- In the following you may use Fatou Lemma, Lebesgue Dominated Convergence Theorem, Hölder Inequality, Hahn-Banach Theorem, Open Mapping Theorem and Borel-Cantelli Lemma.

1. [15] Let X be a Hausdorff space and $B(X)$ be the vector space of all bounded real-valued functions on X. For each $f \in B(X)$, let $\|f\|_\infty = \sup\{|f(x)| \mid x \in X\}$. Decide whether each of the following spaces is a Banach space with norm $\|\cdot\|_\infty$.

   (a) $C_c(X) = \{f \in B(X) | \{x \in X | f(x) \neq 0\} \text{ is compact}\}$.

   (b) $C_\circ(X) = \{f \in B(X) | \{x \in X \mid |f(x)| \geq \epsilon\} \text{ is compact for any } \epsilon > 0\}$.

   (c) $C^2(X) = \{f \in B(X) | f'' \text{ exists and is continuous on } X\}$ where $X = [0, 1]$.

2. [15] Let X be a normed vector space and S be a proper closed linear subspace of X. Let $0 < a < 1$. Show that there is $x_\circ \in X$ with $\|x_\circ\| = 1$ such that $d(x_\circ, S) \geq a$. Use this to prove that if $\{x \in X \mid \|x\| \leq 1\}$ is compact, then X is finite dimensional.

3. [20] Let $(X, \mathcal{A}, \mu)$ be a measure space and $\{f_n\}$ be a sequence of measurable functions on $(X, \mathcal{A})$ that is Cauchy in measure. Show that it has a subsequence $\{f_{n_k}\}$ that converges almost uniformly to a measurable function f on $(X, \mathcal{A})$. Show that if $f_n \in L^p(\mu)$ for all $n \in \mathbb{N}$ with $1 < p < \infty$ and $\lim_{k\to\infty} \||f_{n_k}|^p - |f|^p\|_1 = 0$, then $\lim_{k\to\infty} \|f_{n_k} - f\|_p = 0$.

4. [15] Let $(X, \mathcal{A}, \mu)$ be a measure space and $0 < q < p < s \leq \infty$. Let f be a function in $L^p(\mu)$ and $E = \{x \in X | f(x) \neq 0\}$. Show that $\lim_{r\to 0^+} \|f\|_r^r = \mu(E)$. Prove that there exist $g \in L^q(\mu)$ and $h \in L^s(\mu)$ such that $f = g + h$ on X.

5. [15] Let $(X, \mathcal{A}, \mu)$ be a measure space and $\|\cdot\|$ be a norm on $L^\infty(\mu)$ in which $L^\infty(\mu)$ is complete. Show that if $\|f\|_\infty \leq \|f\|$ for all $f \in L^\infty(\mu)$, then there is $C \in \mathbb{R}$ such that $\|f\| \leq C\|f\|_\infty$ for all $f \in L^\infty(\mu)$.

6. [10] Let H be the set of all absolutely continuous real-valued functions f on $[0, 1]$ such that $f(0) = 0$ and $f' \in L^2[0, 1]$. For $f, g \in H$, define $(f, g) = \int_0^1 f'g'$. Let $0 < t \leq 1$. For each $f \in H$, let $F(f) = f(t)$. Show that F is a bounded linear functional on the Hilbert space H. Find $\|F\|$. Find $f_\circ \in H$ such that $F(f) = (f, f_\circ)$ for all $f \in H$.

7. [10] Let $1 \leq p, q \leq \infty$ with $1/p + 1/q = 1$. For each $g \in L^q[0, 1]$, let $F_g(f) = \int_0^1 fg$ for $f \in L^p[0, 1]$. Let $F \in L^p[0, 1]^*$. Is there $g \in L^q[0, 1]$ such that $F = F_g$? Explain.

Algebraic Topology I: Ph.D. Qualifying Exam
August 7, 2013

Instructions

1) Use only ONE-SIDE of each answer sheet.
2) Answers without concrete justification will not be graded at all.
3) TWENTY points for each.

