# 대수학 I 박사 자격시험

2013. 2. 6

1. (10 Points) Let R be an integral domain. Prove that if R is finite, then R is a field.

2. (10 Points) Let F be a field and $f(x)(\neq 0) \in F[x]$ with deg $f(x) = n$. Prove that $f(x)$ has at most n roots in $F$.

3. (20 Points) Show that $f(x) = x^4 - 5x^2 + 1$ is irreducible in $\mathbb{Q}[x]$.

4. (20 Points) Find all abelian groups of order 72, up to isomorphism, in terms of both elementary divisors and invariant factors.

5. (20 Points) Prove that every group is isomorphic to a group of permutations.

6. (20 Points) Let $p$, $q$ be distinct primes with $p < q$.
   (a) Determine whether any group G of order $pq$ is simple.
   (b) Prove that if $q \not\equiv 1$ mod p, G is abelian and cyclic.

# Doctoral Qualifying Exam

## Differential Geometry (Spring, 2013)

1. **(30 points)** Answer the following questions:
   (a) If $N$ is a closed embedded submanifold of a smooth manifold $M$, $U$ is any open neighberhood of $N$, and $f : N \to \mathbb{R}$ is a real-valued smooth function, show that there is a smooth function $\tilde{f} : M \to \mathbb{R}$ with $\tilde{f} = f$ on $N$ such that the support of $\tilde{f}$ is contained in $U$.
   (b) Prove or disprove (a) above if $M = \mathbb{R}$ and $N = (0,1) \subset M = \mathbb{R}$.
   (c) Prove or disprove (a) above if $M = \mathbb{R}^2$, $N = S^1$, the unit circle in $\mathbb{R}^2$, and $\mathbb{R}$, the codomain of $f$, is replaced by $S^1$.
2. **(30 points)** Answer the following questions:
   (a) Give a definition of a *complete* smooth vector field on a smooth manifold.
   (b) Let $f : \mathbb{R}^2 \to \mathbb{R}$ be a differentiable function given by $f(x,y) = \sqrt{1+x^2+y^2}$. Then define a smooth vector field $\nabla f$ on $\mathbb{R}^2$ by the relation
   $$df = \frac{xdx+ydy}{\sqrt{1+x^2+y^2}} = \frac{4}{\sqrt{(1+x^2+y^2)^3}}\langle \nabla f, \cdot \rangle,$$
   where $\langle \cdot, \cdot \rangle$ denotes the standard inner product on $\mathbb{R}^2$. Find explicitly the vector field $\nabla f$ on $\mathbb{R}^2$.
   (c) Prove or disprove that $\nabla f$ from (b) is complete.
3. **(20 points)** Answer the following questions:
   (a) Prove that for a smooth 1-form $\alpha$ on a smooth manifold $M$
   $$d\alpha(X,Y) = X(\alpha(Y)) - Y(\alpha(X)) - \alpha([X,Y])$$
   for smooth vector fields $X$ and $Y$.
   (b) Let $\omega = \sum_{i<j<k} \omega_{ijk} dx^i \wedge dx^j \wedge dx^k$ on $\mathbb{R}^n$ for $n > 3$. Find a necessary and sufficient condition for $\omega$ to be closed.

4. **(10 points)** Let $f : M \to N$ be a proper map between oriented $n$-manifolds such that the differential $f_* : T_pM \to T_{f(p)}N$ is orientation-preserving whenever $p$ is a regular point of $f$. Assume that $N$ is connected and that $f$ is not surjective. Show that all points of $M$ are critical points of $f$.

5. **(10 points)** Let $f : S^k \times S^{n-k} \to T^n$ be a smooth map for integers $n \geq 4$ and $2 \leq k \leq n-2$, where $S^k$ (resp. $S^{n-k}$) is the unit sphere in $\mathbb{R}^{k+1}$ (resp. $\mathbb{R}^{n-k+1}$) and $T^n$ is the $n$-dimensional torus $S^1 \times \cdots \times S^1$ ($n$ times). Determine the degree of $f$.

**– The End –**

2013 Spring KAIST Qualifying Exam for Ph. D program
Algebraic Topology I

Choose 5 problems out of the following 6 problems to solve. Mark the number of the problems that you choose to solve with black circle.

