QUALIFYING EXAM (ALGEBRA)

Each problem has 10 points.

1. Let G be an abelian group. Prove that $\{g \in G \mid |g| < \infty\}$ is a subgroup of G (called the torsion subgroup of G). Give an explicit example where this set is not a subgroup when G is non-abelian.

2. Let G be a group. Prove that $N =< x^{-1}y^{-1}xy \mid x, y \in G >$ is a normal subgroup of $G(N$ is called the commutator subgroup of $G)$ and G/N is abelian.

3. Prove that the number of Sylow p-subgroups of $GL_2(\mathbb{F}_p)$ is $p + 1$.

4. Let x be a nilpotent element (i.e, $x^m = 0$ for some $m \in \mathbb{Z}^+$) of the commutative ring R with an identity. Prove that $1 + rx$ is a unit in R for all $r \in R$.

5. Prove that $x^{n-1} + x^{n-2} + \cdots + x + 1$ is irreducible over $\mathbb{Z}$ if and only if n is a prime.

6. (a) An element m of the R-module M is called a torsion element if $rm = 0$ for some nonzero element $r \in R$. Let $\mathrm{Tor}(M)$ be the set of torsion elements of M. Prove that if R is an integral domain then $\mathrm{Tor}(M)$ is a submodule of M.
(b) Let V be a 2-dimensional vector space over $\mathbb{R}$. Show that there is an element in $V \otimes_{\mathbb{R}} V$ cannot be written as a simple tensor.

7. Let R be a discrete valuation ring. Show that R is an Euclidean domain.

8. Prove that $\mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}/n\mathbb{Z}, \mathbb{Z}/m\mathbb{Z}) \cong \mathbb{Z}/(n, m)\mathbb{Z}$.

9. (a) Find all similarity classes of 6×6 matrices over $\mathbb{Q}$ with minimal polynomial $(x + 2)^2(x - 1)$.
(b) Determine all possible Jordan canonical forms for a linear transformation with characteristic polynomial $(x - 2)^3(x - 3)^2$.

10. Prove that an $n \times n$ matrix A with entries in $\mathbb{C}$ satisfying $A^3 = A$ can be diagonalized.

Ph.D Qualifying Exam
Complex Analysis
Aug 2014
(3 hours)

Problem 1 (10pt). *Find a conformal map of a horizontal strip* $\{-A < Im z < A\}$ *onto the left half-plane* $\{Re w < 0\}$.

Problem 2 (20pt). *Evaluate the following integrals*

(1)
$$\int_0^\infty \frac{\log x}{x^2+a^2}\,dx = \frac{\pi}{2a}\log a, \qquad a>0$$

(2)
$$\int_0^\infty \sin x^2\,dx = \int_0^\infty \cos x^2\,dx = \frac{\sqrt{\pi}}{2\sqrt{2}}$$

(Hint. Perform a contour integral on a sector cornered at $0, R,$ *and* $Re^{i\pi/4}$.*)*

Problem 3 (10pt). *Show that if* f *is analytic in the unit disc, is bounded, and converges uniformly to zero in the sector* $0 < \arg z < \pi/4$ *as* $|z| \to 1$, *then* $f = 0$.

Problem 4 (20pt).

(1) *Find and classify the singularities of*
$$f(z) = \frac{(z-1)^2}{(z-2)^2 \sin \pi z} e^{\frac{1}{z^2}}.$$
(eg. removable, poles (and its order), or essential singularities.)

(2) *Suppose that* $f(z)$ *has an isolated singularity at* $z = z_0$, *and that* $\lim_{z\to z_0}(z-z_0)^\alpha f(z) = M \neq 0$. *Prove that* α *must be an integer.*

Problem 5 (30pt). *Let* $f(z)$ *be analytic in a neighborhood of* $|z| \leq 1$ *and assume that* $|f(z)| = 1$ *for* $|z| = 1$.

(1) *Assume that* $f(z)$ *has no zeros in* $|z| < 1$. *Show that* $f(z) = e^{i\theta}$ *for some* $\theta \in \mathbb{R}$.

(2) *Show that if* $f(z)$ *may have zeros in* $|z| < 1$, *then they are at most finitely many.*

(3) *Let* $a_1, \cdots, a_k$ *be zeros in* $|z| < 1$ *of* $f(z)$.*(repeated with multiplicities.) Show that* $f(z)$ *is expressed as*
$$f(z) = e^{i\theta}\prod_{j=1}^{k}\frac{z-a_j}{1-\overline{a_j}z}.$$

Problem 6 (10pt). *Suppose that* $f(z)$ *is a meromorphic function in* $\mathbb{C}$. *Show that if* $|f(z)|$ *is bounded for* $|z| > R$ *for some* $R > 0$, *then* $f(z)$ *is a rational function.*

Numerical Analysis Qualifying Exam

August 2014

1. (7 points each) We consider uniform approximation to a continuous function f on an interval $[a, b]$ from the space Π_n of all polynomials of degree $\leq n$.