Problems

1. Let X be the subset of $\mathbb{R}^3$ obtained by deleting the three coordinate axes. In other words,

$X = \mathbb{R}^3 \setminus (\{(t,0,0) : t \in \mathbb{R}\} \cup \{(0,t,0) : t \in \mathbb{R}\} \cup \{(0,0,t) : t \in \mathbb{R}\})$.

Compute $\pi_1(X)$.

2. Let S be a closed surface with Euler characteristic -1. Compute the singular homology groups of S.

3. For a map $\phi : X \to Y$ between abelian groups X and Y, we define $\operatorname{coker}\phi = Y/\phi(X)$. Consider the following commutative diagram of abelian groups such that the rows are exact:

$$\begin{array}{ccccccc} & & A & \xrightarrow{i} & B & \xrightarrow{j} & C & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle g} & & \downarrow{\scriptstyle h} & & \\ 0 & \longrightarrow & A & \xrightarrow{i'} & B & \xrightarrow{j'} & C & & \end{array}$$

Prove that there exists a homomorphism $d : \ker h \to \operatorname{coker} f$ such that the following sequence is exact:

$$\ker g \xrightarrow{j} \ker h \xrightarrow{d} \operatorname{coker} f \xrightarrow{i'} \operatorname{coker} g$$

4. Prove that an m-dimensional manifold is not homeomorphic to an n-dimensional manifold if $m \neq n$.

5. Define

$$\begin{aligned} X_1 &= \{(x,y) : y = |\sin(2\pi/x)|, 0 < x < 2\} \\ X_2 &= \{(0,y) : 0 < y < 1\} \\ X_3 &= \{(x,y) : y = -\sqrt{1-(1-x)^2}, 0 < x < 2\} \end{aligned}$$

Find the singular homology groups of $X = X_1 \cup X_2 \cup X_3$.

Qualifying Exam-2 2013 in Advanced Statistics

1. [10 pts] Let T be a chi-square random variable with k degrees of freedom. Find the limiting distribution of
$$\frac{T-k}{\sqrt{2k}}$$
as $k \to \infty$.

2. [10 pts each] Let $X_1, \cdots, X_n$ be iid Poisson(λ) random variables. Let $Y = \sum_{i=1}^n X_i$.

   (a) Find a $1-\alpha$ confidence interval of λ using the fact that
   $$P_\lambda(Y \le y) = P(W > 2n\lambda)$$
   where $W \sim \chi^2_{2(y+1)}$.

   (b) Assume that $\lambda \sim Gamma(a, b)$. Find a $1-\alpha$ credible interval of λ.

   (c) Denote the confidence interval obtained in question (a) by C, and denote the posterior distribution of λ considered in question (b) by $\pi(\lambda|x_1, \cdots, x_n)$. Find the limit of $\pi(\lambda \in C|x_1, \cdots, x_n)$ as $\sum_{i=1}^n x_i \to \infty$.

3. [10 pts] Suppose that a minimal sufficient statistic exists for a parameter θ of a distribution F. Then is it true that any complete sufficient statistic for θ is also minimally sufficient? Why?

4. [10 pts] Let $X_1, \cdots, X_n$ be a random sample from a distribution with pdf f. We know that the sample mean ($\bar{X}$) and the sample variance (S^2) are independent when the distribution is Normal. Is it also true when the distribution is an exponential distribution with parameter λ? Why?

5. [10 pts] For two random variables, X and Y, suppose that $|\rho_{XY}| = 1$. What is the relationship between X and Y? Why?

6. Let $X_1, \cdots, X_n$ be a random sample from an exponential distribution with parameter λ. Let $\bar{X}$ be the sample mean.

   (a) [10 pts] Find the distribution of $X_1 - \bar{X}$ and the limit of the distribution.

   (b) [7 pts] Find a best unbiased estimator of λ and its variance.

   (c) [13 pts] For two real constants, $0 < a < b < \infty$, find a best, if any, unbiased estimator of $P(a < X_1 < b)$ and its variance.