1. Answer the following question with reasons. (20 points)

   (a) Suppose $n \geq 2$. Does there exist a continuous map $f : S^n \to S^1$ which is not homotopic to a constant?

   (b) Suppose $n \geq 2$. Does there exist a continuous map $f : \mathbb{R}P^n \to S^1$ which is not homotopic to a constant?

   (c) Let $T = S^1 \times S^1$ be the torus. Does there exist a continuous map $f : T \to S^1$ which is not homotopic to a constant?

2. Let $X$ be the pinched torus, that is, $X$ is the quotient space $(S^1 \times S^1)/(\{pt.\} \times S^1)$. Compute $\pi_1(X)$ and $H_*(X)$. (20 points)

3. Prove the following $3 \times 3$ lemma. Consider the following commutative diagram of abelian groups.

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & A_3 & \xrightarrow{\alpha_2} & A_2 & \xrightarrow{\alpha_1} & A_1 & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle \delta_2} & & \downarrow{\scriptstyle \epsilon_2} & & \downarrow{\scriptstyle \zeta_2} & & \\
0 & \longrightarrow & B_3 & \xrightarrow{\beta_2} & B_2 & \xrightarrow{\beta_1} & B_1 & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle \delta_1} & & \downarrow{\scriptstyle \epsilon_1} & & \downarrow{\scriptstyle \zeta_1} & & \\
0 & \longrightarrow & C_3 & \xrightarrow{\gamma_2} & C_2 & \xrightarrow{\gamma_1} & C_1 & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}$$

   If all three columns and the first two rows are short exact sequences, then show that the last row is also a short exact sequence. (20 points)

4. Let $X = \{(x, y, z) \in \mathbb{R}^3 \mid xy = 0\}$. (20 points)

   (a) Compute $H_1(X - (0, 0, 0))$.

(b) Using part (a) prove that $X$ is not homeomorpic to $\mathbb{R}^3$.

(c) Prove or disprove: $X$ is homotopy equivalent to $\mathbb{R}^2$.

5. Let $X$ be a CW-complex with exactly one $(n+1)$-cell. Prove that if $H_n(X;\mathbb{Z})$ is a nontrivial finite group, then $H_{n+1}(X;\mathbb{Z}) = 0$. (20 points)

6. It is known that if $C$ is a homeomorphic cop of the circle in $S^3$, then $H_*(S^3 - C) \cong H_*(S^1)$. Assuming this fact compute the homology groups of the following spaces. (20 points)

   (a) $Y = \mathbb{R}^3 - C$ where $C$ is a homeomorphic copy of a circle in $\mathbb{R}^3$.

   (b) $Z = \mathbb{R}^3 - X$ where $X$ os a homeomorphic copy of the figure-eight space (i.e., the one-point union of two circles).

# Real Analysis, Ph.D. Qualifying Exam (February 2013)

1. **[10]** Let $(X, \mathcal{A}, \mu)$ be a measure space. Let $f \in L^\infty(X, \mathcal{A}, \mu)$ and $g \in L^1(X, \mathcal{A}, \mu)$ with $\|fg\|_1 = \|f\|_\infty \|g\|_1$. Show that $|f| = \|f\|_\infty$ a.e. on the set $\{x \in X | g(x) \neq 0\}$.

2. **[10]** Let $(X, \mathcal{A}, \mu)$ be a semifinite measure space and $f$ be a measurable function on $(X, \mathcal{A})$. Let $\Gamma$ be the set of all simple functions $\varphi$ on $(X, \mathcal{A})$ with $\int_X |\varphi| d\mu \leq 1$. Show that if $|\int_X f\varphi d\mu| \leq 2$ for all $\varphi \in \Gamma$, then $|f| \leq 2$ a.e..

3. **[15]** For each polynomial $f$ on $[0, 1]$, let $\|f\| = \|f\|_\infty + \|f'\|_\infty$. Let $X$ be the normed vector space of polynomials on $[0, 1]$ with the norm $\|\cdot\|_\infty$ and $Y$ be the normed vector space of polynomials on $[0, 1]$ with the norm $\|\cdot\|$. Let $A$ be the linear operator from $X$ to $Y$ defined by $A(f) = f'$ for $f \in X$. Show that $A$ is unbounded. Does $A$ have a closed graph?