   (a) Assuming f is even and is monotone decreasing on $[0, \infty)$, find the best uniform approximation to f from Π_1 on the interval $[-a, a]$. Justify your choice.

   (b) Prove that for every n, there exists a sufficiently large a such that the best approximation to the function $f(t) := \cos t$ on $[-a, a]$ from Π_n is the zero polynomial.

   (c) Prove or disprove the following statement: if, for some $K > 0$, 0 is the best approximation to $\sin t$ from Π_n on $[0, K]$, then 0 is also the best approximation from Π_n on that interval to $\cos t$.

2. (7 points each) We consider numerical approximations to the integral

$$\int_0^1 f(x)\, dx\,.$$

   (a) Derive an error formula for the composite trapezoidal rule (using a uniform partition).

   (b) Derive from the composite trapezoidal rule another rule by a one-step application of Richardson extrapolation. What is the order of the new rule?

   (c) Compare the composite trapezoidal rule to the composite midpoint rule (in terms of the error estimates, and in any other terms you choose).

3. (3 points each) Consider a differential equation

$$y' = f(x, y) \qquad y(x_0) = Y_0$$

   with $f(x, y)$ continuous and satisfying the Lipschitz condition

$$|f(x, y_1) - f(x, y_2)| \leq K|y_1 - y_2| \qquad -\infty < y_1, y_2 < \infty \qquad x_0 \leq x \leq b$$

   for some $K \geq 0$. Which of the following Linear Multistep Methods are convergent? For the ones that are not, are they inconsistent, or not stable, or both?

   i. $y_{n+2} = \frac{1}{2}y_{n+1} + \frac{1}{2}y_n + 2hf(y_{n+1})$

   ii. $y_{n+1} = y_n$

   iii. $y_{n+4} = y_n + \frac{4}{3}h\,(f(y_{n+3}) + f(y_{n+2}) + f(y_{n+1}))$

   iv. $y_{n+3} = -y_{n+2} + y_{n+1} + y_n + 2h\,(f(y_{n+2}) + f(y_{n+1}))$

4. (7 points each) Consider the iteration method

$$x^{(k+1)} = Mx^{(k)} + b\,,$$

   where $x^{(k)}$ and b are vectors in $\mathbf{R}^n, M \in \mathbf{R}^{n\times n}$, and $x^{(0)}$ is a given initial guess. Assume that $\|M\| < 1$, where $\|M\|$ is a matrix norm induced by the vector norm $\|x\|$. Show that

(a) The process is convergent to the unique solution of the linear system $x = Mx + b$.

(b) Prove that

$$\|x^{(k)} - x\| \leq \|(I - M)^{-1}\| \cdot \|x^{(k+1)} - x^{(k)}\| .$$

(c) Show also that

$$\|x^{(k)} - x\| \leq \|M\|^k \|x^{(0)}\| + \frac{\|M\|^k \|b\|}{1 - \|M\|} .$$

5. (a) (5 points) Define "a Householder matrix". Give an example of a 3×3 Householder matrix whose entries are all non-zero.

   (b) (5 points) Prove that every Householder matrix H is symmetric, orthogonal, and self-invertible (i.e., $H^2 = I$).

   (c) (10 points) Discuss *briefly* a numerical algorithm that employs Householder matrices.

   (d) (5 points) Prove or disprove: every symmetric, orthogonal, self-invertible matrix is a Householder matrix.

Real Analysis (Summer 2014)

1. (15 points) Suppose E is a measurable subset of $\mathbb{R}$ with $m(E) > 0$. Prove that the difference set of E, which is defined by

$$\{z \in \mathbb{R} : z = x - y \text{ for some } x, y \in E\},$$

contains an open interval centered at the origin.

2. (20 points) Find the values of the following limits:

   (a) $\displaystyle\lim_{n\to\infty} \int_0^n \left(1 - \frac{x}{n}\right)^n e^{x/2} dx$

   (b) $\displaystyle\lim_{n\to\infty} \int_0^n \left(1 + \frac{x}{n}\right)^n e^{-2x} dx$

3. (15 points) Give an example of an increasing function on $\mathbb{R}$ whose set of discontinuities is precisely $\mathbb{Q}$.

4. (20 points) Let F be an increasing function on $[0, 1]$ with $F(0) = 0$ and $F(1) = 1$. Let μ be a Borel measure defined by $\mu((a, b)) = F(b^-) - F(a^+)$ and $\mu(\{0\}) = \mu(\{1\}) = 0$. Suppose that the function F satisfies a Lipschitz condition $|F(x) - F(y)| \leq A|x - y|$ for some $A > 0$. Let m be the Lebesgue measure on $[0, 1]$.