THE END

For problems 1 through 4: **ALL WORK MUST BE SHOWN.**

Assume all hypergraphs are finite and simple (meaning no repeated edges). Let $\tau(H)$ denote the *transversal number* (blocking number) of the hypergraph H, and $\tau^*(H)$ the *fractional transversal number*. A hypergraph H is *τ-critical* if $\tau(H \setminus E) < \tau(H)$ for any edge $E \in H$. A hypergraph is *k-colorable* if the vertices can be spit into k classes such that each edge contain vertices from at least two distinct classes. The hypergraph H is *k-chromatic* if k is the minimum integer for which H is k-colorable. Let $R_k(c_1, c_2, \ldots, c_k)$ denote the *Ramsey number* for k-edge coloring of the complete graph with color class sizes $c_1, c_2, \ldots, c_k$, in other words, the minimum integer m such that any k-coloring of the edges of K_m contains a monochromatic clique of size c_i with color i for some $1 \leq i \leq k$.

1 (30 Points)

(a) Suppose that any k edges of an r-uniform hypergraph have at least one vertex in common. Show that

$$\tau(H) \leq \frac{r-1}{k-1} + 1$$

(b) Show that if a hypergraph is τ-critical, then either H is a collection of pairwise disjoint edges or $\tau(H) > \tau^*(H)$.

(c) Determine the maximum number of vertices of a 5-uniform τ-critical hypergraph with $\tau = 2$, and find all hypergraphs (up to isomorphism) which attain the maximum number of vertices.

2 (20 Points)

Let H be an r-uniform hypergraph, where $|H|$ denote the number of edges of H.

(a) Prove that the vertices of V can be colored red and blue such that the number of monochromatic edges is at most $\frac{|H|}{2^{r-1}}$. (An edge is monochromatic if all its vertices have the same color.)

(b) Prove that the vertex set can be colored by r colors such that there are at least $\frac{r!}{r^r}|H|$ multicolored edges. (An edge is multicolored if all its vertices have distinct colors.)

3 (30 Points)

(a) Let $H = \{A_1, \ldots, A_n\}$ be a 3-chromatic hypergraph on n vertices, without isolated vertices, such that for any vertex v the hypergraph

$$H - v = \{A_1 \setminus \{v\}, \ldots, A_n \setminus \{v\}\}$$

is 2-colorable. Prove that the incidence vectors of the edges of G generate the whole n-dimensional space.

(b) Show that if a hypergraph H has the property, that for all $k \geq 1$, the union of any k-edges has at least $k+1$ vertices, then H is 2-chromatic.

(c) Construct an r-uniform hypergraph such that the union of any k edges has at least k vertices, but H is not 2-chromatic.

4 (20 Points)

(a) Prove that the Ramsey number $R_k(3, \ldots, 3) \leq \lfloor e \cdot k! \rfloor + 1$, for all $k \geq 2$.

(b) Show that the Ramsey number $R_2(5, 3) = 14$.

Ph.D Qualifying Exam
Complex Analysis
Aug 2013
(3 hours)

Problem 1. *Assume that $f : \Omega \to \mathbb{C}$ is a C^1 analytic function, and $\gamma \subset \Omega$ is a simple closed and piecewise C^1 contour. Prove the Cauchy Theorem i.e.*

$$\oint_\gamma f(z)\, dz = 0.$$

Problem 2. *Evaluate the following integral*

$$\int_{-\infty}^{\infty} \frac{\cos x}{x^2 + a^2}\, dx, \qquad \text{where} \quad a > 0.$$

Problem 3. *Assume that $f(z)$ is an entire function so that for sufficiently large z,*

$$|f(z)| \le C_1 + C_2 |z|^{1/2},$$

for some C_1, C_2. Show that $f(z)$ is a constant function.