4. **[15]** Let $(X, \mathcal{A}, \mu)$ be a measure space and $f \in L^1(X, \mathcal{A}, \mu)$. Let $\{f_n\}$ be a sequence in $L^1(X, \mathcal{A}, \mu)$ that converges pointwise to $f$ on $X$ and $\sup_{n \in \mathbb{N}} f_n \leq f$ a.e.. Show that if $\lim_{n\to\infty} \int_E |f_n| d\mu = \int_E |f| d\mu$ for each $E \in \mathcal{A}$, then $\lim_{n\to\infty} \int_X f_n d\mu = \int_X f d\mu$. You may use Fatou Lemma.

5. **[15]** Let $(X, \mathcal{A}, \mu)$ be a measure space and $f$ be a measurable function on $(X, \mathcal{A})$. Let $\nu$ be a finite measure on $\mathcal{A}$ such that $\nu(E) = \int_E f d(\mu + \nu)$ for all $E \in \mathcal{A}$. For each $E \in \mathcal{A}$, let $\rho(E) = \nu(\{x \in E | f(x) < 1\})$. Show that $\rho \ll \mu$ and $(\nu - \rho) \perp \mu$.

6. **[15]** Let $X$ be a nonempty set and $\Omega$ be a collection of subsets of $X$ with $\emptyset, X \in \Omega$. Let $\rho$ be a nonnegative set function on $\Omega$ with $\rho(\emptyset) = 0$. For each subset $E$ of $X$, define

$$\mu^*(E) = \inf \left\{ \sum_{n=1}^{\infty} \rho(E_n) \middle| E \subseteq \bigcup_{n=1}^{\infty} E_n \text{ with } E_n \in \Omega \right\}.$$

Show that $\mu^*$ is an outer measure on $X$.

7. **[20]** Let $X$ be a normed vector space and let $\mathcal{F}$ be a nonempty subset of $X^*$ such that $\sup_{f \in \mathcal{F}} |f(x)| < \infty$ for each $x \in X$. Let $\Gamma$ be the set of all real-valued functions $\gamma$ on $\mathcal{F}$ such that $\|\gamma\| = \sup_{f \in \mathcal{F}} |\gamma(f)| < \infty$. Let $A$ be the linear operator from $X$ to the Banach space $\Gamma$ with the norm $\|\cdot\|$ given by $A(x)(f) = f(x)$ for $x \in X$ and $f \in \mathcal{F}$. Must $A$ be bounded? Show that if $X$ is complete, then $A$ is bounded and $\sup_{f \in \mathcal{F}} \|f\| < \infty$. You may use Closed Graph Theorem.

**MAS 541 Complex Function Theory - February 2012**
**Ph.D. Qualifying Exam**

*No document, cell phone or any electronic device allowed. If you can not solve a problem, it is better to write clear and relevant ideas than irrelevant computations. Do not claim that you solved a problem unless you think you did. Good luck! =)*

We denote by $\mathbb{D} = \{ z \in \mathbb{C} \mid |z| < 1 \}$ the unit disk in $\mathbb{C}$.

1. -10 pts- Give the statement (not the proof) of Liouville Theorem.

2. -10 pts- Give the definition of a meromorphic function on $\mathbb{C}$.

3. -10 pts- Does there exist a holomorphic bijection between $\mathbb{D}$ and $\{ z = x+iy \in \mathbb{C} \mid x > 0,\ y > 0,\ x+y < 1\}$? *Justify your answer in no more than two lines.*

4. -10 pts- Prove or disprove: Any holomorphic function $f : \mathbb{C} \to \mathbb{C}$ has a dense image.

5. -10 pts- Recall that a family of functions $\mathcal{F}$ is called *normal* when: for every sequence in $\mathcal{F}$, there is a subsequence that converges uniformly on compact sets (the limit need not be in $\mathcal{F}$). Let $\mathcal{A}$ be a family of holomorphic functions on $\mathbb{D}$. Denote $\mathcal{A}' = \{ f' \mid f \in \mathcal{A}\}$. Assume $\mathcal{A}'$ is normal and

$$\sup \{ |f(0)| \mid f \in \mathcal{A}\} < \infty.$$

Show that $\mathcal{A}$ is normal.

