when $E\tau = \infty$.)

THE END