Problem 4. *Let $f(z)$ be analytic in a domain Ω containing $|z| \le 1$, with the only zeros of f being the distinct points $a_1, a_2, \cdots, a_n$, of multiplicities $m_1, m_2, \cdots, m_n$, respectively, and each a_j lies in the disk $|z| < 1$. Given that g is analytic in Ω, derive a formula of*

$$\oint_{|z|=1} \frac{f'(z)g(z)}{f(z)}\, dz$$

Problem 5. *Let $u(z)$ be a nonconstant, real valued, harmonic function on $\mathbb{C}$. Prove that there exists a sequence $\{z_n\}$ such that $z_n \to \infty$ for which $u(z_n) \to 0$.*

Problem 6. *Let $\{f_n(z)\}$ be a sequence of functions analytic in the connected open set D and assume they converge uniformly on every compact subset of D. Show that the sequence of derivatives $\{f_n'(z)\}$ also converges uniformly on every compact subset of D.*

Problem 7. *Let $a > 1$ be given. Show that the equation*

$$a - z - e^{-z} = 0$$

has exactly one root in the half plane $\{z : Re\, z > 0\}$, and moreover, this root is real.

Numerical analysis, Qualifying Exam. 2013

Each problem is worth 10 points.

1. (20 pts)
   (a) Define a Lagrange interpolation polynomial with data $\{(x_i, f(x_i))\}_{i=0}^n$. x_i all distinct.
   (b) What is the error form in the above ? Derive it.
   (c) Define a Newton's form of interpolation polynomial using the same data.
   (d) Explain what happens if some x_i are repeated, and in this case what is the correct data corresponding to the repeated points?
2. (15 pts) Prove the Newton's method to solve a scalar nonlinear equation $f(x) = 0$(assume f is differentiable) is second order convergent under appropriate condition. What happens if the condition is not satisfied?
3. (15 pts) Describe Newton's method to solve a system of nonlinear equations $\mathbf{F}(\mathbf{x}) := A\mathbf{x} + g(\mathbf{x})\mathbf{x} = 0$ starting from some initial points $\mathbf{x}_0$. Here $\mathbf{x} = (x_1, \cdots, x_n)$, A is $n \times n$ constant matrix and $g(\mathbf{x})$ is a scalar C^1 function of $\mathbf{x}$.
4. (10 pts) Suggest at least one more method to solve above system(Problem 3) and provide a sufficient condition for the convergence.
5. (15pts) Explain Runge phenomena and show how one can avoid it.
6. (15 pts)
   (a) Let $\mathbf{u}, \mathbf{v}$ are any vectors in $\mathbb{R}^n$. Find an $n \times n$ matrix of the form $H_{\mathbf{w}} = I - 2\mathbf{w}\mathbf{w}^*$, for some unit vector $\mathbf{w} \in \mathbb{R}^n$ such that $H_{\mathbf{w}}\mathbf{u} = \mathbf{v}$.
   (b) Show that $H_{\mathbf{w}}$ is symmetric and orthogonal
   (c) Explain how to transform A into an upper triangular matrix R using above transformations.(Including how to avoid instability)
7. (10 pts) Describe Euler's method(explicit and implicit) to solve an ODE. $\dot{x} = f(t, x(t))$, $x(0) = x_0$. Discuss advantages and disadvantages.

Qualifying Exam in Probability Theory (August 2013)

1. (10 pts) Suppose $\{X_n, n \geq 1\}$ are random variables with a common unit exponential distribution, i.e. $\mathbb{P}(X_n > x) = e^{-x}$, $x > 0$ and set $M_n = \max\{X_1, \ldots, X_n\}$, $n \geq 1$. Find the distribution of Y which satisfies $M_n - \log n \Rightarrow Y$. $\Rightarrow$ denotes the convergence in distribution.

2. (10 pts) Suppose X and Y are independent random variables and $h : \mathbb{R}^2 \to [0, \infty)$ is measurable. Define $g(x) = \mathbb{E}[h(x, Y)]$. Show that $g(\cdot)$ is measurable and $\mathbb{E}[g(X)] = \mathbb{E}[h(X, Y)]$.

3. (10 pts) Compute

$$\lim_{n\to\infty} \int_0^1 \int_0^1 \cdots \int_0^1 \left(\frac{x_1 + x_2 + \cdots + x_n}{n}\right)^p dx_1\, dx_2 \cdots dx_n$$

   where p is a positive integer.