6. (a) -5 pts- Let $f$ be a holomorphic function with a zero of order $m \geq 1$ at the origin. Show that on some neighborhood of the origin, we can write $f(z) = z^m h(z)$ where $h$ is a nonvanishing holomorphic function.

   (b) -10 pts- Let $f$ be a holomorphic function with a zero of order $m \geq 1$ at the origin. Show that on a small neighborhood $D$ of the origin, there exists $g \in \mathcal{O}(D)$ such that $g'(0) \neq 0$ and $f(z) = \big(g(z)\big)^m$ on $D$. *Hint: use the fact that a nonvanishing holomorphic function has a local logarithm.*

   (c) -10 pts- Let $f : \Omega \to \mathbb{C}$ be a nonconstant holomorphic map. Show that $f$ is open, i.e., the image of any open set is open. *Hint: you can use the complex version of the inverse function theorem.*

   (d) -10 pts- Prove or disprove: If a holomorphic function $f : \Omega \to \mathbb{C}$ is one-to-one (i.e., injective), then its derivative never vanishes.

7. -15 pts- Let $f : z \mapsto \sum a_n z^n$ be an entire function. Assume that for some positive constants $A, a$ and $\rho$, for all $z$,

$$|f(z)| \leq A e^{a|z|^\rho}.$$

Show that $(|a_n|^{1/n} n^{1/\rho})_n$ is bounded.

# KAIST DMS Qualifying Exam in Probability Theory, February 2013

**Student ID:** **Name:**

1. (10 pts) If $\{A_n : n \geq 0\}$ is an independent sequence of events, show
$$\mathbb{P}(\cap_{n=1}^{\infty} A_n) = \prod_{n=1}^{\infty} \mathbb{P}(A_n).$$

2. (10 pts) A finite family $\mathcal{F}_i, i \in I$ of $\sigma$-algebras is independent iff for every choice of non-negative $\mathcal{F}_i$-measurable random variables $X_i, i \in I$, we have
$$\mathbb{E}[\prod_{i\in I} X_i] = \prod_{i \in I} \mathbb{E}[X_i].$$

3. (10 pts) Let $\{X_n\}$ be independent random variables and $S_n = X_1 + \ldots + X_n$. Show that the event $\{\omega : S_n(\omega)/n \to 0\}$ has probability 0 or 1.

4. (10 pts) Suppose $X$ and $Y$ are independent random variables and $h : \mathbb{R}^2 \to [0, \infty)$ is measurable. Define $g(x) = \mathbb{E}[h(x, Y)]$. Show that $g(\cdot)$ is measurable and $\mathbb{E}[g(X)] = \mathbb{E}[h(X, Y)]$.

5. (10 pts) Suppose that $X_1, \ldots, X_n$ are independent random variables with $\mathbb{E}X_i = 0$ and $\mathrm{Var}(X_i) < \infty$. Set $S_n = X_1 + \ldots + X_n$. Show that
$$\mathbb{P}\Big(\max_{1\leq k\leq n} |S_k| \geq \lambda\Big) \leq \lambda^{-2}\mathrm{Var}(S_n).$$

6. (10 pts) Let $\{X_i, i \geq 1\}$ be iid with $\mathbb{E}X_i = 0$ and $\mathbb{E}X_i^2 = \sigma^2 \in (0, \infty)$. Set $S_n = X_1 + \ldots + X_n$. Let $R_n$ be a sequence of non-negative random variables and $a_n$ a sequence of integers with $a_n \to \infty$ and $R_n/a_n \to 1$ in probability. Show that
$$\frac{S_{R_n}}{\sigma\sqrt{a_n}} \Rightarrow N(0, 1).$$

7. (10 pts) Suppose $\{X_n, n \geq 1\}$ are random variables with a common unit exponential distribution, i.e. $\mathbb{P}(X_n > x) = e^{-x}$, $x > 0$ and set $M_n = \max\{X_1, \ldots, X_n\}$, $n \geq 1$. Find the distribution of $Y$ which satisfies $M_n - \log n \Rightarrow Y$. $\Rightarrow$ denotes the convergence in distribution.