   (a) Prove that $\mu \ll m$.

   (b) Prove that $\dfrac{d\mu}{dm} \leq A$ almost everywhere.

5. (15 points) Assume that $f \in L^q(\mathbb{R})$ for some $1 \leq q < \infty$. Prove that

$$\lim_{p\to\infty} \|f\|_p = \|f\|_\infty.$$

6. (15 points) Consider the operator on $L^2([0, 1])$ defined by $T(f)(t) = tf(t)$. Prove that T is a bounded operator with $T = T^*$, but that T is not compact.

Qualifying Exam 2014 in Advanced Statistics

September, 2014

1. (10pt) Suppose that the number Y of lifetime events of domestic violence for a typical woman in Daejeon is assumed to follow a distribution with a pdf as

$$p_Y(y; \lambda) = \lambda^y e^{-\lambda}/y!, \quad y = 0, 1, \ldots, \infty, \quad \lambda > 0.$$

Let $Y_1, Y_2, \ldots, Y_n$ constitute a random sample of size n (where n is large) from this population.

   (a) (5pt) Suppose you believe that reported values of Y greater than zero (especially large reported values) are not very accurate, but that reported values of Y equal to zero are accurate. As a possible remedy, you want to analyze the data by converting each Y_i to a dichotomous random variable X_i (i.e., $X_i = 1$ if $Y_i \geq 1$ and $X_i = 0$ if $Y_i = 0$). Using the n mutually independent dichotomous random variables $X_1, \ldots, X_n$, find an explicit expression for the MLE $\hat{\lambda}^*$ of λ and its large sample variance.

   (b) (5pt) Make a quantitative comparison between the properties of $\hat{\lambda}^*$ and $\hat{\lambda}$, where $\hat{\lambda}$ is the MLE for λ obtained by using $Y_1, \ldots, Y_n$ in two situations: (1) $\{Y_i\}_{i=1}^n$ are accurate; (2) $\{Y_i\}_{i=1}^n$ are not accurate but $\{X_i\}_{i=1}^n$ are accurate. Be sure to comment on issues of validity (i.e., bias) and precision (i.e., variability).

2. (10pt) Let $Y_1, \ldots, Y_n$ constitute a random sample of size n from a N(0, σ^2) population. A certain statistician proposes to estimate the unknown parameter $\theta = \sigma^2$ using the estimator

$$\hat{\theta} = k_1 \frac{\sum_{i=1}^n Y_i^2}{n} + k_2 \frac{(\sum_{i=1}^n Y_i)^2}{n},$$

where $k_1 + k_2 = 1$. First, prove that $\hat{\theta}$ is an unbiased estimator for θ. Then, find specific values for k_1 and k_2 that minimize $V(\hat{\theta})$. How do these choices for k_1 and k_2 relate to the MVUE and MVBUE estimators for θ?

3. (20pt) Suppose that $X_1, X_2, \ldots, X_n$ constitute a random sample of size n from a N(μ, σ^2) population. Then, consider the n random variables $Y_1, Y_2, \ldots, Y_n$, where $Y_i = e^{X_i}$, $i = 1, \ldots, n$.

   (a) (3pt) Find a density function of the random variable Y_i.

   (b) (2pt) Prove that $E(Y_i) = exp(r\mu + \frac{r^2\sigma^2}{2})$, $-\infty < r < \infty$.

   (c) (5pt) Using the n observations $Y_1, \ldots, Y_n$, find two statistics that are jointly sufficient for μ and σ^2.

(d) (10pt) Consider the following two statistics: the arithmetic mean $\bar{Y}_a = \frac{\sum_{i=1}^n Y_i}{n}$ and the geometric mean $\bar{Y}_g = (\prod_{i=1}^n Y_i)^{1/n}$. Derive an explicit expression for Corr $(\bar{Y}_a,\ \bar{Y}_g)$. Then, find the limiting value of this correlation as $n \to \infty$, and then comment on your finding.

4. (30pt) Let $X_1, \ldots, X_{n_1}$ be a random sample from a Poisson population with mean μ_1 and $Y_1, \ldots, Y_{n_2}$ be a random sample from a Poisson population with mean μ_2. Further, let $\hat{\mu}_1 = \bar{X} = \frac{\sum_{i=1}^{n_1} X_i}{n_1}$, $\hat{\mu}_2 = \bar{Y} = \frac{\sum_{i=1}^{n_2} Y_i}{n_2}$, and $\hat{\mu} = (n_1\bar{X} + n_2\bar{Y})/(n_1 + n_2)$. Testing $H_0 : \mu_1 = \mu_2$ vs $H_A : \mu_1 \neq \mu_2$ is of your interest.