4. (10 pts) Suppose X is an integrable random variable on $(\Omega, \mathcal{B}, P)$, and A_n are sets in $\mathcal{B}$. Show that if $P\{A_n\} \longrightarrow 0$ as n goes to ∞, then $\int_{A_n} X\, dP \longrightarrow 0$ as n goes to ∞.

5. (10 pts) If for each $i = 1, \cdots, n$, $\mathcal{C}_i$ is a non-empty class of events satisfying

   (a) $\mathcal{C}_i$ is a π-system

   (b) $\mathcal{C}_i, i = 1, \cdots, n$ are independent,

   then show that $\sigma(\mathcal{C}_1), \cdots, \sigma(\mathcal{C}_n)$ are independent.

6. (10 pts) Compute the characteristic function of $N(\mu, \sigma^2)$, $\exp(\lambda)$, and Poisson(λ).

7. (10 pts) Consider a probability space $(\Omega = [0,1], \mathcal{B}([0,1]), \mathbb{P})$ where $\mathbb{P}$ is the Lebesgue measure. Define the following random variables on Ω.

$$X(\omega) = \begin{cases} 1, & \text{if } \omega \in [0, \frac{1}{2}] \\ 0, & \text{if } \omega \in (\frac{1}{2}, 1] \end{cases}, Y(\omega) = \begin{cases} 1, & \text{if } \omega \in [0, \frac{3}{4}] \\ 0, & \text{if } \omega \in (\frac{3}{4}, 1] \end{cases}, Z(\omega) = \begin{cases} 1, & \text{if } \omega \in [\frac{1}{4}, \frac{3}{4}] \\ 0, & \text{if } \omega \notin (\frac{1}{4}, \frac{3}{4}] \end{cases}.$$

   Compute $\mathbb{E}[X|Y]$ and $\mathbb{E}[X|Y, Z]$. Check if $\mathbb{E}[X|Y] = \mathbb{E}[X|Y, Z]$ is true. What can you say about your result ?

8. (10 pts) Let $\{X_n\}$ be independent random variables and $S_n = X_1 + \ldots + X_n$. Show that the event $\{\omega : S_n(\omega)/n \to 0\}$ has probability 0 or 1.

9. (10 pts) Suppose $\{A_n\}$ are independent events satisfying $P\{A_n\} < 1$, for all n. Show that

$$P\{\cup_{n=1}^{\infty} A_n\} = 1 \text{ if and only if } P\{A_n \text{ i.o.}\} = 1.$$

10. (10 pts) If $X_n \xrightarrow{P} X$ and F is continuous at x, show that $\lim_{n\to\infty} F_n(x) = F(x)$. Here, $F_n(x)$ denotes the distribution function of X_n, and $F(x)$ is the distribution function of X. (Hint: First show that

$$F(x - \epsilon) - P\{|X_n - X| \geq \epsilon\} \leq F_n(x) \leq P\{|X_n - X| \geq \epsilon\} + F(x + \epsilon).$$

    and then prove the statement in the problem.)

QUALIFYING EXAM (ALGEBRA)

1. Let $A = \begin{bmatrix} 4 & 2 & 2 \\ 2 & 4 & 2 \\ 2 & 2 & 4 \end{bmatrix}$.Find an orthogonal matrix P so that such that $P^{-1}AP$ is a diagonal matrix.

2. Show that no group of order 36 is simple.

3. Show that every group of order $255 = 3 \cdot 5 \cdot 17$ is abelian.

4. Let R be a commutative ring with identity. Let $\mathfrak{m}_1, \mathfrak{m}_2, \dots, \mathfrak{m}_r$ be distinct maximal ideals of R. Let $a_1, a_2, \dots, a_r$ be elements of R. Show that there exists an element a of R satisfying

$$a \equiv a_i \mod \mathfrak{m}_i \quad \text{for any } i = 1, 2, \dots, r.$$

5. Let R be a commutative Nötherian ring with identity. Show that the polynomial ring $R[X]$ is also Nötherian.