8. (10 pts) Find the characteristic functions of $N(\mu, \sigma^2)$, $\exp(\lambda)$, and Poisson$(\lambda)$.

9. (10 pts) Consider a Markov chain $\{X_n, n \geq 0\}$ on the state space $\{0, 1, 2, \ldots\}$ with transition probabilities $p_{ij} = \frac{e^{-i} i^j}{j!}$, $i, j \geq 0$ and $p_{00} = 1$. Show that $\{X_n\}$ is a martingale with respect to $\mathcal{F}_n = \sigma(X_0, X_1, \ldots, X_n)$ and that
$$\mathbb{P}(\sup\{X_n, n \geq 0\} \geq x | X_0 = i) \leq i/x.$$

10. (10 pts) Suppose that $\{(X_n, \mathcal{F}_n), n \geq 0\}$ is an $L_1$-bounded martingale. If there exists an integrable random variable $Y$ such that $X_n \leq \mathbb{E}[Y|\mathcal{F}_n]$ then $X_n \leq \mathbb{E}[X_\infty|\mathcal{F}_n]$ for all $n \geq 0$ where $X_\infty = \lim_{n\to\infty} X_n$ almost surely.

# Qualifying Exam-1 2013 in Advanced Statistics

1. [10 pts] Define a curved exponential family and give an example of it with the graph of its corresponding parameter space.

2. [6 pts] For two random variables, $X$ and $Y$, find $g^*(X)$ so that

$$\min_{g(x)} E(Y - g(X))^2 = E(Y - g^*(X))^2.$$

3. [10 pts] Let $X_1, \cdots, X_n$ be a random sample from $N(\theta, 1)$. Find, if any, the best unbiased estimator of $\theta^2$ and check if its variance is equal to the Cramer-Rao lower bound.

4. It may be that a person cannot actually discriminate between the two choices (can you tell Coke from Pepsi?), but the setup of the experiment is such that a choice must be made between two products A and B. Therefore, there is a confounding between discriminating correctly and guessing correctly. Consider the following parameters:
   $p$ = probability that a person can actually discriminate,
   $c$ = probability that a person discriminates correctly.

   Suppose that $n$ people participate the discrimination experiment and that the probability $p$ varies across the participants according to a beta distribution, $beta(\alpha, \beta)$. Let $X_i$ be the indicator of a correct discrimination by person $i$, $i = 1, 2, \cdots, n$.

   (a) [3 pts] Express the probability $c$ as a function of $p$.
   (b) [8 pts] Find the mean and variance of $\sum_{i=1}^n X_i$.
   (c) [4 pts] Is the distribution of $\sum_{i=1}^n$ a beta-binomial? Why?

5. Let there be a random sample of size $n$, $\mathbf{X}_i = (X_{i1}, X_{i2}, X_{i3})'$, $i = 1, 2, \cdots, n$, from a tri-variate normal distribution $N(\mu, \Sigma)$ where $\mu$ is a 3-dimensional mean vector and $\Sigma$ a $3 \times 3$ covariance matrix. Let $\rho_{jk}$ be the correlation coefficient between the $j$th and $k$th component of $\mathbf{X}_i$.

   (a) [10 pts] Construct a procedure for testing if all the $\rho_{jk}$'s are zero at the significance level $\alpha$.
   (b) [7 pts] Let $U = aX_{11} + BX_{12}$ and $V = CX_{12} + dX_{13}$, where $a$ and $d$ are fixed constants and $(1-B)/2 \sim beta(\alpha_1, \beta_1)$ and $(1-C)/2 \sim beta(\alpha_2, \beta_2)$. $B$ and $C$ are independent of each other and they are also independent of $\mathbf{X}_i$'s. Find $corr(U, V)$.