   (a) (6pt) Show that the generalized likelihood ratio statistic $-2ln\hat{\lambda}$ can be expressed as a function of $\hat{\mu}_1$ and $\hat{\mu}_2$.

   (b) (6pt) Show that the Wald statistic $\hat{W}$ is

$$\hat{W} = (\hat{\mu}_1 - \hat{\mu}_2)^2/(\frac{\hat{\mu}_1}{n_1} + \frac{\hat{\mu}_2}{n_2})$$

   (c) (6pt) Show that the Score statistic $\hat{S}$ is

$$\hat{S} = (\hat{\mu}_1 - \hat{\mu}_2)^2/(\frac{\hat{\mu}}{n_1} + \frac{\hat{\mu}}{n_2})$$

   (d) (6pt) Suppose that $n_1 = n_2 = 30$, $\hat{\mu}_1 = 6$ and $\hat{\mu}_2 = 5$. Compute the values of $-2ln\hat{\lambda}$, $\hat{W}$, $\hat{S}$; do you reject H_0 at the significance level $\alpha = 0.05$ based on each of the three statistics?

   (e) (6pt) Find an appropriate 95% confidence interval for $\mu_1 - \mu_2$. Does this CI support the conclusions of the hypothesis tests in (d)?

5. (30pt) A researcher gathers data (x_i, Y_i) on each of a large number n of randomly chosen sparsely populated cities in Korea, where $x_i (> 0)$ is the known population size (in millions of people) in city i and Y_i is the random variable denoting the number of people in city i with a certain disease. It is reasonable to assume that Y_i has a Poisson distribution with mean $E(Y_i) = \theta x_i$ where θ is an unknown parameter. Let $Y_1, Y_2, \ldots, Y_n$ constitute a set of mutually independent random variables.

   (a) (16pt) Derive the following 4 estimators for θ; (1) the unweighted least squares estimator ($\hat{\theta}_{uls}$); (2) the weighted least square estimator with an appropriate choice of weights ($\hat{\theta}_{wls}$); (3) the method of moments estimator ($\hat{\theta}_{mm}$); (4) the maximum likelihood estimator ($\hat{\theta}_{ml}$).

   (b) (6pt) Use rigorous arguments to provide what you consider to be the best choice for a $100(1-\alpha)\%$ confidence interval for the unknown parameter θ.

   (c) (8pt) If $\sum_{i=1}^n x_i = 0.82$, what is the power of the UMP test for testing $H_0 : \theta = 1$ vs $H_1 : \theta > 1$ when the probability of a Type 1 error is approximately equal to 0.05 and when, in reality, $\theta = 5$?

Combinatorics: Qualifying Exam

Name:

August 6, 2014

1. (20pts) Let $\mathcal{F}$ be a family of subsets of $\{1, 2, \ldots, n\}$. Prove that if $|X \cap Y| = k$ for all distinct $X, Y \in \mathcal{F}$, then $|F| \leq n$.

2. (20pts) Let G be a simple graph of average degree k. Then G contains an induced subgraph of minimum degree at least $k/2$.

3. (10pts) Let $A_1, A_2, \ldots, A_m$ be distinct subsets of $\{1, 2, \ldots, n\}$.

   (a) Prove that if $m \geq n + 1$, there exist disjoint subsets I, J of $\{1, 2, \ldots, m\}$ such that

   $$\bigcup_{i \in I} A_i = \bigcup_{j \in J} A_j \text{ and } I \cup J \neq \emptyset.$$

   (b) Prove that if $m \geq n + 2$, there exist disjoint subsets I, J of $\{1, 2, \ldots, m\}$ such that

   $$\bigcup_{i \in I} A_i = \bigcup_{j \in J} A_j, \ \bigcap_{i \in I} A_i = \bigcap_{j \in J} A_j, \text{ and } I \cup J \neq \emptyset.$$

4. (10pts) Let $A_1, A_2, \ldots, A_m, B_1, B_2, \ldots, B_m$ be sets such that $|A_i| = r$ and $|B_j| = s$ for all $i, j \in \{1, 2, \ldots, m\}$. Assume that

   (a) $A_i \cap B_i = \emptyset$ for all $i \in \{1, 2, \ldots, m\}$ and

   (b) $A_i \cap B_j = \emptyset$ for all $i < j$.

   Prove that $m \leq \binom{r+s}{r}$.

   (There'll be a partial credit, if a solution proves for $i \neq j$ in (b) instead of $i < j$.)

5. (20pts) Prove that every simple graph G admits a 3-colorable subgraph with at least $2|E(G)|/3$ edges.

6. (20pts) Prove that for every k, there exists N such that in every k-coloring of $\{1, 2, \ldots, N\}$, there are 3 integers a, b, c (not necessarily distinct) such that a, b, c, $a + b$, $b + c$, $a + b + c$ have the same color.