6. [10 pts] For any bounded pdf $f(x)$ on $[a, b]$, define $c = \max_{a \le x \le b} f(x)$. Let $X$ and $Y$ be independent, with $X \sim uniform(a, b)$ and $Y \sim uniform(0, c)$. For a number $d > b$, define a new random variable

$$W = \begin{cases} X & \text{if } Y < f(X) \\ d & \text{if } Y \ge f(X). \end{cases}$$

   Show that

$$P(W \le w) = \frac{1}{c(b-a)} \int_a^w f(t)dt \quad \text{for } a \le w \le b.$$

7. Let $(X_i, Y_i)$, $i = 1, 2, \cdots, n$, be a random sample from the distribution with the joint pdf

$$f(x, y) = \exp\{-(\theta x + y/\theta)\}, \quad x > 0, \ y > 0.$$

(a) [6 pts] Find the Fisher information, $I(\theta)$, in the sample.

(b) [10 pts] Let

$$T = \sqrt{\sum_{i=1}^{n} Y_i \Big/ \sum_{i=1}^{n} X_i} \text{ and } U = \sqrt{\sum_{i=1}^{n} X_i \sum_{i=1}^{n} Y_i}.$$

Find the Fisher information in $(T, U)$.

(c) [6 pts] Show that $(T, U)$ is jointly sufficient but not complete.

8. [10 pts] Prove the invariance property of MLE's. Namely, prove that, if $\hat{\theta}$ is the MLE of $\theta$, then for any function $g(\theta)$, the MLE of $g(\theta)$ is $g(\hat{\theta})$.

**THE END**

# Numerical Analysis Qualifying Exam

**February 2013**

1. (10 points each)

   (a) State Newton's method to find zeros of $f(x) = 0$ and prove the convergence.

   (b) When $f'(\alpha) = 0$ for some zero $\alpha$ of $f$, what do you think of the Newton's method? Modify the Newton's method in this case.

2. (20 points) What is the Chebyshev zeros and give a motive for them.

3. (15 points) Consider the difference equation $U^{n+4} = U^n$ with four starting values $U^0, U^1, U^2, U^3$. Use the roots of the characteristic polynomial to find a representation of the solution to this equation.

4. (10 points each) Consider the conjugate gradient method for the minimization of $\frac{1}{2}(Ax, x) - (b, x)$. ($A$ is a symmetric and positive definite matrix) in the form: Starting with $x_0 = 0, r_0 = b$ and $p_1 = r_0$, the successive approximations to the minimizer are computed by

$$x_k = x_{k-1} + \alpha_k p_k, \quad r_k = r_{k-1} - \alpha_k A p_k, \quad p_{k+1} = r_k + \beta_{k+1} p_k$$

   where $\alpha_k = (r_k, p_k)/\|p_k\|_A^2$ and $\beta_{k+1} = -(r_k, p_k)_A/\|p_k\|_A^2$.

   (a) Show that for $k = 0, 1, 2, \cdots$, the following relations are true:

$$\text{span}\{p_1, p_2, \cdots, p_{k+1}\} = \text{span}\{r_0, r_1, \cdots, r_k\} = \text{span}\{r_0, Ar_0, \cdots, A^k r_0\}.$$

   (b) Show that if $A \in \mathbf{R}^{n\times n}$ then for some $m \leq n$, $r_m = 0$ (assume that all operations are performed exactly).

5. (25 points) Suppose that an $n \times n$ matrix $A$ has eigenvalues $\lambda_1, \cdots, \lambda_n$ ordered by

$$|\lambda_1| > |\lambda_2| > |\lambda_3| \geq \cdots \geq |\lambda_n|$$

   with linearly independent eigenvectors $v^{(1)}, v^{(2)}, \cdots, v^{(n)}$.

   (a) Show how, and prove why, the Power method can be applied to an initial vectors $x^{(0)}$ given by

$$x^{(0)} = \beta_2 v^{(2)} + \beta_3 v^{(3)} + \cdots + \beta_n v^{(n)},$$

   to get a sequence that converges to $\lambda_2$.

   (b) Show that for any vector $x = \sum_{i=1}^{n} \beta_i v^{(i)}$, the vector $x^{(0)} = (A - \lambda_1 I)x$ satisfies the property given in part (a).

   (c) Show how one can continue this method to find $\lambda_3$ by using a suitably chosen $x^{(0)}$.